Interaction-Driven Dynamics for Collective Behavior: Derivation, Model Hierarchies and Pattern Stability

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Outline

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Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

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Definition of the distance

Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d) \, .$$

Random variables:

Say that X is a random variable with law given by μ, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right] .$$

Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{ \mathbb{E}\left[|X - Y|^p\right] \right\}$$

where (X, Y) are couples of random variables with μ and ν as respective laws.

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Well-posedness in probability measures[1]

Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with P^t the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions f and g with initial data f_0 and g_0, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \le \alpha(t)\, W_1(f_0, g_0)$$

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Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta |v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij} (v_j - v_i)}^{\text{orientation}}. \end{cases}$$

  then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

  is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time T as an alternative derivation of the kinetic models.

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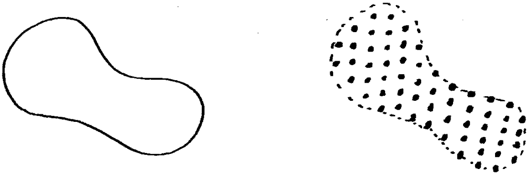
Mean-Field Limit

Just take as many particles as needed in order to have

$$W_1(f_t, f_t^N) \leq \alpha(t)\, W_1(f_0, f_0^N) \to 0 \qquad \text{as } N \to \infty$$

by sampling the initial data in a suitable way.

The sequences of particle solutions becomes a Cauchy sequence with the distance W_1 converging to the solution of the kinetic equation.



Hauray-Jabin 2011: mean field limit for Vlasov with potentials such that $|\nabla U| \leq r^{-\alpha}$, with $\alpha < 1$ with initial data for Vlasov in $L^1 \cap L^\infty$.

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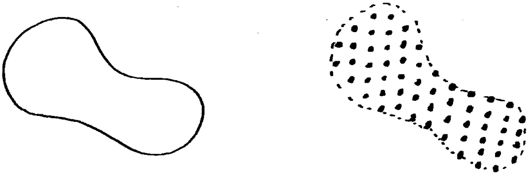

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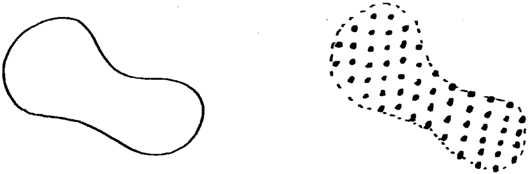

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Outline

Proof of the Theorem

Conditions on E:

1. E is continuous on $[0, T] \times \mathbb{R}^d$,
2. For some $C > 0$,

$$|E(t,x)| \leq C_E(1 + |x|), \quad \text{for all } t, x \in [0,T] \times \mathbb{R}^d, \text{ and}$$

3. E is locally Lipschitz with respect to x, i.e., for any compact set $K \subseteq \mathbb{R}^d$ there is some $L_K > 0$ such that

$$|E(t,x) - E(t,y)| \leq L_K|x - y|, \qquad t \in [0,T], \quad x, y \in K.$$

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$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta |v|^2) v f) = 0,$$

which is a linear first-order equation. The associated characteristic system of ODE's is

$$\frac{d}{dt} X = V,$$
$$\frac{d}{dt} V = E(t, X) + V(\alpha - \beta |V|^2).$$

Flow Map:

Given $(X_0, V_0) \in \mathbb{R}^d \times \mathbb{R}^d$ there exists a unique solution (X, V) to the ODE system in $\mathcal{C}^1([0, T]; \mathbb{R}^d \times \mathbb{R}^d)$ satisfying $X(0) = X_0$ and $V(0) = V_0$. In addition, there exists a constant C which depends only on T, $|X_0|$, $|V_0|$, α, β and the constant C_E, such that

$$|(X(t), V(t))| \le |(X_0, V_0)| \, e^{Ct} \quad \text{for all } t \in [0, T].$$

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We can thus consider the flow at time $t \in [0,T)$ of ODE's equations

$$\mathcal{T}_E^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d.$$

Again by basic results in ode's, the map $(t,x,v) \mapsto \mathcal{T}_E^t(x,v) = (X,V)$ with (X,V) the solution at time t to the ODE system with initial data (x,v), is jointly continuous in (t,x,v).

For a measure $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, the function

$$f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d), \quad t \mapsto f_t := \mathcal{T}_E^t \# f_0$$

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Estimates on characteristics:

- Taking characteristics with initial data inside a fixed ball then there exists $R > 0$ depending on T, in which the whole trajectories are inside a possibly larger ball of radius R for all times $t \in [0, T]$.
- For some constant C which depends only on α, β, R and $\mathrm{Lip}_R(E^i)$, for all P^0 in B_R

$$\left|\mathcal{T}^t_{E^1}(P^0) - \mathcal{T}^t_{E^2}(P^0)\right| \leq \frac{e^{Ct} - 1}{C} \sup_{s\in[0,T)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)}.$$

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- For some constant C as before

$$\left|\mathcal{T}_E^t(P_1) - \mathcal{T}_E^t(P_2)\right| \leq |P_1 - P_2|\, e^{C \int_0^t (\mathrm{Lip}_R(E_s)+1)\, ds}, \quad t \in [0, T].$$

Proof of the Theorem

Error on transported measures through different flows:

Let $\mathcal{T}_1, \mathcal{T}_2 : \mathbb{R}^d \to \mathbb{R}^d$ be two Borel measurable functions. Also, take $f \in \mathcal{P}_1(\mathbb{R}^d)$. Then,

$$W_1(\mathcal{T}_1\#f, \mathcal{T}_2\#f) \leq \|\mathcal{T}_1 - \mathcal{T}_2\|_{L^\infty(\text{supp} f)} .$$

Continuity in time for solutions of the linear transport:

$$W_1(\mathcal{T}_E^s\#f, \mathcal{T}_E^t\#f) \leq C\,|t-s|\,, \qquad \text{for any } t, s \in [0, T].$$

Error on transported measures through different initial data:

Take a locally Lipschitz map $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ and $f, g \in \mathcal{P}_1(\mathbb{R}^d)$, both with compact support contained in the ball B_R. Then,

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W_1(f_t, g_t) &= W_1(\mathcal{T}_f^t \# f_0, \mathcal{T}_g^t \# g_0) \\
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&\leq \left\| \mathcal{T}_f^t - \mathcal{T}_g^t \right\|_{L^\infty(\mathrm{supp} f_0)} + L_t\, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} \, ds + L_t\, W_1(f_0, g_0) \\
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Outline

Rigorous Statement of the Mean-Field Limit

Aggregation Equation:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \text{ with } u(t,x) := -\nabla U * \rho, & t > 0, \quad x \in \mathbb{R}^d, \\ \rho(0,x) := \rho_0(x), & x \in \mathbb{R}^d, \end{cases}$$

Particle Approximation and Empirical distribution $\mu_N(t)$:

$$\begin{cases} \dot{X}_i(t) = -\sum_{j \neq i} m_j \nabla U(X_i(t) - X_j(t)), \\ X_i(0) = X_i^0, \qquad i = 1, \ldots, N. \end{cases}$$

$$\mu_N(t) = \sum_{i=1}^N m_i \delta_{X_i(t)}, \quad \sum_{i=1}^N m_i = \int_{\mathbb{R}^d} \rho_0(x)dx = 1,$$

with $m_i > 0$, $i = 1, \ldots, N$. We set $\nabla U(0) = 0$ even if singular at the origin.

The convergence:

"$\mu_N^0 \rightharpoonup \rho^0$ weakly-$*$ as measures $\Longrightarrow \mu_N(t) \rightharpoonup \rho(t)$ weakly-$*$ as measures for small time or for every time?"

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Rigorous Statement of the Mean-Field Limit

Quantities to Control: the W_∞-distance between $\rho(t)$ and $\mu_N(t)$, and the minimum inter-particle distance:

$$\eta(t) := W_\infty(\mu_N(t), \rho(t)), \quad \eta_m(t) := \min_{1\le i\ne j\le N} \left(|X_i(t) - X_j(t)|\right),$$

with $\eta^0 := \eta(0)$ and $\eta_m^0 := \eta_m(0)$.

Assumptions on the potential U: it is C^2 except at the origin, where it might have a singularity. We set $U(0) = 0$ by definition, and

$$|\nabla U(x)| \le \frac{C}{|x|^\alpha}, \quad \text{and} \quad |D^2 U(x)| \le \frac{C}{|x|^{1+\alpha}}, \quad \forall\, x \in \mathbb{R}^d \setminus \{0\},$$

for $-1 \le \alpha < d - 1$.

Note that due to the assumptions on U, we can always find $1 < p < \infty$ such that $(\alpha + 1)p' < d$, and thus ∇U belongs to $\mathcal{W}^{1,p'}_{loc}(\mathbb{R}^d)$.

Weak Solutions: $\rho \in L^\infty(0, T; (L^1 \cap L^p)(\mathbb{R}^d)) \cap \mathcal{C}([0, T], \mathcal{P}_1(\mathbb{R}^d))$, with initial data $\rho^0 \in (\mathcal{P}_1 \cap L^p)(\mathbb{R}^d)$.

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Main Result.-

Let ρ be a solution to the aggregation equation up to time $T > 0$, such that $\rho \in L^\infty(0,T;(L^1 \cap L^p)(\mathbb{R}^d)) \cap \mathcal{C}([0,T],\mathcal{P}_1(\mathbb{R}^d))$, with initial data $\rho^0 \in (\mathcal{P}_1 \cap L^p)(\mathbb{R}^d)$, $0 \le \alpha < -1 + d/p'$, and $1 < p \le \infty$. Furthermore, we assume μ_N^0 converges to ρ^0 for the distance d_∞ as the number of particles N goes to infinity,

$$W_\infty(\mu_N^0, \rho^0) \to 0 \quad \text{as} \quad N \to \infty,$$

and that the initial quantities η^0, η_m^0 satisfy

$$\lim_{N\to\infty} \frac{(\eta^0)^{d/p'}}{(\eta_m^0)^{1+\alpha}} = 0.$$

Under the previous assumptions on the potential, for N large enough the associated particle system is well-defined up to time T, in the sense that there is no collision between particles before that time, and moreover

$$\mu_N(t) \rightharpoonup \rho(t) \quad \text{weakly-}* \text{ as measures as} \quad N \to \infty, \quad \text{for all} \quad t \in [0,T].$$

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Outline

Strategy of the Proof

- In Step A, we estimate the growth of the W_∞ Wasserstein distance between the continuum and the discrete solutions η that involves η itself and η_m in the form:

$$\frac{d\eta}{dt} \leq C\eta\|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right).$$

- In Step B, we estimate the decay of the minimum inter-particle distance η_m, which also involves the terms η and η_m in the form:

$$\frac{d\eta_m}{dt} \geq -C\eta_m\|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right).$$

- In Step C, under the assumption of a well prepared initial approximation, we combine the estimates above to conclude the desired result.

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Step A: Well-defined characteristics

The assumptions on the potential lead to

$$|\nabla U(x) - \nabla U(y)| \leq \frac{2|x-y|}{\min(|x|,|y|)^{\alpha+1}}\,.$$

Given the velocity fields $u(x,t) = -\nabla U * \rho$ and "$u_N := -\nabla U * \mu_N$". We define the flows:

$$\begin{cases} \dfrac{d}{dt}(\Psi(t;s,x)) = u(t;s,\Psi(t;s,x)), \\ \Psi(s;s,x) = x, \end{cases}$$

for all $s,t \in [0,T]$, and

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Step A: Well-defined Flows

$$
\begin{aligned}
|u(t,x) - u(t,y)| &\le \int_{\mathbb{R}^d} |\nabla U(x-z) - \nabla U(y-z)| \rho(t,z)\, dz \\
&\le 2|x-y| \int_{\mathbb{R}^d} \frac{1}{\min(|x-z|, |y-z|)^{\alpha+1}} \rho(t,z)\, dz \\
&\le 4|x-y| \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}} \rho(t,z)\, dz\,.
\end{aligned}
$$

Now, splitting the last integral into the near- and far-field sets $\mathcal{A} := \{z : |x-z| \ge 1\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$ and estimating the two terms, we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}} \rho(t,z)\, dz &\le \|\rho(t)\|_1 + \left(\int_{\mathcal{B}} \frac{1}{|x-y|^{(1+\alpha)p'}} dy \right)^{1/p'} \|\rho(t)\|_p \\
&\le C\|\rho\|\,,
\end{aligned}
$$

for all $x \in \mathbb{R}^d$ due to the assumption $(1+\alpha)p' < d$.

Step A: Well-defined Flows

$$
\begin{aligned}
|u(t,x)-u(t,y)| &\le \int_{\mathbb{R}^d} |\nabla U(x-z)-\nabla U(y-z)|\rho(t,z)\,dz \\
&\le 2|x-y| \int_{\mathbb{R}^d} \frac{1}{\min(|x-z|,|y-z|)^{\alpha+1}}\rho(t,z)\,dz \\
&\le 4|x-y| \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}}\rho(t,z)\,dz\,.
\end{aligned}
$$

Now, splitting the last integral into the near- and far-field sets $\mathcal{A} := \{z : |x-z| \ge 1\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$ and estimating the two terms, we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}}\rho(t,z)\,dz &\le \|\rho(t)\|_1 + \left(\int_{\mathcal{B}} \frac{1}{|x-y|^{(1+\alpha)p'}}\,dy\right)^{1/p'} \|\rho(t)\|_p \\
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Step A: Estimate of the evolution of W_∞

Fixed $0 \le t_0 < \min(T, T_0^N)$ and choose an optimal transport map for W_∞ denoted by $\mathcal{T}^0$ between $\rho(t_0)$ and $\mu_N(t_0)$; $\mu_N(t_0) = \mathcal{T}^0 \# \rho(t_0)$.

The solution of the aggregation equation is given by $\rho(t) = \Psi(t; t_0, \cdot\,)\#\rho(t_0)$ and obviously $\mu_N(t) = \Psi_N(t; t_0, \cdot\,)\#\mu_N(t_0)$ for $t \ge t_0$. We also notice that for $t \ge t_0$

$$\mathcal{T}^t \# \rho(t) = \mu_N(t), \quad \text{where} \quad \mathcal{T}^t = \Psi_N(t; t_0, \cdot) \circ \mathcal{T}^0 \circ \Psi(t_0; t, \cdot).$$

By Definition of the W_∞ Wasserstein distance, we get

$$\eta(t) = W_\infty(\mu_N(t), \rho(t)) \le \|\Psi(t; t_0, \cdot) - \Psi_N(t; t_0, \cdot) \circ \mathcal{T}^0\|_\infty.$$

We notice that

$$\frac{d}{dt}\left(\Psi_N(t; t_0, \mathcal{T}^0(x)) - \Psi(t; t_0, x)\right)\Big|_{t=t_0} = u_N(t_0, \mathcal{T}^0(x)) - u(t_0, x).$$

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$$\frac{d}{dt}\|\Psi_N(t; t_0, \cdot) \circ \mathcal{T}^0 - \Psi(t; t_0, \cdot)\|_\infty\Big|_{t=t_0^+} \le \|u_N(t_0, \cdot) \circ \mathcal{T}^0 - u(t_0, \cdot)\|_\infty.$$

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We now note that

$$u_N(t_0, \mathcal{T}^0(x)) - u(t_0, x) = -\int_{\mathbb{R}^d} \nabla U(\mathcal{T}^0(x) - y) d\mu_N(t_0, y) + \int_{\mathbb{R}^d} \nabla U(x-y)\rho(t_0, y) dy$$
$$= -\int_{\mathbb{R}^d} \left(\nabla U(\mathcal{T}^0(x) - \mathcal{T}^0(y)) - \nabla U(x-y)\right) \rho(t_0, y) dy.$$

and thus $$\frac{d^+\eta}{dt} \le C \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla U(\mathcal{T}(x) - \mathcal{T}(y)) - \nabla U(x-y)| \rho(y) dy.$$

We decompose the integral on $\mathbb{R}^d$ into the near- and the far-field parts as $\mathcal{A} := \{z : |x - z| \ge 4\eta\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$, to get

$$\mathcal{I}_1 \le \int_{\mathcal{A}} \frac{2\left(|x - \mathcal{T}(x)| + |y - \mathcal{T}(y)|\right)}{\min(|x-y|, |\mathcal{T}(x) - \mathcal{T}(y)|)^{\alpha+1}} \rho(y) dy \le C\eta \|\rho\|.$$

and

$$\mathcal{I}_2 \le \int_{\mathcal{B}} \frac{\rho(y)}{|x-y|^\alpha} dy + \int_{\mathcal{B}} \frac{\rho(y)}{\eta_m^\alpha} dy \le C(\eta^{d/p'-\alpha} + \eta^{d/p'} \eta_m^{-\alpha}) \|\rho\|.$$

Using that $\eta_m \le 2\eta$, one obtains the estimate in Step A.

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Using that $\eta_m \le 2\eta$, one obtains the estimate in Step A.

Step B: Estimate of the evolution of η_m

Pick i, j so that $|X_i - X_j| = \eta_m$.

$$\begin{aligned}\frac{d}{dt}|X_i - X_j| &\geq -|u_N(X_i) - u_N(X_j)| \\ &\geq -\int_{\mathbb{R}^d} |\nabla U(X_i - y) - \nabla U(X_j - y)|\, d\mu_N(y) \\ &= -\int_{\mathbb{R}^d} |\nabla U(X_i - \mathcal{T}(y)) - \nabla U(X_j - \mathcal{T}(y))|\, \rho(y) dy\,,\end{aligned}$$

where $\mathcal{T}$ is the optimal map satisfying $\mu_N(t) = \mathcal{T}\#\rho(t)$.

Decomposing in near- and far-field parts the domain $\mathbb{R}^d$ as $\mathcal{A} := \{y : |X_i - y| \geq 2\eta \text{ and } |X_j - y| \geq 2\eta\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$, we can estimate

$$\begin{aligned}&\int_{\mathcal{A}} |\nabla U(X_i - \mathcal{T}(y)) - \nabla U(X_j - \mathcal{T}(y))|\, \rho(y) dy \\ &\leq \int_{\mathcal{A}} \frac{2|X_i - X_j|}{\min(|X_i - \mathcal{T}(y)|, |X_j - \mathcal{T}(y)|)^{\alpha+1}}\, \rho(y) dy \\ &\leq 2^{2+\alpha}|X_i - X_j| \int_{\mathcal{A}} \left(\frac{1}{|X_i - y|^{\alpha+1}} + \frac{1}{|X_j - y|^{\alpha+1}} \right) \rho(y) dy \leq C\eta_m \|\rho\|,\end{aligned}$$

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$$
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Step B: Estimate of the evolution of W_∞

For the integral over $\mathcal{B}$, we use that as soon as $X_i \neq \mathcal{T}(y)$, then

$$|\nabla U(X_i - \mathcal{T}(y))| \leq \frac{1}{|X_j - \mathcal{T}(y)|^\alpha} \leq \frac{1}{\eta_m^\alpha},$$

and $\nabla U(X_i - \mathcal{T}(y)) = 0$ otherwise, and similarly for X_j.

A simple Hölder computation implies that

$$\int_{\mathcal{B}} |\nabla U(X_i - \mathcal{T}(y)) - \nabla U(X_j - \mathcal{T}(y))| \, \rho(y) dy \leq C \eta^{d/p'} \eta_m^{-\alpha} \|\rho\|.$$

Putting together we finally conclude the estimate in Step B.

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Putting together we finally conclude the estimate in Step B.

Step B: Estimate of the evolution of W_∞

For the integral over $\mathcal{B}$, we use that as soon as $X_i \neq \mathcal{T}(y)$, then

$$|\nabla U(X_i - \mathcal{T}(y))| \leq \frac{1}{|X_j - \mathcal{T}(y)|^\alpha} \leq \frac{1}{\eta_m^\alpha},$$

and $\nabla U(X_i - \mathcal{T}(y)) = 0$ otherwise, and similarly for X_j.

A simple Hölder computation implies that

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Putting together we finally conclude the estimate in Step B.

Step C: Closing the Argument

$$\begin{cases} \dfrac{d^+\eta}{dt} & \leq C\eta\|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right), \\ \dfrac{d\eta_m}{dt} & \geq -C\eta_m\|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right), \end{cases}$$

for $t \in [0, \min(T, T_0^N))$.

For this, we set

$$f(t) := \frac{\eta(t)}{\eta^0}, \quad g(t) := \frac{\eta_m(t)}{\eta_m^0} \quad \text{and} \quad \xi_N := (\eta^0)^{d/p'} (\eta_m^0)^{-(1+\alpha)}.$$

Note that ξ_N depends on the number of particles N. It yields

$$\frac{d^+f}{dt} \leq C\|\rho\| f \left(1 + \xi_N f^{d/p'} g^{-(1+\alpha)}\right),$$
$$\frac{dg}{dt} \geq -C\|\rho\| g \left(1 + \xi_N f^{d/p'} g^{-(1+\alpha)}\right).$$

Observe that $f(0) = g(0) = 1$ and $\xi_N \to 0$ as N goes to infinity.

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Then, there exists a positive constant $T_*^N \leq T_0^N$ for sufficiently large N such that

$$\xi_N f^{d/p'} g^{-(1+\alpha)} \leq 1 \quad \text{for} \quad t \in [0, T_*^N] \, ,$$

Then it follows that $f(t) \leq e^{2\|\rho\|t}$ and $g(t) \geq e^{-2\|\rho\|t}$.
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$$\xi_N f^{d/p'} g^{-(1+\alpha)} \leq 1 \quad \text{holds for} \quad t \leq -\frac{\log(\xi_N)}{2(d/p' + (1+\alpha))\|\rho\|},$$

so that

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Our assumption for the initial data finally implies

$$\liminf_{N\to\infty} T_*^N \geq \lim_{N\to\infty} -\frac{\log(\xi_N)}{2(d/p' + (1+\alpha))\|\rho\|} = \infty \, ,$$

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Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns. More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Mean-field limit under reasonable conditions leads to rigorous derivation of the mesoscopic/kinetic models with/without noise.
- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011).
3. C.-Klar-Martin-Tiwari (M3AS 2010).
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