Workshop on

"Frontiers in Nonlocal Nonlinear PDEs"'

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Book of abstracts

ALL RELATIVE ENTROPIES FOR GENERAL NONLINEAR FOKKER-PLANCK EQUATIONS

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We shall revisit the entropy method for quasilinear Fokker-Planck equations with confinement to deduce exponential convergence to the equilibrium. Even for prototypical examples like the porous-medium equation, only one relative entropy has been known so far - the Ralston-Newman entropy, which is the analog of the logarithmic entropy in the linear case. We shall give a complete characterization of all admissible relative entropies for each quasilinear Fokker-Planck equation. In particular we find that fast-diffusion equations with power-law nonlinearities admit only one entropy, while porous medium equations give rise to a whole family of admissible relative entropies (similar to linear Fokker-Planck equations). These additional entropies then imply also new moment-control estimates on the porous-medium solution.
This is a joint work with José Antonio Carrillo and Daniel Matthes.

LONG-TIME BEHAVIOR OF THE INTEGRATE AND FIRE NEURON POPULATION MODEL WITH LONG DELAYS

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We study the asymptotic behavior of the nonlinear integrate-and-fire mean-field model for the electrical activity in a population of neurons. This model is quite interesting since it shows blow up in cases without delay for certain initial conditions, has stable and unstable equilibria, and is understood essentially only in perturbative regimes. When the delay d of the interaction effect between neurons is large, we show that the system behaves essentially like the linear system, without interaction, in each time interval of duration d. The background firing rate is adjusted at the end of each interval to be the equilibrium firing rate of the previous step. Hence, a good approximation of the dynamic nonlinear behavior is given by the iteration of equilibria obtained from the initial data. This shows rigorously, for large delays, that there are stable and unstable equilibria for some values of the connectivity parameter b. The proof uses the relative entropy to these "transient states", together with some new estimates of the equation in strong norma. This work is a collaboration with MarÃa J. CÃ¡ceres and Alejandro Ramos-Lora.

MEAN FIELD SPARSE OPTIMAL CONTROL OF SYSTEMS WITH ADDITIVE NOISE

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We analyze the problem of controlling a multi-agent system with additive white noise through parsimonious interventions on a selected subset of the agents (leaders). For such a controlled system with a SDE constraint, we introduce a rigorous limit process towards an infinite dimensional optimal control problem constrained by the coupling of a system of ODE for the leaders with a McKean-Vlasov-type SDE, governing the dynamics of the prototypical follower. The latter is, under some assumptions on the distribution of the initial data, equivalent with a (nonlinear parabolic) PDE-ODE system. The derivation of the limit mean-field optimal control problem is achieved by linking the mean-field limit of the governing equations together with the Γ-limit of the cost functionals for the finite dimensional problems. Joint work with Giacomo Ascione (SSM Napoli) and Francesco Solombrino (Napoli Federico II).

SINGULARITY FORMATION FOR THE KELLER-SEGEL SYSTEM IN THE PLANE

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The classical model for chemotaxis is the planar Keller-Segel system

$$u_t = \Delta u - \nabla \cdot (u \nabla v), \quad v(\cdot, t) = \frac{1}{2\pi} \log \frac{1}{|\cdot|} * u(\cdot, t).$$

in $\mathbb{R}^2 \times (0, \infty)$. Blow-up of finite mass solutions is expected to take place by aggregation, which is a concentration of bubbling type, common to many geometric flows. We build with precise profiles solutions in the critical-mass case 8π, in which blow-up in infinite time takes place. We establish stability of the phenomenon detected under arbitrary mass-preserving small perturbations and present new constructions in the finite time blow-up scenario.

MANY-PARTICLE LIMIT FOR A SYSTEM OF INTERACTION EQUATIONS DRIVEN BY NEWTONIAN POTENTIALS

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We consider a discrete particle system of two species coupled through nonlocal interactions driven by the one-dimensional Newtonian potential, with repulsive self-interaction and attractive cross-interaction. After providing a suitable existence theory in a finite-dimensional framework, we explore the behaviour of the particle system in case of collisions and analyse the behaviour of the solutions with initial data featuring particle clusters. Subsequently, we prove that the empirical measure associated to the particle system converges to the unique 2-Wasserstein gradient flow solution of a system of two partial differential equations (PDEs) with nonlocal interaction terms in a proper measure sense. The latter result uses uniform estimates of the L^p-norms of a piecewise constant reconstruction of the density using the particle trajectories.

HYPOCOERCIVITY IN KINETIC EQUATIONS

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Hypocoercivity in kinetic equations relies on the construction of a norm which involves a kind of "twist" between position and velocity directions. If the norm is equivalent to an L^2 norm, it is very convenient to use a non-local perturbation which relies on properties of the diffusion limit. This approach is very robust and covers mean field potentials (electrostatic Poisson interaction) as well as anomalous diffusion limits.

POROUS MEDIUM EQUATION AS LIMIT OF NONLOCAL INTERACTION

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I will discuss a recent result in collaboration with Martin Burger (FAU Erlangen), where we connect the quadratic porous medium equation with a nonlocal interaction equation. We prove the convergence of solutions of a nonlocal interaction equation to the solution of the quadratic porous medium equation in the limit of a localising interaction kernel. The analysis is carried out at the level of the (nonlocal) partial differential equations and we use the gradient flow structure of the equations to derive bounds on energy, second order moments, and logarithmic entropy. The dissipation of the latter yields sufficient regularity to obtain compactness results and pass to the limit in the localised convolutions. The strategy we propose relies on a discretisation scheme which could be slightly modified in order to extend our result to PDEs with no gradient flow structure. Our analysis allows to treat the case of limiting weak solutions of the non-viscous porous medium equation at relevant low regularity, assuming the initial value to have finite energy and entropy. However, the latter excludes particle solutions of the nonlocal interaction equation.

RECENT DEVELOPMENTS ON SOME NONLINEAR ANISOTROPIC DIFFUSION EQUATIONS

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In this talk, I will expose several recent results concerning the study of some nonlinear anisotropic evolution equations. In particular, the main model which will be discussed is an anisotropic, possibly non-homogeneous version of the evolution p-Laplacian equation,when fast diffusion holds in all directions. In particular, the existence of a self-similar fundamental solution to this equation is shown, while uniqueness is proven in a smaller range. Moreover, asymptotic behavior of finite mass solutions in terms of the self-similar solution will be sketched. The results are based on a recent joint work with J. L. Vázquez and B. Volzone.

SYMMETRIZATION FOR FRACTIONAL ELLIPTIC PROBLEMS

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We describe symmetrization results in the form of mass concentration (*i.e.* integral) comparison for fractional elliptic equations of the type $(-\Delta)^s u = f$ $(0 < s < 1)$ in a bounded domain Ω, equipped with homogeneous Dirichlet boundary conditions. We use a new direct method which recovers, in the limit $s \to 1$, the classical pointwise Talenti rearrangement inequality. Some possible applications of the method to nonlinear equations and to equations with lower order terms will be discussed.

THE SINGULAR SET IN THE STEFAN PROBLEM

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The Stefan problem describes phase transitions such as ice melting to water, and it is among the most classical free boundary problems. It is well known that the free boundary consists of a smooth part (the regular part) and singular points. In this talk, I will describe a recent result with Ros-Oton and Serra, where we analyze the singular set in the Stefan problem and prove a series of fine results on its structure.

CONCENTRATION VERSUS SIMPLIFICATION IN AGGREGATION-DIFFUSION EQUATIONS

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Over the last two decades, intense work has been devoted to the Aggregation-Diffusion equation

$$\partial_t \rho = \operatorname{div}\Big(\rho \nabla (U'(\rho) + V + W * \rho)\Big).$$

This family of problem models, amongst other phenomena, the mean-field limit of systems with a large number of interacting particles arising in biology. It includes, for example, the famous model by Keller-Segel for chemotaxis. From the mathematical point of view, they benefit from having a gradient flow structure. There is a long literature discussing the existence of gradient-flow solutions, characterising the existence/non-existence of minimisers of the associated free-energy functional and, in particular, discussing the existence or not of delta Deltas. The presence of a Delta is usually described as a concentration phenomena. In the absence of candidate stationary state, the solutions diffuse.In this talk, I will discuss two behaviours at opposite ends of the spectrum. On the one hand, the asymptotic formation of Dirac deltas in the case of Fast Diffusion. On the other, a general result of asymptotic simplification to the heat kernel when W is bounded, with suitably integrable derivatives. The talk presents joint work with J.A. Carrillo, J.L. Vázquez, Y. Yao, and C. Zeng.

THE FRACTIONAL POROUS MEDIUM EQUATION ON CLASSES OF NONCOMPACT RIEMANNIAN MANIFOLDS

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We consider porous medium type equations on noncompact manifolds M, the evolution being driven by the fractional Laplacian on M, such operator being meant in the spectral sense. The manifold is assumed to satisfy analytic-geometric properties like: the validity of Faber-Krahn (or Gagliardo-Nirenberg, or Sobolev) inequalities on M, the fact that M is Cartan-Hadamard, namely that M is simply connected with nonpositive curvature, and the condition $\sec \leq -c < 0$, sec denoting sectional curvature. Under the sole assumption that the Faber-Krahn inequality holds on M, we prove well-posedness of the problem, in the weak dual sense, for data belonging to a weighted L^1 space where the weight is, roughly speaking, the fractional Green function on the manifold, assumed to exists. Various kind of smoothing effects are then proved, with stronger results as soon as further assumptions are required. This is a joint work with E. Berchio, M. Bonforte, M. Muratori.

AN SPDE APPROACH TO LIMIT THEOREMS FOR STOCHASTIC GRADIENT DESCENT

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We study limit theorems (LLN and CLT) for stochastic gradient descent (SGD) in the over-parameterised ($N \to \infty$) and small learning rate ($\sigma \to 0$) regimes. We approach the problem by rigorously associating to SGD the solution of a Dean–Kawasaki-like SPDE with conservative, multiplicative noise of the form conjectured by Rotskoff and van Den Eijnden. Since the coefficients of the noise in this equation are irregular and nonlocal, we approach it through its associated martingale problem. The main contribution of this work is associating to this martingale problem an associated "coupling SPDE", for which we establish probabilistically strong existence and uniqueness via an appropriate nonlinear superposition principle. In contrast to previous work in this direction, this allows us to obtain quantitative (and sharp) limit theorems for SGD and also to decouple the limits $M \to \infty$ and $\sigma \to 0$. This joint work with Benjamin Gess (Bielefeld & MPI-MiS) and Vitalii Konarovskyi (Bielefeld).

OPTIMAL CONTROL PROBLEMS FOR INTERACTION EQUATIONS: EQUIVALENCE OF LAGRANGIAN, EULERIAN AND KANTOROVICH FORMULATIONS, GAMMA-CONVERGENCE AND FINITE PARTICLES APPROXIMATION

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In this talk I will describe a family of optimal control problems for a class of interaction equations. I will describe the Lagrangian, Eulerian and Kantorovich formulation of the problem and I will discuss a convex relaxation and an equivalence result for the various formulations. Finally, a finite particle approximation and a Gamma-convergence result from the discrete to the continuous will be illustrated. The talk is based on works in collaboration with G. Cavagnari, C. Orrieri, G. Savaré.

DATA AND (KINETIC) MODELS FOR THE COVID EPIDEMICS, WHAT CAN THEY EXPLAIN, WHAT CAN THEY PREDICT

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What is now called age-structured population dynamics, was developed in the context of epidemiology by A.G. McKendrick in a series of papers between 1914 - 1926. McKendrick's inspiration was L. Boltzmann, and the models were meant to give a better description of the statistics of an epidemic than the earlier ODE compartment models. One of the difficulties is to model the infection process, including the evolution of infectiousness in the individual. I will discuss data from Vo, Gangelt, Zuhai, Geneva, a study on the risk of infection in public transport in the Frankfurt region, and the data on the symptomatic infection among the placebo group in the Pfizer-BionTech study. I will also explain a modification of the so-called Kermack-McKendrick model proposed by A. Stevens and myself, and the consequences on qualitative behavior of these modifications.

AGGREGATION-DIFFUSION MODELS WITH RIESZ POTENTIALS

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Aggregation-diffusion models describe the motion of interacting agents towards states of overall balance between diffusion effects and mutual attraction. We consider evolution models with porous medium type diffusion and with aggregation governed by the Riesz potential of order s, and we discuss their gradient flow structure. We analyze the stationary states of the dynamics, which are characterized by fractional elliptic PDEs: we focus on their uniqueness and regularity properties, as well as their behavior in the most singular regime, as s goes to 0.

LIMIT PROFILES FOR A CLASS OF CHOQUARD EQUATIONS

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We discuss the asymptotic behaviour of groundstates for a class of singularly perturbed Choquard type equations with a local repulsion term. We identify seven different asymptotic regimes and provide a characterisation of the limit profiles of the groundstates when perturbation parameter is small. We also outline the behaviour of groundstates when perturbation is strong. In some of the regimes the limit profile is given by a compactly supported discontinuous minimizer of a Thomas-Fermi type variational problem. This is joint work with Zeng Liu (Suzhou, China) and Damiano Greco (Swansea).

STABILITY OF THE POROUS MEDIUM EQUATION ON RIEMANNIAN MANIFOLDS WITH RESPECT TO THE 2-WASSERSTEIN METRIC

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Given a complete and connected Riemannian manifold M with Ricci curvature bounded from below, we study the stability properties of nonnegative solutions of the porous medium equation (or slightly more general diffusion equations) with respect to the 2-Wasserstein distance. Under an additional non-collapse condition on the volume measure of M, we show a stability estimate that works under possibly negative curvature bounds, which to some extent generalizes important results by Sturm and Otto-Westdickenberg that were proved under nonnegative Ricci curvature. The overall strategy of proof strongly relies on a quantitative smoothing effect that yields instantaneous boundedness of the solutions even if the initial datum is a finite measure, combined with the abstract Hamiltonian approach recently developed by Ambrosio, Mondino and Savaré. By means of an explicit construction in the special case of the hyperbolic space, we are also able to show that our stability estimate is optimal for small times. This has some important consequences with respect to the EVI gradient-flow interpretation of the equation.

The talk is based on a joint work with N. De Ponti and C. Orrieri.

LEAPFROGGING FOR EULER EQUATIONS AND THE VORTEX FILAMENT CONJECTURE

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We consider the Euler equations for incompressible fluids in 3-dimension. A classical question that goes back to Helmholtz is to describe the evolution of vorticities with a high concentration around a curve. The work of Da Rios in 1906 states that such a curve must evolve by the so-called "binormal curvature flow". Existence of true solutions whose vorticity is concentrated near a given curve that evolves by this law is a long-standing open question that has only been answered for the special case of a circle travelling with constant speed along its axis, the thin vortex-rings. In this talk I will discuss the construction of helical filaments, associated to a translating-rotating helix, and of two vortex rings interacting between each other, the so-called leapfrogging. The results are in collaboration with J. Davila (U. of Bath), M. del Pino (U. of Bath) and J. Wei (U. of British Columbia).

SEGREGATED SOLUTIONS FOR SOME NONLINEAR SCHRÖDINGER SYSTEMS

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I will focus on the existence and multiplicity of positive non-radial solutions for some systems of Schrödinger equations in a weak fully attractive or repulsive regime in presence or in absence of an external radial trapping. The results have been obtained in collaboration with Haixia Chen (Central China Normal University, Wuhan), Maria Medina (Universidad Autonoma de Madrid) and Giusi Vaira (Università di Bari).

LARGE-TIME BEHAVIOUR FOR ANISOTROPIC STABLE NONLOCAL DIFFUSION PROBLEMS WITH CONVECTION

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We study the large-time behaviour of nonnegative solutions to the Cauchy problem for a nonlocal heat equation with a nonlinear convection term. The diffusion operator is the infinitesimal generator of a stable Lévy process, which may be highly anisotropic. The initial data are assumed to be bounded and integrable. The mass of the solution is conserved along the evolution, and the large-time behaviour is given by the source-type solution with this mass of a limit equation that depends on the relative strength of convection and diffusion. When diffusion is stronger than convection the original equation simplifies asymptotically to the purely diffusive nonlocal heat equation. When convection dominates, it does so only in the direction of convection, and the limit equation is still diffusive in the subspace orthogonal to this direction, with a diffusion operator that is a "projection" of the original one onto the subspace. The determination of this projection is one of the main issues of the paper. When convection and diffusion are of the same order the limit equation coincides with the original one. Most of our results are new even in the isotropic case in which the diffusion operator is the fractional Laplacian. We assume in principle that the convection nonlinearity is locally Lipschitz, but we are also able to cover some situations in which it is only locally Hölder. This possibility has never been considered before in the nonlocal diffusion setting. This is a joint work with Jørgen Endal (U. Autónoma de Madrid) and Liviu Ignat (Simion Stoilow Institute of Mathematics of the Romanian Academy).

DISSIPATIVE EVOLUTIONS OF PROBABILITY MEASURES

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The theory of Otto-Wasserstein gradient flows driven by displacement convex functionals in the space of probability measures have been developed in many directions with new and interesting applications, including nonlocal models of interacting particles. Thanks to it, it is possible to construct a semigroup of contractions whose orbits are approximated by the solutions of the JKO scheme with an explicit estimate of the error in terms of the discretization step. We aim to extend the theory to the case of more general evolutions, which are not necessarily generated by a functional but still retain the contraction (or λ-contraction) property with respect to the L^2-Kantorovich-Rubinstein-Wasserstein distance. They are induced by a general (possibly multivalued) dissipative probability vector field, i.e. a set of probability measures on the tangent space $\mathrm{Tan}(\mathbb{R}^d)$ (which my be identified with $\mathbb{R}^d \times \mathbb{R}^d$) which satisfy a suitable notion of dissipativity. We show that this extension is coherent with the case of gradient flows and we discuss some conditions which ensure the generation of contraction semigroups and the convergence of the explicit or the implicit Euler scheme. (in collaboration with Giulia Cavagnari and Giacomo Sodini)

MATHEMATICAL MODELS IN BIOLOGICAL MODEL ORGANISMS

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Flatworms are among the biological model organisms for regeneration. Severe cutting and grafting experiments still result in the regeneration of a complete and functioning flatworm. So far, most mathematical models could not account for these basic experiments, without adaptations for each setting. A new model is suggested and analyzed, which covers the central regeneration experiments, while not changing the structure of the equations. Dynamic boundary conditions play a crucial role here.
(Joint work with A. Scheel and C. Tenbrock)

The p-LAPLACIAN EVOLUTION EQUATION IN SEVERAL SCENARIOS

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We will present several lines of progress in the study of p-Laplacian evolution equations both of local or nonlocal type. Existence, regularity, selfsimilar solutions and asymptotic behaviour are covered.

ASYMPTOTIC BEHAVIOR OF AGGREGATION-DIFFUSION EQUATIONS WITH BOUNDED INTERACTION POTENTIAL

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In this work, we consider the aggregation-diffusion equation with linear diffusion and a bounded interaction potential. It is well-known that there is no steady state with finite mass for such potential, but the long time behavior of solutions was unclear. We show that if the interaction potential is bounded and its first and second derivatives decay fast enough at infinity, then all solutions converge to the fundamental solution of the heat equation with some rate as times goes to infinity, without any smallness assumption on the initial data. (Joint work with José Carrillo, David Gómez-Castro, and Chongchun Zeng).