

**Elastic models for growing tissues:
scaling laws and derivation by
Gamma convergence**

Reza Pakzad
University of Pittsburgh

University of Oxford – 9 Nov. 2009

Nonlinear elasticity: variational formulation

$$J(u) = \int_{\Omega} W(\nabla u) - \int_{\Omega} f u$$

$u = u(x) \in \mathbb{R}^n$ deformation, $x \in \Omega \subset \mathbb{R}^n$ ref configuration, f external body force, W elastic energy density.

Assumptions on energy density $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_+$:

- $W(RF) = W(F) \quad \forall R \in SO(3)$ **frame indifference**
- $W(Id) = \min W = 0$ **normalisation**
- $W(F) \geq c \operatorname{dist}^2(F, SO(3)), \quad c > 0$ **non-degeneracy**

Many open problems: existence of minimizers, regularity
(see survey by [Ball](#), 'Some open problems in elasticity')

Nonlinear elasticity 3d \rightarrow 2d

Major question since the beginning of elasticity theory:

- 2d simpler to understand, visualise
- important in engineering
- qualitatively new behaviour: crumpling, collapse
- subtle influence of geometry (isometries of mid-surface)

Rigorous derivation via Γ -convergence (De Giorgi)

X a metric space, $F_n : X \rightarrow [-\infty, +\infty]$.

$F_n \xrightarrow{\Gamma} F$ whenever

(The Γ -liminf inequality) $\forall x_n \rightarrow x$ in X , $F(x) \leq \liminf_{n \rightarrow \infty} F_n(x_n)$.

(The Γ -limsup inequality)

$\forall x \in X$, $\exists x_n \in X$, s. t. $\limsup_{n \rightarrow \infty} F_n(x_n) \leq F(x)$.

(Compactness) A sequence of approximate minimizers $\{x_n\}$ has a converging subsequence in X .

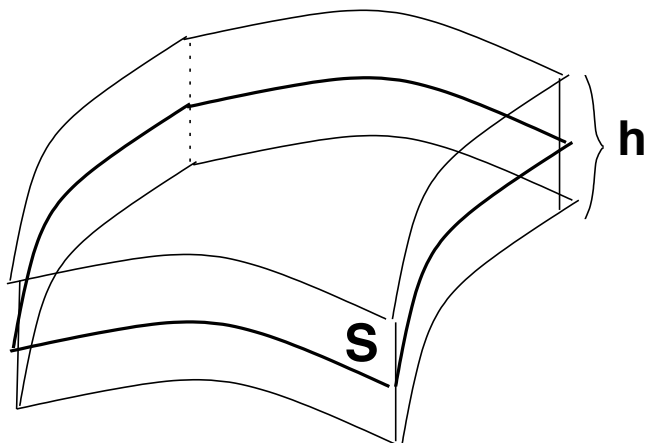
Metatheorem Γ -convergence \Leftrightarrow conv. of minimizers

The set-up for thin shells

S : 2d surface in \mathbb{R}^3 compact connected oriented

\vec{n} : unit normal, $T_x S$: tangent space

$\Pi = \nabla \vec{n}$: shape operator



$$S^h = \{x + t\vec{n}(x); x \in S, -\frac{h}{2} < t < \frac{h}{2}\}$$

$$u^h \in W^{1,2}(S^h, \mathbb{R}^3)$$

applied force $f^h \in L^2(S^h, \mathbb{R}^3)$

study minimizers of:

$$J^h(u^h) = \frac{1}{h} \int_{S^h} W(\nabla u^h) - \frac{1}{h} \int f^h u^h$$

Given α such that $\frac{1}{h^\alpha} f^h \rightarrow f$, let $\beta := \max\{\alpha, 2\alpha - 2\}$,

then $\inf J^h \sim h^\beta$

Goal: Identify I_β such that $\frac{1}{h^\beta} E^h \xrightarrow{\Gamma} I_\beta$.

Different scaling limits

- plates $S \subset \mathbb{R}^2$

Le Dret-Raoult (95) for $\beta = 0$ (stretching theory) (95)

Friesecke-James-Müller (02, 05)

$\beta = 2$ Kirchhoff theory, $2 < \beta < 4$ linearised Kirchhoff,

$\beta = 4$ von Kármán theory, $\beta > 4$ linear

Conti-Maggi (08) $0 < \beta < 5/3$

$5/3 \leq \beta < 2$ open \sim paper crumpling

- non-flat S :

Le Dret-Raoult (00) for $\beta = 0$ (stretching theory) (00)

FJM-Mora (03) for $\beta = 2$ (bending theory)

Lewicka-Mora-P. (08) for $\beta = 4$ (generalized von Kármán)

Morphogenesis in soft growing tissues

Observation: residual stress at free equilibria in leaves.

A model incorporating growth by [Rodriguez-Hoger-McCulloch](#)

(J. Biomech 1994)

studied by [Ben Amar](#) et al., [Marder-Papanicolau](#):

$$\text{decomposition } \nabla u = FA$$

$A : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ = growth tensor; change in mass. $\det A > 0$.

F = elastic tensor; reorganization of body ($= (\nabla u)A^{-1}$)

$$\boxed{\text{Energy } \int W(F)} \geq \int \text{dist}^2((\nabla u)A^{-1}, SO(3)) \, dx$$
$$\approx \int_{\Omega} \text{dist}^2(\nabla u, SO(3)A) \, dx$$

3d non-Euclidean elasticity I

$$E(u) = \int_{\Omega} \text{dist}^2(\nabla u, \mathcal{F}(x)), \quad \mathcal{F}(x) = SO(3)A(x).$$

$$E(u) = 0 \Leftrightarrow \nabla u(x) \in SO(3)A(x)$$

$$\Leftrightarrow (\nabla u)^T \nabla u = A^T A \text{ and } \det \nabla u > 0$$

$G := A^T A$ is a **Riemannian metric** on Ω ,

$(\nabla u)^T \nabla u$ is the **pull back metric**.

$$E(u) \approx \int_{\Omega} |(\nabla u)^T \nabla u - G|^2 \text{ for } \nabla u \text{ close to } SO(3)A.$$

Observation: $E_0(u) := \int_{\Omega} |(\nabla u)^T \nabla u - G|^2$ is not a suitable energy functional

- $\exists u \in W^{1,\infty}$ $E_0(u) = 0$ (see **Gromov** 'Partial Diff Rel')

- If $\exists x$ $R_{ijkl}(x) \neq 0$ then this u cannot be orientation

preserving/reversing in any ball $B(x, \varepsilon) \Rightarrow$ **folding structure**

3d non-Euclidean elasticity II

Given metric G , define $\mathcal{F}(x) = SO(3)A(x)$,

$A(x) = \text{positive def } \sqrt{G}$

Lemma (Lewicka-P., '09) $\inf_{u \in W^{1,2}} E(u) > 0 \Leftrightarrow R_{ijkl} \neq 0$

Proof. Assume that $E(u_n) \rightarrow 0$. **Split** $u_n = w_n + z_n$:

$\Delta_G z^k = \Delta_G u^k$ in Ω , $z^k|_{\partial\Omega} = 0$, $k = 1..3$. Integrate by parts:

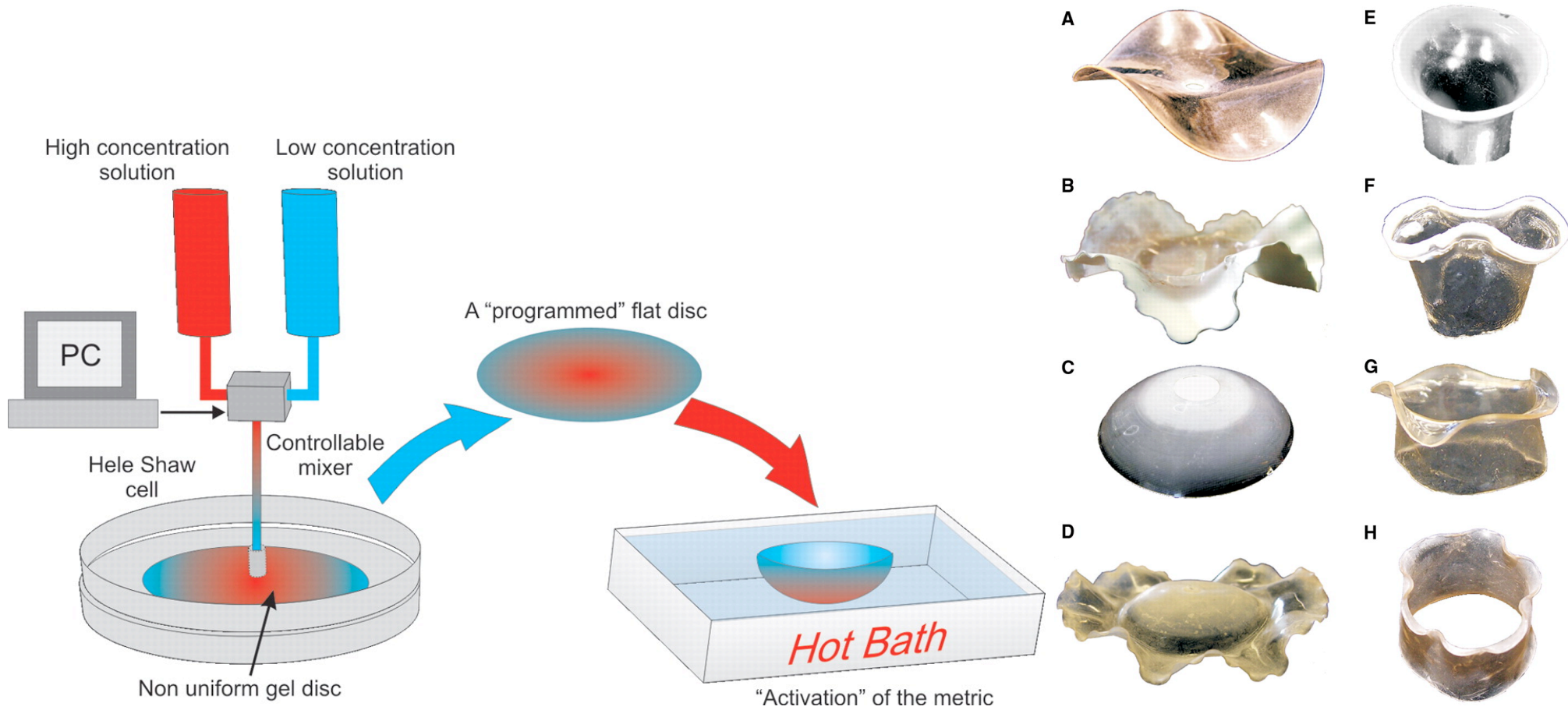
$$\begin{aligned} \bullet \int \sqrt{|G|} |\nabla z^k|_G^2 &= \int \partial_j z^k (G^{ij} \sqrt{|G|} \partial_i u^k - (\text{cof} \nabla u)_{kj}) \\ &\leq \|\nabla z^k\|_{L^2} \|\sqrt{|G|} (\nabla u) G^{-1} - \text{cof} \nabla u\|_{L^2} \leq \|\nabla z^k\|_{L^2} E(u)^{1/2} \end{aligned}$$

because: $F \in \mathcal{F}(x) \Leftrightarrow \text{cof} F = \sqrt{|G|} F G^{-1}$

$\bullet \Rightarrow z_n \rightarrow 0$ in $W^{1,2}(\Omega)$. Also $\|w_n\|_{W^{1,2}} \leq C$ and $\Delta_G w_n^k = 0$

$\Rightarrow w_n \rightarrow u$ in $W_{loc}^{1,2}(\Omega) \Rightarrow u_n \rightarrow u$ so $E(u) = 0$ so $R \equiv 0$.

3d non-Euclidean elasticity: an experiment



Shaping of elastic sheets by prescription of Non-Euclidean metrics

Klein, Efrati, Sharon, Science (2007)

Reprinted with permission of AAAS

3d non-Euclidean elasticity of thin plates

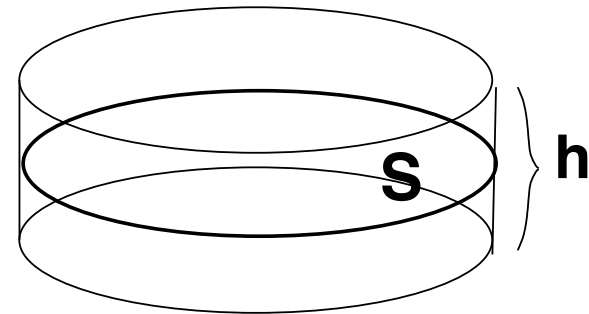
$$\inf_{u \in W^{1,2}} E(u) > 0 \Leftrightarrow R_{ijkl} \neq 0$$

Thin plates: $\Omega = S^h = S \times (-h/2, h/2)$, $S \subset \mathbb{R}^2$

Assume that $G(\tilde{x}, x_3) = \begin{bmatrix} [g_{\alpha\beta}(\tilde{x})] & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

given: $[g_{\alpha\beta}]$ metric on S

$$E^h(u) = \frac{1}{h} \int_{S^h} \text{dist}^2(\nabla u, \mathcal{F}(x))$$



- $R_{ijkl}(G) \equiv 0 \Leftrightarrow K_g \equiv 0$
- If $K_g \neq 0 \Rightarrow \inf E^h > 0$,

Question: How do $\inf E^h$ scale?

Non-Euclidean elasticity: 3d \rightarrow 2d

$$E^h(u) = \frac{1}{h} \int_{S^h} \text{dist}^2(\nabla u, \mathcal{F}(x)), \quad \mathcal{F}(x) = SO(3) \sqrt{G(x)}$$

$$y(x_1, x_2, x_3) := u(x_1, x_2, hx_3) : S^1 \rightarrow \mathbb{R}^3$$

THM (Lewicka-P. '09) Suppose that $E^h(u^h) \leq Ch^2$.

Then, (for a subsequence): $y^h \rightarrow y(x_1, x_2) \in W^{1,2}$ (strongly!),

$y \in W^{2,2}$ and $(\nabla y)^T \nabla y = [g_{\alpha\beta}]$.

THM (L.-P.) $\frac{1}{h^2} E^h \xrightarrow{\Gamma} I$ Γ -converges:

- $\liminf \frac{1}{h^2} E^h(u^h) \geq I(y)$
- $\forall y \in W^{2,2}(S) \quad (\nabla y)^T \nabla y = [g_{\alpha\beta}] \quad \exists u^h \in W^{1,2}(S^h)$

such that $y^h \rightarrow y$ and $\lim \frac{1}{h^2} E^h(u^h) = I(y)$

The limit functional

Admissible maps (isom. immersions of $[g_{\alpha\beta}]$):

$$\mathcal{A} = \left\{ y \in W^{2,2}(S, \mathbb{R}^3); \quad (\nabla y)^T \nabla y = [g_{\alpha\beta}] \right\}$$

Normal $\vec{n} = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|}$, 2nd fund form $\Pi = (\nabla y)^T \nabla \vec{n}$

$$A_{\alpha\beta} = \sqrt{[g_{\alpha\beta}]}$$

$$I(y) = \frac{1}{24} \int_S Q_2(x, A_{\alpha\beta}^{-1} \Pi)$$

$Q_2(x, \cdot)$ is a quadratic form related to $\nabla^2 \text{dist}^2(\cdot, \mathcal{F}(x))(A)$

- OK for other energy densities $W \leftrightarrow \text{dist}^2(\cdot, \mathcal{F}(x))$ frame invariant, energy well at \mathcal{F} , bound from below (!)

Isometric immersions of 2d metrics in 3d

Corollary (L.-P.)

- $\inf E^h \leq Ch^2 \Leftrightarrow [g_{\alpha\beta}]$ has a $W^{2,2}$ iso.imm. in \mathbb{R}^3
- $\frac{1}{h^2} \inf E^h \rightarrow 0 \Leftrightarrow K_g \equiv 0$
($[g_{\alpha\beta}]$ has a 2d pull back realization)
- In particular $K_g \neq 0 \Rightarrow \inf E^h \geq ch^2$ for $c > 0$.

Recall:

- (Nash-Kuiper) Every n dim G has C^1 isom imm in \mathbb{R}^{n+1}
- (Nirenberg) Every $[g_{\alpha\beta}]$ on $S \subset \mathbb{R}^2$ with $K_g > 0$ has a smooth isom imm in \mathbb{R}^3
- (Han-Hong) Same true for $K_g < 0$ (bounded domain!)
- (Pogorelov) Example of g with no local C^2 isom. imm.

Rigidity estimate / Nonlinear Korn / Qualitative Liouville

Theorem (Friesecke-James-Muller '02) $\exists C_\Omega \forall u \in W^{1,2}(\Omega, \mathbb{R}^n)$:

$$\min_{R \in SO(n)} \int_{\Omega} |\nabla u - R|^2 \leq C_\Omega \int_{\Omega} \text{dist}^2(\nabla u, SO(n))$$

- Nonlinear Korn, since $T_{Id}SO(n) = so(n)$ skew matrices:

$$\min_{A \in so(n)} \int |\nabla u - A|^2 \leq C \int |\text{sym} \nabla u|^2$$

- Quantitative Liouville: if $\nabla u \in SO(n)$ a.e. $\Rightarrow \nabla u \equiv \text{const.}$

THM (Lewicka-P.) $\forall u \in W^{1,2}(\Omega, \mathbb{R}^n) \exists B \in \mathbb{R}^{n \times n}$:

$$\int |\nabla u - B|^2 \leq C \left(\int \text{dist}^2(\nabla u, \mathcal{F}(x)) + \|\nabla G\|_\infty^2 (\text{diam } \Omega)^2 |\Omega| \right)$$

C depends on $\|G\|_\infty, \|G^{-1}\|_\infty$ and on Ω : uniformly for bilipschitz equivalent Ω with controlled Lip constant

Application: compactness

$$E^h(u^h) = \frac{1}{h} \int_{S^h} \text{dist}^2(\nabla u^h, \mathcal{F}(x)) \leq Ch^2$$

- divide S^h into cubes of size h , apply rigidity estimate:

$$\int_{cube} |\nabla u^h - B^h|^2 \leq C(\int_{cube} \text{dist}^2(\nabla u^h, \mathcal{F}(x)) + \|\nabla G\|_\infty^2 h^2 |cube|)$$

- Sum over all cubes \Rightarrow approx of ∇u^h by $B^h : S \rightarrow \mathbb{R}^{3 \times 3}$:

$$\frac{1}{h} \int_{S^h} |\nabla u^h - B^h|^2 \leq C(E^h(u^h) + h^2 \|\nabla G\|_\infty^2) \leq Ch^2$$

- apply rigidity est on union of 2 neighbouring cubes

$$\Rightarrow \text{difference quotient estimate} \Rightarrow \|\nabla B^h\|_{L^2(S)}^2 \leq C$$

- compactness: $B^h \rightharpoonup B$ in $W^{1,2}$, hence $\nabla u^h \rightarrow B$ in L^2 ,

\Rightarrow higher differentiability of the limit $y \in W^{2,2}$

- $B = [\partial_1 y, \partial_2 y, \vec{n}] \in \mathcal{F}(x) \Rightarrow y$ is an isom. imm.

Energy scalings for perturbations of Euclidean metric

- **Main Goal** : Write equilibrium eqns. of a thin elastic body

$\Omega = S^h$, subject to growth tensor $A = Id + \text{perturbation}$:

$$G := A^T A = Id + \text{perturbation}$$

relating to a small time step in the dynamic problem.

- **Secondary Goal**: Compare the infimum of energies with the magnitude of the curvature tensor and dimensions of the domain.

$G^h = Id + h^\gamma \varepsilon_g$ metric on S^h , $\varepsilon_g : S \rightarrow \mathbb{R}_{sym}^{3 \times 3}$

- Gauss curvature of $G|_S^h = -\frac{1}{2} h^\gamma \text{curl}^T \text{curl}(\varepsilon_g)_{tan} + O(h^{2\gamma})$

THM (Lewicka-P. '09) Assume $\text{curl}^T \text{curl}(\varepsilon_g)_{tan} \neq 0$, then:

$$\frac{1}{h^\beta} \inf E^h \rightarrow \infty \quad \forall \beta > \max\{2\gamma, \gamma + 2\}.$$

Von Kármán - like morphogenesis model

$$G^h = Id + h^2 \varepsilon_g + hx_3 \kappa_g \text{ metric on } S^h, \quad \varepsilon_g, \kappa_g : S \rightarrow \mathbb{R}_{sym}^{3 \times 3}$$

THM (Lewicka-P. '09) Assume $\text{curl}(\kappa_g)_{tan} \neq 0$

or $\text{curl}^T \text{curl}(\varepsilon_g)_{tan} + \frac{1}{2} \det(\kappa_g)_{tan} \neq 0$. Then:

$$\inf E^h \sim h^4.$$

THM (L.-P.) $\frac{1}{h^4} E^h \xrightarrow{\Gamma} I_g$, where

$$I_g(v, u) = \frac{1}{24} \int_S Q_2(\nabla^2 v + \frac{1}{2}(\kappa_g)_{tan}) \quad \text{bending}$$

$$+ \frac{1}{2} \int_S Q_2(\text{sym} \nabla u + \frac{1}{2} \nabla v \otimes \nabla v - \frac{1}{2}(\varepsilon_g)_{tan}) \quad \text{stretching}$$

Euler-Lagrange equations: (for $Q_2 = Id$)

(Mahadevan and Liang, PNAS 2009)

$$\Delta^2 \Phi = - \det \nabla^2 v - \frac{1}{2} \text{curl}^T \text{curl}(\varepsilon_g)_{tan}$$

$$\Delta^2 v = 12 \text{cof} \nabla^2 \Phi : \nabla^2 v - \frac{1}{2} \text{div}^T \text{div}(\kappa_g)_{tan}$$

Von Kármán theory for shells , $\beta = 4$

THM (Lewicka-Mora-P. '08) Suppose that $E^h(u^h) \leq Ch^4$.

Then (for a subsequence, modulo rigid motions):

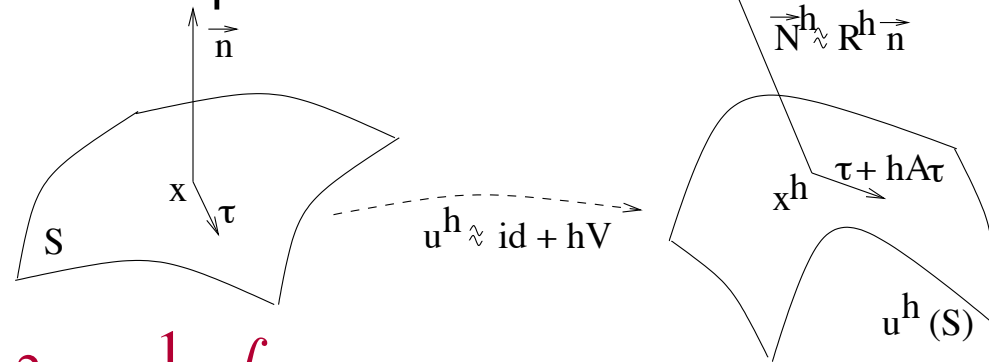
$$y^h \rightarrow \text{id in } W^{1,2}$$

$$\frac{1}{h} \int y^h - \text{id} \rightarrow V \in W^{2,2}(S, \mathbb{R}^3), V \text{ is a first order inf. isometry:}$$

$$\text{a.e. } x \in S, \forall \tau \in T_x S, \partial_\tau V(x) \cdot \tau = 0 \quad (\nabla V = A \in so(3))$$

$$\frac{1}{h^2} \text{sym } \nabla(\int y^h - \text{id}) \rightarrow B \in \text{finite strain space } \mathcal{B}$$

THM (L.-M.-P.) $\frac{1}{h^4} E^h \xrightarrow{\Gamma} I_4$, where



$$I_4(V, B) = \frac{1}{2} \int_S Q_2(B - \frac{1}{2}A^2) + \frac{1}{24} \int_S Q_2(\nabla(A\vec{n}) - A\Pi)$$

Infinite hierarchy of shell models

Critical exponents:

$$\beta_i = 2 + \frac{2}{i-1}$$

Conjecture (L.-P.) Let $N \in \mathbb{N}$ and assume that $\beta \in [\beta_{N+1}, \beta_N)$ and $E^h(u^h) \approx h^\beta$.

The limiting theory is given by I_β defined on the space \mathcal{V}_N of **N-th order infinitesimal isometries**:

$$\mathcal{V}_N = \{(V_1 \dots V_N) \text{ displacements} : S \rightarrow \mathbb{R}^3;$$

$$u_\varepsilon := \text{id} + \sum_{i=1}^N \varepsilon^i V_i \text{ preserve the metric up to order } \varepsilon^N\}$$

- when $\beta = \beta_{N+1}$ then $I_\beta = \int_S Q_2(\delta_{N+1} g_S) + \int_S Q_2(\delta_1 \Pi_S)$
- when $\beta \in (\beta_{N+1}, \beta_N)$ then $I_\beta = \int_S Q_2(\delta_1 \Pi_S)$

Constrained theories for $2 < \beta < 4$

$$E^h(u^h) \leq Ch^\beta, \quad \beta \in [2 + \frac{2}{N}, 2 + \frac{2}{N-1}) \Rightarrow u^h|_S \approx \text{id} + \sum_{i=1}^N \varepsilon^i V^i$$

$$(\nabla u^h)^T \nabla u^h - \text{Id} = O(\varepsilon^{N+1})$$

$\beta \in [3, 4) \Rightarrow$ 2nd order isometry

$\beta \in [8/3, 3) \Rightarrow$ 3rd order isometry, etc

THM (L.-M.-P.) If S is **strictly convex** then $\forall V \in \mathcal{V}_1 \cap C^{2,\alpha} \exists w_\varepsilon$
 equibd. in $C^{2,\alpha}$: $\forall \varepsilon \quad \text{id} + \varepsilon V + \varepsilon^2 w_\varepsilon$ is exact isometry.

THM (L.-M.-P.) If S is strictly convex then $\forall \beta \in (2, 4)$:

$$I_\beta(V) = \frac{1}{24} \int_S Q_2(\nabla(A\vec{n}) - A\Pi), \quad V \in \mathcal{V}_1.$$

Constrained theories for $2 < \beta < 4$

Remarks:

- plates have matching property of \mathcal{V}_2 to exact isom. (FJM)

- \Rightarrow for plates $I_\beta(v) = \int |\nabla^2 v|^2$ again bending,

constraint: $v =$ out-of-plane displacement such that

(V^1, V^2, v) is a 2nd order isometry \Leftrightarrow $\det \nabla^2 v = 0$

- crucial point in the upper bound for S strictly convex:

$S \in C^{m+2, \alpha} \Rightarrow \mathcal{V}_1 \cap C^{m, \alpha}$ is dense in \mathcal{V}_1 with $W^{2,2}$ norm.

- trivially OK for $S \subset \mathbb{R}^2$

The matching property for infinitesimal isometries on S

- $V \in \mathcal{V}_1 \Leftrightarrow \partial_\tau V = A\tau$, skew matrix field $A : S \rightarrow SO(3)$
- $(V, V_2) \in \mathcal{V}_2 \Leftrightarrow \text{sym}\nabla V = 0$ and $\text{sym}\nabla V_2 = \frac{1}{2}(A^2)_{tan}$
- $V = V_{tan} + (V\vec{n})\vec{n}$. Then: $\text{sym}\nabla V = \text{sym}\nabla V_{tan} + (V\vec{n})\Pi$
- Let $S \subset \mathbb{R}^2$ plate.

$V \in \mathcal{V}_1 \Leftrightarrow V_{tan}$ rigid motion

$\exists V_2 : (V, V_2) \in \mathcal{V}_2 \Leftrightarrow \det \nabla^2 V^3 = 0$ nontrivial condition

Theorem (FJM) If $V \in \mathcal{V}_1 \cap W^{1,\infty}$ and $\det \nabla^2 V^3 = 0$ then $\exists w_\varepsilon$

equibd in $W^{2,2}$: $\forall \varepsilon$ $id + \varepsilon V + \varepsilon^2 w_\varepsilon$ is exact isometry

Our result 'The same for convex S and 1st order isom.'

Matching infinitesimal isometries on convex S

Given $V \in \mathcal{V}_1$, then $id + hV + h^2w_h$ is exact isometry \Leftrightarrow

$$\text{sym}\nabla w_h = -\frac{1}{2}(\nabla V + h\nabla w_h)^T (\nabla V + h\nabla w_h)$$

THM 1 \exists linear solution operator $\mathcal{T} : L^2_{\text{sym}}(S) \rightarrow L^2(S)$:

$$\text{sym}\nabla(\mathcal{T}B) = B \quad \forall B \in L^2_{\text{sym}}$$

$$\|(\mathcal{T}B)_{\text{tan}}\|_{W^{1,2}} + \|(\mathcal{T}B)\vec{n}\|_{L^2} \leq C\|B\|_{L^2}$$

THM 2 $\forall \phi, \psi \in C^{2,\alpha}(\bar{S}, \mathbb{R}^3)$

$$\|\mathcal{T}(\text{sym}((\nabla\phi)^T \nabla\psi))\|_{C^{2,\alpha}} \leq C\|\phi\|_{C^{2,\alpha}}\|\psi\|_{C^{2,\alpha}}$$

Proof: $\mathcal{G}_h(w) = -\frac{1}{2}\mathcal{T}((\nabla V + h\nabla w_h)^T (\nabla V + h\nabla w_h))$

By Thm 2: $\|\mathcal{G}_h(w)\|_{C^{2,\alpha}} \leq C\|V + hw\|_{C^{2,\alpha}}^2$

For $h \ll 1$ \mathcal{G}_h is a contraction on a closed ball in $C^{2,\alpha}$

$\Rightarrow \exists$ fixed points w_h with uniform bound.

Proof of Thm 1: solution operator to $\text{sym}\nabla(\mathcal{T}B) = B$

We know $\text{sym}\nabla(\mathcal{T}B)$, look for $\text{skew}\nabla(\mathcal{T}B) \approx \text{curl}(\mathcal{T}B) =: \omega$

$$\mathcal{L}\omega = -\partial_i(\sqrt{|g|}h^{ij}\partial_j\omega) - 2\sqrt{|g|}H\omega = \mathcal{D}(B)$$

where: $[h^{ij}] = \Pi^{-1}$, $\mathcal{D}(B) = 2\text{nd order operator}$,

$$H = \text{mean curvature} > 0 \quad \Rightarrow \quad \underline{\mathcal{L} \text{ not positive!}}$$

- Extend coefficients of \mathcal{L} , \mathcal{D} on $\tilde{S} \supset \supset S$. Solve:

$$\tilde{\mathcal{L}}\omega = \tilde{\mathcal{D}}(B) + f_0(B), \quad \omega|_{\partial\tilde{S}} = 0$$

\exists linear $f_0 : L^2_{\text{sym}} \rightarrow C_0^\infty(\tilde{S} \setminus S)$ and $\tilde{\mathcal{D}}(B) + f_0(B) \in (\ker \tilde{\mathcal{L}})^\perp$

Lemma \exists linear S solution operator to $\mathcal{L}(S(B)) = \mathcal{D}(B)$:

$$\|S(B)\|_{L^2} \leq C\|B\|_{L^2}.$$

Proof of Thm 1: solution operator to $\text{sym}\nabla(\mathcal{T}B) = B$

• We obtain linear $\mathcal{T}_1 : B \mapsto w$, $\text{sym}\nabla w_{tan} + (w\vec{n})\Pi = B$

• Modification $\mathcal{T}B := w - (\mathcal{P}(w_{tan}) + \alpha\vec{n})$

\mathcal{P} = projection onto tangent fields whose $\nabla_{tan} \in \text{span } \Pi$

$$\Rightarrow 0 = \text{sym}\nabla(\mathcal{P}(w_{tan})) + \alpha\Pi = \text{sym}\nabla(\mathcal{P}(w_{tan}) + \alpha\vec{n})$$

• Korn-type inequality: $\forall v \in W^{1,2}(S)$ tangent field

$$\|\nabla v\|_{L^2} \leq C(\|v\|_{L^2} + \|\mathbf{P}_{(\text{span } \Pi)^\perp}(\nabla v)_{tan}\|_{L^2})$$

• Usual argument by contradiction \Rightarrow

$$\|w_{tan} - \mathcal{P}(w_{tan})\|_{W^{1,2}} \leq C\|\mathbf{P}(\nabla w_{tan})_{tan}\|_{L^2}$$

• $\|(\mathcal{T}B)_{tan}\|_{W^{1,2}} \leq C(\|\mathbf{P}(\text{sym}\nabla w_{tan})\|_{L^2} + \|\mathbf{P}(\text{skew}\nabla w_{tan})\|_{L^2})$

$\leq C(\|\mathbf{P}(B)\| + \|S(B)\|) \leq C\|B\|_{L^2}$ by Lemma.

Conclusions

- **Lively interaction:**
Nonlinear elasticity, Analysis, Geometry
- **Reduction 3d to 2d:**
key point is rigidity of surfaces of various geometries,
small energy \Rightarrow isometries / infinitesimal isometries
- **Non-Euclidean elasticity:** morphogenesis of soft tissues, shape formation by growth, non-zero stress at equilib.
- **Rainbow of open problems:**
limiting theories for $\beta < 2$,
Sobolev isometries immersions
infinitesimal isometries on (e.g. hyperbolic) surfaces,
dynamic growth,
convergence of weak solutions of balance laws.

Infinite hierarchy of shell models, $\beta > 2$

- If $E^h(u^h) \leq Ch^\beta$ then $\frac{1}{h^2}E^h \rightarrow 0 \Rightarrow \int_S Q_2(\Pi(y) - \Pi) = 0$
 $\Rightarrow y$ is rigid motion and $u|_S = \text{id} + \sum_{i=1}^{\infty} \varepsilon^i V^i$
- formal calculations $\Rightarrow \frac{1}{h^\beta}E^h(u^h) \approx \frac{1}{h^\beta} \int_S |\delta g_S|^2 + \frac{h^2}{h^\beta} \int_S |\delta \Pi_S|^2$

change of metric change of Π

- energy distributed equipart. in stretching and bending:

$$\frac{1}{h^\beta} |\delta g_S|^2, \quad \frac{1}{h^{\beta-2}} |\delta \Pi_S|^2 = O(1) \Rightarrow \varepsilon = h^{\beta/2-1}$$

- $\delta g_S = (\nabla u^h)^T \nabla u^h - \text{Id} = \sum_{i=1}^{\infty} \varepsilon^i A_i$

i -th order change in metric $A_i = \sum_{j+k=i} \text{sym}((\nabla V_j)^T \nabla V_k)$

$$\Rightarrow \int_S |A_i|^2 \approx h^{(i-1)(\beta_i-\beta)} \quad \forall i$$

$$\beta_i = 2 + \frac{2}{i-1}$$