Specificity of dimension two in high conductivity problems

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    - A two-dimensional extension of the div-curl lemma
    - A two-dimensional H-convergence result
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In a bounded open set Ω of $\mathbb{R}^2$, we consider the conductivity problem

$$\begin{cases} -\operatorname{div}(A_n \nabla u_n) & = & f & \text{in } \Omega \\ u_n & = & 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{for } f \in H^{-1}(\Omega), \tag{1}$$

where A_n is a symmetric matrix-valued function satisfying the equi-coercivity assumption $A_n \geq \alpha\, I_2$, for $\alpha > 0$.

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where A_n is a symmetric matrix-valued function satisfying the equi-coercivity assumption $A_n \geq \alpha I_2$, for $\alpha > 0$. Problem (1) is associated with the energy F_n defined by

$$F_n(u) := \left\{\begin{array}{ll} \displaystyle\int_\Omega A_n \nabla u \cdot \nabla u \, dx & \text{if } u \in H_0^1(\Omega). \\ \infty & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega). \end{array}\right. \tag{2}$$

Aim

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- *Mosco (94) proved that the Γ-limits F of F_n satisfy the Beurling-Deny (58) representation formula: for $u \in C_c^1(\Omega)$,*

$$F(u) = \underbrace{\int_\Omega A(dx)\nabla u \cdot \nabla u}_{\textit{diffusion}} + \underbrace{\int_\Omega u^2 k(dx)}_{\textit{local term}} + \underbrace{\int_{\Omega^2\setminus\Delta} (u(x) - u(y))^2 j(dx, dy)}_{\textit{nonlocal term}}. \tag{3}$$

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- *Camar-Eddine & Seppecher (02) proved that any Dirichlet form (3) is the Γ-limit of a suitable sequence of diffusions F_n with A_n isotropic.*

Theorem (2d extension of the Murat-Tartar lemma (76))

Let $\alpha > 0$, let $\bar{a} \in L^\infty(\Omega)$ and let A_n be a sequence of symmetric matrix-valued functions in $L^\infty(\Omega)^{2\times 2}$ satisfying

$$A_n \geq \alpha\, I_2 \ \ \text{a.e. in}\ \Omega \quad \text{and} \quad |A_n| \rightharpoonup \bar{a} \ \ \text{in}\ \mathcal{M}(\Omega)*\,. \tag{4}$$

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Let $\xi_n \in L^2(\Omega)^2$ *and* $v_n \in H^1(\Omega)$ *satisfying*

$$\begin{cases} \displaystyle\int_\Omega A_n^{-1}\xi_n \cdot \xi_n\, dx + \|v_n\|_{H^1(\Omega)} \leq c, \\ \operatorname{div} \xi_n \ \text{is compact in } H^{-1}(\Omega). \end{cases} \tag{5}$$

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Then, there exist $\xi \in L^2(\Omega)^2$ and $v \in H^1(\Omega)$ such that

$$\left\{ \begin{array}{l} \xi_n \rightharpoonup \xi \ \text{ in } \mathcal{M}(\Omega)^2 * \quad \text{and} \quad \nabla v_n \rightharpoonup \nabla v \ \text{ in } L^2(\Omega)^2, \\ \xi_n \cdot \nabla v_n \rightharpoonup \xi \cdot \nabla v \quad \text{in } \mathcal{D}'(\Omega). \end{array} \right. \tag{6}$$

Sketch of the proof

Without loss of generality we can assume that Ω is a ball.
Consider $u_n := (-\Delta)^{-1}\big(\mathrm{div}\,(\xi_n)\,\big)$ which strongly converges to u in $H_0^1(\Omega)$, so that $\xi_n - \nabla u_n$ is divergence free.

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$$\xi_n - \nabla u_n = J\nabla\tilde{u}_n, \quad \text{where } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which weakly converges to $\tilde{u}$ in $L^2(\Omega)$, with $\xi = \nabla u + J\nabla\tilde{u}$.

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But the weak limit of $|A_n|$ in $\mathcal{M}(\Omega)*$ is a function, so is the limit of $|\nabla\tilde{u}_n|$. Thus, by the Lions concentration compactness $\tilde{u}_n$ strongly converges in $L^2(\Omega)$.

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Writing $\xi_n \cdot \nabla v_n = \nabla u_n \cdot \nabla v_n - \operatorname{div}\big(\tilde{u}_n\, J\nabla v_n\big)$, we get the desired convergence.

The Fenchenko-Khruslov fibers reinforcement

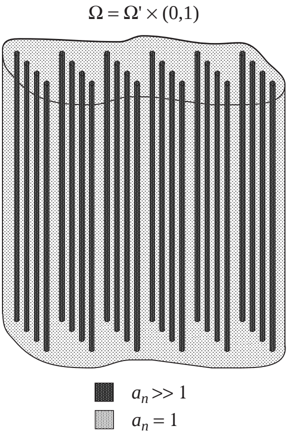

Figure: High-conductivity thin cylinders

Counter-example

Set $\Omega := \Omega' \times (0,1)$ and $Y := (-1,1)^2$. Let ω_n be a $\frac{1}{n}$-periodic lattice of vertical cylinders of radius $\frac{1}{n}e^{-n^2}$. Consider the conductivity a_n defined by

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Let u_n be the solution in $H^1_0(\Omega)$ of the equation

$$-\operatorname{div}(a_n \nabla u_n) = f \quad \text{in } \mathcal{D}'(\Omega), \quad \text{for } f \in L^2(\Omega).$$

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Let $\hat{V}_n$ be the Y-periodic function on $\mathbb{R}^2$ defined, for $R \in (0,1)$, by

$$\hat{V}_n(y) := \begin{cases} \dfrac{\ln r + n^2}{\ln R + n^2} & \text{if } r := \sqrt{y_1^2 + y_2^2} \in (e^{-n^2}, R) \\ 0 & \text{if } r \leq e^{-n^2} \text{ (region of high conductivity)} \\ 1 & \text{if } r \geq R. \end{cases}$$

The sequences $\xi_n := a_n \nabla u_n$ and $\hat{v}_n(x) := \hat{V}_n(nx)$ satisfy the assumptions of the $2d$ div-curl theorem, and $\hat{v}_n$ weakly converges to 1 in $H^1(\Omega)$.

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$$\xi_n \cdot \nabla \hat{v}_n \rightharpoonup 2\pi \, (u - v) \quad \text{in } \mathcal{D}'(\Omega),$$

where the limit u of u_n in $H^1_0(\Omega)$ and $v \in H^1_0\big(0,1;L^2(\Omega')\big)$ are solutions of the coupled system

$$\begin{cases} -\Delta u + 2\pi\,(u - v) & = f \quad \text{in } \Omega \\ -\dfrac{\partial^2 v}{\partial x_3^2} + v - u & = 0 \quad \text{in } \Omega. \end{cases}$$

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Therefore, convergence (6) does not hold true. Here, v reads as an integral in u, which yields a nonlocal term in the limit equation satisfied by u.

$M(\alpha, \beta; \Omega)$ denotes the set of the matrix-valued functions $A : \Omega \longrightarrow \mathbb{R}^{2\times 2}$ such that

$$\forall\, \xi \in \mathbb{R}^2, \quad A(x)\xi\cdot\xi \geq \alpha\, |\xi|^2 \quad \text{and} \quad A^{-1}(x)\xi\cdot\xi \geq \beta^{-1}\, |\xi|^2, \quad \text{a.e. } x \in \Omega.$$

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Definition

Let $\alpha>0$ and β_n be a positive sequence, with $\alpha\leq\beta_n$. A sequence A_n in $M(\alpha,\beta_n;\Omega)$ is said to H_2-converge to $A_*\in M(\alpha,\beta;\Omega)$, with $0<\alpha\leq\beta$, if for any $f\in H^{-1}(\Omega)$, the solution $u_n\in H^1_0(\Omega)$ of $-\operatorname{div}(A_n\nabla u_n)=f$ in $\mathcal{D}'(\Omega)$, satisfies

$$\begin{cases} u_n & \rightharpoonup & u & \text{weakly in } H^1_0(\Omega)\\ A_n\nabla u_n & \rightharpoonup & A_*\nabla u & \text{weakly in } \mathcal{M}(\Omega)^2*,\end{cases}$$

where u solves $-\operatorname{div}(A_*\nabla u)=f$ in $\mathcal{D}'(\Omega)$.

We extend in $2d$ the classical Murat-Tartar H-convergence (78) for high conductivities:

Theorem

Let Ω be regular open set of $\mathbb{R}^2$. Let $\alpha > 0$, let β_n be a sequence with $\beta_n \geq \alpha$, and let A_n be a sequence in $M(\alpha, \beta_n; \Omega)$. Assume that there exists $a \in L^\infty(\Omega)$ such that

$$\frac{\det A_n}{\det A_n^s}\,|A_n^s| \rightharpoonup a \quad \text{weakly in } \mathcal{M}(\bar{\Omega})*, \qquad A_n^s := \frac{1}{2}\left(A_n + A_n^T\right).$$

Then, there exists a subsequence of n, still denoted by n, and a matrix-valued function A_* in $M(\alpha, \beta; \Omega)$, with $\beta = 2\,\|a\|_{L^\infty(\Omega)}$, such that A_n H_2-converges to A_*.

Uniform convergence of optimal sequences

Theorem

Let Ω *be a regular bounded open set of* $\mathbb{R}^2$, *and let* A_n *be a sequence of symmetric matrices in* $M(\alpha, \beta_n; \Omega)$, *with* $0 < \alpha \leq \beta_n$.
Let $u \in H^1(\Omega) \cap C(\Omega)$ *and let* $\hat{u}_n$ *be a sequence in* $H^1(\Omega) \cap C(\Omega)$ *which strongly converges to* u *in* $L^2(\Omega)$ *and satisfies*

$$\exists \lim_{n \to \infty} \int_{\Omega} A_n \nabla \hat{u}_n \cdot \nabla \hat{u}_n \, dx < \infty.$$

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Then, up to a subsequence of n, still denoted by n, there exists $u_n \in H^1(\Omega)$ which satisfies

$$\limsup_{n\to\infty} \int_\Omega A_n \nabla u_n \cdot \nabla u_n \, dx \le \lim_{n\to\infty} \int_\Omega A_n \nabla \hat{u}_n \cdot \nabla \hat{u}_n \, dx,$$

and $\quad u_n \longrightarrow u \quad$ *strongly in* $L^\infty_{\mathrm{loc}}(\Omega)$.

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Remark

By density $\hat{u}_n$ *can be chosen continuous. Moreover, the continuity of* u *is not restrictive since we can prove that the continuous functions are dense in the domain of the* Γ*-limit of the energies.*

Sketch of the proof

Let $p \in (1,2)$. Since $\hat{u}_n$ weakly converges in $H^1(\Omega)$, it converges uniformly to u in a compact set K such that the p-capacity $C_p(\Omega \setminus K)$ is arbitrary small, where for $E \subset \mathbb{R}^2$,

$$C_p(E) := \inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^p : \ u \in W^{1,p}(\mathbb{R}^2) \cap L^{p^*}(\mathbb{R}^2), \ u \geq 1 \text{ around } E \right\}.$$

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We then use the following result:

Lemma

There exists $R_p > 0$ *such that for any continuous curve* L *of extremities* a, b, *we have* $C_p(L) \geq R_p \, |a-b|^{\frac{1}{2-p}}$.

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So, any connected subset O of $\Omega \setminus K$ satisfies

$$\mathrm{diam}\,(O) \leq c\,\big(C_p(O)\big)^{2-p} \ll 1.$$

Then, define u_n by: for any connected component O of $\Omega \setminus K$,

$$\begin{cases} u_n := \hat{u}_n \ \text{ in } K, \quad u_n - \hat{u}_n \in H_0^1(O), \\ \displaystyle\int_O A_n \nabla u_n \cdot \nabla u_n \, dx \leq \int_O A_n \nabla v \cdot \nabla v \, dx, \quad \forall\, v,\ v - \hat{u}_n \in H_0^1(O). \end{cases}$$

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This combined with the uniform convergence of $\hat{u}_n$ in K, the continuity of u and $\operatorname{diam}(O) \ll 1$, implies that

$$\|u_n - u\|_{L^\infty_{\rm loc}(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

Counter-example

Go back to the previous fibers structure with the conductivity a_n. Let $f \in L^2(\Omega)$ and let u_n be the solution in $H^1_0(\Omega)$ of the equation $-\operatorname{div}(a_n \nabla u_n) = f$ in $\mathcal{D}'(\Omega)$. We have

$$u_n \rightharpoonup u \text{ in } H^1_0(\Omega) \quad \text{and} \quad v_n := \frac{1_{\omega_n}}{\pi {r_n}^2}\, u_n \rightharpoonup v \text{ in } \mathcal{M}(\Omega)\,*,$$

where u and $v \in H^1_0\big((0,1); L^2(\Omega')\big)$ solve the system

$$\begin{cases} -\Delta u + 2\pi\,(u - v) & = f \quad \text{in } \Omega \\ -\dfrac{\partial^2 v}{\partial x_3^2} + v - u & = 0 \quad \text{in } \Omega. \end{cases}$$

Counter-example

Go back to the previous fibers structure with the conductivity a_n. Let $f \in L^2(\Omega)$ and let u_n be the solution in $H^1_0(\Omega)$ of the equation $-\operatorname{div}\left(a_n \nabla u_n\right) = f$ in $\mathcal{D}'(\Omega)$. We have

$$u_n \rightharpoonup u \text{ in } H^1_0(\Omega) \quad \text{and} \quad v_n := \frac{1_{\omega_n}}{\pi {r_n}^2}\, u_n \rightharpoonup v \text{ in } \mathcal{M}(\Omega)\,*,$$

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Note that u_n is optimal for the previous construction. Then, assume that u_n converges uniformly to u locally in Ω. Since u_n is continuous in Ω, so is its limit u. Hence, we obtain

$$\frac{1_{\omega_n}}{\pi {r_n}^2}\, u \rightharpoonup u \quad \text{in } \mathcal{M}(\Omega)\,*\,.$$

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We thus deduce that for any $\varphi \in C_c(\Omega)$ with support K,

$$\begin{aligned}\int_\Omega \varphi\,(v-u)\,dx &= \lim_{n\to\infty}\left[\frac{1}{\pi {r_n}^2}\int_{\omega_n}\varphi\,(u_n-u)\,dx\right]\\ &\le \lim_{n\to\infty}\left[\frac{|\omega_n|}{\pi {r_n}^2}\,\|\varphi\|_{L^\infty(K)}\,\|u_n-u\|_{L^\infty(K)}\right]=0,\end{aligned}$$

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Remark

In $3d$, thin cylinders may have both a small capacity and a fixed diameter, which makes possible nonlocal effects.

Compactness of equi-coercive diffusions by Γ-convergence

Theorem

Let A_n be a sequence of symmetric matrix-valued functions in Ω such that $A_n \geq \alpha\, I_2$, for $\alpha > 0$. Let F_n be the associated diffusion and let F be its Γ-limit for the strong topology of $L^2(\Omega)$. Assume that $C_c^1(\Omega)$ is contained in the domain $D(F)$ of F.

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Then, there exist $\mu \in \mathcal{M}(\Omega)$, $\mu \geq 0$, and $A \in L^\infty_\mu(\Omega)^{2\times 2}$ the regular part of which is α-coercive, such that

$$F(u) = \int_\Omega A\nabla u \cdot \nabla u \, d\mu, \quad \forall\, u \in C^1_c(\Omega). \tag{7}$$

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For any $u \in C^1_c(\Omega)$ *and any optimal sequence* u_n, *i.e. strongly converging to* u *in* $L^2(\Omega)$ *and such that* $F_n(u_n)$ *tends to* $F(u)$,

$$A_n\nabla u_n \cdot \nabla u_n \longrightarrow\ A\nabla u \cdot \nabla u\, d\mu \quad \text{in } \mathcal{M}(\Omega) * .$$

Sketch of the proof

By Γ-convergence combined with the previous result, there exists $w_n = (w_n^1, w_n^2) \in H_0^1(\Omega)^2$ such that w_n^i is a optimal sequence for F_n which converges uniformly to x_i in $\omega \Subset \Omega$.

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Defining μ and A by

$$\begin{cases} A_n\nabla w_n^1 \cdot \nabla w_n^1 + A_n\nabla w_n^2 \cdot \nabla w_n^2 \rightharpoonup \mu & \text{in } \mathcal{M}(\Omega)* \\ A_n\nabla w_n^i \cdot \nabla w_n^j \, dx \rightharpoonup A_{ij}\, d\mu & \text{in } \mathcal{M}(\Omega)*, \text{ for } i,j \in \{1,2\}, \end{cases}$$

we get that $F(u) = \lim_{n\to\infty} F_n\big(u(w_n)\big)$ has the desired form (7).

The periodic case

Set $Y := (0,1)^2$. Consider the case $A_n(x) = A^\sharp_n(nx)$, where $A^\sharp_n$ is a sequence of Y-periodic symmetric matrix-valued functions.

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$$A_n^*\lambda\cdot\lambda := \min\left\{\int_Y A_n^\sharp(\lambda+\nabla\varphi)\cdot(\lambda+\nabla\varphi)\,dy : \varphi\in H^1_\sharp(Y)\right\},\ \lambda\in\mathbb{R}^2.$$

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Theorem

The diffusion energy F_n associated with $A_n^\sharp(nx)$ Γ-converges to

$$F(u) = \int_\Omega A^*\nabla u\cdot\nabla u\,dx,\quad \textit{for } u \in H_0^1(\Omega),$$

where $$A^* = \lim_{n\to\infty} A_n^*.$$

A complete analysis for 2d periodic conduction problems

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- *If* $|A_n^*| \to \infty$, u_n *strongly converges to* 0 *in* $H_0^1(\Omega)$.

A div-curl result in dimension $d \geq 2$

Theorem (Casado-Díaz, Murat, B. (08))

Let Ω *be an open set of* $\mathbb{R}^d$, $d \geq 2$. *Consider* $v_n \in \mathcal{M}(\Omega)^d$ *and* $w_n \in L^d(\Omega)^d$ *such that*

$$\begin{cases} v_n \rightharpoonup v \ \ in\ \mathcal{M}(\Omega)^d *, & |v_n - v| \rightharpoonup \mu \ \ in\ \mathcal{M}(\Omega)* \\ w_n \rightharpoonup w \ \ in\ L^d(\Omega)^d, & |w_n - w|^d \rightharpoonup \nu \ \ in\ \mathcal{M}(\Omega)*, \end{cases}$$

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Then, there exist two sequences $x_k \in \Omega$ *and* $r_k \in \mathbb{R}^d$ *such that*

$$\begin{aligned} & v_n \cdot w_n \rightharpoonup v \cdot w + \sum_{k=1}^{\infty} \operatorname{div}\left(r_k\, \delta_{x_k}\right) \quad \text{in } \mathcal{D}'(\Omega), \\ & \text{with} \quad \forall\, k \geq 1, \quad |r_k| \leq c\, \mu(\{x_k\})\, \nu(\{x_k\})^{\frac{1}{d}}, \end{aligned} \tag{8}$$

where c *only depends on* d.

Back to the 2d div-curl lemma

Remark

- *The divergence term with the measures* δ_{x_k} *in* (8) *corresponds to the default of compactness in the div-curl result.*

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- *The two-dimensional div-curl result is a particular case of the previous one for* $d = 2$ *under the assumption*

$$\int_\Omega A_n^{-1} v_n \cdot v_n \, dx \leq c,$$

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where $|A_n|$ *weakly converges in* $\mathcal{M}(\Omega)*$ *to a function in* $L^1(\Omega)$. *Indeed, the limit* μ *of* $|v_n - v|$ *is also a function and does not load the points of* Ω, *so that the divergence term of* (8) *is equal to zero.*

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This combined with the estimate of r_k *implies that there is no measure term in* (8). *Thus, the div-curl convergence holds.*

Homogenization of unbounded monotone operators of d-Laplacian type

The previous div-curl result allows us to homogenize any sequence of monotone operators of the type

$$u \in W_0^{1,d}(\Omega) \longmapsto \mathrm{div}\big(a_n(x, \nabla u)\big),$$

where $a_n : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and any $\xi, \eta \in \mathbb{R}^d$,

$$\begin{cases} a_n(x,0) = 0, \\ M_n(x,\xi,\eta) := \big(a_n(x,\xi) - a_n(x,\eta)\big) \cdot (\xi - \eta) \geq \alpha\, |\xi - \eta|^d, \\ \big|a_n(x,\xi) - a_n(x,\eta)\big| \leq \beta_n(x)^{\frac{1}{d}}\, M_n(x,\xi,\eta)^{\frac{1}{2}} \big(a_n(x,\xi)\cdot\xi + a_n(x,\eta)\cdot\eta\big)^{\frac{d-2}{2d}}, \\ \text{where} \quad 1 \leq \beta_n \longrightarrow \beta \in L^\infty(\Omega) \quad \text{weakly in } \mathcal{M}(\Omega)*. \end{cases}$$

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The model example is $a_n(x,\xi) = \beta_n(x)\,|\xi|^{d-2}\xi$.

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This extends partially (for d-Laplacian type operators) the bounded case of Chiadò-Piat, Dal Maso, Defranceschi (90).

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The uniform convergence approach can be extended to homogenize any sequence of energies of the type

$$F_n(u) := \int_\Omega f_n(nx, \nabla u)\, dx, \quad \text{for } u \in W^{1,p}(\Omega),$$

where the energy density $f_n : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ satisfies

- $f_n(\cdot,\xi)$ is Y-periodic,
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Consider for a fixed n the homogenized density defined by

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Theorem (Braides, Casado-Díaz and B. (08))

Assume there exists $f^*:\mathbb{R}^2\to[0,\infty)$ *such that*

$$\forall\,\xi\in\mathbb{R}^2,\qquad \lim_{n\to\infty} f_n^*(\xi)=f^*(\xi)\le c\,(1+|\xi|^p)\,. \tag{9}$$

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Remark

Let F *be the* Γ*-limit of* F_n*, which exists up to a subsequence, and assume the existence of a continuous minimizer for each* $f_n^*(\xi)$*. Due to the convexity assumption, if there exists a non-zero function* $u\in W^{1,p}(\Omega)$ *with compact support in* Ω *satisfying* $F(u)<\infty$*, point-wise convergence* (9) *holds for a subsequence.*

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This talk was based on the following works:

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