Toward self–similar solutions for the Navier–Stokes equations

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local Leray solutions fot the Navier–Stokes equations I

The Cauchy problem

Find $\vec{u}(t,x)$ $(t \in (0, T),\ x \in \mathbb{R}^3)$ such that

$$\partial_t \vec{u} + \vec{u}.\vec{\nabla}\vec{u} = \Delta\vec{u} - \vec{\nabla} p$$

$$\mathrm{div}\ \vec{u} = 0$$

$$\vec{u}(0,x) = \vec{u}_0$$

local Leray solutions fot the Navier–Stokes equations II

Helmholtz decomposition :

If $\vec{u} \in (L^2)^3$, then $\vec{u} = \vec{v} + \vec{w}$ with $\mathrm{div}\ \vec{v} = 0$ and $\vec{\nabla} \wedge \vec{w} = 0$.

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$\mathrm{div}\ \vec{u} = \mathrm{div}\ \vec{w}$ $\qquad\qquad$ $\vec{\nabla} \wedge \vec{w} = 0 \Rightarrow \vec{w} = \vec{\nabla} q$ with $q \in \dot{H}^1$

local Leray solutions fot the Navier–Stokes equations II

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$\mathrm{div}\ \vec{u} = \mathrm{div}\ \vec{w}$ $\qquad$ $\vec{\nabla} \wedge \vec{w} = 0 \Rightarrow \vec{w} = \vec{\nabla} q$ with $q \in \dot{H}^1$

Leray's projection operator

- Helmholtz decomposition $\vec{u} = \mathbb{P}\vec{u} + \vec{\nabla} q$ with $\mathrm{div}\ \mathbb{P}\vec{u} = 0$
- $\Delta q = \mathrm{div}\ \vec{u}$
- $\mathbb{P}\vec{u} = \vec{u} - \vec{\nabla} \frac{1}{\Delta} \mathrm{div}\ \vec{u}$

local Leray solutions fot the Navier–Stokes equations II

Helmholtz decomposition :

If $\vec{u} \in (L^2)^3$, then $\vec{u} = \vec{v} + \vec{w}$ with $\mathrm{div}\ \vec{v} = 0$ and $\vec{\nabla} \wedge \vec{w} = 0$.

$\mathrm{div}\ \vec{u} = \mathrm{div}\ \vec{w}$ $\qquad$ $\vec{\nabla} \wedge \vec{w} = 0 \Rightarrow \vec{w} = \vec{\nabla} q$ with $q \in \dot{H}^1$

Leray's projection operator

- Helmholtz decomposition $\vec{u} = \mathbb{P}\vec{u} + \vec{\nabla} q$ with $\mathrm{div}\ \mathbb{P}\vec{u} = 0$
- $\Delta q = \mathrm{div}\ \vec{u}$
- $\mathbb{P}\vec{u} = \vec{u} - \vec{\nabla}\frac{1}{\Delta}\mathrm{div}\ \vec{u}$

Rewriting the Cauchy problem

The Cauchy problem may be rewritten as : to find $\vec{u}$ such that

$$\partial_t \vec{u} + \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) = \Delta \vec{u}$$
$$\vec{u}(0, x) = \vec{u}_0 \text{ with } \mathrm{div}\ \vec{u}_0 = 0$$

local Leray solutions fot the Navier–Stokes equations III

Leray solutions

Let $\vec{u}_0 \in L^2$ with $\mathrm{div}\ \vec{u}_0 = 0$. $\vec{u}(t,x)$ ($t \in (0,+\infty)$, $x \in \mathbb{R}^3$) is a Leray solution if it satisfies Leray's energy inequality :

$\vec{u} \in L^\infty_t L^2_x$ and $\vec{\nabla} \otimes \vec{u} \in L^2_t L^2_x$

$\partial_t \vec{u} + \mathbb{P}(\vec{u}.\vec{\nabla}\vec{u}) = \Delta \vec{u}$

$\vec{u}(0,x) = \vec{u}_0$

$\int_{\mathbb{R}^3} |\vec{u}(t,x)|^2\ dx \leq \int_{\mathbb{R}^3} |\vec{u}_0(t,x)|^2\ dx - 2\int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes \vec{u}(s,x)|^2\ dx\ ds$

local Leray solutions fot the Navier–Stokes equations III

Leray solutions

Let $\vec{u}_0 \in L^2$ with $\operatorname{div} \vec{u}_0 = 0$. $\vec{u}(t,x)$ ($t \in (0,+\infty)$, $x \in \mathbb{R}^3$) is a Leray solution if it satisfies Leray's energy inequality :

$\vec{u} \in L^\infty_t L^2_x$ and $\vec{\nabla} \otimes \vec{u} \in L^2_t L^2_x$

$\partial_t \vec{u} + \mathbb{P}(\vec{u}.\vec{\nabla}\vec{u}) = \Delta \vec{u}$

$\vec{u}(0,x) = \vec{u}_0$

$\int_{\mathbb{R}^3} |\vec{u}(t,x)|^2 \, dx \le \int_{\mathbb{R}^3} |\vec{u}_0(t,x)|^2 \, dx - 2 \int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes \vec{u}(s,x)|^2 \, dx \, ds$

- ◇ Leray solution exists (Leray 1934)
- ◇ the set of Leray solutions is compact
  - the family $(\vec{u}_n)_{n\in\mathbb{N}}$ is (locally in t and x) uniformly bounded in $L^2 H^1$
  - the family $(\partial_t \vec{u}_n)_{n\in\mathbb{N}}$ is (locally in t and x) uniformly bounded in $L^2 H^{-2}$

    $\Rightarrow$ we can find a sequence $n_k \to +\infty$ such that $\vec{u}_{n_k}$ is (locally) convergent strongly in $L^2 L^2$ and weakly in $L^2 H^1$

local Leray solutions fot the Navier–Stokes equations IV

local Leray solutions

Let $\vec{u}_0 \in L^2_{uloc}$ with $\operatorname{div} \vec{u}_0 = 0$. $\vec{u}(t,x)$ ($t \in (0,T)$, $x \in \mathbb{R}^3$) is a local Leray solution if it satisfies Scheffer's local energy inequality :

$\vec{u} \in L^\infty_t L^2_x$ locally and $\vec{\nabla} \otimes \vec{u} \in L^2_t L^2_x$ locally

$\partial_t \vec{u} + \mathbb{P}(\vec{u}.\vec{\nabla}\vec{u}) = \Delta \vec{u}$

$\vec{u}(0,x) = \vec{u}_0$

$\partial_t |\vec{u}|^2 = \Delta |\vec{u}|^2 - 2|\vec{\nabla} \otimes \vec{u}|^2 - \vec{\nabla}.((|\vec{u}|^2 + 2p)\vec{u}) - \mu \qquad (\mu \geq 0)$

local Leray solutions fot the Navier–Stokes equations IV

local Leray solutions

Let $\vec{u}_0 \in L^2_{uloc}$ with $\mathrm{div}\ \vec{u}_0 = 0$. $\vec{u}(t,x)$ $(t \in (0,T),\ x \in \mathbb{R}^3)$ is a local Leray solution if it satisfies Scheffer's local energy inequality :

$$\vec{u} \in L^\infty_t L^2_x \text{ locally and } \vec{\nabla} \otimes \vec{u} \in L^2_t L^2_x \text{ locally}$$

$$\partial_t \vec{u} + \mathbb{P}(\vec{u}.\vec{\nabla}\vec{u}) = \Delta \vec{u}$$

$$\vec{u}(0,x) = \vec{u}_0$$

$$\partial_t |\vec{u}|^2 = \Delta |\vec{u}|^2 - 2|\vec{\nabla} \otimes \vec{u}|^2 - \vec{\nabla}.((|\vec{u}|^2 + 2p)\vec{u}) - \mu \qquad (\mu \geq 0)$$

◇ local Leray solution exists for $T = O(\|\vec{u}_0\|^{-2}_{L^2_{uloc}})$ with

$$(*) \qquad \sup_{0<t<T} \|\vec{u}\|_{L^2_{uloc}} + \|(\int_0^T |\vec{\nabla} \otimes \vec{u}|^2\, dt)^{1/2}\|_{L^2_{uloc}} \leq C_{\|\vec{u}_0\|_{L^2_{uloc}}}$$

◇ the set of local Leray solutions satisfying $(*)$ is compact

local Leray solutions fot the Navier–Stokes equations V

Scaling :

If $\vec{u}$, p solution on $(0, T) \times \mathbb{R}^3$ with initial data $\vec{u}_0$, then

$$\lambda \vec{u}(\lambda^2 t, \lambda x), \quad \lambda^2 p(\lambda^2 t, \lambda x)$$

is solution on $(0, T/\lambda^2) \times \mathbb{R}^3$ with initial data $\lambda \vec{u}_0(\lambda x)$

local Leray solutions fot the Navier–Stokes equations V

Scaling :

If $\vec{u}$, p solution on $(0, T) \times \mathbb{R}^3$ with initial data $\vec{u}_0$, then

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local Leray solutions

Let $\vec{u}_0 \in \dot{M}^{2,3}$ with $\operatorname{div} \vec{u}_0 = 0$. $\vec{u}(t,x)$ $(t \in (0,+\infty),\ x \in \mathbb{R}^3)$ is a local Leray solution if it satisfies Scheffer's local energy inequality :

$$\begin{aligned}
&\vec{u} \in L^\infty_t L^2_x \text{ and } \vec{\nabla} \otimes \vec{u} \in L^2_t L^2_x \\
&\partial_t \vec{u} + \mathbb{P}(\vec{u}.\vec{\nabla}\vec{u}) = \Delta\vec{u} \\
&\vec{u}(0,x) = \vec{u}_0 \\
&\partial_t |\vec{u}|^2 = \Delta|\vec{u}|^2 - 2|\vec{\nabla} \otimes \vec{u}|^2 - \vec{\nabla}.((|\vec{u}|^2 + 2p)\vec{u}) - \mu \qquad (\mu \geq 0)
\end{aligned}$$

local Leray solutions fot the Navier–Stokes equations VI

◇ local Leray solution exists with

$$(*) \quad \sup_{T>0} T^{-1/2} \sup_{0<t<T} \sup_{x_0 \in \mathbb{R}^3} \int_{\|x-x_0\| \le \sqrt{T}} |\vec{u}(t,x)|^2 \, dx \le C_{\|\vec{u}_0\|_{\dot{M}^{2,3}}}$$

and

$$(**) \quad \sup_{T>0} T^{-1/2} \sup_{x_0 \in \mathbb{R}^3} \int_0^T \int_{|x-x_0| \le \sqrt{T}} |\vec{\nabla} \otimes \vec{u}|^2 \, dx \, dt \le C_{\|\vec{u}_0\|_{\dot{M}^{2,3}}}$$

◇ the set of local Leray solutions satisfying $(*)$ and $(**)$ is compact

local Leray solutions fot the Navier–Stokes equations VI

◇ local Leray solution exists with

$$(*)\quad \sup_{T>0} T^{-1/2} \sup_{0<t<T} \sup_{x_0\in\mathbb{R}^3} \int_{\|x-x_0\|\leq\sqrt{T}} |\vec{u}(t,x)|^2\, dx \leq C_{\|\vec{u}_0\|_{\dot{M}^{2,3}}}$$

and

$$(**)\quad \sup_{T>0} T^{-1/2} \sup_{x_0\in\mathbb{R}^3} \int_0^T \int_{|x-x_0|\leq\sqrt{T}} |\vec{\nabla}\otimes\vec{u}|^2\, dx\, dt \leq C_{\|\vec{u}_0\|_{\dot{M}^{2,3}}}$$

◇ the set of local Leray solutions satisfying $(*)$ and $(**)$ is compact

◇ if $C_{\|\vec{u}_0\|_{\dot{M}^{2,3}}}$ is small enough, then $|\vec{u}(t,x)| \leq D_{\|\vec{u}_0\|_{\dot{M}^{2,3}}} \frac{1}{\sqrt{t}}$ (Caffarelli-Kohn-Nirenberg)
$\Rightarrow$ uniqueness for $C_{\|\vec{u}_0\|_{\dot{M}^{2,3}}}$ small enough (and self–similarity if $\vec{u}_0$ homogeneous)

Mild solutions

With the integral formulation, we seek $\vec{u}$ as a fixed point of

$$\vec{v} \mapsto e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{v}\otimes\vec{v}\ ds = e^{t\Delta}\vec{u}_0 - B(\vec{v},\vec{v})$$

Picard's iterative method (a.k.a. Banach's contraction principle)

selfsimilar solutions

Mild solutions

With the integral formulation, we seek $\vec{u}$ as a fixed point of

$$\vec{v} \mapsto e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{v}\otimes\vec{v}\;ds = e^{t\Delta}\vec{u}_0 - B(\vec{v},\vec{v})$$

Picard's iterative method (a.k.a. Banach's contraction principle)

- on the plus side : uniqueness of the solution
- on the minus side : only for small times or small data

selfsimilar solutions

Mild solutions

With the integral formulation, we seek $\vec{u}$ as a fixed point of

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Picard's iterative method (a.k.a. Banach's contraction principle)

- on the plus side : uniqueness of the solution
- on the minus side : only for small times or small data

selfsimilar solutions

- uniqueness, existence and homogeneous data provide selfsimilar solutions
- allowed norms : Lorentz $L^{3,\infty}$, Morrey–Campanato $\dot{M}^{p,3}$, Besov $\dot{B}_p^{3/p-1,\infty}$ $(p < +\infty)$, Koch-Tataru BMO^{-1}

Approximations I : Leray

equation

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.(\omega_\epsilon * \vec{u}_\epsilon) \otimes \vec{u}_\epsilon - \vec{\nabla} p_\epsilon$$

- $\mathrm{div}\ \vec{u}_\epsilon = 0$
- $p_\epsilon = \mathbb{Q}(\omega_\epsilon * \vec{u}_\epsilon, \vec{u}_\epsilon)$ with $\mathbb{Q}(\vec{f}, \vec{g}) = -\sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial_i \partial_j}{\Delta}(f_i g_j)$

Local energy equality

$$\partial_t |\vec{u}_\epsilon|^2 = \Delta |\vec{u}_\epsilon|^2 - 2|\vec{\nabla} \otimes \vec{u}_\epsilon|^2 - \vec{\nabla}.((\vec{\omega}_\epsilon * \vec{u}_\epsilon.\vec{u}_\epsilon + 2p_\epsilon)\vec{u}_\epsilon)$$

Approximations II : Friedrich's mollification

equation

P_N projection in Fourier variable on $|\xi| \leq 2^N$

$$\partial_t \vec{u}_N = \Delta \vec{u}_N - P_N\big(\mathbb{P}\vec{\nabla}.\vec{u}_N \otimes \vec{u}_N\big)$$

$\vec{u}_{0,N} = P_N \vec{u}_0$

$\longrightarrow$ Galerkin on bounded domains ...

- $\mathrm{div}\ \vec{u}_N = 0$
- $p_N = P_N \mathbb{Q}(\vec{u}_N, \vec{u}_N)$

Local energy equality

$$\partial_t |\vec{u}_N|^2 = \Delta |\vec{u}_N|^2 - 2|\vec{\nabla} \otimes \vec{u}_N|^2 - \vec{\nabla}.((|\vec{u}_N|^2 + 2p_N)\vec{u}_N)$$

Approximations III : Beiraõ da Vega

equation

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \mathbb{P}\vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon \Delta^2 \vec{u}_\epsilon$$

- $\mathrm{div}\ \vec{u}_\epsilon = 0$
- $p_\epsilon = \mathbb{Q}(\vec{u}_\epsilon, \vec{u}_\epsilon)$

Local energy equality

$$\partial_t |\vec{u}_\epsilon|^2 = \Delta |\vec{u}_\epsilon|^2 - 2|\vec{\nabla} \otimes \vec{u}_\epsilon|^2 - \vec{\nabla}.((|\vec{u}_\epsilon|^2 + 2p_\epsilon)\vec{u}_\epsilon) - 2\epsilon|\Delta \vec{u}_\epsilon|^2 + \dots$$

$$\dots + 4\epsilon \sum_{j=1}^{3} \partial_j(\Delta \vec{u}_\epsilon.\partial_j \vec{u}_\epsilon) - 2\epsilon \Delta(|\vec{\nabla} \otimes \vec{u}_\epsilon|^2) - 2\epsilon \Delta(\vec{u}_\epsilon.\Delta \vec{u}_\epsilon)$$

Approximations IV : Vishik & Fursikov

equation

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon |\vec{u}_\epsilon|^4 \vec{u}_\epsilon + \frac{1}{\epsilon} \vec{\nabla}(\mathrm{div}\ \vec{u}_\epsilon)$$

- $\mathrm{div}\ \vec{u}_\epsilon \neq 0$
- $p_\epsilon = -\frac{1}{\epsilon} \mathrm{div}\ \vec{u}_\epsilon$

Local energy equality

$$\partial_t |\vec{u}_\epsilon|^2 = \Delta |\vec{u}_\epsilon|^2 - 2|\vec{\nabla} \otimes \vec{u}_\epsilon|^2 - \vec{\nabla}.((|\vec{u}_\epsilon|^2 + 2p_\epsilon)\vec{u}_\epsilon) + \dots$$

$$-2\epsilon |\vec{u}_\epsilon|^6 - 2\epsilon p_\epsilon^2 - \epsilon p_\epsilon |\vec{u}_\epsilon|^2$$

Approximations V

- **Leray :** uniqueness, energy equality, convergence to Navier-Stokes (L^2, L^2_{uloc}, $\dot{M}^{2,3}$) but no scaling
- **Friedrich :** uniqueness, energy equality, convergence to Navier-Stokes (L^2) but no scaling
- **Beiraõ da Vega :** uniqueness, energy equality, convergence to Navier-Stokes (L^2; probably L^2_{uloc}, $\dot{M}^{2,3}$) but no scaling
- **Vishik & Fursikov :** uniqueness, energy equality, convergence to Navier-Stokes (L^2; $\dot{M}^{2,3}$) but no scaling

Approximations V

- **Leray :** uniqueness, energy equality, convergence to Navier-Stokes (L^2, L^2_{uloc}, $\dot{M}^{2,3}$) but no scaling
- **Friedrich :** uniqueness, energy equality, convergence to Navier-Stokes (L^2) but no scaling
- **Beiraõ da Vega :** uniqueness, energy equality, convergence to Navier-Stokes (L^2; probably L^2_{uloc}, $\dot{M}^{2,3}$) but no scaling
- **Vishik & Fursikov :** uniqueness, energy equality, convergence to Navier-Stokes (L^2; $\dot{M}^{2,3}$) but no scaling
- **Navier-Stokes :**
  - no uniqueness (compactness of the set of solutions)
  - no energy equality (energy inequality)
  - scaling

Approximations VI: modified Vishik & Fursikov

equation

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon|\vec{u}_\epsilon|^2 \vec{u}_\epsilon + \frac{1}{\epsilon}\vec{\nabla}(\mathrm{div}\ \vec{u}_\epsilon)$$

- no uniqueness
- scaling
- $\mathrm{div}\ \vec{u}_\epsilon \neq 0$
- $p_\epsilon = -\frac{1}{\epsilon}\mathrm{div}\ \vec{u}_\epsilon$

Local energy equality

$$\partial_t|\vec{u}_\epsilon|^2 = \Delta|\vec{u}_\epsilon|^2 - 2|\vec{\nabla} \otimes \vec{u}_\epsilon|^2 - \vec{\nabla}.((|\vec{u}_\epsilon|^2 + 2p_\epsilon)\vec{u}_\epsilon) + \dots$$
$$\dots - 2\epsilon|\vec{u}_\epsilon|^4 - 2\epsilon p_\epsilon^2 - \epsilon p_\epsilon|\vec{u}_\epsilon|^2$$

Convergence in the finite energy case I

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon|\vec{u}_\epsilon|^2 \vec{u}_\epsilon - \vec{\nabla} p_\epsilon$$

$$p_\epsilon = -\frac{1}{\epsilon}\mathrm{div}\ \vec{u}_\epsilon$$

$\vec{u}_0 \in L^2$

- energy equality

$$\partial_t \|\vec{u}_\epsilon\|_2^2 = -2\|\vec{\nabla} \otimes \vec{u}_\epsilon\|_2^2 - 2\epsilon\|\vec{u}_\epsilon\|_4^4 - 2\epsilon\|p_\epsilon\|_2^2 - \epsilon \int p_\epsilon |\vec{u}_\epsilon|^2\ dx$$

- energy inequality

$$\|\vec{u}_\epsilon\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{u}_\epsilon\|_2^2\ ds + \epsilon \int_0^t \|\vec{u}_\epsilon\|_4^4\ ds + \epsilon\|p_\epsilon\|_2^2\ ds \leq \|\vec{u}_0\|_2^2$$

Convergence in the finite energy case II

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon |\vec{u}_\epsilon|^2 \vec{u}_\epsilon - \vec{\nabla} p_\epsilon, \qquad p_\epsilon = -\frac{1}{\epsilon} \mathrm{div}\ \vec{u}_\epsilon$$

We then write $\vec{u}_\epsilon = \vec{v}_\epsilon + \vec{w}_\epsilon$ with $\vec{v}_\epsilon = \mathbb{P}\vec{u}_\epsilon$

$\vec{v}_\epsilon$

- $(\vec{v}_\epsilon)_{\epsilon>0}$ is (locally in t and x) uniformly bounded in $L^2 H^1$
- $(\partial_t \vec{v}_\epsilon)_{\epsilon>0}$ is (locally in t and x) uniformly bounded in $L^2 H^{-2}$
  $\Rightarrow$ we can find a sequence $\epsilon_k \to 0$ such that $\vec{v}_{\epsilon_k}$ is (locally) convergent strongly in $L^2 L^2$ and weakly in $L^2 H^1$

$\vec{w}_\epsilon$

- $\vec{w}_\epsilon = -\epsilon \frac{1}{\Delta} \vec{\nabla} p_\epsilon$ is (locally) convergent strongly in $L^2 H^1$

Convergence in the finite energy case III

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon |\vec{u}_\epsilon|^2 \vec{u}_\epsilon - \vec{\nabla} p_\epsilon, \qquad p_\epsilon = -\frac{1}{\epsilon} \mathrm{div}\ \vec{u}_\epsilon$$

Behaviour of p_ϵ

- $\partial_t p_\epsilon = (1 + \frac{1}{\epsilon}) \Delta p_\epsilon + \frac{1}{\epsilon} \mathrm{div}\ (\vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon + \epsilon |\vec{u}_\epsilon|^2 \vec{u}_\epsilon)$
- $p_\epsilon = \frac{1}{1+\epsilon} \int_0^t e^{(t-s)(1+\frac{1}{\epsilon})\Delta} (1+\frac{1}{\epsilon}) \Delta \frac{1}{\Delta} \mathrm{div}\ (\vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon + \epsilon |\vec{u}_\epsilon|^2 \vec{u}_\epsilon)\, ds$
- p_ϵ is bounded in $L^p L^q$ for $1 < q < 3/2$ and $\frac{2}{p} + \frac{3}{q} = 4$ (maximal regularity)

$\Rightarrow$ the weak limit $\vec{u}$ satisfies the local energy inequality

Convergence in the infinite energy case I.

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon |\vec{u}_\epsilon|^2 \vec{u}_\epsilon - \vec{\nabla} p_\epsilon, \qquad p_\epsilon = -\frac{1}{\epsilon} \mathrm{div}\ \vec{u}_\epsilon$$

The model is not good for L^2_{uloc} : for $\varphi(x) = \varphi_0(x - x_0)$ ($x_0 \in \mathbb{R}^3$) we have

$\partial_t \int |\vec{u}_\epsilon|^2 \varphi(x)\ dx = \int |\vec{u}_\epsilon|^2 \Delta\varphi(x)\ dx - 2\int |\vec{\nabla} \otimes \vec{u}_\epsilon|^2\ \varphi(x)\ dx +$
$\int (|\vec{u}_\epsilon|^2 + 2p_\epsilon)\vec{u}_\epsilon.\vec{\nabla}.\varphi(x)\ dx - 2\epsilon \int |\vec{u}_\epsilon|^4\ \varphi(x)\ dx -$
$2\epsilon \int p_\epsilon^2\ \varphi(x)\ dx - \epsilon \int p_\epsilon |\vec{u}_\epsilon|^2\ \varphi(x)\ dx$

Bad term : $\int p_\epsilon\ \vec{u}_\epsilon.\vec{\nabla}.\varphi(x)\ dx$: maximal regularity is not valid in $(L^p L^q)_{uloc}$
$\Rightarrow$ existence of $\vec{u}_\epsilon$ can be proved only for $t < T_\epsilon$ with $\lim_{\epsilon \to 0} T_\epsilon = 0$!

Convergence in the infinite energy case II.

$$\partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla}.\vec{u}_\epsilon \otimes \vec{u}_\epsilon - \epsilon |\vec{u}_\epsilon|^2 \vec{u}_\epsilon - \vec{\nabla} p_\epsilon, \qquad p_\epsilon = -\frac{1}{\epsilon} \mathrm{div}\ \vec{u}_\epsilon$$

Let $\omega(x) = (1+|x|^2)^{-3/4}$ and $\varpi(x) = (1+|x|^2)^{-5/4}$.

- $\vec{u}_0 \in \dot{M}^{2,3} \Rightarrow \int |\vec{u}_0|^2\ \omega(x)\ dx < +\infty$.
- (local energy) we define
  - $\alpha(t) = \int |\vec{u}_\epsilon(t,x)|^2 \omega(x)\ dx$
  - $\beta(t) = \int_0^t \int |\vec{\nabla} \otimes \vec{u}_\epsilon(s,x)|^2 \omega(x)\ ds\ dx$
  - $\gamma(t) = \int_0^t \int \epsilon |\vec{\nabla} \otimes \vec{u}_\epsilon(s,x)|^4 \omega(x)\ ds\ dx$
  - $\delta(t) = \int_0^t \int \frac{1}{\epsilon} |\mathrm{div}\ \vec{u}_\epsilon(s,x)|^2\ \omega(x)\ ds\ dx$
  - $\eta(t) = \int_0^t \int \frac{1}{\epsilon^{3/2}} |\mathrm{div}\ \vec{u}_\epsilon(s,x)|^{3/2} \varpi(x)\ ds\ dx = \int_0^t \int |p_\epsilon(s,x)|^{3/2} \varpi(x)\ ds\ dx$

- $\alpha(t) + 2\beta(t) + \frac{3}{2}\gamma(t) + \frac{3}{2}\delta(t) \leq \alpha(0) + Z$
  $Z = \int_0^t \int |\vec{u}|^2 \Delta\omega\ ds\ dx + \int_0^t \int |\vec{u}|^2 \sum_{i=1}^3 u_i \partial_i\ \omega\ ds\ dx - \int_0^t \int \frac{2}{\epsilon} \mathrm{div}\ \vec{u} \sum_{i=1}^3 u_i \partial_i \omega\ ds\ dx$

Convergence in the infinite energy case III.

$Z = \int_0^t\int |\vec{u}|^2 \Delta\omega \; ds \; dx + \int_0^t\int |\vec{u}|^2 \sum_{i=1}^3 u_i \partial_i \; \omega \; ds \; dx -$
$\int_0^t\int \frac{2}{\epsilon} \mathrm{div} \; \vec{u} \sum_{i=1}^3 u_i \partial_i \omega \; ds \; dx$

$Z = Z_1 + Z_2 + Z_3$

- $|\Delta\omega(x)| \leq C\omega(x) \Rightarrow Z_1 \leq C \int_0^t \alpha(s) \; ds.$
- $|\vec{\nabla}\omega(x)| \leq \frac{3}{2}\omega(x)^{3/2} \Rightarrow Z_2 \leq C \int_0^t \int |\vec{u}_\epsilon(s,x)|^3 \omega^{3/2} \; ds \; dx.$
  $Z_2 \leq \frac{1}{4}\beta(t) + C \int_0^t \alpha(s) \; ds + C \int_0^t \alpha^3(s) \; ds.$
- $|\vec{\nabla}\omega(x)| \leq \frac{3}{2}\omega^{1/2}\varpi^{2/3} \Rightarrow Z_3 \leq$
  $C \int_0^t \|\sqrt{\omega}\vec{u}_\epsilon\|_3 \|\varpi^{2/3} \frac{1}{\epsilon} \mathrm{div} \; \vec{u}_\epsilon\|_{3/2} \; ds$
  $Z_3 \leq \frac{1}{4}\beta(t) + C \int_0^t \alpha(s) \; ds + C \int_0^t \alpha^3(s) \; ds + C\eta(t).$

Convergence in the infinite energy case IV.

$$p_\epsilon = \frac{1}{1+\epsilon}\int_0^t e^{(1+\frac{1}{\epsilon})(t-s)\Delta}(1+\frac{1}{\epsilon})\Delta\frac{\vec\nabla}{\Delta}.(\vec u_\epsilon.\vec\nabla\vec u_\epsilon + \epsilon|\vec u_\epsilon|^2\vec u_\epsilon)\ ds$$

$\varpi \in A_{3/2}$ (Muckenhoupt class) $\Rightarrow$ maximal regularity of the heat kernel holds in $L^{3/2}_t L^{3/2}_x(\varpi(x)\ dx)$

$\eta(t) \leq C\int_0^t\int \left|\frac{\vec\nabla}{\Delta}.(\vec u_\epsilon.\vec\nabla\vec u_\epsilon)\right|^{3/2}\ \varpi(x)\ ds\ dx +$
$C\int_0^t\int \left|\frac{\vec\nabla}{\Delta}.(\epsilon|\vec u_\epsilon|^2\vec u_\epsilon)\right|^{3/2}\ \varpi(x)\ ds\ dx$

$\sqrt{\omega}f \in L^2$, $\sqrt{\omega}g \in H^1$, $h = \frac{1}{\sqrt{-\Delta}}(fg)$

- $\int |h|^{3/2}\varpi\ dx \leq C\|\sqrt{\omega}f\|_2^{3/2}(\|\sqrt{\omega}g\|_2^{3/2} + \|\sqrt{\omega}g\|_{\frac{17}{8}}^{3/2})$

Convergence in the infinite energy case V.

$$\|\sqrt{\omega}f\|_2^{3/2}|\sqrt{\omega}g\|_2^{3/2} \leq \frac{3}{4}\kappa\|\sqrt{\omega}f\|_2^2 + \frac{1}{4\kappa^3}\|\sqrt{\omega}g\|_2^6$$

$$\|\sqrt{\omega}f\|_2^{3/2}\|\sqrt{\omega}g\|_{\frac{17}{8}}^{3/2} \leq \kappa\|\sqrt{\omega}f\|_2^2 + \kappa\|\sqrt{\omega}g\|_6^2 + C_{\kappa,q}\|\sqrt{\omega}g\|_2^{\frac{186}{25}}$$

$\sqrt{\omega}f \in L^2$, $\sqrt{\omega}g \in H^1$, $h = \frac{1}{\sqrt{-\Delta}}(fg)$

$$Z \leq \beta(t) + C\int_0^t \alpha(s) + \alpha^3(s) + \alpha^{\frac{93}{25}}(s)\ ds$$

so that the control is uniform with respect to ϵ :

$\alpha(t)+\beta(t)+\frac{3}{2}\gamma(t)+\frac{3}{2}\delta(t) \leq \alpha(0)+C\int_0^t \alpha(s)+\alpha^3(s)+\alpha^{\frac{93}{25}}(s)\ ds$

Conclusion?

PROBLEM

We have no uniqueness, so that the problem of existence of self–similar solutions for large homogeneous initial data is still open. . .