Global regular solutions to the Navier-Stokes equations in a cylinder with slip boundary conditions

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We consider the initial-boundary value problem

$$
(1)\qquad
\begin{aligned}
& v_t + v\cdot\nabla v - \mathrm{div}\mathbb{T}(v,p) = f && \text{in } \ \Omega^{kT} = \Omega\times(kT,(k+1)T),\\
& \mathrm{div} v = 0 && \\
& v\cdot\bar{n} = 0 && \text{on } \ S^{kT} = S\times(kT,(k+1)T),\\
& \nu\bar{n}\cdot\mathbb{D}(v)\cdot\bar{\tau}_\alpha + \gamma v\cdot\bar{\tau}_\alpha = 0 && \\
& v|_{t=kT} = v(kT) && \text{in } \ \Omega,
\end{aligned}
$$

$\Omega\subset\mathbb{R}^3$ is a cylindrical domain, $S=\partial\Omega$
$\bar{n}$ – unit outward normal vector to S,
$\bar{\tau}_\alpha$, $\alpha=1,2$, unit tangent vectors to S,
$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-pI$ – stress tensor
$\mathbb{D}(v)=\nabla v+(\nabla v)^T$ – dilatation tensor
$\nu>0$ – constant viscosity coefficient
$\gamma>0$ – slip coefficient
$k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Main Theorem

Assume that $v(0), v_{,x_3}(0) \in H^1(\Omega)$, $f \in L_2(\mathbb{R}_+, L_2(\Omega))$, $f_{,x_3} \in L_2(\mathbb{R}_+; L_2(\Omega))$, $\|f(t)\|_{L_2(\Omega)} \le \|f(0)\|_{L_2(\Omega)} e^{-\delta t}$, $\|f_{,x_3}(t)\|_{L_2(\Omega)} \le \|f_{,x_3}(0)\|_{L_2(\Omega)} e^{-\delta t}$, $\delta > 0$.
Assume that the quantity

$$\begin{aligned} &\|f_{,x_3}\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|f_3\|_{L_2(kT,(k+1)T;L_2(S_2))} \\ &+ \|v_{,x_3}(kT)\|_{L_2(\Omega)}, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \end{aligned}$$

is sufficiently small.
Then there exists a global solution to problem (1) such that

$$v \in W_2^{2,1}(\Omega \times (kT,(k+1)T)), \quad \nabla p \in L_2(kT,(k+1)T; L_2(\Omega))$$

for any $k \in \mathbb{N}_0$.

Lemma 1 (the Korn inequality)
Let $E_\Omega(v) = \|\mathbb{D}(v)\|^2_{L_2(\Omega)}$, $\mathrm{div} v = 0$, $v \cdot \bar{n}|_S = 0$ Ω *is not axially symmetric.*
There exists $c_1 = c_1(\Omega, S)$ *such that*

$$\|v\|^2_{H^1(\Omega)} \le c_1 E_\Omega(v). \tag{2}$$

Lemma 2 (weak solutions)

Assume $f \in L_\infty(\mathbb{R}_+; L_{6/5}(\Omega))$, $v(0) \in L_2(\Omega)$, Ω *is not axially symmetric*

There exists a weak solution to (1) such that

$$(3) \qquad \|v(t)\|_{L_2(\Omega)} \le \frac{1}{\sqrt{2}}(c_2\|f\|_{L_\infty(\mathbb{R}_+;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \equiv d_1,$$

where $t \in \mathbb{R}_+$, c_2 *– constant from* $H^1(\Omega) \subset L_6(\Omega)$,

$$(4) \qquad \begin{aligned} \|v\|_{V_2^0(\Omega\times(kT,t))} &\le \varphi(c_1,\nu)(\|f\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))}+ \\ &+ \|v(kT)\|_{L_2(\Omega)}) \equiv d_2, \end{aligned}$$

$t \in (kT,(k+1)T]$.

$$\|v\|_{V_2^0(\Omega\times(T_1,T_2))} = \operatorname{esssup}_{t\in(T_1,T_2)}\|v(t)\|_{L_2(\Omega)} + \left(\int_{T_1}^{T_2} \|\nabla v(t)\|^2_{L_2(\Omega)}dt\right)^{1/2}.$$

φ *is always increasing positive function.*

Lemma 3 *Assume that v is given. Then (h,q), $h=v_{,x_3}$, $q=p_{,x_3}$, $g=f_{,x_3}$, is a solution to the problem*

$$
\begin{aligned}
& h_{,t}-\mathrm{div}\mathbb{T}(h,q)=-v\cdot\nabla h-h\cdot\nabla v+g && \text{in } \Omega^{kT},\\
& \mathrm{div} h=0 \\
& h\cdot\bar n=0,\ \ \nu\bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha+\gamma h\cdot\bar\tau_\alpha=0,\ \ \alpha=1,2, && \text{on } S^{kT},\\
& h_i=0,\ \ i=1,2,\ \ h_{3,x_3}=0 && \text{on } S_2^{kT},\\
& h|_{t=kT}=h(kT) && \text{in } \Omega.
\end{aligned}
\tag{5}
$$

Lemma 4 *Let* $F_3 = (\mathrm{rot} f)_3$, h, v *be given. Then* $\chi = (\mathrm{rot} v)_3$ *is a solution to the problem*

$$
(6) \qquad \begin{aligned}
&\chi_{,t} + v \cdot \nabla \chi - h_3 \chi + h_2 v_{3,x_1} - h_1 v_{3,x_3} - \nu \Delta \chi = F_3 && \text{in } \Omega^{kT}, \\
&\chi = \sum_{i=1}^{2} a_i v_i && \text{on } S_1^{kT}, \\
&\chi_{,x_3} = 0 && \text{on } S_2^{T}, \\
&\chi|_{t=0} = \chi(kT) && \text{in } \Omega,
\end{aligned}
$$

where a_i, $i = 1, 2$, *depends on the tangent and normal vectors to* S_1 *and their derivatives.*

Estimates

Lemma 5 *Assume v is the weak solution,* $h \in L_\infty(kT,(k+1)T;L_3(\Omega))$*,*
$g \in L_2(kT,(k+1)T;L_{6/5}(\Omega))$*,* $f_3 \in L_2(S_2 \times (kT,(k+1)T))$*,*
$h(kT) \in L_2(\Omega)$*.*
Then

$$
\begin{aligned}
(7) \qquad & \|h\|_{V_2^0(\Omega\times(kT,t))} \le \varphi(c_1,c_2,\nu,d_2)(\|h\|_{L_\infty(kT,t;L_3(\Omega))} \\
& \quad + \|g\|_{L_2(kT,t;L_{6/5}(\Omega))} + \|f_3\|_{L_2(S_2\times(kT,t))}) + \|h(kT)\|_{L_2(\Omega)},
\end{aligned}
$$

$t \in (kT,(k+1)T]$.

Lemma 6 *Let the assumptions of Lemma 5 hold. Let*
$v \in L_2(kT,(k+1)T;L_3(\Omega))$*.*
Then

$$
\begin{aligned}
(8) \qquad & \|h\|^2_{V_2^0(\Omega\times(kT,t))} \le \varphi(c_1,c_2,\nu)\exp(\|\nabla v\|^2_{L_2(kT,(k+1);L_3(\Omega))}) \\
& \quad \cdot [\|g\|^2_{L_2(kT,t;L_3(\Omega))} + \|f_3\|^2_{L_2(kT,t;L_2(S_2))} + \|k(kT)\|^2_{L_2(\Omega)}],
\end{aligned}
$$

$t \in (kT,(k+1)T]$.

Lemma 7 *Assume that* $h \in L_\infty(kT,(k+1)T;L_3(\Omega))$, $v' = (v_1, v_2) \in L_\infty(kT,(k+1)T;H^1(\Omega)) \cap L_2(kT,(k+1)T;H^2(\Omega)) \cap L_2(\Omega;H^{1/2}(kT,(k+1)T))$, $\chi(kT) \in L_2(\Omega)$, $F_3 \in L_2(kT,(k+1)T;L_{6/5}(\Omega))$. *Then solutions to (6) satisfy the inequality*

$$\begin{aligned} \|\chi\|_{V_2^0(\Omega\times(kT,t))} &\le \varepsilon(\|v'\|_{L_\infty(kT,t;H^1(\Omega))} + \|v'\|_{L_2(kT,t;H^2(\Omega))}) \\ &+ c_3\|v'\|_{L_2(\Omega;H^{1/2}(kT,t))} + \frac{1}{\varepsilon}\varphi(c_1,c_3,d_2)(1+\sup_t\|h(t)\|_{L_3(\Omega)}) \\ &+ \varphi(c_2\nu)\|F_3\|_{L_2(kT,t;L_{6/5}(\Omega))} + \|\chi(kT)\|_{L_2(\Omega)}, \end{aligned} \tag{9}$$

where $t \in (kT,(k+1)T]$, c_3 *does not depend on* T, $\varepsilon \in (0,1)$.
Let us consider the elliptic problem

$$\begin{aligned} & v_{1,x_2} - v_{2,x_1} = \chi && \text{in } \Omega' = \Omega \cap \{\text{plane } : \ x_3 = \text{const}\} \\ & v_{1,x_1} + v_{2,x_2} = -h_3 && \text{in } \Omega' = \Omega \cap \{\text{plane } \ x_3 = \text{const}\} \\ & v' \cdot \bar{n}' = 0 && \text{on } S_1' = S_1 \cap \{\text{plane} : \ x_3 = \text{const}\}, \end{aligned} \tag{10}$$

where x_3 *and* t *are treated as parameters.*

Lemma 8 *Let the assumptions of Lemmas 4 and 7 be satisfied. Then for solutions to (10) the inequality is valid*

$$\|v'\|_{V_2^1(\Omega\times(kT,t))} \leq \varphi(c_1, c_2, c_3, d_1, d_2) \cdot [\|h\|_{L_\infty(kT,t;L_3(\Omega))} + 1 + \|v'\|_{L_2(\Omega,H^{1/2}(kT,t))} + K_1(k,T)], \tag{11}$$

where

$$K_1(k,T) = \|g\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|F_3\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|f_3\|_{L_2(S_2\times(kT,(k+1)T))} + \|h(kT)\|_{L_2(\Omega)} + \|\chi(kT)\|_{L_2(\Omega)}, \tag{12}$$

and

$$\|v'\|_{V_2^1(\Omega\times(kT,(k+1)T))} = \operatorname{esssup}_t \|v'\|_{H^1(\Omega)} + \left(\int_{kT}^{(k+1)T} \|\nabla v'\|^2_{H^1(\Omega)} dt \right)^{1/2}.$$

Let us consider problem (1) in the form

$$
(13)\qquad
\begin{aligned}
& v_t - \mathrm{div}\mathbb{T}(v,p) = -v' \cdot \nabla v - v_3 h + f \\
& \mathrm{div} v = 0 \\
& v \cdot \bar{n}|_S = 0, \quad \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha|_S = 0, \quad \alpha = 1,2, \\
& v|_{t=kT} = v(kT).
\end{aligned}
$$

In view of imbedding

$$
(14)\qquad \|v'\|_{L_{10}(\Omega^{kT})} \le c_4 \|v'\|_{V_2^1(\Omega^{kT})}
$$

with c_4 independent of T, we obtain for solutions to (13) the inequality

$$
(15)\qquad \|v\|_{W^{2,1}_{5/3}(\Omega^{kT})} + \|\nabla p\|_{L_{\frac{5}{3}}(\Omega^{kT})} \le c_5(d_2 H(k,T) + K_2(k,T)),
$$

with c_5 independent of T, where

$$
(16)\qquad H(k,T) = \|h\|_{L_\infty(kT,(k+1)T;L_3(\Omega))} + \|h\|_{L_{\frac{10}{3}}(\Omega^{kT})},
$$

$$K_2(k,T) = K_1(k,T) + \varphi(d_1,d_2)(d_1+d_2) + \|f\|_{L_{5/3}(\Omega^{kT})} + \|v(kT)\|_{W^{4/5}_{5/3}(\Omega)}. \tag{17}$$

Increasing regularity by using the estimates

$$\|v'\cdot\nabla v\|_{L_2(\Omega^{kT})} \le \|v'\|_{L_{10}(\Omega^{kT})}\|\nabla v\|_{L_{5/2}(\Omega^{kT})}$$
$$\le c\|v'\|_{V_2^1(\Omega^{kT})}\|v\|_{W^{2,1}_{5/3}(\Omega^{kT})} \le \varphi(d_1,d_2)(H+K_2)^2$$

and

$$\|v_3h\|_{L_2(\Omega^{kT})} \le \|v_3\|_{L_5(\Omega^{kT})}\|h\|_{L_{\frac{10}{3}}(\Omega^{kT})}$$
$$\le \|v\|_{W^{2,1}_{5/3}(\Omega^{kT})}\|h\|_{L_{\frac{10}{3}}(\Omega^{kT})}$$
$$\le \varphi(d_1,d_2)(H+K_2)^2$$

we obtain the inequality

$$\begin{aligned}(18)\qquad &\|v\|_{W_2^{2,1}(\Omega^{kT,t})} + \|\nabla p\|_{L_2(\Omega^{kT,t})} \\ &\le \varphi(d_1, d_2)(H + K_2)^2 + c(\|f\|_{L_2(\Omega^{kT,t})} + \|v(kT)\|_{H^1(\Omega)}),\end{aligned}$$

where $\Omega^{kT,t} = \Omega \times (kT, t)$.
In view of (18) we obtain for solutions to (5) the inequality

$$\begin{aligned}(19)\qquad &\|h\|_{W_2^{2,1}(\Omega^{kT,t})} + \|\nabla q\|_{L_2(\Omega^{kT,t})} \\ &\le (\varphi(d_1, d_2)H^8 + K_3^4)\|h\|_{L_2(\Omega^{kT,t})} + K_4(kT),\end{aligned}$$

where

$$K_3 = \varphi(d_1, d_2)K_2^2 + \|f\|_{L_2(\Omega^{kT})} + \|v(kT)\|_{H^1(\Omega)},$$

$$K_4 = \|g\|_{L_2(\Omega^{kT})} + \|h(kT)\|_{H^1(\Omega)}.$$

Let

$$d(kT) = \|g\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|f_3\|_{L_2(kT,(k+1)T;L_2(S_2))} + \|h(kT)\|_{L_2(\Omega)}.$$

Then Lemma 6 implies

$$\|h\|_{L_2(\Omega^{kT})} \leq \exp(d_2 H + K_2) d(kT). \tag{20}$$

Since

$$H(kT) \leq c\|h\|_{W_2^{2,1}(\Omega^{kT})} \equiv cX(kT)$$

we obtain from (19) and (20) the inequality

$$X(kT) \leq c[\varphi(d_1, d_2) X^8(kT) + K_3^4(kT)] \cdot \exp(d_2 X(kT) + K_2(kT)) d(kT) + K_4(kT). \tag{21}$$

There exists a constant $A = A(k,T)$ such that

$$X(kT,t) \leq A(k,T) \tag{22}$$

if

$$c[\varphi(d_1,d_2)A^8 + K_3^8]\exp(d_2A + K_2)d + K_4 \leq A.$$

The above inequality holds for d sufficiently small.
Next:

1. we prove the existence on each interval $[kT,(k+1)T]$,
2. to prove global existence we have to show that $A = A(T)$, does not depend on k.

Existence in $[kT,(k+1)T]$, $k \in \mathbb{N}_0$.

To prove the existence of solutions we use the Leray-Schauder fixed point theorem. For this purpose we construct the mappings

$$
\begin{aligned}
& v_t - \mathrm{div}\mathbb{T}(v,p) = -\lambda\tilde{v}\cdot\nabla\tilde{v} + f \\
& \mathrm{div} v = 0 \\
& v\cdot\bar{n}|_S = 0, \quad \nu\bar{n}\cdot\mathbb{D}(v)\cdot\bar{\tau}_\alpha + \gamma v\cdot\bar{\tau}_\alpha|_S = 0, \quad \alpha = 1,2, \\
& v|_{t=kT} = v(kT).
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
& h_t - \mathrm{div}\mathbb{T}(h,q) = -\lambda(\tilde{v}\cdot\nabla\tilde{h} + \tilde{h}\cdot\nabla\tilde{v}) + g \\
& \mathrm{div} h = 0, \\
& h\cdot\bar{n}|_{S_1} = 0, \quad \nu\bar{n}\cdot\mathbb{D}(h)\cdot\bar{\tau}_\alpha + \gamma h\cdot\bar{\tau}_\alpha|_{S_1} = 0, \quad \alpha = 1,2, \\
& h_i|_{S_2} = 0, \quad i = 1,2, \quad h_{3,x_3}|_{S_2} = 0, \\
& h|_{t=kT} = h(kT),
\end{aligned} \tag{24}
$$

where $\lambda\in[0,1]$, $\tilde{v},\tilde{h}$ are treated as given.

Problems (23) and (24) imply the mappings

$$\Phi_1 : \ (\tilde{v}, \lambda) \to (v, p),$$
$$\Phi_2 : \ (\tilde{v}, \tilde{h}, \lambda) \to (h, q).$$

Let $\Phi = (\Phi_1, \Phi_2)$.
Estimate (22) is the appriori estimate for a fixed point of Φ for $\lambda = 1$.
For $\lambda = 0$ we have a unique existence of solutions to problems (23) and (24).
It remains to show compact continuity of Φ.
Let $\mathcal{M}(\Omega^{kT}) = L_{2r}(kT, (k+1)T; W^1_{\frac{6\eta}{3+\eta}}(\Omega))$, $r \geq 2$, $\eta \geq 2$.
Then

$$\|\tilde{v} \cdot \nabla \tilde{v}\|_{L_r(kT,(k+1)T;L_\eta(\Omega))} \leq c\|\tilde{v}\|^2_{\mathcal{M}(\Omega^{kT})}.$$

Hence

$$\Phi_1 : \ \mathcal{M}(\Omega^{kT}) \to W^{2,1}_{\eta,r}(\Omega^{kT}) \subset \mathcal{M}(\Omega^{kT})$$

where the imbedding is compact for

$$1 < \frac{3}{2\eta} + \frac{1}{r}. \tag{25}$$

In the same way compactness of Φ_2 can be shown.
To show continuity of Φ we consider (23) and (24) for v_i, h_i, $\tilde{v}_i$, $\tilde{h}_i$, $i = 1, 2$.
Then the functions

$$V = v_1 - v_2, \quad \tilde{V} = \tilde{v}_1 - \tilde{v}_2, \quad H = h_1 - h_2, \quad \tilde{H} = \tilde{h}_1 - \tilde{h}_2,$$
$$P = p_1 - p_2, \quad Q = q_1 - q_2$$

are solutions to the problems

$$\begin{aligned} & V_t - \mathrm{div}\mathbb{T}(V, P) = -\lambda(\tilde{V} \cdot \nabla \tilde{v}_1 + \tilde{v}_2 \cdot \nabla \tilde{V}), \\ & \mathrm{div} V = 0, \\ & V \cdot \bar{n}|_S = 0, \quad \nu \bar{n} \cdot \mathbb{D}(V) \cdot \bar{\tau}_\alpha + \gamma V \cdot \bar{\tau}_\alpha|_S = 0, \\ & V|_{t=0} = 0 \end{aligned} \tag{26}$$

and

$$
\begin{aligned}
& H_t - \mathrm{div}\mathbb{T}(H,Q) = -\lambda(\tilde{\tilde{H}} \cdot \nabla \tilde{v}_1 + \tilde{\tilde{h}}_2 \cdot \nabla \tilde{\tilde{V}} + \tilde{\tilde{V}} \cdot \nabla \tilde{\tilde{h}}_1 + \tilde{v}_2 \cdot \nabla \tilde{\tilde{H}}) \\
& \mathrm{div} H = 0 \\
(27) \quad & H \cdot \bar{n}|_{S_1} = 0, \ \ \nu \bar{n} \cdot \mathbb{D}(H) \cdot \bar{\tau}_\alpha + \gamma H \cdot \bar{\tau}_\alpha|_{S_1} = 0, \ \ \alpha = 1,2, \\
& H_i|_{S_2} = 0, \ \ i = 1,2, \ \ H_{3,x_3}|_{S_2} = 0, \\
& H|_{t=0} = 0.
\end{aligned}
$$

Let us consider problem (26). Then continuity of Φ_1 follows from the inequalitites $(\lambda \neq 0)$

$$
\begin{aligned}
\|V\|_{\mathcal{M}(\Omega^{kT})} &= \|V\|_{L_{2r}(kT;(k+1)T;W^1_{\frac{6\eta}{3+\eta}}(\Omega))} \\
&\le c\|V\|_{W^{2,1}_2(\Omega^{kT})} \le c\|V\|_{W^{2,1}_{\eta,r}(\Omega^{kT})} \\
&\le \varphi(A(k,T))\|\tilde{V}\|_{L_{2r}(kT,(k+1)T;W^2_{\frac{6\eta}{3+\eta}}(\Omega))} = \varphi(A(k,T))\|\tilde{V}\|_{\mathcal{M}(\Omega^{kT})},
\end{aligned}
$$

for $r \ge 2$, $\eta \ge 2$.
Similar considerations imply continuity of Φ_2.

To prove global existence we have to show that constant $A(k,T)$ appearing in (22) does not depend on k. $A(k,T)$ depends on k by norms $\|v(kT)\|_{H^1(\Omega)}$ and $\|h(kT)\|_{H^1(\Omega)}$ only.
Therefore, we have to show that

$$\|v((k+1)T)\|_{H^1(\Omega)} \leq \|v(kT)\|_{H^1(\Omega)} \tag{28}$$

and

$$\|h((k+1)T)\|_{H^1(\Omega)} \leq \|h(kT)\|_{H^1(\Omega)} \tag{29}$$

for any $k \in \mathbb{N}_0$.

Multiplying $(1)_1$ by $\mathrm{div}\mathbb{T}(v,p)$ and integrating over Ω yields

$$
\begin{aligned}
\|\mathbb{D}(v(k+1)T)\|^2_{L_2(\Omega)} &\le e^{-\nu c_6 T + c_7} \int\limits_{kT}^{(k+1)T} \|v(t)\|^2_{L_\infty(\Omega)} dt \\
&\cdot \left[\int\limits_{kT}^{(k+1)T} \|f(t)\|^2_{L_2(\Omega)} e^{\nu c_6 t} dt + \|\mathbb{D}(v(kT))\|^2_{L_2(\Omega)} \right],
\end{aligned}
\tag{30}
$$

where c_6 and c_7 do not depend on T.
By the Korn inequality

$$
c_1 \|v\|_{H^1(\Omega)} \le \|\mathbb{D}(v)\|_{L_2(\Omega)} \le c_1' \|v\|_{H^1(\Omega)}
$$

the decay

$$
\|f(t)\|_{L_2(\Omega)} \le e^{-\delta t}
$$

and T sufficiently large, inequality (30) implies (28).

Similarly, we obtain

$$\|\mathbb{D}(h((k+1)T))\|^2_{L_2(\Omega)} \leq c^{-\nu c_6 T + c_7} \int\limits_{kT}^{(k+1)T} (\|v(t)\|^2_{L_\infty(\Omega)} + \|\nabla v(t)\|^2_{L_3(\Omega)}) dt \cdot \left[\int\limits_{kT}^{(k+1)T} \|g(t)\|^2_{L_2(\Omega)} e^{\nu c_6 t} dt + \|\mathbb{D}(h(kT))\|^2_{L_2(\Omega)} \right] \tag{31}$$

Then (31), the decay

$$\|g(t)\|_{L_2(\Omega)} \leq e^{-\delta t}$$

and T sufficiently large imply (29).