PROPERTIES OF C^1-SMOOTH FUNCTIONS WITH CONSTRAINTS ON THE GRADIENT RANGE

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Using Gromov method of convex integration, many mathematicians (J. M. Ball, S. Müller, V. Šverák, B. Kirchheim, M. A. Sychev, et al.; see [Müller98], [Kirchheim&Székelyhidi08], [Sychev06] for instance) studied the following problem: What conditions must a set K satisfy so that the differential relation $\nabla v \in K$ have nontrivial Lipschitz solutions? We study the similar problem for C^1-smooth (not only Lipschitz) solutions to the differential relations. In a particular, we prove that if $v : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^1-smooth function and the interior of the gradient range $\nabla v(\Omega)$ is empty, then this gradient range has the Lebesgue measure zero and it is a curve locally. Furthermore, this curve has tangents in a weak sense and the direction of these tangents is a function of bounded variation. We proved also that in this case the level sets of the gradient mapping $\nabla v : \Omega \to \mathbb{R}^2$ are straight lines. It implies that the graph of v should be a ruled developing surface.

As a corollary of our results, we prove that the gradient range of every C^1-smooth bump of two or three variables is the closure of its interior. (Recall that a C^1-smooth function is called a bump if its support is bounded and nonempty.)

Also we apply our results to geometry and to mappings with bounded distortion.

Our results give some information about analytical and geometrical properties of the gradient ranges of C^1-smooth functions of two variables. Geometrical properties of the gradient ranges for the case of differentiable (nonsmooth) functions were studied in [Maly96],[K00].

Also we include in this talk some generalizations of the above results to the multidimensional case, see [K09].

We will use the following notation.

Henceforth ∇v stands for the gradient $\nabla v = \left(\frac{\partial v_i}{\partial x_j}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in \mathbb{R}^{m\times n}$ of a mapping $v = (v_1,\dots,v_m) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$.

Throughout the sequel, $\operatorname{Int} E$ is the interior of a set E, $\operatorname{Cl} E$ is the closure of a set E, ∂E is the boundary of E, and $\operatorname{meas}(E)$ is the Lebesgue measure of E. By a domain we mean an open connected set. The inner product of vectors a and b is denoted by $a \cdot b$. By a curve we mean a continuous mapping $\gamma : \mathbb{R} \to \mathbb{R}^k$. If a mapping $\gamma : \mathbb{R} \to \mathbb{R}^k$ is continuous and injective, then it is also called an arc. Connectedness is understood in the sense of general topology.

1 On necessary and sufficient conditions for a curve to be the gradient range of a C^1-smooth function [K&Panov06], [K&Panov07]

1.1 Analytical formulation of the result.

Theorem 1.1.1. *Let* $\gamma : \mathbb{R} \ni u \mapsto (\gamma_1(u), \gamma_2(u)) \in \mathbb{R}^2$ *be a continuous injective function and* $v : \Omega \to \mathbb{R}$ *be a* C^1-*smooth function of a domain* $\Omega \subset \mathbb{R}^2$. *Suppose that the following inclusions are fulfilled:*

$$\nabla v(\Omega) \subset \gamma(\mathbb{R}), \tag{1}$$

$$\gamma(J) \subset \nabla v(\Omega), \tag{2}$$

where J *is a connected subset of* $\mathbb{R}$. *Then* γ *has the following property:*

(Γ_1) *for any point* $u_0 \in J$ *there exist a neighborhood* $V = V(u_0)$ *and a left continuous function* $l : V \to \mathbb{R}$ *of bounded variation such that after a linear transformation of coordinates of the plane* $\mathbb{R}^2$ *the following formula holds:*

$$\forall [u_1, u_2] \subset V \cap J \quad \gamma_2(u) \mid_{u_1}^{u_2} = \gamma_1(u) l(u) \mid_{u_1}^{u_2} - \int\limits_{u_1}^{u_2 - 0} \gamma_1(u)\, dl(u), \tag{3}$$

where we integrate in the sense of Lebesgue-Stieltjes, and we use the standard notation $f(s) \mid_{s_1}^{s_2} := f(s_2) - f(s_1)$.

Remark 1.1.2. (i) In the above Theorem, if γ is the graph of a function $\varphi : \mathbb{R} \to \mathbb{R}$, i.e., $\gamma(s) = (s, \varphi(s))$, where $\varphi \in C^2(\mathbb{R})$, then $l(s) \equiv \varphi'(s)$, and formula (3) turns to the usual formula of integration by parts.

(ii) If we fix the coordinate system, then we can rewrite the property (Γ_1) in the following equivalent form:

$(\tilde{\Gamma}_1)$ for every $u_0 \in J$ there exist a neighborhood $V = V(u_0)$, a left continuous function $l : V \to \mathbb{R}$ of bounded variation, and a unit vector $\bar{e} \in \mathbb{R}^2$ such that the following formula holds:

$$\forall [s_1, s_2] \subset V \cap J \;\; \bar{e} \cdot \gamma(s) \,|_{s_1}^{s_2} = \bar{e}^{\perp} \cdot \gamma(s) l(s) \,|_{s_1}^{s_2} - \int\limits_{s_1}^{s_2 - 0} \bar{e}^{\perp} \cdot \gamma(s) \, dl(s),$$

where we denote by $\bar{e}^{\perp}$ the unit vector that is orthogonal to $\bar{e}$.

Theorem 1.1.3 (the converse to Theorem 1.1.1). *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a continuous function. Suppose the function γ has the property (Γ_1) on a connected set $J \subset \mathbb{R}$. Then there exist a domain $\Omega \subset \mathbb{R}^2$ and a C^1-smooth function $v : \Omega \to \mathbb{R}$ such that the inclusions (1)-(2) are fulfilled. More precisely, there exists a function $u : \Omega \to \mathbb{R} \in C(\Omega)$ such that*

$$\nabla v(z) \equiv \gamma(u(z)) \quad \textit{for } z \in \Omega,$$

$$u(\Omega) = J.$$

Examples.

Corollary 1.1.4 [K08]. *There exist a continuous injective function* $\gamma : \mathbb{R} \to \mathbb{R}^2$ *and a* C^1*-smooth function* $v : \Omega \to \mathbb{R}$ *of a domain* $\Omega \subset \mathbb{R}^2$ *such that* $\nabla v(\Omega) = \gamma(\mathbb{R})$ *and the arc* $\gamma(\mathbb{R})$ *has no tangents at any points.*

Corollary 1.1.5 [K09]. *There exists a* C^1*-smooth function* $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma \neq \mathrm{const}$, *such that there is no* C^1*-smooth function* $v : \Omega \to \mathbb{R}$ *of a domain* $\Omega \subset \mathbb{R}^2$ *satisfying the conditions* $\nabla v(\Omega) \subset \gamma(\mathbb{R})$ *and* $\nabla v \neq \mathrm{const}$.

1.2 Existing and continuity of tangents.

We need some notions from [K&Panov07]. Denote by $\mathbb{R}P^1$ the set of straight lines of the plane $\mathbb{R}^2$ passed through a point 0, i.e., $\mathbb{R}P^1$ is a one-dimensional real projective space.

Definition 1.2.1. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a continuous function (a plane curve). We shall say that a straight line $p \in \mathbb{R}P^1$ is *right-hand σ-tangent* to the curve γ at a point $s_0 \in \mathbb{R}$ (denote $p = \gamma'_+(s_0)$), if for any sequence $s_\nu \to s_0 + 0$ such that

$$\sup_{\nu} \sup_{s \in [s_0, s_\nu]} \frac{|\gamma(s) - \gamma(s_0)|}{|\gamma(s_\nu) - \gamma(s_0)|} < \infty,$$

the convergence $\frac{\gamma(s_\nu) - \gamma(s_0)}{|\gamma(s_\nu) - \gamma(s_0)|} \to p$ is fulfilled (we understand the convergence in the natural sense).

In a similar way we define the left-hand σ-tangent $\gamma'_-(u_0)$ and the σ-tangent $\gamma'(u_0)$. Evidently, if a curve has a usual tangent at a point, then this tangent is the σ-tangent at the same point. But the converse is false, it follows from Corollary 1.1.4 and Theorem 1.2.2 below.

Theorem 1.2.2. *Suppose that the conditions of Theorem 1.1.1 are satisfied and denote* $a = \inf J$, $b = \sup J$. *Then* γ *has also the following property:*

(Γ_2) *for any point* $u_0 \in J$ *there exist a neighborhood* $V = V(u_0)$ *and a left continuous function* $l : V \to \mathbb{R}$ *of bounded variation such that after a linear transformation of coordinates*[1] *of the plane* $\mathbb{R}^2$ *the following equalities hold:*

$$\forall s \in V \cap J \setminus \{b\} \quad \gamma'_+(s) = (1, l(s+0)), \tag{4}$$

$$\forall s \in V \cap J \setminus \{a\} \quad \gamma'_-(s) = (1, l(s)), \tag{5}$$

i.e., these σ*-tangents exist and they are parallel to the vectors* $(1, l(s+0))$, $(1, l(s))$ *respectively*[2]. *Consequently,* $\gamma'_+(s)$ *is right-continuous at any point* $s \in J \setminus \{b\}$, *and* $\gamma'_-(s)$ *is left-continuous at any point* $s \in J \setminus \{a\}$, *and* $\gamma'_+(s) = \gamma'_-(s) = \gamma'(s)$ *for all* $s \in J \setminus E_\sigma$, *where the exceptional set* E_σ *is at most countable.*

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[1] Here the neighborhood V, the function $l : V \to \mathbb{R}$ and the transformation of coordinates are the same as in the property (Γ_1)

[2] We denote by $l(s+0)$ the limit on the right of the function l at the point s.

1.3 On conditions on a function φ, which are necessary and sufficient for existence of nontrivial C^1-smooth solutions for the PDE $\frac{\partial v}{\partial t} = \varphi(\frac{\partial v}{\partial x})$.

In this subsection we consider the case when

$$\gamma(u) \equiv (u, \varphi(u)), \tag{6}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function. In this case the previous inclusions $\gamma(J) \subset \nabla v(\Omega) \subset \gamma(\mathbb{R})$ are equivalent to the following relations:

$$v_t = \varphi(v_x) \ \text{ in } \Omega, \tag{7}$$

$$J \subset v_x(\Omega). \tag{8}$$

From Theorem 1.1.1, 1.1.3 we derive the following results.

Theorem 1.3.1. *Let $v : \Omega \to \mathbb{R}$ be a C^1-smooth function of a domain $\Omega \subset \mathbb{R}^2$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose $v_t = \varphi(v_x)$ in Ω and $J = v_x(\Omega)$. Then the function φ has the following property:*

(Γ_3) *there exists a measure-zero closed set $F \subset \operatorname{Cl} J$ such that φ satisfies Lipschitz condition locally in $U = J \setminus F$. Moreover, the function φ is differentiable on U except for at most countable set $E_{\sigma,U} \subset U$. Furthermore, if the derivative φ' is formally extended to the entire interval J according to the rule*

$$\varphi'(u) = \begin{cases} \varphi'(u), \ u \in U \setminus E_{\sigma,U}; \\ \lim\limits_{\tau \to u-0} \varphi'(\tau), \ u \in E_{\sigma,U}; \\ \infty, \ u \in J \cap F, \end{cases}$$

then the resulting function φ' has a locally bounded variation on U. Moreover, for any point $u_0 \in J$ there is a neighborhood $V = V(u_0)$ and a number $\alpha \in \mathbb{R}$ such that $\frac{1}{\varphi' - \alpha} : V \cap J \to \mathbb{R}$ is a function of bounded variation on $V \cap J$.

Theorem 1.3.2. *Let a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ have the property (Γ_3). Then there exists a C^1-smooth function $v : \Omega \to \mathbb{R}$ of a domain $\Omega \subset \mathbb{R}^2$ such that the relations $v_t = \varphi(v_x)$ in Ω and $J = v_x(\Omega)$ are fulfilled.*

2 Properties of C^1-smooth functions whose gradient range has no interior points [K07]

2.1 Main results

Theorem 2.1.1 *Let* $v : \Omega \to \mathbb{R}$ *be a* C^1*-smooth function of a domain* $\Omega \subset \mathbb{R}^2$. *Suppose*

$$\operatorname{Int} \nabla v(\Omega) = \emptyset. \tag{9}$$

Then for any point $z \in \Omega$ *with* $\operatorname{meas} \nabla v^{-1}(\nabla v(z)) = 0$ *there is a straight line* $L \ni z$ *such that* $\nabla v \equiv \mathrm{const}$ *on the connected component of the set* $L \cap \Omega$ *containing* z.

Corollary 2.1.2. *Let* $v : \Omega \to \mathbb{R}$ *be a* C^1*-smooth function of a domain* $\Omega \subset \mathbb{R}^2$. *Suppose the equality* (9) *is fulfilled. Then for any point* $z_0 \in \Omega$ *there exist an open connected neighborhood* Ω_0, *continuous functions* $u : \Omega_0 \to \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}^2$ *such that the function* γ *is nonconstant on any interval and*

$$\nabla v(z) \equiv \gamma(u(z)) \quad \textit{for } z \in \Omega_0. \tag{10}$$

The next Theorem follows from Corollary 2.1.2 and the results of the first section.

Theorem 2.1.3. *Suppose that the conditions of Corollary 2.1.2 are satisfied. Then the function* γ *has additionally the properties* (Γ_1) *and* (Γ_2) *with* $J = u(\Omega_0)$.

Corollary 2.1.4. *Let* $v : \Omega \to \mathbb{R}$ *be a* C^1*-smooth function on a domain (open connected set)* $\Omega \subset \mathbb{R}^2$. *Suppose the equality* $\operatorname{Int} \nabla v(\Omega) = \emptyset$ *is fulfilled. Then* $\operatorname{meas} \nabla v(\Omega) = 0$.

Finally we have the following result.

Theorem 2.1.5. *Let* $v : \Omega \to \mathbb{R}$ *be a* C^1*-smooth function of a domain* $\Omega \subset \mathbb{R}^2$*. Then the following assertions are equivalent:*

(i) $\operatorname{Int} \nabla v(\Omega) = \emptyset$*;*

(ii) $\operatorname{meas} \nabla v(\Omega) = 0$*;*

(iii) for any point $z \in \Omega$ *there exists a rectilinear segment* $I \ni z$ *(the point* z *is an interior point of* $I \subset \Omega$*) such that* $\nabla v \equiv \text{const}$ *on* I*.*

Remark 2.1.6. *In Theorem 2.1.5 the Hausdorff dimension of the set* $\nabla v(\Omega)$ *may be any number* $s \in [1, 2]$*.*

Remark 2.1.7. The implication (ii)⇒(iii) in Theorem 2.1.5 was proved in [Pogorelov56].

Example 2.1.8. For the case of functions of three and more variables the implication (iii)⇒(ii) is false. Namely there exists a C^1-smooth homogeneous function $v : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ such that $\operatorname{meas} \nabla v(\Omega) > 0$.

2.2 Applications to bumps

Theorem 2.2.1. *Let* $v : \mathbb{R}^n \to \mathbb{R}$ *be a* C^1*-smooth bump,* $n = 2, 3$. *Then the gradient range* $\nabla v(\mathbb{R}^n)$ *is regularly closed, i.e.,* $\nabla v(\mathbb{R}^n) = \operatorname{Cl}\operatorname{Int}\nabla v(\mathbb{R}^n)$.

Recall that a C^1-smooth function $v : \mathbb{R}^n \to \mathbb{R}$ is called *a bump* if its support, defined as the closure of the set $\{z \in \mathbb{R}^n \mid v(z) \neq 0\}$, is bounded and nonempty.

The plane case was proved also in [Kolar&Kristensen02],[Kolar&Kristensen05] under additional assumptions on modulus of continuity of ∇v. For C^2-smooth functions the assertion is true in any dimensions (see the classical paper [Hartman&Nirenberg59]).

2.3 Applications to Geometry

The next theorem is reformulation of the Theorem 2.1.5.

Theorem 2.3.1. *Let $S \subset \mathbb{R}^3$ be a C^1-smooth manifold. Then the following assertions are equivalent:*

(i) the spherical image of S has no interior points.

(ii) the spherical image of S has the area (the Lebesgue measure) zero.

(iii) S is a normal developing surface.

Recall that *the spherical image of a surface S* is the set $\{\mathbf{n}(x) \mid x \in S\}$, where $\mathbf{n}(x)$ is the unit normal vector to S at the point x. Recall also that a C^1-smooth manifold $S \subset \mathbb{R}^3$ is called *a normal developing surface* [Shefel′74] if for any $x_0 \in S$ there exists a straight segment $I \subset S$ (the point x_0 is an interior point of I) such that the tangent plane to S is stationary along I.

C^1-smooth surfaces satisfying the condition (ii) are the partial case of surfaces of *bounded extrinsic curvature* studied in [Pogorelov56]. It has been already mentioned that A.V. Pogorelov proved the implication (ii)⇒(iii). He proved also that the intrinsic metric of such surface is isometric to Euclidian metric.

3 Multidimensional case [K09], [K10]

In this section we extend the previous results to the case of C^1-smooth mappings $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that the gradient range is 1-dimensional (in a sense).

According to the classical result of Hartman and Nirenberg [Hartman&Nirenberg59], if the Hessian $D^2 v$ of some C^2-smooth mapping $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfies $\operatorname{rank} D^2(x) \equiv 1$, then the level sets of the gradient mapping $Dv : \Omega \to \mathbb{R}^n$ are hyperplanes. We prove analogous statements for C^1-smooth mappings $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$.

We will often use the following concept.

Definition 3.0.0. A set $E \subset \mathbb{R}^k$ is called $*$-*one-dimensional* whenever for every linear mapping $L : \mathbb{R}^k \to \mathbb{R}^2$ the image $L(E)$ has no interior points in the topology of $\mathbb{R}^2$; i.e., $\operatorname{Int} L(E) = \emptyset$.

In case $\mathbb{R}^k$ is the plane (i.e., $k = 2$), the $*$-one-dimensionality of a set E is equivalent to the property that the topological dimension of E is at most 1. Furthermore, for arbitrary dimensions k if $\mathcal{H}^2(E) = 0$ then E is $*$-one-dimensional. However, if $k > 2$ then in $\mathbb{R}^k$ there exist non-$*$-one-dimensional sets homeomorphic to an interval of $\mathbb{R}$. For instance, the graphs of Peano curves are of this type.

3.1 Properties of the level sets of the gradient mapping.

Theorem 3.1.1. *Take a C^1-smooth function $v : \Omega \to \mathbb{R}^m$ on a domain $\Omega \subset \mathbb{R}^n$. Suppose that $\nabla v(\Omega)$ is $*$-one-dimensional. Then for every point $x \in \Omega$ with* $\operatorname{meas} \nabla v^{-1}(\nabla v(x)) = 0$ *there is a hyperplane $H = H(x) \ni x$ such that* $\operatorname{comp}_x(H \cap \Omega) = \operatorname{comp}_x \nabla v^{-1}(\nabla v(x))$.

Here the symbol $\operatorname{comp}_z E$ stands for the connected component of E containing the point z.

In the particular case that v belongs to the C^2 smoothness class and $m = 1$ Theorem 3.1.1 was proved in [Hartman&Nirenberg59].

Corollary 3.1.2. *Take a C^1-smooth function $v : \Omega \to \mathbb{R}^m$ on a domain $\Omega \subset \mathbb{R}^n$. Suppose that $\nabla v(\Omega)$ is $*$-one-dimensional. Then for every point $x_0 \in \Omega$ there exist a connected open neighborhood Ω_0 and continuous functions $u : \Omega_0 \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}^{m\times n}$ such that γ is nonconstant on any interval and*

$$\nabla v(x) \equiv \gamma(u(x)) \quad \textit{for } x \in \Omega_0.$$

3.2 Necessary and sufficient conditions (in analytical form) for a curve to be a gradient range.

Given vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, denote by $a \otimes b$ the $m \times n$ matrix $(a_i b_j)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$.

Theorem 3.2.1. *Take a C^1-smooth function $v : \Omega \to \mathbb{R}^m$ on a domain $\Omega \subset \mathbb{R}^n$. Suppose that $\nabla v(\Omega)$ is $*$-one-dimensional. Take a subdomain Ω_0 of Ω and continuous functions $u : \Omega_0 \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}^{m\times n}$ satisfying the conclusions of Corollary* 3.1.2 *(i.e., γ is nonconstant on any interval and the identity $\nabla v(x) \equiv \gamma(u(x))$ holds). Then γ enjoys on the interval $J = u(\Omega_0)$ the following property:*

(MΓ_1) *there is a left continuous function $l = (l_1, \dots, l_n) : J \to S(0,1)$ of locally bounded variation such that for all $\bar{e} \in S(0,1)$ and $[s_1, s_2] \subset J$ if $0 \notin \mathrm{Cl}\{l(s)\cdot\bar{e} \mid s \in [s_1,s_2]\}$ then*

$$\gamma(s)\,|_{s_1}^{s_2} = [\gamma(s)\bar{e}] \otimes \frac{l(s)}{l(s)\cdot\bar{e}}\bigg|_{s_1}^{s_2} - \int_{s_1}^{s_2-0} [\gamma(s)\bar{e}] \otimes d\frac{l(s)}{l(s)\cdot\bar{e}},$$

where the integration is carried out in the sense of Lebesgue–Stieltjes over the half-open interval $[s_1, s_2)$, the standard notation $f(s)|_{s_1}^{s_2} := f(s_2) - f(s_1)$ is used, and $S(x,r)$ stands for the sphere in $\mathbb{R}^n$ of radius r centered at x.

Moreover, if $u(x) = s \in J$ and $\operatorname{meas} u^{-1}(s) = 0$ then the hyperplane $H(x)$ of Theorem 3.1.1 *is orthogonal to the vector $l(s)$.*

Example 3.2.2. Take a C^1-smooth function $\gamma(s) : \mathbb{R} \to \mathbb{R}^{n\times m}$ whose derivative is nonvanishing everywhere. Suppose that $\nabla v(\Omega) = \gamma(\mathbb{R})$ for some C^2-smooth function $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. It is well known that $\gamma'(s)$ must be a rank 1 matrix for all $s \in \mathbb{R}$; i.e., $\gamma'(s) = a \otimes b = (a_i b_j)$, where $a = a(s) \in \mathbb{R}^m$ and $b = b(s) \in \mathbb{R}^n$. We can always achieve (by switching to $\tilde{a}(s) = |b(s)|a(s)$ and $\tilde{b}(s) = b(s)/|b(s)|$) the equality $|b(s)| \equiv 1$. Then we can take $b(s)$ as the function $l(s)$ in Theorem 3.2.1.

Theorem 3.2.3 (the converse to Theorem 3.2.1). *Take a continuous function* $\gamma : \mathbb{R} \to \mathbb{R}^{m\times n}$ *nonconstant on any interval. Suppose that* γ *enjoys* $(\mathrm{M}\Gamma_1)$ *on a connected subset* $J \subset \mathbb{R}$*. Then there are a domain* $\Omega \subset \mathbb{R}^n$*, a* C^1*-smooth function* $v : \Omega \to \mathbb{R}^m$*, and a continuous function* $u : \Omega \to \mathbb{R}$ *such that*

$$\nabla v(x) \equiv \gamma(u(x)) \quad \textit{for } x \in \Omega,$$

$$u(\Omega) = J.$$

3.3 Existing and continuity of tangents.

Although the curves under consideration may lack classical tangents at all points (see previous examples), they have some weak analogs of tangents at every point.

Denote by $\mathbb{R}P^{n-1}$ the $(n-1)$-dimensional real projective space; i.e., $\mathbb{R}P^{n-1}$ is the set of lines in $\mathbb{R}^n$ passing through 0. Sometimes we naturally identify a line of $\mathbb{R}P^{n-1}$ with a nonzero vector of $\mathbb{R}^n$ parallel to this line.

Definition 3.3.1 Take a continuous mapping (curve) $\gamma : \mathbb{R} \to \mathbb{R}^{m\times n}$ nonconstant on any interval. Say that a line $p \in \mathbb{R}P^{n-1}$ is a *right σ-tangent* to γ at s_0 (and write $p = \gamma'_{\sigma+}(s_0)$) if for every sequence $s_\nu \to s_0 + 0$ satisfying

$$\sup_{\nu} \sup_{s\in[s_0,s_\nu]} \frac{|\gamma(s)-\gamma(s_0)|}{|\gamma(s_\nu)-\gamma(s_0)|} < \infty$$

there is a sequence of vectors $a_\nu \in \mathbb{R}^m$ such that

$$\frac{\gamma(s_\nu)-\gamma(s_0)}{|\gamma(s_\nu)-\gamma(s_0)|} - a_\nu \otimes l \to 0,$$

where $l \in S(0,1)$ is a vector parallel to p.

Similarly we introduce the *left σ-tangent* $\gamma'_{\sigma-}(s_0)$ at s_0 and simply *σ-tangent* $\gamma'_\sigma(s_0)$ at s_0. It is obvious that if γ has the usual tangent at the point and this tangent is a rank 1 matrix $a \otimes b$ then the curve also has the σ-tangent parallel to b. However, the converse is false.

Theorem 3.3.2. *Assume the hypotheses of Theorems* 3.1.1 *and* 3.2.1. *Put* $J = u(\Omega_0)$, $a = \inf J$, *and* $b = \sup J$. *Then aside from* $(\mathrm{M}\Gamma_1)$ *the function* γ *enjoys the following property:*

$(\mathrm{M}\Gamma_2)$ *there exists a left continuous function* $l = (l_1, \dots, l_n) : J \to S(0,1)$ *of locally bounded variation*[3] *such that*

$$\forall s \in J \setminus \{b\} \quad \gamma'_{\sigma+}(s) = l(s+0),$$
$$\forall s \in J \setminus \{a\} \quad \gamma'_{\sigma-}(s) = l(s);$$

i.e., the σ*-tangents exist and are parallel to the vectors* $l(s+0)$ *and* $l(s)$ *respectively. Therefore,* $\gamma'_{\sigma+}(s)$ *is a right continuous function at every point* $s \in J \setminus \{b\}$ *and* $\gamma'_{\sigma-}(s)$ *is a left continuous function at every point* $s \in J \setminus \{a\}$, *while* $\gamma'_{\sigma+}(s) = \gamma'_{\sigma-}(s) = \gamma'_{\sigma}(s)$ *for all points* $s \in (a,b) \setminus E_\sigma$, *where the exceptional set* E_σ *is at most countable.*

Moreover, $E_\sigma \subset E_u$, *where* $E_u = \{s \in J \mid \mathrm{meas}\, u^{-1}(s) > 0\}$.

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[3] Here l is the same function as in $(\mathrm{M}\Gamma_1)$.

3.4 Functional dependence of partial derivatives.

Theorem 3.4.1 *Take a continuous mapping* $\gamma : \mathbb{R} \ni s \mapsto (\gamma_{ij}(s)) \in \mathbb{R}^{m\times n}$ *with*

$$\gamma_{11}(s) \equiv s \tag{8}$$

and a C^1*-smooth function* $v : \Omega \to \mathbb{R}^m$ *on a domain* $\Omega \subset \mathbb{R}^n$*. Suppose that*

$$\forall i = 1, \ldots, m \ \forall j = 1, \ldots, n \quad \frac{\partial v_i}{\partial x_j} = \gamma_{ij}\left(\frac{\partial v_1}{\partial x_1}\right) \quad \text{in } \Omega. \tag{9}$$

Put

$$J = \frac{\partial v_1}{\partial x_1}(\Omega). \tag{10}$$

Then assertions of Theorems 3.1.1, 3.2.1 *and* 3.3.2 *hold; i.e.,* γ *enjoys* (MΓ_1) *and* (MΓ_2) *on* J *and the level sets of* ∇v *are hyperplanes.*

We stress that in Theorem 3.4.1 we do not require $\nabla v(\Omega)$ to be a $*$-one-dimensional set.

Functions γ_{i1} from the above theorem should be twice differentiable a.e. by Theorem 1.3.1, but it is not true for γ_{ij} with $j > 1$.

Example 3.4.2 Take an arbitrary continuous function $f : \mathbb{R} \to \mathbb{R}$. Then there are a continuous mapping $\gamma : \mathbb{R} \ni u \mapsto (\gamma_{ij}(u)) \in \mathbb{R}^{m\times n}$ with $\gamma_{11}(s) \equiv s$ and $\gamma_{21}(s) \equiv f(s)$, and a C^1-smooth function $v : \mathbb{R}^n \to \mathbb{R}^m$ satisfying (9) and (10) with $J = \mathbb{R}$ and $\Omega = \mathbb{R}^n$. We can determine $v = (v_1, \ldots, v_m) : \mathbb{R}^n \to \mathbb{R}^m$ as $v_1(x_1, \ldots, x_n) = \frac{(x_1)^2}{2}$ and $v_2(x_1, \ldots, x_n) = F(x_1)$, where $F : \mathbb{R} \to \mathbb{R}$ is the antiderivative of f, and $v_i = 0$ for $i > 2$. Clearly, $\gamma_{ij} = 0$ for $i > 2$ or $j > 1$.

3.5 Applications to mappings with bounded distortion

Theorem 3.5.1. *Let $K \subset \mathbb{R}^{2\times 2}$ be a compact set and the topological dimension of K equals 1. Suppose there exists $\lambda > 0$ such that $\forall A, B \in K \quad |A - B|^2 \le \lambda \det(A - B)$. Then for any Lipschitz mapping $v : \Omega \to \mathbb{R}^2$ on a domain $\Omega \subset \mathbb{R}^2$ such that $\nabla v(x) \in K$ a.e. the identity $\nabla v \equiv \mathrm{const}$ holds.*

Recall that a set K has the topological dimension at most 1 if for any $x \in K$ there exists a sequence of neighborhoods $U_\nu(x)$ such that $\operatorname{diam} U_\nu(x) \to 0$ and $K \cap \partial U_\nu(x)$ is a totally disconnected set, i.e., all the connected components of $K \cap \partial U_\nu(x)$ are singletons.

Many partial cases of Theorem 3.5.1 (for instance, when $K = SO(2)$ or K is a segment) are well-known (see, for example, [Müller98]).

In conditions of above Theorem 3.5.1 we can not change the uniform boundedness of the relations $\frac{|A-B|^2}{\det(A-B)}$ by their finiteness because of the following simple

Example 3.5.2. Consider the compact set

$$K = \left\{ \begin{pmatrix} s, & \frac{1}{2}s^2 \\ s^2, & \frac{2}{3}s^3 \end{pmatrix} \mid s \in [0,1] \right\}$$

and the mapping $v : \Omega \ni (x,y) \mapsto \left(-\frac{x^2}{2y}, \frac{x^3}{3y^2}\right) \in \mathbb{R}^2$, where $\Omega = \{(x,y) \mid -y < x < 0\}$. It is easy to see that K has topological dimension 1 (since K is a smooth arc), $\nabla v(x) \in K$ for all $x \in \Omega$, and $\det(A - B) > 0$ for any $A, B \in K$, $A \neq B$.

Theorem 3.5.3. *Let* $v : \Omega \to \mathbb{R}^2$ *be a* C^1*-smooth mapping on a domain* $\Omega \subset \mathbb{R}^2$. *Suppose* $\det(A-B) > 0$ *for any* $A, B \in \nabla v(\Omega)$, $A \neq B$. *Then assertions of Theorems* 3.1.1, 3.2.1 *and* 3.3.2 *hold; i.e., the level sets of* ∇v *are hyperplanes,* ∇v *is generated by a curve* $\gamma : \mathbb{R} \to \mathbb{R}^{2\times 2}$ *locally and* γ *enjoys* $(\mathrm{M}\Gamma_1)$ *and* $(\mathrm{M}\Gamma_2)$.

3.6 Some analogs of Sard's theorem.

Given a function $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, denote by Z_v the set $Z_v = \{x \in \Omega \mid \operatorname{rank} \nabla v(x) < m\}$ of critical points of v.

Theorem 3.6.1 *Take a C^1-smooth mapping $v : \Omega \to \mathbb{R}^m$ on a domain $\Omega \subset \mathbb{R}^n$ satisfying the hypotheses of Theorems* 3.1.1 *or* 3.4.1 (*i.e.,* $\nabla v(\Omega)$ *is a* $*$*-one-dimensional set or level sets of the gradient mapping* ∇v *are hyperplanes or all the partial derivatives of function* v *depend on* $\frac{\partial v_1}{\partial x_1}$ *). Then*

$$\operatorname{meas} v(Z_v) = 0. \tag{15}$$

Recall that in the classical Sard's theorem the validity of (15) for $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ requires C^r-smoothness, where $r = \max(0, n - m) + 1$.

In the dimensions $n = 2$ and $m = 1$ Theorem 3.6.1 was prtoved in [K06]. However, it came out later that for these values of the dimension Theorem 3.6.1 follows easily from the earlier results of [Pogorelov56].

Theorem 3.6.2 *Take a C^1-smooth mapping $v : \Omega \to \mathbb{R}^m$ on a domain $\Omega \subset \mathbb{R}^n$. Suppose that for any point $z \in \Omega$ there exists a hyperplane $H \ni z$ such that $\nabla v \equiv \mathrm{const}$ on $H \cap B(z, r)$ for some $r = r(z) > 0$. Then $\operatorname{meas} \nabla v(\Omega) = 0$.*

Remark 3.6.3. *In Theorem 3.6.2 the Hausdorff dimension of the set $\nabla v(\Omega)$ may be any number $s \in [1, nm]$.*

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Corollary 3.1.2. *Take a* C^1*-smooth function* $v : \Omega \to \mathbb{R}^m$ *on a domain* $\Omega \subset \mathbb{R}^n$*. Suppose that* $\nabla v(\Omega)$ *is* $*$*-one-dimensional. Then every point* $x_0 \in \Omega$ *has a convex open neighborhood* Ω_0 *such that for every point* $x \in \Omega_0$ *there is a hyperplane* $H = H(x) \ni x$ *satisfying the condition* $\nabla v \equiv \mathrm{const}$ *on* $H \cap \Omega_0$*.*