Interaction-Driven Dynamics for Collective Behavior: Derivation, Model Hierarchies and Pattern Stability

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Imperial College London

Lecture 1, L'Aquila 2015

Outline

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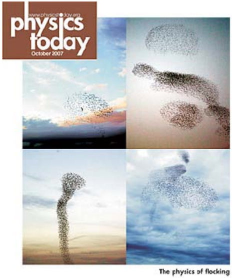
Swarming by Nature or by design?



Fish schools and Birds flocks.

Individual Based Models (Particle models)

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
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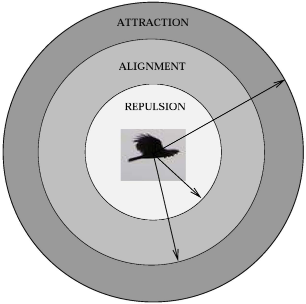

Metric versus Topological Interaction

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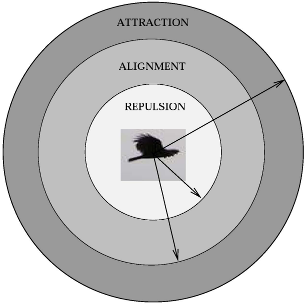

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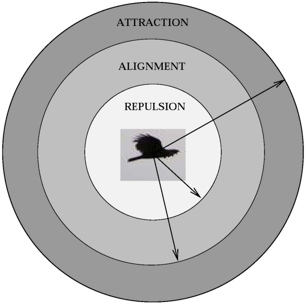

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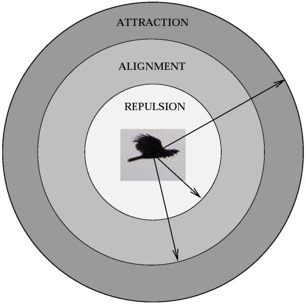
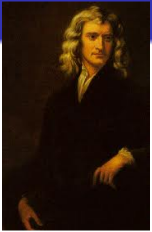


Metric versus Topological Interaction

2nd Order Model: Newton's like equations



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$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

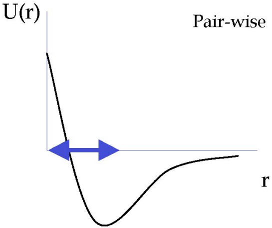
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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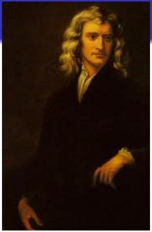

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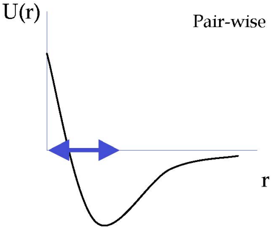

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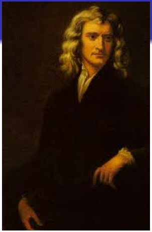

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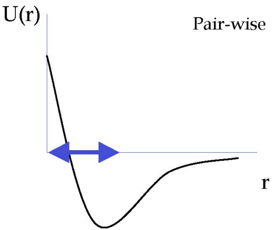

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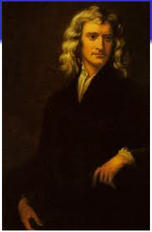

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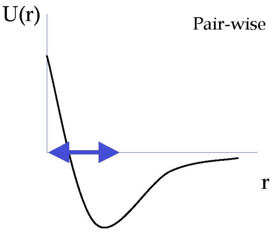
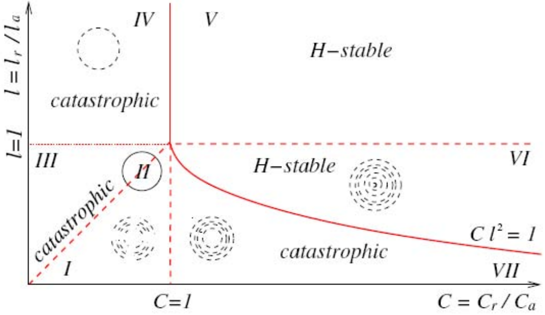


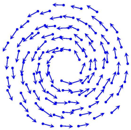
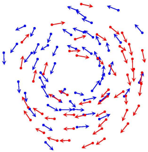
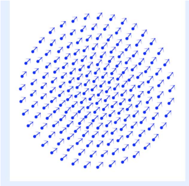
Model with an asymptotic velocity

Classification of possible patterns: Morse potential. D'Orsogna, Bertozzi et al. model (PRL 2006).



Model with an asymptotic speed

Typical patterns: milling, double milling or flocking:

Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\left(v_j - v_i\right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^\gamma}.$$

Asymptotic flocking: $\gamma < 1/2$; Cucker-Smale.
General Proof for $0 < \gamma \leq 1/2$; C.-Fornasier-Rosado-Toscani.

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Leadership, Geometrical Constraints, and Cone of Influence

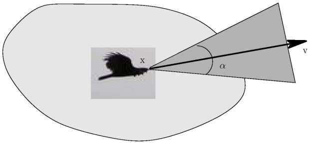
Cucker-Smale with local influence regions:

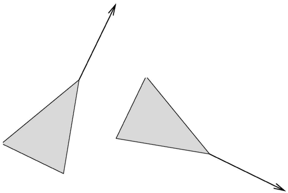
$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases}$$

where $\Sigma_i(t) \subset \{1, \ldots, N\}$ is the set of dependence, given by

$$\Sigma_i(t) := \left\{ 1 \leq \ell \leq N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \geq \alpha \right\}.$$

Cone of Vision:





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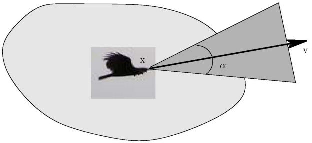
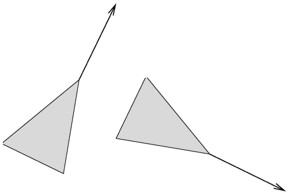




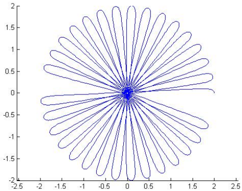
Roosting Forces

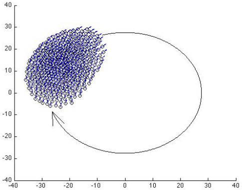
Adding a roosting area to the model:

$$\left\{\begin{aligned} \frac{dx_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= (\alpha - \beta\,|v_i|^2)v_i - \sum_{j\neq i} \nabla U(|x_i - x_j|) - v_i^{\perp}\,\nabla_{x_i}\left[\phi(x_i)\cdot v_i^{\perp}\right]. \end{aligned}\right.$$

with the roosting potential ϕ given by $\phi(x) := \dfrac{b}{4}\left(\dfrac{|x|}{R_{\text{Roost}}}\right)^4$.

Roosting effect: milling flocks $N = 400, R_{\text{roost}} = 20$.





Adding Noise

Self-Propelling/Friction/Interaction with Noise Particle Model:

$$\begin{cases} \dot{x}_i = v_i, \\ dv_i = \left[(\alpha - \beta\, |v_i|^2) v_i - \nabla_{x_i} \sum_{j \neq i} U(|x_i - x_j|) \right] dt + \sqrt{2\sigma}\, d\Gamma_i(t) \,, \end{cases}$$

where $\Gamma_i(t)$ are N independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. The Cucker–Smale Particle Model with Noise:

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Vicsek's model

Assume N particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i\, dt, \\ dV_t^i = \sqrt{2}\, P(V_t^i) \circ dB_t^i - P(V_t^i) \left(\dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j) \right) dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2}.$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-C.

Main issue: phase transition? Degond-Liu-Frouvelle.

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1st Order Friction Model:

Edelshtein-Keshet, Mogilner (JMB 2000): Assume the variations of the velocity and speed are much smaller than spatial variations, then from Newton's equation:

$$m\frac{d^2x_i}{d^2t} + \alpha\frac{dx_i}{dt} + \sum_{j\neq i}\nabla U(|x_i - x_j|) = 0$$

so finally, we obtain

$$\frac{dx_i}{dt} = -\sum_{j\neq i}\nabla U(|x_i - x_j|) \qquad \text{in the continuum setting} \Rightarrow \qquad \begin{cases} \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

Flock Solutions: stationary states x_i^s of the 1st order model are connected to particular solutions of the Bertozzi etal 2nd order model of the form

$$x_i(t) = x_i^s + tv_0$$

with v_0 fixed with $|v_0|^2 = \frac{\alpha}{\beta}$.

For which potentials do we evolve towards some nontrivial steady states/patterns?
Is there any implication of the stability from first to 2nd order models?

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Edelshtein-Keshet, Mogilner (JMB 2000): Assume the variations of the velocity and speed are much smaller than spatial variations, then from Newton's equation:

$$m\frac{d^2x_i}{d^2t} + \alpha\frac{dx_i}{dt} + \sum_{j\neq i}\nabla U(|x_i - x_j|) = 0$$

so finally, we obtain

$$\frac{dx_i}{dt} = -\sum_{j\neq i}\nabla U(|x_i - x_j|) \qquad \text{in the continuum setting} \Rrightarrow \qquad \begin{cases} \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

Flock Solutions: stationary states x_i^s of the 1st order model are connected to particular solutions of the Bertozzi etal 2nd order model of the form

$$x_i(t) = x_i^s + tv_0$$

with v_0 fixed with $|v_0|^2 = \frac{\alpha}{\beta}$.

For which potentials do we evolve towards some nontrivial steady states/patterns?
Is there any implication of the stability from first to 2nd order models?

Mathematical Questions:

- What are the continuum models associated to these systems as the number of individuals gets larger and larger? Mean-field limits.
- What is the good analytical framework to deal with the possible concentration of mass in finite/infinite time in space or in velocity?
- What is the good analytical framework to deal with particles and continuum solutions at the same time?
- How to deal with the stability of patterns, which perturbations?

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Schedule:

- Lecture 2: Second order Models - Kinetic Equations for Swarming: measure solutions - mean field limit with/without noise.
- Lectures 3-4: First order Models - Aggregation Equations: derivation and mean-field limit, stability/instability of steady states for repulsive/attractive potentials. Qualitative properties of Steady States.
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Outline

Definition of the distance[1]

Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu$$

for all $\varphi \in C_o(\mathbb{R}^d)$.

Random variables:

Say that X is a random variable with law given by μ, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right].$$

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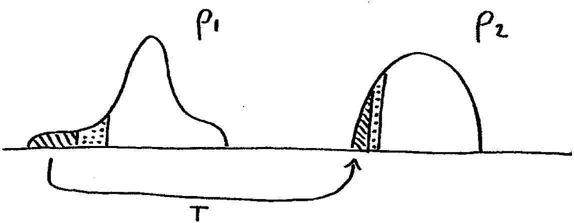
Two piles of sand!

Energy needed to transport m kg of sand from $x = a$ to $x = b$:

$$\boxed{\text{energy} = m\,|a-b|^2}$$



$W_2^2(\rho_1, \rho_2)$ = Among all possible ways to transport the mass from ρ_1 to ρ_2, find the one that minimizes the total energy

$$W_2^2(\rho_1, \rho_2) = \int_{\mathbb{R}^d} |x - T(x)|^2 \, d\rho_1(x)$$

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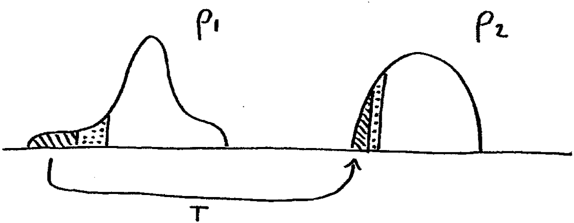

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Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2, \infty$:

$$W_p^p(\mu,\nu) = \inf_\pi \left\{ \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \, d\pi(x,y) \right\} = \inf_{(X,Y)} \left\{ \mathbb{E}\left[|X-Y|^p \right] \right\}$$

where the transference plan π runs over the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and $\nu \in \mathcal{P}_p(\mathbb{R}^d)$ and (X, Y) are all possible couples of random variables with μ and ν as respective laws.

$$W_\infty(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \sup_{(x,y) \in \operatorname{supp}(\pi)} |x-y| \,,$$

Monge's optimal mass transport problem:

Find

$$I := \inf_T \left\{ \int_{\mathbb{R}^d} |x - T(x)|^p \, d\mu(x);\ \nu = T\#\mu \right\}^{1/p}.$$

Take $\gamma_T = (1_{\mathbb{R}^d} \times T)\#\mu$ as transference plan π.

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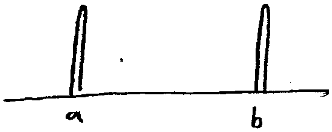
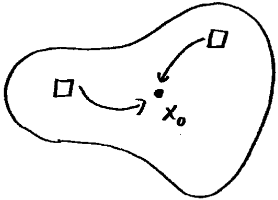
Three examples



$$W_2^2(\delta_a, \delta_b) = |a - b|^2$$



$$W_2^2(\rho, \delta_{X_0}) = \int |X_0 - y|^2 \, d\rho(y)$$
$$= \mathrm{Var}\,(\rho)$$

Outline

Euclidean Wasserstein Distance

Convergence Properties

1. **Convergence of measures:** $W_2(\mu_n, \mu) \to 0$ is equivalent to $\mu_n \rightharpoonup \mu$ weakly-* as measures and convergence of second moments.
2. **Weak lower semicontinuity:** Given $\mu_n \rightharpoonup \mu$ and $\nu_n \rightharpoonup \nu$ weakly-* as measures, then
$$W_2(\mu, \nu) \leq \liminf_{n \to \infty} W_2(\mu_n, \nu_n).$$
3. **Completeness:** The space $\mathcal{P}_2(\mathbb{R}^d)$ endowed with the distance W_2 is a complete metric space.

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One dimensional Case

Distribution functions:

In one dimension, denoting by $F(x)$ the distribution function of μ,

$$F(x) = \int_{-\infty}^{x} d\mu,$$

we can define its pseudo inverse:

$$F^{-1}(\eta) = \inf\{x : F(x) > \eta\} \qquad \text{for } \eta \in (0,1),$$

we have $F^{-1} : ((0,1), \mathcal{B}_1), d\eta) \longrightarrow (\mathbb{R}, \mathcal{B}_1)$ is a random variable with law μ, i.e., $F^{-1}\#d\eta = \mu$

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Wasserstein distance:

In one dimension, it can be checked[a] that given two measures μ and ν with distribution functions $F(x)$ and $G(y)$ then, $(F^{-1} \times G^{-1})\#d\eta$ has joint distribution function $H(x,y) = \min(F(x), G(y))$. Therefore, in one dimension, the optimal plan is given by $\pi_{opt}(x,y) = (F^{-1} \times G^{-1})\#d\eta$, and thus

$$W_p(\mu,\nu) = \left(\int_0^1 [F^{-1}(\eta) - G^{-1}(\eta)]^p \, d\eta\right)^{1/p} = \|F^{-1} - G^{-1}\|_{L^p(\mathbb{R})}$$

$1 \le p \le \infty$.

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