A Critical Centre-Stable Manifold for the Focusing Cubic NLS in $\mathbb{R}^{3+1}$

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Introduction

The focusing cubic nonlinear Schrödinger equation in $\mathbb{R}^{3+1}$ is

$$i\partial_t \psi + \Delta\psi = -|\psi|^2\psi, \ \psi(0) \text{ given.}$$

It is $\dot{H}^{1/2}$ critical, with scaling given by

$$\psi(x,t) \mapsto \alpha\psi(\alpha x, \alpha^2 t).$$

Well-posedness

Most important estimates for the solution of the linear equation $i\partial_t\psi + \Delta\psi = 0$ on $\mathbb{R}^3$:

- $\|\psi(t)\|_2 = \|\psi(0)\|_2$ (unitarity of the evolution)
- $\|\psi(t)\|_\infty \leq Ct^{-3/2}\|\psi(0)\|_1$ ($L^1 \to L^\infty$ decay)
- $\|\chi(x)D^{1/2}\psi\|_{L^2_{t,x}} \leq C\|\psi(0)\|_2$ (local smoothing)
- $\|\psi\|_{L^2_t L^6_x} \leq C\|\psi(0)\|_2$ (endpoint Strichartz estimates).

Small $\dot{H}^{1/2}$ data $\implies$ global dispersive solutions.
Large $\dot{H}^{1/2}$ data $\implies$ local well-posedness in the critical sense; the time of existence depends on the profile, not only on the norm (Cazenave–Weissler 1988).
Short proof: consider an interval $[0, T]$ on which $\|\psi\|_{L^2_{[0,T]}L^6_x}$ is sufficiently small; apply a fixed point argument.

Finite-time blowup

Conserved quantities:

$$M[\psi(t)] = \int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx \text{ (mass)},$$
$$E[\psi(t)] = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\psi(x,t)|^2 - \frac{1}{4}|\psi(x,t)|^4 \, dx \text{ (energy)},$$
$$P[\psi(t)] = \mathrm{Re} \int_{\mathbb{R}^3} \psi \; i\nabla\overline{\psi} \, dx \text{ (momentum)}.$$

Finite-time blowup can occur (for $E < 0$; Glassey 1977). Virial identity:

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x|^2 |\psi(x,t)|^2 \, dx = E[\psi(t)] - \int_{\mathbb{R}^3} |\psi(x,t)|^4 \, dx \leq C < 0.$$

Solitons

The equation also admits soliton-type solutions:

$$\psi(x,t) = e^{it\alpha^2}\phi(x,\alpha),$$

where

$$-\Delta\phi + \alpha^2\phi = \phi^3.$$

Ground states are solitons characterized by $\phi \geq 0$. They are unique up to symmetries (Coffman '72, Kwong '89), strictly positive, radially symmetric, smooth, and exp. decaying (Berestycki–Lions '80s); the Derrick–Pohozaev identity holds:

$$\phi(x,\alpha) = \alpha\phi(\alpha x, 1).$$

The equation is invariant under the eight-dimensional Lie group of symmetry transformations generated by:

1. Galilean coordinate changes:
   $f(x,t) \mapsto e^{i(vx - t|v|^2)} f(x - 2tv - D, t)$.
2. Complex phase change: $f(x,t) \mapsto e^{i\gamma} f(x,t)$.
3. Scaling: $f(x,t) \mapsto \alpha f(\alpha x, \alpha^2 t)$.

Applying these to the ground state, we obtain an eight-parameter family of solutions, also called solitons.

Stability of solitons

Are solitons stable under small perturbations?
Berestycki–Lions and Berestycki–Cazenave showed in the '80s that blow-up can occur for arbitrarily small perturbations of ground state solitons.
The linearized Hamiltonian evolution has two kinds of instabilities:

- linear, corresponding to the soliton manifold itself
- exponential, corresponding to two exponentially unstable modes.

The first kind is recoverable, the second one is not.

Energy trapping

Below the level of the ground state ϕ, energy and mass determine the behavior of the solution by energy trapping: Kenig–Merle $\dot H^1$, Duyckaerts–Holmer–Roudenko $\dot H^{1/2}$,

$$M[\psi]E[\psi] < M[\phi]E[\phi] \implies \begin{cases} \text{either} & M[\psi]\|\nabla\psi\|_2^2 < M[\phi]E[\phi] \\ \text{or} & M[\psi]\|\nabla\psi\|_2^2 > M[\phi]E[\phi]. \end{cases}$$

Two cases ensue — plus three more if we allow for equality.

1. ψ disperses to zero, has finite Strichartz norm
2. ψ blows up in finite time, both as $t \to \infty$ and as $t \to -\infty$

3'. ψ is the soliton, $\psi = \phi$

4'. ψ converges exponentially fast to the soliton on one side, disperses on the other

5'. ψ converges exponentially fast to the soliton on one side, blows up in finite time on the other.

However, these quantities are only defined for H^1 solutions.

Main result

There exists a $\dot{H}^{1/2}$ codimension-one *centre-stable* real-analytic manifold $\mathcal{N}$ for the equation, locally near the soliton manifold.
A solution that starts on $\mathcal{N}$

1. exists globally for positive time
2. stays on the manifold for infinite positive time and finite negative time
3. is asymptotically stable: decomposes into a moving soliton (that converges to a final state as $t \to \infty$) and an error term that scatters like the solution of the free Schrödinger equation.

The solution depends analytically on the initial data.
This result is new for the scaling-invariant norm.
Another codimension-one stable manifold can be constructed for $t \to -\infty$. Their intersection is a codimension-two manifold of solutions stable both at $+\infty$ and at $-\infty$.

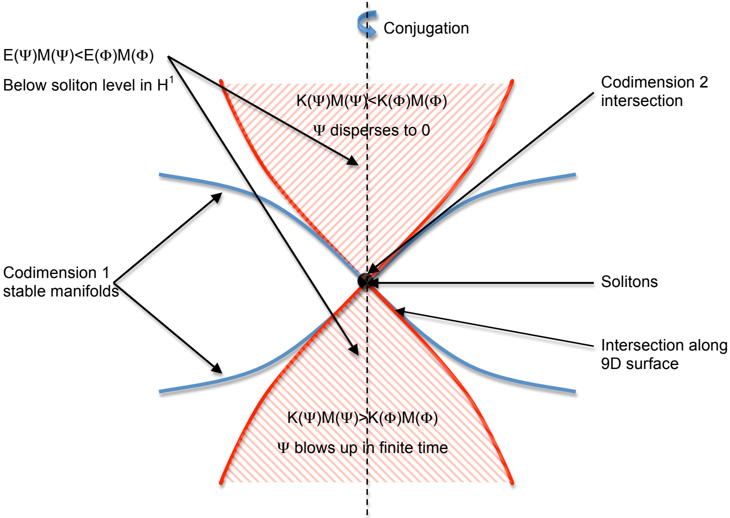

Figure: Solutions to the 3D cubic NLS near the soliton

Brief history

1989 Bates–Jones proved that the space of solutions decomposes into an unstable and a centre-stable manifold, for a large class of semilinear equations.

2000 Gesztesy–Jones–Latushkin–Stanislavova proved that the result of Bates–Jones applies to the semilinear Schrödinger equation.

2004 Schlag proved that, for initial data on a codimension-one submanifold of $W^{1,1} \cap W^{1,2}$, the solution exists globally in time and decomposes into a soliton and a dispersive term.
The space $W^{1,1} \cap W^{1,2}$ is not invariant. To ask the question of invariance one needs to weaken the norm to $\Sigma^{5/2+\epsilon}$.

2007 Global existence and centre-stable manifold in $\Sigma = \langle x \rangle^{-3/4-\epsilon} L^2 \cap H^{3/4+\epsilon}$.

2008 Current result.

Description of the manifold

The manifold is the graph of an analytic function defined on a small ball in a codimension-nine linear subspace of $\dot{H}^{1/2}$, to which we then apply the symmetry transformations, recovering eight codimensions.
For some soliton ϕ and Schwartz functions $f_k(\phi)$, $1 \leq k \leq 9$,

$$\mathcal{N}_0(\phi) = \{r_0 \in \dot{H}^{1/2} \mid \langle r_0, f_k(\phi)\rangle = 0, \|r_0\|_{\dot{H}^{1/2}} < \epsilon\}$$

is a small ball in a codimension-nine linear subspace of $\dot{H}^{1/2}$. To this we add the constant ϕ and a quadratic correction, obtaining

$$\mathcal{N}(\phi) = \{\psi \in \dot{H}^{1/2} \mid \psi = \phi + r_0 + h(r_0)f^+,\ r_0 \in \mathcal{N}_0(\phi)\}.$$

$h(r_0)$ is real-analytic, meaning it has a Taylor series expansion.
The full manifold is given by

$$\mathcal{N} = \bigcup_{\phi \text{ soliton}} \mathcal{N}(\phi).$$

Sketch of the proof

The given equation is

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi = 0.$$

We seek solutions $\psi = w(\pi(t)) + r$, where $w(\pi(t))$ is a moving soliton whose movement is described by the modulation path π:

$$\pi(t) = (\alpha(t), \Gamma(t), v_k(t), D_k(t)),$$
$$w(\pi(t))(x) = e^{i\theta(t,x)}\phi(x - y(t), \alpha(t))$$
$$\theta(t,x) = v(t)x - \int_0^t (|v(s)|^2 - \alpha^2(s))\, ds + \Gamma(t)$$
$$y(t) = 2\int_0^t v(s)\, ds + D(t).$$

We obtain yet another nonlinear equation in r, which we then linearize. Imposing orthogonality conditions that give rise to modulation equations, we arrive at the system

$$i\partial_t R + \mathcal{H}_{\pi^0}(t) R = F,\ \ F = -iL_{\pi^0} R + N(R^0, W_{\pi^0}) - N_{\pi^0}(R^0, W_{\pi^0})$$
$$\dot f = 4\alpha^0 \|W_{\pi^0}\|_2^{-2} (\langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle), f \in \{\alpha, \Gamma\}$$
$$\dot f = 2\|W_{\pi^0}\|_2^{-2} (\langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle - i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle), f \in \{v_k, D_k\}.$$

Here $R = \begin{pmatrix} r \\ \overline{r} \end{pmatrix}$, $W_\pi = \begin{pmatrix} w_\pi \\ \overline{w_\pi} \end{pmatrix}$ etc. and

$$\mathcal{H}_{\pi^0}(t) = \begin{pmatrix} \Delta + 2|w_{\pi^0}(t)|^2 & (w_{\pi^0}(t))^2 \\ -(\overline{w}_{\pi^0}(t))^2 & -\Delta - 2|w_{\pi^0}(t)|^2 \end{pmatrix},$$
$$N(R^0, W_{\pi^0}) = \begin{pmatrix} -|r^0|^2 r^0 - (r^0)^2 \overline{w}_{\pi^0} - 2|r^0|^2 w_{\pi^0} \\ |r^0|^2 \overline{r^0} + (\overline{r^0})^2 w_{\pi^0} + 2|r^0|^2 \overline{w}_{\pi^0} \end{pmatrix},$$
$$\begin{aligned} L_{\pi^0} R &= 4\alpha^0 \sum_{f \in \alpha, \Gamma} \|W_{\pi^0}\|_2^{-2} \langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle \partial_f W_{\pi^0} \\ &\quad + 2 \sum_{f \in \{v_k, D_k\}} \|W_{\pi^0}\|_2^{-2} \langle R, (d_\pi \Xi_f(W_{\pi^0}))\dot\pi^0\rangle \partial_f W_{\pi^0}, \\ N_{\pi^0}(R^0, W_{\pi^0}) &= 4\alpha^0 \sum_{f \in \{\alpha, \Gamma\}} \|W_{\pi^0}\|_2^{-2} i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle \partial_f W_{\pi^0} \\ &\quad + 2 \sum_{f \in \{v_k, D_k\}} \|W_{\pi^0}\|_2^{-2} i\langle N(R^0, W_{\pi^0}), \Xi_f(W_{\pi^0})\rangle \partial_f W_{\pi^0}. \end{aligned}$$

Contraction argument

We solve the linearized equation first and then use a contraction argument to obtain a solution of the nonlinear equation.
We show that a sufficiently small neighborhood of zero in

$$X = \{(R, \pi) \mid R \in L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6}_x,\ \dot{\pi} \in L^1\}$$

is stable under the solution map. Then we prove that, for solutions of small X norm, the solution map acts as a contraction in

$$Y = \{(Z, \pi) \mid e^{-t\rho} Z(t) \in L^\infty_t \dot{H}^{1/2}_x \cap L^2_t \dot{W}^{1/2,6}_x,\ e^{-t\rho} \dot{\pi}(t) \in L^1_t\},$$

for some small $\rho > 0$.
For analyticity, we show that each derivative is well-defined in a polynomially weighted space in time and that these weights are uniformly bounded by an exponential.

The potential is given as a function of the soliton w, which has an oscillation and a translation movement. We perform a transformation that eliminates both:

$$r(x,t) = \mathfrak{g}(t) z(x,t) = e^{i \int_0^t (\alpha^2(s) - v^2(s))\, ds} z\Big(x - 2\int_0^t v(s)\, ds\Big),$$

$$w(x,t) = \mathfrak{g}(t) w_z(x,t),$$

$$w_z(x,t) = e^{i(v(t)\cdot x + \Gamma(t))} \phi(x - D(t), \alpha(t)).$$

Importantly, note that $\mathfrak{g}_z$ is not a symmetry transformation. The natural symmetry transformation to consider is the one that takes $w(\pi(t))$ to $e^{it}\phi(x,1)$. However, it gives rise to commutation terms that we cannot control, e.g.

$$\alpha(x\nabla)r.$$

$\mathfrak{g}_z$ is the closest equivalent we can afford, keeping the desirable features with less problematic commutator terms.

Spectrum of the Hamiltonian

The linearized Hamiltonian is, up to conjugation by a time-dependent unitary operator,

$$\mathcal{H} = \begin{pmatrix} \Delta - 1 + 2\phi^2 & \phi^2 \\ -\phi^2 & -\Delta + 1 - 2\phi^2 \end{pmatrix}.$$

$\sigma(\mathcal{H})$ consists of three components:

1. the a.c. part $(-\infty, -1] \cup [1, \infty)$
2. one pair of imaginary eigenvalues $\pm i\sigma$
3. the generalized eigenspace at zero ($\mathrm{Ker}(\mathcal{H}) \neq \mathrm{Ker}(\mathcal{H}^2)$).

This description depends on ingredients that have been proved numerically by Demanet–Schlag and Marzuola–Simpson.

Let Z be a solution of the linearized equation, subject to this transformation. We decompose it into Riesz projections on the three components of the spectrum:

$$Z = P_c Z + P_0 Z + P_{im} Z.$$

We need to control each component in the Strichartz norm, with half a derivative.

1. In the zero eigenspace, $P_0 Z$ can grow polynomially in t. The method of modulation (Soffer–Weinstein) fixes that.
2. The imaginary eigenspace component $P_{im} Z$ is exponentially unstable. This requires passing to a stable submanifold (Schlag).
3. Concerning the a.c. spectrum projection, we prove Strichartz estimates, for a time-dependent potential (this is the main new ingredient of the proof).

The imaginary component

The potential instability due to $P_{im}Z$ is dealt with as follows:

Lemma

Consider the equation $\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} - \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$, *where* $f \in L^1 \cap L^\infty$*. Then* x *is bounded on* $[0, \infty)$ *if and only if*

$$x_1(0) = -\int_0^\infty e^{-t\sigma} f_1(t)\, dt.$$

This leads to a unique choice of $x_1(0)$ that makes x bounded.
Since x here stands for $P_{im}Z$, this is our codimension-one submanifold condition.
We then get exponential decay for $P_{im}Z$.

The zero eigenspace component

To bound the zero eigenspace component, we impose the orthogonality condition

$$P_0(t)(Z(t)) = 0 \iff \langle Z(t), \sigma_3 \partial_F W(t) \rangle = 0,$$

where $\partial_F W(t)$, for F running over the parameters of W, span the tangent space at W to the soliton manifold. This leads to a system of eight *modulation equations* for the path π of the form

$$\dot{\pi} = L_{\pi^0}(\pi^0, Z) + N_{\pi^0}(\pi^0, Z^0).$$

At $t = 0$, imposing this condition leads to the loss of eight codimensions.

Strichartz estimates for the a.c. part

Rather than proving Strichartz estimates for constant $\mathcal{H}$, we have to do so for a time-dependent equation:

$$i\partial_t Z - iv(t)\nabla Z + \alpha(t)\sigma_3 Z + \mathcal{H}Z = F,\ Z(0) \text{ given.}$$

Here

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \begin{pmatrix} W_1 & W_2 \\ -\overline{W}_2 & -W_1 \end{pmatrix}.$$

Treating $iv(t)\nabla Z - \alpha(t)\sigma_3 Z$ as a perturbation and moving this term to the right-hand side leads to the requirement

$$\langle t \rangle \dot\alpha(t) \in L^1_t,$$

which is too strong ($v\nabla$ is worse, hopeless by this method). We only have

$$\dot\alpha, \dot v \in L^1_t.$$

Strichartz estimates in this regime are the essence of the proof.

History

Goldberg–Schlag: $t^{-3/2}$ decay estimates for the high energy part.
Erdogan–Schlag: L^2 boundedness of the evolution, an analysis in the presence of threshold resonances or eigenvalues.
Schlag: $t^{-3/2}$ decay estimates, nonendpoint Strichartz estimates.
B.: Endpoint Strichartz estimates.
Cuccagna–Mizumatchi: All the above follow from wave operator estimates.
Cuccagna obtained, using wave operators, a "manifold" for the $1D$ NLS, for even initial data.
The current problem: Cuccagna's approach also works (it was my initial approach as well), but only for even initial data.

Time-dependent Strichartz estimate

Theorem

Consider the equation

$$i\partial_t Z - iv(t)\nabla Z + A(t)\sigma_3 Z + \mathcal{H} Z = F,\ Z(0) \text{ given},$$

$$\mathcal{H} = \mathcal{H}_0 + V,\ \mathcal{H}_0 = \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix},\ V = \begin{pmatrix} W_1 & W_2 \\ -W_2 & -W_1 \end{pmatrix}.$$

Assume that V is sufficiently smooth and decaying, $\|A\|_\infty$ and $\|v\|_\infty$ are small, and $\mathcal{H}$ has no embedded eigenvalues or resonances at the thresholds. Then

$$\|P_c Z\|_{L^\infty_t \dot H^{1/2}_x \cap L^2_t \dot W^{1/2,6}_x} \leq C\Big(\|Z(0)\|_{\dot H^{1/2}} + \|F\|_{L^1_t \dot H^{1/2}_x + L^2_t \dot W^{1/2,6/5}_x}\Big).$$

Here P_c is the projection on the continuous spectrum of $\mathcal{H}$.

Concluding the proof

We obtain global asymptotically stable solutions for initial data on a codim-9 submanifold of $\dot{H}^{1/2}$. One imaginary eigenvalue accounts for one codimension; the orthogonality condition accounts for the other 8. Letting the symmetries of the equation act (transversally) on the codim-9 manifold, we recover 8 codimensions, obtaining a codim-1 manifold $\mathcal{N}$. Points on the manifold are characterized by two properties:

1. being close in norm to solitons
2. *the existence of a global asymptotically stable solution having them as initial data*.

Both properties are preserved by the nonlinear evolution. Therefore, $\mathcal{N}$ is invariant under the nonlinear evolution.

Thank you for your attention!