Interaction-Driven Dynamics for Collective Behavior: Derivation, Model Hierarchies and Pattern Stability

J. A. Carrillo

Imperial College London

Lecture 3, L'Aquila 2015

Outline

Outline

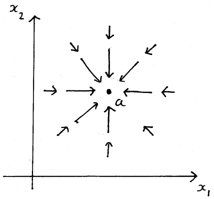
Aggregation for particles - Continuum Model

One particle attracted/repelled by a fixed location $x = a$

$$\dot{X} = -\nabla U(X - a) \qquad U(x) = U(-x), U(0) = 0\,, U \in C(\mathbb{R}^d, \mathbb{R}) \cap C^1(\mathbb{R}^d/\{0\}, \mathbb{R})$$

Multiple particles attracted by one another

$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$



$\rho(t, x)$ = density of particle at time t

$$v(x) = -\int_{\mathbb{R}^d} \nabla U(x - y) \, \rho(y) dy$$

So $\boxed{v = -\nabla U * \rho}$:

$$\begin{cases} \rho_t + \text{div}\partial\rho v = 0 \\ v = -\nabla U * \rho \end{cases}$$

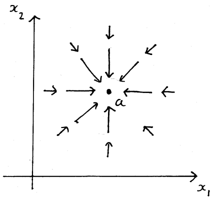
Aggregation for particles - Continuum Model

One particle attracted/repelled by a fixed location $x = a$

$$\dot{X} = -\nabla U(X - a) \qquad U(x) = U(-x), U(0) = 0\,, U \in C(\mathbb{R}^d, \mathbb{R}) \cap C^1(\mathbb{R}^d/\{0\}, \mathbb{R})$$

Multiple particles attracted by one another

$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$



$\rho(t,x)$ = density of particle at time t

$$v(x) = -\int_{\mathbb{R}^d} \nabla U(x-y)\, \rho(y)dy$$

So $\boxed{v = -\nabla U * \rho}$:

$$\begin{cases} \rho_t + \mathrm{div}\partial\rho v = 0 \\ v = -\nabla U * \rho \end{cases}$$

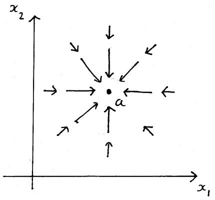
Aggregation for particles - Continuum Model

One particle attracted/repelled by a fixed location $x = a$

$$\dot{X} = -\nabla U(X - a) \qquad U(x) = U(-x), U(0) = 0\,, U \in C(\mathbb{R}^d, \mathbb{R}) \cap C^1(\mathbb{R}^d/\{0\}, \mathbb{R})$$

Multiple particles attracted by one another

$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$



$\rho(t, x) =$ density of particle at time t

$$v(x) = -\int_{\mathbb{R}^d} \nabla U(x - y)\, \rho(y) dy$$

So $\boxed{v = -\nabla U * \rho}$:

$$\begin{cases} \rho_t + \text{div } \rho v = 0 \\ \text{v} = -\nabla U * \rho \end{cases}$$

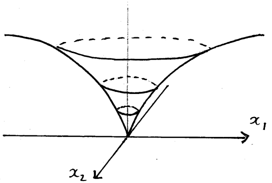
Aggregation Equation

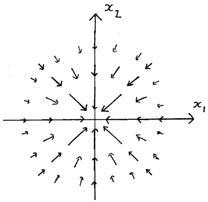
$$\begin{cases} \rho_t + \operatorname{div} \rho v = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction repulsive/attractive potentials do we get convergence towards some nontrivial steady states?
How can we find these stationary states and what are their properties?

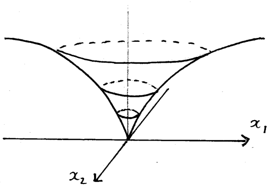
Aggregation Equation

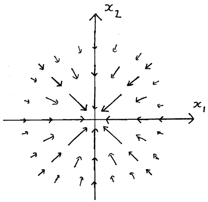
$$\begin{cases} \rho_t + \text{div } \rho v = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction repulsive/attractive potentials do we get convergence towards some nontrivial steady states?
How can we find these stationary states and what are their properties?

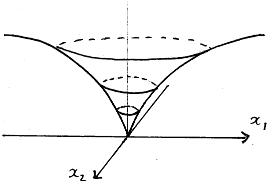
Aggregation Equation

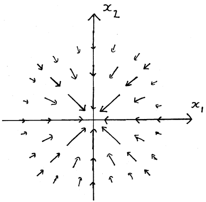
$$\begin{cases} \rho_t + \text{div}\, \rho v = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction repulsive/attractive potentials do we get convergence towards some nontrivial steady states?

How can we find these stationary states and what are their properties?

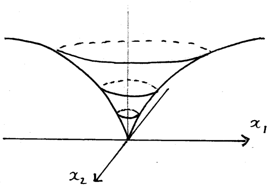
Aggregation Equation

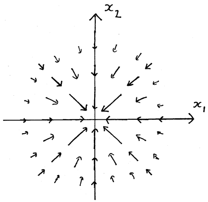
$$\begin{cases} \rho_t + \text{div } \rho v = 0 \\ v = -\nabla U * \rho \end{cases}$$

$\rho(t,x)$: density
$v(t,x)$: velocity field
$x \in \mathbb{R}^d$, $t > 0$

$U : \mathbb{R}^d \to \mathbb{R}$
"interaction potential"

$-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$
"attracting/repulsing field"





For which interaction repulsive/attractive potentials do we get convergence towards some nontrivial steady states?
How can we find these stationary states and what are their properties?

Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2} \iint U(x-y)\,\rho(x)\,\rho(y)\,dxdy$$

with respect to the Wasserstein distance W_2.
(C., McCann, Villani; RMI 2003, ARMA 2006).

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\nabla\left[\frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right]\right).$$

with entropy dissipation:

$$\frac{d}{dt}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^2} \rho(t,x)\left|\nabla \frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right|^2 dx.$$

Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2} \iint U(x-y)\, \rho(x)\, \rho(y)\, dxdy$$

with respect to the Wasserstein distance W_2.
(C., McCann, Villani; RMI 2003, ARMA 2006).

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \mathrm{div}\left(\rho(t,x)\nabla\left[\frac{\delta\mathcal{F}}{\delta\rho}(t,x)\right]\right) .$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^2} \rho(t,x) \left|\nabla\frac{\delta\mathcal{F}}{\delta\rho}(t,x)\right|^2 dx .$$

Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2} \iint U(x-y)\ \rho(x)\ \rho(y)\ dxdy$$

with respect to the Wasserstein distance W_2.
(C., McCann, Villani; RMI 2003, ARMA 2006).

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\nabla\left[\frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right]\right) .$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^2} \rho(t,x)\left|\nabla \frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right|^2 dx .$$

Minimization of the Interaction Energy

Minimization Problem

We want to find local minimizers of the total interaction energy

$$\mathcal{F}[\mu] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\mu(x) d\mu(y) \, .$$

in some set of probability measures $\mathcal{P}(\mathbb{R}^d)$.

What is the right topology to talk about measures being close?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
- Economic Applications: Mean Field Games, Cournot-Nash Equilibria.
- Fractional Diffusion: More singular than Newtonian repulsion but still locally integrable potentials. Levy Flights.
- Random Matrices: Eigenvalue distributions.

Minimization of the Interaction Energy

Minimization Problem

We want to find local minimizers of the total interaction energy

$$\mathcal{F}[\mu] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\mu(x) d\mu(y)\,.$$

in some set of probability measures $\mathcal{P}(\mathbb{R}^d)$.

What is the right topology to talk about measures being close?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
- Economic Applications: Mean Field Games, Cournot-Nash Equilibria.
- Fractional Diffusion: More singular than Newtonian repulsion but still locally integrable potentials. Levy Flights.
- Random Matrices: Eigenvalue distributions.

Minimization of the Interaction Energy

Minimization Problem

We want to find local minimizers of the total interaction energy

$$\mathcal{F}[\mu] := \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} U(x-y)\, d\mu(x)d\mu(y)\,.$$

in some set of probability measures $\mathcal{P}(\mathbb{R}^d)$.

What is the right topology to talk about measures being close?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
- Economic Applications: Mean Field Games, Cournot-Nash Equilibria.
- Fractional Diffusion: More singular than Newtonian repulsion but still locally integrable potentials. Levy Flights.
- Random Matrices: Eigenvalue distributions.

Minimization of the Interaction Energy

Minimization Problem

We want to find local minimizers of the total interaction energy

$$\mathcal{F}[\mu] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\mu(x) d\mu(y)\,.$$

in some set of probability measures $\mathcal{P}(\mathbb{R}^d)$.

What is the right topology to talk about measures being close?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
- Economic Applications: Mean Field Games, Cournot-Nash Equilibria.
- Fractional Diffusion: More singular than Newtonian repulsion but still locally integrable potentials. Levy Flights.
- Random Matrices: Eigenvalue distributions.

Minimization of the Interaction Energy

Minimization Problem

We want to find local minimizers of the total interaction energy

$$\mathcal{F}[\mu] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\mu(x) d\mu(y)\,.$$

in some set of probability measures $\mathcal{P}(\mathbb{R}^d)$.

What is the right topology to talk about measures being close?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
- Economic Applications: Mean Field Games, Cournot-Nash Equilibria.
- Fractional Diffusion: More singular than Newtonian repulsion but still locally integrable potentials. Levy Flights.
- Random Matrices: Eigenvalue distributions.

Minimization of the Interaction Energy

Minimization Problem

We want to find local minimizers of the total interaction energy

$$\mathcal{F}[\mu] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\mu(x) d\mu(y) \,.$$

in some set of probability measures $\mathcal{P}(\mathbb{R}^d)$.

What is the right topology to talk about measures being close?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
- Economic Applications: Mean Field Games, Cournot-Nash Equilibria.
- Fractional Diffusion: More singular than Newtonian repulsion but still locally integrable potentials. Levy Flights.
- Random Matrices: Eigenvalue distributions.

Minimization of the Interaction Energy

Minimization Problem

We want to find local minimizers of the total interaction energy

$$\mathcal{F}[\mu] := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\mu(x) d\mu(y)\,.$$

in some set of probability measures $\mathcal{P}(\mathbb{R}^d)$.

What is the right topology to talk about measures being close?

Recurrent Question in many fields:

- Statistical Mechanics & Crystallization: Typically very singular potentials at zero: Lennard-Jones.
- Semiconductors - Astrophysics - Chemotaxis: Macroscopic model obtained from Vlasov Equation under certain limits. Newtonian Potential.
- Economic Applications: Mean Field Games, Cournot-Nash Equilibria.
- Fractional Diffusion: More singular than Newtonian repulsion but still locally integrable potentials. Levy Flights.
- Random Matrices: Eigenvalue distributions.

Outline

Individual Based Models (Particle models)

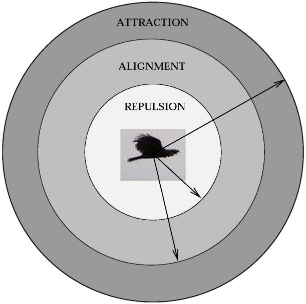
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms,... and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



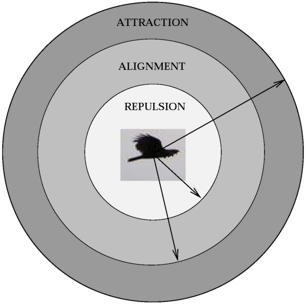
Individual Based Models (Particle models)

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.
Highly developed social organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms,... and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



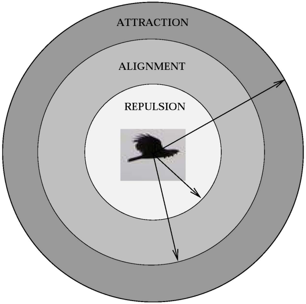
Individual Based Models (Particle models)

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.
Highly developed social organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms,... and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

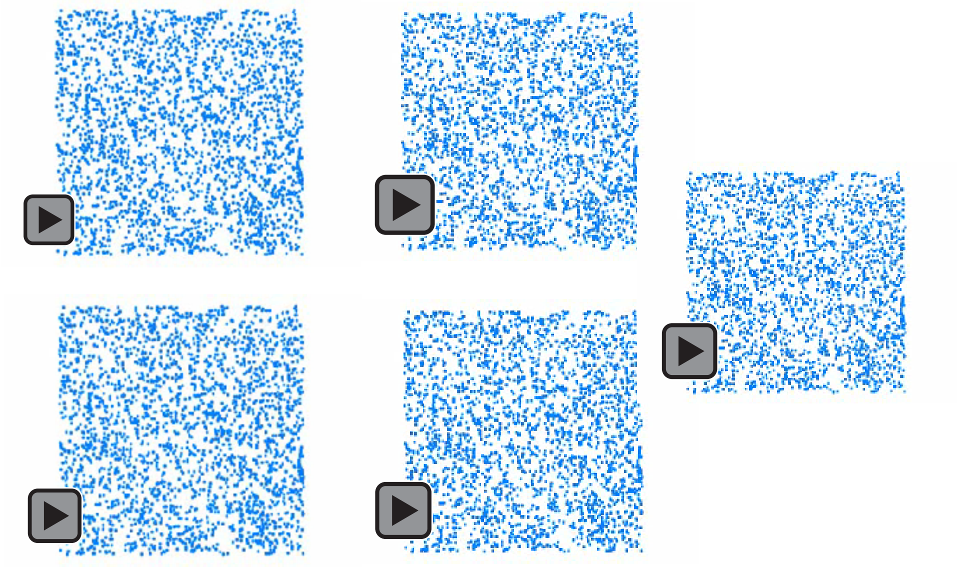
[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



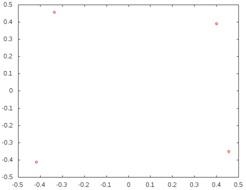
Outline

Nontrivial patterns? - Particle Simulations

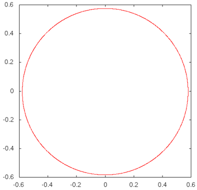


Summary: Particle Simulations $d = 2$

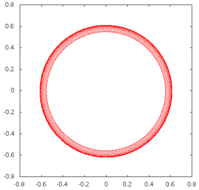
Potential $a = 4$, $b = 2.1$



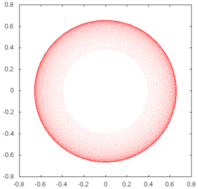
Potential $a = 4$, $b = 1.85$



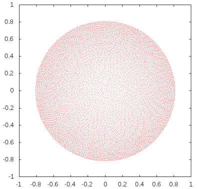
Potential $a = 4$, $b = 1.1$



Potential $a = 4$, $b = 0.85$



Potential $a = 4$, $b = 0.05$



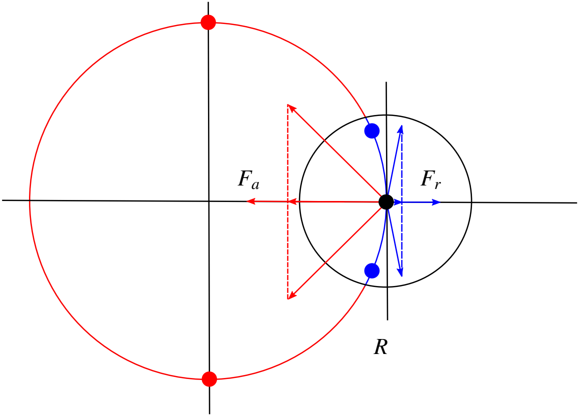
$$\dot{X}_i = -\sum_{j \neq i} m_j \, \nabla U(X_i - X_j)$$

$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$$

$$2 - d \leq b < a$$

Spherical shell

A spherical shell for some radius R is a stationary state for the aggregation equation for radial potentials.



Outline

W_∞-Topology

The W_∞-distance is defined as the optimal maximal mass displacement given by

$$W_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \sup_{(x,y)\in\mathrm{supp}(\pi)} |x-y|\,,$$

- It is a good topology since it is closer to linearization around equilibrium of dynamical systems.
- It is the coarser topology among Wasserstein distances since all of them are ordered.
- Then, a local minimizer in W_2 is a local minimizer in W_∞ but not viceversa.

Basic Hypotheses:
(H1) U is a bounded from below lower semi-continuous function in $L^1_{loc}(\mathbb{R}^d)$.

W_∞-Topology

The W_∞-distance is defined as the optimal maximal mass displacement given by

$$W_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \sup_{(x,y)\in \text{supp}(\pi)} |x-y| \,,$$

- It is a good topology since it is closer to linearization around equilibrium of dynamical systems.
- It is the coarser topology among Wasserstein distances since all of them are ordered.
- Then, a local minimizer in W_2 is a local minimizer in W_∞ but not viceversa.

Basic Hypotheses:
(H1) U is a bounded from below lower semi-continuous function in $L^1_{loc}(\mathbb{R}^d)$.

W_∞-Topology

The W_∞-distance is defined as the optimal maximal mass displacement given by

$$W_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \sup_{(x,y)\in\mathrm{supp}(\pi)} |x-y|\,,$$

- It is a good topology since it is closer to linearization around equilibrium of dynamical systems.
- It is the coarser topology among Wasserstein distances since all of them are ordered.
- Then, a local minimizer in W_2 is a local minimizer in W_∞ but not viceversa.

Basic Hypotheses:
(H1) U is a bounded from below lower semi-continuous function in $L^1_{loc}(\mathbb{R}^d)$.

W_∞-Topology

The W_∞-distance is defined as the optimal maximal mass displacement given by

$$W_\infty(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \sup_{(x,y)\in\operatorname{supp}(\pi)} |x-y|,$$

- It is a good topology since it is closer to linearization around equilibrium of dynamical systems.
- It is the coarser topology among Wasserstein distances since all of them are ordered.
- Then, a local minimizer in W_2 is a local minimizer in W_∞ but not viceversa.

Basic Hypotheses:

(H1) U is a bounded from below lower semi-continuous function in $L^1_{loc}(\mathbb{R}^d)$.

W_∞-Topology

The W_∞-distance is defined as the optimal maximal mass displacement given by

$$W_\infty(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \sup_{(x,y)\in \operatorname{supp}(\pi)} |x-y|,$$

- It is a good topology since it is closer to linearization around equilibrium of dynamical systems.
- It is the coarser topology among Wasserstein distances since all of them are ordered.
- Then, a local minimizer in W_2 is a local minimizer in W_∞ but not viceversa.

Basic Hypotheses:
(H1) U is a bounded from below lower semi-continuous function in $L^1_{loc}(\mathbb{R}^d)$.

Euler-Lagrange Conditions

W_∞ EL-Conditions

Assume that U satisfies (H1) and let μ be a local compactly supported minimizer of the energy $\mathcal{F}[\mu]$ in the W_∞ ball or radius ε. Then any point $x_0 \in \text{supp}(\mu)$ is a local minimimum of $\psi = U * \mu$ in the sense that

$$\psi(x_0) \le \psi(x) \text{ for a.e. } x \in B_\varepsilon(x_0).$$

Note that ε is uniform on the support of μ.

W_2 EL-Conditions

Under the same assumptions, if μ is a W_2-local minimizer of the energy, then the potential ψ satisfy

(i) $\psi(x) = (U * \mu)(x) = 2\mathcal{F}[\mu]$ μ-a.e.

(ii) $\psi(x) = (U * \mu)(x) \ge 2\mathcal{F}[\mu]$ for a.e. $x \in \mathbb{R}^d$.

Regularity??

Euler-Lagrange Conditions

W_∞ EL-Conditions

Assume that U satisfies (H1) and let μ be a local compactly supported minimizer of the energy $\mathcal{F}[\mu]$ in the W_∞ ball or radius ε. Then any point $x_0 \in \text{supp}(\mu)$ is a local minimimum of $\psi = U * \mu$ in the sense that

$$\psi(x_0) \leq \psi(x) \text{ for a.e. } x \in B_\varepsilon(x_0).$$

Note that ε is uniform on the support of μ.

W_2 EL-Conditions

Under the same assumptions, if μ is a W_2-local minimizer of the energy, then the potential ψ satisfy

(i) $\psi(x) = (U * \mu)(x) = 2\mathcal{F}[\mu]$ μ-a.e.

(ii) $\psi(x) = (U * \mu)(x) \geq 2\mathcal{F}[\mu]$ for a.e. $x \in \mathbb{R}^d$.

Regularity??

Euler-Lagrange Conditions

W_∞ EL-Conditions

Assume that U satisfies (H1) and let μ be a local compactly supported minimizer of the energy $\mathcal{F}[\mu]$ in the W_∞ ball or radius ε. Then any point $x_0 \in \text{supp}(\mu)$ is a local minimimum of $\psi = U * \mu$ in the sense that

$$\psi(x_0) \leq \psi(x) \text{ for a.e. } x \in B_\varepsilon(x_0).$$

Note that ε is uniform on the support of μ.

W_2 EL-Conditions

Under the same assumptions, if μ is a W_2-local minimizer of the energy, then the potential ψ satisfy

(i) $\psi(x) = (U * \mu)(x) = 2\mathcal{F}[\mu]$ μ-a.e.

(ii) $\psi(x) = (U * \mu)(x) \geq 2\mathcal{F}[\mu]$ for a.e. $x \in \mathbb{R}^d$.

Regularity??

Euler-Lagrange Conditions

W_∞ EL-Conditions

Assume that U satisfies (H1) and let μ be a local compactly supported minimizer of the energy $\mathcal{F}[\mu]$ in the W_∞ ball or radius ε. Then any point $x_0 \in \mathrm{supp}(\mu)$ is a local minimimum of $\psi = U * \mu$ in the sense that

$$\psi(x_0) \leq \psi(x) \text{ for a.e. } x \in B_\varepsilon(x_0).$$

Note that ε is uniform on the support of μ.

W_2 EL-Conditions

Under the same assumptions, if μ is a W_2-local minimizer of the energy, then the potential ψ satisfy

(i) $\psi(x) = (U * \mu)(x) = 2\mathcal{F}[\mu]$ μ-a.e.

(ii) $\psi(x) = (U * \mu)(x) \geq 2\mathcal{F}[\mu]$ for a.e. $x \in \mathbb{R}^d$.

Regularity??

Mild versus Strong Repulsive potentials

Support is essentially 0-dimensional for $b > 2$.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which behaves like $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ cannot have a s-dimensional smooth components for any $1 \leq s \leq d$.

Dimension of the Support depends on $2 - d < b < 2$.

Assume that μ is a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ such that U is radial with $U(x) \sim -|x|^b$ near zero and $2 - d < b < 2$. If μ contains s-Haussdorff dimensional connected components in its support, then $s \geq 2 - b$.

(Balagué, C., Laurent, Raoul; ARMA 2013)

Strategy: Pure variational approach: by contradiction we build better competitors.

Mild versus Strong Repulsive potentials

Support is essentially 0-dimensional for $b > 2$.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which behaves like $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ cannot have a s-dimensional smooth components for any $1 \leq s \leq d$.

Dimension of the Support depends on $2 - d < b < 2$.

Assume that μ is a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ such that U is radial with $U(x) \sim -|x|^b$ near zero and $2 - d < b < 2$. If μ contains s-Haussdorff dimensional connected components in its support, then $s \geq 2 - b$.

(Balagué, C., Laurent, Raoul; ARMA 2013)

Strategy: Pure variational approach: by contradiction we build better competitors.

Mild versus Strong Repulsive potentials

Support is essentially 0-dimensional for $b > 2$.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which behaves like $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ cannot have a s-dimensional smooth components for any $1 \leq s \leq d$.

Dimension of the Support depends on $2 - d < b < 2$.

Assume that μ is a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ such that U is radial with $U(x) \sim -|x|^b$ near zero and $2 - d < b < 2$. If μ contains s-Haussdorff dimensional connected components in its support, then $s \geq 2 - b$.

(Balagué, C., Laurent, Raoul; ARMA 2013)

Strategy: Pure variational approach: by contradiction we build better competitors.

Mild versus Strong Repulsive potentials

Support is essentially 0-dimensional for $b > 2$.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which behaves like $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ cannot have a s-dimensional smooth components for any $1 \leq s \leq d$.

Dimension of the Support depends on $2 - d < b < 2$.

Assume that μ is a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ such that U is radial with $U(x) \sim -|x|^b$ near zero and $2 - d < b < 2$. If μ contains s-Haussdorff dimensional connected components in its support, then $s \geq 2 - b$.

(Balagué, C., Laurent, Raoul; ARMA 2013)

Strategy: Pure variational approach: by contradiction we build better competitors.

Mild versus Strong Repulsive potentials

Support is essentially 0-dimensional for $b > 2$.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which behaves like $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ cannot have a s-dimensional smooth components for any $1 \leq s \leq d$.

Dimension of the Support depends on $2 - d < b < 2$.

Assume that μ is a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ such that U is radial with $U(x) \sim -|x|^b$ near zero and $2 - d < b < 2$. If μ contains s-Haussdorff dimensional connected components in its support, then $s \geq 2 - b$.

(Balagué, C., Laurent, Raoul; ARMA 2013)

Strategy: Pure variational approach: by contradiction we build better competitors.

Outline

Macroscopic Models: Repulsive-Attractive Potentials

Assume the following hypotheses on U hold.

(H1) $U \in C^2(\mathbb{R}^d)$ and U is **radially symmetric**.

(H2) U is **bounded from below** and $U(0) = 0$.

(H3) There exists $R > 0$ with $U(x) < 0$ for all $|x| < R$ and $U(x) \geq 0$ for all $|x| \geq R$.

(H4) Fix $\alpha > 2$. We write $\widetilde{U}(|x|) := U(x)$ and $\widetilde{U}_p(r) := \frac{\widetilde{U}(pr)}{p^\alpha}$ for any $p > 0$ and $r \geq 0$. There exists a constant $C > 0$ such that

$$\begin{cases} \widetilde{U}_p(r) \to -Cr^\alpha \\ \widetilde{U}_p'(r) \to -C\alpha r^{\alpha-1} \end{cases} \quad \text{as } p \to 0 \text{ for all } r \geq 0.$$

Second variation of energy.- Let μ be a d_∞-local minimizer of E with $E(\mu) < +\infty$. There exists $\delta > 0$ such that for all $x_0 \in \operatorname{supp}\mu$ we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\nu(x)\, d\nu(y) \geq 0$$

for any measure ν with $\operatorname{supp}\nu \subset \operatorname{supp}\mu \cap B(x_0, \delta)$ and $\nu(\mathbb{R}^d) = 0$.

Macroscopic Models: Repulsive-Attractive Potentials

Assume the following hypotheses on U hold.

(H1) $U \in C^2(\mathbb{R}^d)$ and U is **radially symmetric**.

(H2) U is **bounded from below** and $U(0) = 0$.

(H3) There exists $R > 0$ with $U(x) < 0$ for all $|x| < R$ and $U(x) \geq 0$ for all $|x| \geq R$.

(H4) Fix $\alpha > 2$. We write $\widetilde{U}(|x|) := U(x)$ and $\widetilde{U}_p(r) := \frac{\widetilde{U}(pr)}{p^\alpha}$ for any $p > 0$ and $r \geq 0$. There exists a constant $C > 0$ such that

$$\begin{cases} \widetilde{U}_p(r) \to -Cr^\alpha \\ \widetilde{U}_p'(r) \to -C\alpha r^{\alpha-1} \end{cases} \quad \text{as } p \to 0 \text{ for all } r \geq 0.$$

Second variation of energy.- Let μ be a d_∞-local minimizer of E with $E(\mu) < +\infty$. There exists $\delta > 0$ such that for all $x_0 \in \operatorname{supp}\mu$ we have

$$\boxed{\int_{\mathbb{R}^d \times \mathbb{R}^d} U(x-y)\, d\nu(x)\, d\nu(y) \geq 0}$$

for any measure ν with $\operatorname{supp}\nu \subset \operatorname{supp}\mu \cap B(x_0, \delta)$ and $\nu(\mathbb{R}^d) = 0$.

Macroscopic Models: Repulsive-Attractive Potentials

Theorem.[1] Let μ be a W_∞-local minimizer of E. Then each point of $\operatorname{supp}\mu$ is isolated; in particular μ is atomic.

Steps of proof.

- Suppose $0, x_1, -x_2 \in \operatorname{supp}\mu \cap B(0,\delta)$. Choose $\nu_\lambda = -\delta_0 + \lambda\delta_{x_1} + (1-\lambda)\delta_{-x_2}$ in place of ν in the second variation and get, for an appropriate choice of λ,

$$\sqrt{-U(x_1)} + \sqrt{-U(x_2)} \geq \sqrt{-U(x_1+x_2)}.$$

- Assume, by homogeneity, that $x_1 + x_2 = pe_1$, where e_1 is the first unit vector of the orthonormal base of $\mathbb{R}^d$, and $p > 0$ is a small rescaling parameter. From the above inequality, get

$$\sqrt{-U(x_1)} + \sqrt{-U(pe_1 - x_1)} \geq \sqrt{-U(pe_1)}.$$

[1] Carrillo, Figalli, Patacchini, in preparation.

Macroscopic Models: Repulsive-Attractive Potentials

Theorem.[1] Let μ be a W_∞-local minimizer of E. Then each point of supp μ is isolated; in particular μ is atomic.

Steps of proof.

- Suppose $0, x_1, -x_2 \in \operatorname{supp} \mu \cap B(0,\delta)$. Choose $\nu_\lambda = -\delta_0 + \lambda\delta_{x_1} + (1-\lambda)\delta_{-x_2}$ in place of ν in the second variation and get, for an appropriate choice of λ,

$$\sqrt{-U(x_1)} + \sqrt{-U(x_2)} \geq \sqrt{-U(x_1+x_2)}.$$

- Assume, by homogeneity, that $x_1 + x_2 = pe_1$, where e_1 is the first unit vector of the orthonormal base of $\mathbb{R}^d$, and $p > 0$ is a small rescaling parameter. From the above inequality, get

$$\sqrt{-U(x_1)} + \sqrt{-U(pe_1 - x_1)} \geq \sqrt{-U(pe_1)}.$$

[1] Carrillo, Figalli, Patacchini, in preparation.

Macroscopic Models: Repulsive-Attractive Potentials

Theorem.[1] Let μ be a W_∞-local minimizer of E. Then each point of supp μ is isolated; in particular μ is atomic.

Steps of proof.

- Suppose $0, x_1, -x_2 \in \operatorname{supp}\mu \cap B(0,\delta)$. Choose $\nu_\lambda = -\delta_0 + \lambda\delta_{x_1} + (1-\lambda)\delta_{-x_2}$ in place of ν in the second variation and get, for an appropriate choice of λ,

$$\sqrt{-U(x_1)} + \sqrt{-U(x_2)} \geq \sqrt{-U(x_1+x_2)}.$$

- Assume, by homogeneity, that $x_1 + x_2 = pe_1$, where e_1 is the first unit vector of the orthonormal base of $\mathbb{R}^d$, and $p > 0$ is a small rescaling parameter. From the above inequality, get

$$\sqrt{-U(x_1)} + \sqrt{-U(pe_1 - x_1)} \geq \sqrt{-U(pe_1)}.$$

[1] Carrillo, Figalli, Patacchini, in preparation.

Macroscopic Models: Repulsive-Attractive Potentials

- Write $x_1 = p(te_1 + y)$, where $y \in \mathbb{R}^d$ with zero first coordinate, and, by homogeneity, $t \in [0,1]$. Then, using that $|x_1| \leq pt + p|y|$ and $|pe_1 - x_1| \leq p(1-t) + p|y|$, and that, for any $x \in \mathbb{R}^d$ and p small enough, $\sqrt{-U(px)}$ is radially non-decreasing as a function of $x \in \mathbb{R}^d$, get

$$\sqrt{-\widetilde{U}(p(t+|y|))} + \sqrt{-\widetilde{U}(p((1-t)+|y|))} \geq \sqrt{-\widetilde{U}(p)}.$$

- Divide the inequality above by $p^{\alpha/2}$ and obtain

$$\sqrt{-\widetilde{U}_p(t+|y|)} + \sqrt{-\widetilde{U}_p((1-t)+|y|)} \geq \sqrt{-\widetilde{U}_p(1)}.$$

- By (H4) get, as $p \to 0$,

$$(t+|y|)^{\alpha/2} + ((1-t)+|y|)^{\alpha/2} \geq 1$$

Macroscopic Models: Repulsive-Attractive Potentials

- Write $x_1 = p(te_1 + y)$, where $y \in \mathbb{R}^d$ with zero first coordinate, and, by homogeneity, $t \in [0,1]$. Then, using that $|x_1| \leq pt + p|y|$ and $|pe_1 - x_1| \leq p(1-t) + p|y|$, and that, for any $x \in \mathbb{R}^d$ and p small enough, $\sqrt{-U(px)}$ is radially non-decreasing as a function of $x \in \mathbb{R}^d$, get

$$\sqrt{-\widetilde{U}(p(t+|y|))} + \sqrt{-\widetilde{U}(p((1-t)+|y|))} \geq \sqrt{-\widetilde{U}(p)}.$$

- Divide the inequality above by $p^{\alpha/2}$ and obtain

$$\sqrt{-\widetilde{U}_p(t+|y|)} + \sqrt{-\widetilde{U}_p((1-t)+|y|)} \geq \sqrt{-\widetilde{U}_p(1)}.$$

- By (H4) get, as $p \to 0$,

$$(t+|y|)^{\alpha/2} + ((1-t)+|y|)^{\alpha/2} \geq 1$$

Macroscopic Models: Repulsive-Attractive Potentials

- Write $x_1 = p(te_1 + y)$, where $y \in \mathbb{R}^d$ with zero first coordinate, and, by homogeneity, $t \in [0,1]$. Then, using that $|x_1| \leq pt + p|y|$ and $|pe_1 - x_1| \leq p(1-t) + p|y|$, and that, for any $x \in \mathbb{R}^d$ and p small enough, $\sqrt{-U(px)}$ is radially non-decreasing as a function of $x \in \mathbb{R}^d$, get

$$\sqrt{-\widetilde{U}(p(t+|y|))} + \sqrt{-\widetilde{U}(p((1-t)+|y|))} \geq \sqrt{-\widetilde{U}(p)}.$$

- Divide the inequality above by $p^{\alpha/2}$ and obtain

$$\sqrt{-\widetilde{U}_p(t+|y|)} + \sqrt{-\widetilde{U}_p((1-t)+|y|)} \geq \sqrt{-\widetilde{U}_p(1)}.$$

- By (H4) get, as $p \to 0$,

$$(t+|y|)^{\alpha/2} + ((1-t)+|y|)^{\alpha/2} \geq 1$$

Macroscopic Models: Repulsive-Attractive Potentials

- For all $s \in [0,1]$ and $z \in \mathbb{R}^d$, define

$$s_\alpha(s,z) = (s + |z|)^{\alpha/2} + ((1-s) + |z|)^{\alpha/2} - 1,$$

  and define, for any two distinct points $v, v' \in \mathbb{R}^d$, the open set

$$S_\alpha(v,v') := \left\{ w \in \mathbb{R}^d \mid s_\alpha\left(\frac{|\pi w - v|}{|v - v'|}, \pi w - w \right) < 0 \right\},$$

  where π denotes the orthogonal projection on the segment $[v, v']$.

- What we have shown: for any $y_0, y_1 \in \operatorname{supp} \mu$, asymptotically close, there cannot be a third point in $\operatorname{supp} \mu \cap S_\alpha(y_0, y_1)$.

Macroscopic Models: Repulsive-Attractive Potentials

- For all $s \in [0,1]$ and $z \in \mathbb{R}^d$, define

$$s_\alpha(s,z) = (s + |z|)^{\alpha/2} + ((1-s) + |z|)^{\alpha/2} - 1,$$

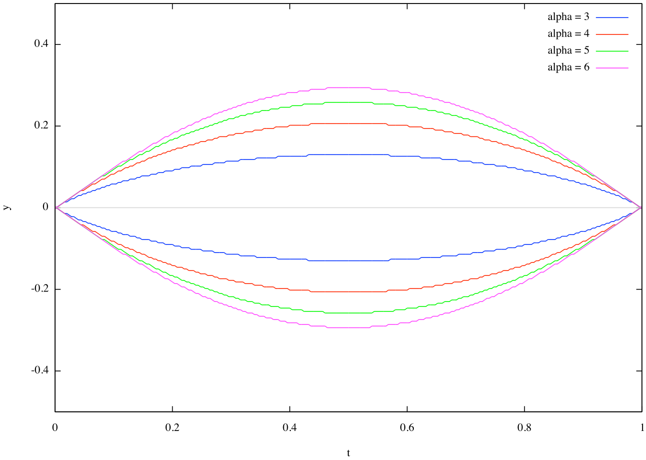
and define, for any two distinct points $v, v' \in \mathbb{R}^d$, the open set

$$S_\alpha(v,v') := \left\{ w \in \mathbb{R}^d \mid s_\alpha\left(\frac{|\pi w - v|}{|v - v'|}, \pi w - w \right) < 0 \right\},$$

where π denotes the orthogonal projection on the segment $[v, v']$.

- What we have shown: for any $y_0, y_1 \in \operatorname{supp}\mu$, asymptotically close, there cannot be a third point in $\operatorname{supp}\mu \cap S_\alpha(y_0, y_1)$.

Macroscopic Models: Repulsive-Attractive Potentials



Macroscopic Models: Repulsive-Attractive Potentials

- For any two distinct points $v, v' \in \mathbb{R}^d$, define the open "double cone" with opening $\tau > 0$ by

$$C_\tau(v, v') := \left\{ w \in \mathbb{R}^d \mid \frac{\mathrm{dist}(w, [v, v'])}{\min\{|\pi w - v|, |\pi w - v'|\}} < \tau \right\},$$

where $[v, v']$ denotes the segment joining v to v' and π denotes the orthogonal projection on the segment $[v, v']$.

- Since $\alpha > 2$, $r \mapsto r^{\alpha/2}$ is a convex function on $[0, +\infty)$, and so $S_\alpha(y_0, y_1)$ is a convex set. Therefore we can fit a double cone generated by y_0 and y_1 inside it.
- We can actually compute the opening $\gamma(\alpha)$ of the cone that fits in $S_\alpha(y_0, y_1)$ with maximal volume:

$$\gamma(\alpha) = \frac{1}{2^{\alpha/2-1}} - 1.$$

- Finish the proof by contradiction. Suppose y_0 is not an isolated point, then it can be approached by a sequence of points in $\mathrm{supp}\mu$ in some direction. Therefore, using (H4) we know that, close enough to y_0, one can find two points belonging to this sequence, say x_k and x_{k+1}, such that $x_{k+1} \in C_{\gamma(\alpha)}(y_0, x_k)$.

Macroscopic Models: Repulsive-Attractive Potentials

- For any two distinct points $v, v' \in \mathbb{R}^d$, define the open "double cone" with opening $\tau > 0$ by

$$C_\tau(v,v') := \left\{ w \in \mathbb{R}^d \mid \frac{\mathrm{dist}(w,[v,v'])}{\min\{|\pi w - v|, |\pi w - v'|\}} < \tau \right\},$$

where $[v,v']$ denotes the segment joining v to v' and π denotes the orthogonal projection on the segment $[v,v']$.

- Since $\alpha > 2$, $r \mapsto r^{\alpha/2}$ is a convex function on $[0,+\infty)$, and so $S_\alpha(y_0,y_1)$ is a convex set. Therefore we can fit a double cone generated by y_0 and y_1 inside it.
- We can actually compute the opening $\gamma(\alpha)$ of the cone that fits in $S_\alpha(y_0,y_1)$ with maximal volume:

$$\gamma(\alpha) = \frac{1}{2^{\alpha/2-1}} - 1.$$

- Finish the proof by contradiction. Suppose y_0 is not an isolated point, then it can be approached by a sequence of points in suppμ in some direction. Therefore, using (H4) we know that, close enough to y_0, one can find two points belonging to this sequence, say x_k and x_{k+1}, such that $x_{k+1} \in C_{\gamma(\alpha)}(y_0,x_k)$.

Macroscopic Models: Repulsive-Attractive Potentials

- For any two distinct points $v, v' \in \mathbb{R}^d$, define the open "double cone" with opening $\tau > 0$ by

$$C_\tau(v, v') := \left\{ w \in \mathbb{R}^d \mid \frac{\mathrm{dist}(w, [v, v'])}{\min\{|\pi w - v|, |\pi w - v'|\}} < \tau \right\},$$

  where $[v, v']$ denotes the segment joining v to v' and π denotes the orthogonal projection on the segment $[v, v']$.
- Since $\alpha > 2$, $r \mapsto r^{\alpha/2}$ is a convex function on $[0, +\infty)$, and so $S_\alpha(y_0, y_1)$ is a convex set. Therefore we can fit a double cone generated by y_0 and y_1 inside it.
- We can actually compute the opening $\gamma(\alpha)$ of the cone that fits in $S_\alpha(y_0, y_1)$ with maximal volume:

$$\gamma(\alpha) = \frac{1}{2^{\alpha/2-1}} - 1.$$

- Finish the proof by contradiction. Suppose y_0 is not an isolated point, then it can be approached by a sequence of points in suppμ in some direction. Therefore, using (H4) we know that, close enough to y_0, one can find two points belonging to this sequence, say x_k and x_{k+1}, such that $x_{k+1} \in C_{\gamma(\alpha)}(y_0, x_k)$.

Macroscopic Models: Repulsive-Attractive Potentials

- For any two distinct points $v, v' \in \mathbb{R}^d$, define the open "double cone" with opening $\tau > 0$ by

$$C_\tau(v, v') := \left\{ w \in \mathbb{R}^d \mid \frac{\mathrm{dist}(w, [v, v'])}{\min\{|\pi w - v|, |\pi w - v'|\}} < \tau \right\},$$

  where $[v, v']$ denotes the segment joining v to v' and π denotes the orthogonal projection on the segment $[v, v']$.
- Since $\alpha > 2$, $r \mapsto r^{\alpha/2}$ is a convex function on $[0, +\infty)$, and so $S_\alpha(y_0, y_1)$ is a convex set. Therefore we can fit a double cone generated by y_0 and y_1 inside it.
- We can actually compute the opening $\gamma(\alpha)$ of the cone that fits in $S_\alpha(y_0, y_1)$ with maximal volume:

$$\gamma(\alpha) = \frac{1}{2^{\alpha/2-1}} - 1.$$

- Finish the proof by contradiction. Suppose y_0 is not an isolated point, then it can be approached by a sequence of points in suppμ in some direction. Therefore, using (H4) we know that, close enough to y_0, one can find two points belonging to this sequence, say x_k and x_{k+1}, such that $x_{k+1} \in C_{\gamma(\alpha)}(y_0, x_k)$.

Outline

Existence Global Minimizers

Non-HStable: Energy at infinity cost more than near the origin, i.e., the potential U satisfies

(H5) There exists $\mu \in \mathcal{P}(\mathbb{R}^d)$ compactly supported such that $\mathcal{F}[\mu] < 0$.

(H5) $\lim_{|x| \to \infty} U(x) \geq 0$.

Main Theorem

Assume that the radial potential U satisfies Hypotheses (H1), (H5), and is increasing outside a large ball. Then there exists a global minimiser for the energy $\mathcal{F}$. Furthermore, any such global minimiser has compact support.

(Cañizo, C., Patacchini; preprint 2014)
Main ideas: Uniform repartition of the mass over the support.

(Simione, Slepcev, Topaloglou; preprint 2014)
Lions Concentration Compactness Principle

Existence Global Minimizers

Non-HStable: Energy at infinity cost more than near the origin, i.e., the potential U satisfies

(H5) There exists $\mu \in \mathcal{P}(\mathbb{R}^d)$ compactly supported such that $\mathcal{F}[\mu] < 0$.

(H5) $\lim_{|x|\to\infty} U(x) \geq 0$.

Main Theorem

Assume that the radial potential U satisfies Hypotheses (H1), (H5), and is increasing outside a large ball. Then there exists a global minimiser for the energy $\mathcal{F}$. Furthermore, any such global minimiser has compact support.

(Cañizo, C., Patacchini; preprint 2014)
Main ideas: Uniform repartition of the mass over the support.

(Simione, Slepcev, Topaloglou; preprint 2014)
Lions Concentration Compactness Principle

Existence Global Minimizers

Non-HStable: Energy at infinity cost more than near the origin, i.e., the potential U satisfies

(H5) There exists $\mu \in \mathcal{P}(\mathbb{R}^d)$ compactly supported such that $\mathcal{F}[\mu] < 0$.

(H5) $\lim_{|x| \to \infty} U(x) \geq 0$.

Main Theorem

Assume that the radial potential U satisfies Hypotheses (H1), (H5), and is increasing outside a large ball. Then there exists a global minimiser for the energy $\mathcal{F}$. Furthermore, any such global minimiser has compact support.

(Cañizo, C., Patacchini; preprint 2014)
Main ideas: Uniform repartition of the mass over the support.

(Simione, Slepcev, Topaloglou; preprint 2014)
Lions Concentration Compactness Principle

Existence Global Minimizers

Non-HStable: Energy at infinity cost more than near the origin, i.e., the potential U satisfies

(H5) There exists $\mu \in \mathcal{P}(\mathbb{R}^d)$ compactly supported such that $\mathcal{F}[\mu] < 0$.

(H5) $\lim_{|x| \to \infty} U(x) \geq 0$.

Main Theorem

Assume that the radial potential U satisfies Hypotheses (H1), (H5), and is increasing outside a large ball. Then there exists a global minimiser for the energy $\mathcal{F}$. Furthermore, any such global minimiser has compact support.

(Cañizo, C., Patacchini; preprint 2014)
Main ideas: Uniform repartition of the mass over the support.

(Simione, Slepcev, Topaloglou; preprint 2014)
Lions Concentration Compactness Principle

Key Estimate

By (H1) for R large enough:

$$E_R := \min \left\{ \mathcal{F}[\mu] \mid \mu \in \mathcal{P}_R(\mathbb{R}^d) \right\} \leq E_* < 0$$

Euler-Lagrange: for ρ_R-almost all $z \in \operatorname{supp}\rho_R$ we have

$$\frac{1}{2} \int_{\mathbb{R}} U(z-x)\, \mathrm{d}\rho_R(x) = E_R.$$

Choose $A \in \mathbb{R}$ with $\frac{1}{2}U_{\min} \leq E_* < A < 0$ and $r' > 0$ with $U(x) \geq 2A$ for $|x| \geq r'$.
Then for ρ_R-almost every z we have

$$\begin{aligned}
2E_R &= \int_{\mathbb{R}} U(z-x)\, \mathrm{d}\rho_R(x) \\
&= \int_{B(z,r')} U(z-x)\, \mathrm{d}\rho_R(x) + \int_{\mathbb{R}^d \setminus B(z,r')} U(z-x)\, \mathrm{d}\rho_R(x) \\
&\geq U_{\min} \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A \int_{\mathbb{R}^d \setminus B(z,r')} \mathrm{d}\rho_R(x) = (U_{\min} - 2A) \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A,
\end{aligned}$$

Rearranging terms:

$$\int_{B(z,r')} \mathrm{d}\rho_R(x) \geq \frac{A - E_R}{A - \frac{1}{2}U_{\min}} \geq \frac{A - E_*}{A - \frac{1}{2}U_{\min}} =: m.$$

Key Estimate

By (H1) for R large enough:

$$E_R := \min \left\{ \mathcal{F}[\mu] \mid \mu \in \mathcal{P}_R(\mathbb{R}^d) \right\} \leq E_* < 0$$

Euler-Lagrange: for ρ_R-almost all $z \in \operatorname{supp}\rho_R$ we have

$$\frac{1}{2} \int_{\mathbb{R}} U(z-x)\, \mathrm{d}\rho_R(x) = E_R.$$

Choose $A \in \mathbb{R}$ with $\frac{1}{2}U_{\min} \leq E_* < A < 0$ and $r' > 0$ with $U(x) \geq 2A$ for $|x| \geq r'$.
Then for ρ_R-almost every z we have

$$\begin{aligned} 2E_R &= \int_{\mathbb{R}} U(z-x)\, \mathrm{d}\rho_R(x) \\ &= \int_{B(z,r')} U(z-x)\, \mathrm{d}\rho_R(x) + \int_{\mathbb{R}^d \setminus B(z,r')} U(z-x)\, \mathrm{d}\rho_R(x) \\ &\geq U_{\min} \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A \int_{\mathbb{R}^d \setminus B(z,r')} \mathrm{d}\rho_R(x) = (U_{\min} - 2A) \int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A, \end{aligned}$$

Rearranging terms:

$$\int_{B(z,r')} \mathrm{d}\rho_R(x) \geq \frac{A - E_R}{A - \frac{1}{2}U_{\min}} \geq \frac{A - E_*}{A - \frac{1}{2}U_{\min}} =: m.$$

Key Estimate

By (H1) for R large enough:

$$E_R := \min\left\{\mathcal{F}[\mu] \mid \mu \in \mathcal{P}_R(\mathbb{R}^d)\right\} \leq E_* < 0$$

Euler-Lagrange: for ρ_R-almost all $z \in \operatorname{supp}\rho_R$ we have

$$\frac{1}{2}\int_{\mathbb{R}} U(z-x)\,\mathrm{d}\rho_R(x) = E_R.$$

Choose $A \in \mathbb{R}$ with $\frac{1}{2}U_{\min} \leq E_* < A < 0$ and $r' > 0$ with $U(x) \geq 2A$ for $|x| \geq r'$.

Then for ρ_R-almost every z we have

$$\begin{aligned} 2E_R &= \int_{\mathbb{R}} U(z-x)\,\mathrm{d}\rho_R(x) \\ &= \int_{B(z,r')} U(z-x)\,\mathrm{d}\rho_R(x) + \int_{\mathbb{R}^d\setminus B(z,r')} U(z-x)\,\mathrm{d}\rho_R(x) \\ &\geq U_{\min}\int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A\int_{\mathbb{R}^d\setminus B(z,r')} \mathrm{d}\rho_R(x) = (U_{\min}-2A)\int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A, \end{aligned}$$

Rearranging terms:

$$\int_{B(z,r')} \mathrm{d}\rho_R(x) \geq \frac{A-E_R}{A-\frac{1}{2}U_{\min}} \geq \frac{A-E_*}{A-\frac{1}{2}U_{\min}} =: m.$$

Key Estimate

By (H1) for R large enough:

$$E_R := \min\left\{\mathcal{F}[\mu] \mid \mu \in \mathcal{P}_R(\mathbb{R}^d)\right\} \le E_* < 0$$

Euler-Lagrange: for ρ_R-almost all $z \in \operatorname{supp}\rho_R$ we have

$$\frac{1}{2}\int_{\mathbb{R}} U(z-x)\,\mathrm{d}\rho_R(x) = E_R.$$

Choose $A \in \mathbb{R}$ with $\frac{1}{2}U_{\min} \le E_* < A < 0$ and $r' > 0$ with $U(x) \ge 2A$ for $|x| \ge r'$. Then for ρ_R-almost every z we have

$$\begin{aligned} 2E_R &= \int_{\mathbb{R}} U(z-x)\,\mathrm{d}\rho_R(x) \\ &= \int_{B(z,r')} U(z-x)\,\mathrm{d}\rho_R(x) + \int_{\mathbb{R}^d\setminus B(z,r')} U(z-x)\,\mathrm{d}\rho_R(x) \\ &\ge U_{\min}\int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A\int_{\mathbb{R}^d\setminus B(z,r')} \mathrm{d}\rho_R(x) = (U_{\min}-2A)\int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A, \end{aligned}$$

Rearranging terms:

$$\int_{B(z,r')} \mathrm{d}\rho_R(x) \ge \frac{A-E_R}{A-\frac{1}{2}U_{\min}} \ge \frac{A-E_*}{A-\frac{1}{2}U_{\min}} =: m.$$

Key Estimate

By (H1) for R large enough:

$$E_R := \min \left\{ \mathcal{F}[\mu] \mid \mu \in \mathcal{P}_R(\mathbb{R}^d) \right\} \le E_* < 0$$

Euler-Lagrange: for ρ_R-almost all $z \in \operatorname{supp}\rho_R$ we have

$$\frac{1}{2}\int_{\mathbb{R}} U(z-x)\,\mathrm{d}\rho_R(x) = E_R.$$

Choose $A \in \mathbb{R}$ with $\frac{1}{2}U_{\min} \le E_* < A < 0$ and $r' > 0$ with $U(x) \ge 2A$ for $|x| \ge r'$.
Then for ρ_R-almost every z we have

$$\begin{aligned} 2E_R &= \int_{\mathbb{R}} U(z-x)\,\mathrm{d}\rho_R(x) \\ &= \int_{B(z,r')} U(z-x)\,\mathrm{d}\rho_R(x) + \int_{\mathbb{R}^d\setminus B(z,r')} U(z-x)\,\mathrm{d}\rho_R(x) \\ &\ge U_{\min}\int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A\int_{\mathbb{R}^d\setminus B(z,r')} \mathrm{d}\rho_R(x) = (U_{\min}-2A)\int_{B(z,r')} \mathrm{d}\rho_R(x) + 2A, \end{aligned}$$

Rearranging terms:

$$\int_{B(z,r')} \mathrm{d}\rho_R(x) \ge \frac{A-E_R}{A-\frac{1}{2}U_{\min}} \ge \frac{A-E_*}{A-\frac{1}{2}U_{\min}} =: m.$$

Examples

Power-laws & Morse Potentials

Consider the following potentials for all $x \in \mathbb{R}^d$ and $C_A, C_R, \ell_A, \ell_R > 0$:

(i) *(Power-law potential)* $U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$ with $-d < b < a$,

(ii) *(Morse potential)* $U(x) = C_R e^{-\frac{|x|}{\ell_R}} - C_A e^{-\frac{|x|}{\ell_A}}$ with either $\ell_A < \ell_R$ and $\frac{C_A}{C_R} < \left(\frac{\ell_R}{\ell_A}\right)^d$,

with the convention $\frac{|x|^0}{0} = \log |x|$.

Discrete To Continuum: Power-law Case

(C., Chipot, Huang; preprint 2014)

Discrete Setting: Find

$$I_N = \inf_{x \in (\mathbb{R}^d)^N} \mathcal{F}_N(x) \,,$$

with

$$\mathcal{F}_N(x_1, \cdots, x_N) = \sum_{i \neq j}^{N} \left(\frac{|x_i - x_j|^a}{a} - \frac{|x_i - x_j|^b}{b} \right) .$$

Uniform Control of the support

Suppose that $1 \leq b < a$. Then the diameter of any global minimizer of $\mathcal{F}_N$ achieving the infimum I_N is bounded independently of N.

Key Idea: use Euler-Lagrange and a convexity argument for the repulsive potential to estimate the distance between the two particles the furthest away.

Discrete To Continuum: Power-law Case

(C., Chipot, Huang; preprint 2014)

Discrete Setting: Find

$$I_N = \inf_{x \in (\mathbb{R}^d)^N} \mathcal{F}_N(x) \,,$$

with

$$\mathcal{F}_N(x_1, \cdots, x_N) = \sum_{i \neq j}^{N} \left(\frac{|x_i - x_j|^a}{a} - \frac{|x_i - x_j|^b}{b} \right) .$$

Uniform Control of the support

Suppose that $1 \leq b < a$. Then the diameter of any global minimizer of $\mathcal{F}_N$ achieving the infimum I_N is bounded independently of N.

Key Idea: use Euler-Lagrange and a convexity argument for the repulsive potential to estimate the distance between the two particles the furthest away.

Discrete To Continuum: Power-law Case

(C., Chipot, Huang; preprint 2014)

Discrete Setting: Find

$$I_N = \inf_{x \in (\mathbb{R}^d)^N} \mathcal{F}_N(x) \,,$$

with

$$\mathcal{F}_N(x_1, \cdots, x_N) = \sum_{i \neq j}^{N} \left(\frac{|x_i - x_j|^a}{a} - \frac{|x_i - x_j|^b}{b} \right) \,.$$

Uniform Control of the support

Suppose that $1 \leq b < a$. Then the diameter of any global minimizer of $\mathcal{F}_N$ achieving the infimum I_N is bounded independently of N.

Key Idea: use Euler-Lagrange and a convexity argument for the repulsive potential to estimate the distance between the two particles the furthest away.

Regularity of Local Minimizers

(H6) The function $U_a(x) := U(x) - V(x)$ with V being the Newtonian potential in dimension d satisfies:

$$\Delta U_a \in L^p_{loc}(\mathbb{R}^d) \quad \text{for some } p \in (d, \infty]$$

with ΔU_a bounded below.

Main Theorem

Assume that the potential U satisfies Hypotheses (H1) and (H6). Then any μ compactly supported W_∞ local minimizer of the energy $\mathcal{F}$ is bounded uniformly, i.e., $\mu = \rho(x) d\mathcal{L}^d$ with $\rho \in L^\infty(\mathbb{R}^d)$.

(C., Delgadino, Mellet; preprint 2014)

Main ideas: Obstacle problems to obtain information out of the Euler-Lagrange conditions (Nash equilibria conditions).

It works for more-singular-than-Newtonian repulsion at the origin.

Regularity of Local Minimizers

(H6) The function $U_a(x) := U(x) - V(x)$ with V being the Newtonian potential in dimension d satisfies:

$$\Delta U_a \in L^p_{loc}(\mathbb{R}^d) \quad \text{for some } p \in (d, \infty]$$

with ΔU_a bounded below.

Main Theorem

Assume that the potential U satisfies Hypotheses (H1) and (H6). Then any μ compactly supported W_∞ local minimizer of the energy $\mathcal{F}$ is bounded uniformly, i.e., $\mu = \rho(x) d\mathcal{L}^d$ with $\rho \in L^\infty(\mathbb{R}^d)$.

(C., Delgadino, Mellet; preprint 2014)
Main ideas: Obstacle problems to obtain information out of the Euler-Lagrange conditions (Nash equilibria conditions).

It works for more-singular-than-Newtonian repulsion at the origin.

Regularity of Local Minimizers

(H6) The function $U_a(x) := U(x) - V(x)$ with V being the Newtonian potential in dimension d satisfies:

$$\Delta U_a \in L^p_{loc}(\mathbb{R}^d) \qquad \text{for some } p \in (d, \infty]$$

with ΔU_a bounded below.

Main Theorem

Assume that the potential U satisfies Hypotheses (H1) and (H6). Then any μ compactly supported W_∞ local minimizer of the energy $\mathcal{F}$ is bounded uniformly, i.e., $\mu = \rho(x)d\mathcal{L}^d$ with $\rho \in L^\infty(\mathbb{R}^d)$.

(C., Delgadino, Mellet; preprint 2014)
Main ideas: Obstacle problems to obtain information out of the Euler-Lagrange conditions (Nash equilibria conditions).

It works for more-singular-than-Newtonian repulsion at the origin.

Regularity of Local Minimizers

(H6) The function $U_a(x) := U(x) - V(x)$ with V being the Newtonian potential in dimension d satisfies:

$$\Delta U_a \in L^p_{loc}(\mathbb{R}^d) \qquad \text{for some } p \in (d, \infty]$$

with ΔU_a bounded below.

Main Theorem

Assume that the potential U satisfies Hypotheses (H1) and (H6). Then any μ compactly supported W_∞ local minimizer of the energy $\mathcal{F}$ is bounded uniformly, i.e., $\mu = \rho(x) d\mathcal{L}^d$ with $\rho \in L^\infty(\mathbb{R}^d)$.

(C., Delgadino, Mellet; preprint 2014)
Main ideas: Obstacle problems to obtain information out of the Euler-Lagrange conditions (Nash equilibria conditions).

It works for more-singular-than-Newtonian repulsion at the origin.

Obstacle Problem

Continuity of the potential

Assume that the potential U satisfies Hypotheses (H1) and (H3). Let μ be a W_∞ local minimizer of E. Then the potential $\psi(x) := U * \mu(x)$ associated to μ is a continuous function in $\mathbb{R}^N$.

Implicit Obstacle Problem

For all $x_0 \in \text{supp}(\mu)$, the potential function ψ is equal, in $B_\varepsilon(x_0)$, to the unique solution of the obstacle problem

$$\begin{cases} \varphi & \geq C_0, & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi & \geq -F(x), & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi & = -F(x), & \text{in } B_\varepsilon(x_0) \cap \{\varphi > C_0\} \\ \varphi & = \psi, & \text{on } \partial B_\varepsilon(x_0), \end{cases}$$

where $C_0 = \psi(x_0)$ and $F(x) = \Delta U_a * \mu \in L^p_{loc}(\mathbb{R}^d)$. Furthermore, the density μ is given by

$$\mu = -\Delta\psi + F.$$

Particular Case: Newtonian repulsion and quadratic confinement, the global minimizer is the characteristic of a ball with unit mass upto translations.

Obstacle Problem

Continuity of the potential

Assume that the potential U satisfies Hypotheses (H1) and (H3). Let μ be a W_∞ local minimizer of E. Then the potential $\psi(x) := U * \mu(x)$ associated to μ is a continuous function in $\mathbb{R}^N$.

Implicit Obstacle Problem

For all $x_0 \in \operatorname{supp}(\mu)$, the potential function ψ is equal, in $B_\varepsilon(x_0)$, to the unique solution of the obstacle problem

$$\begin{cases} \varphi \geq C_0, & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi \geq -F(x), & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi = -F(x), & \text{in } B_\varepsilon(x_0) \cap \{\varphi > C_0\} \\ \varphi = \psi, & \text{on } \partial B_\varepsilon(x_0), \end{cases}$$

where $C_0 = \psi(x_0)$ and $F(x) = \Delta U_a * \mu \in L^p_{loc}(\mathbb{R}^d)$. Furthermore, the density μ is given by

$$\mu = -\Delta\psi + F.$$

Particular Case: Newtonian repulsion and quadratic confinement, the global minimizer is the characteristic of a ball with unit mass upto translations.

Obstacle Problem

Continuity of the potential

Assume that the potential U satisfies Hypotheses (H1) and (H3). Let μ be a W_∞ local minimizer of E. Then the potential $\psi(x) := U * \mu(x)$ associated to μ is a continuous function in $\mathbb{R}^N$.

Implicit Obstacle Problem

For all $x_0 \in \text{supp}(\mu)$, the potential function ψ is equal, in $B_\varepsilon(x_0)$, to the unique solution of the obstacle problem

$$\begin{cases} \varphi \geq C_0, & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi \geq -F(x), & \text{in } B_\varepsilon(x_0) \\ -\Delta\varphi = -F(x), & \text{in } B_\varepsilon(x_0) \cap \{\varphi > C_0\} \\ \varphi = \psi, & \text{on } \partial B_\varepsilon(x_0), \end{cases}$$

where $C_0 = \psi(x_0)$ and $F(x) = \Delta U_a * \mu \in L^p_{loc}(\mathbb{R}^d)$. Furthermore, the density μ is given by

$$\mu = -\Delta\psi + F.$$

Particular Case: Newtonian repulsion and quadratic confinement, the global minimizer is the characteristic of a ball with unit mass upto translations.

Conclusions

- The dimensionality of the support of local minimizers of the interaction energy can be classified in terms of the repulsion strength of the potential near zero.
- If the strength of the repulsion is stronger than or equal to Newtonian, they are bounded uniformly.
- Compactly supported global minimizers exist under the reasonable condition that it costs less energy to be near the origin than to be at infinity.
- References:

1. Balagué-C.-Laurent-Raoul (Physica D & ARMA 2013).
2. Cañizo-C.-Patacchini (to appear in ARMA 2015).
3. C.-Delgadino-Mellet (preprint 2014).
4. C.-Figalli-Patacchini (in preparation).

Conclusions

- The dimensionality of the support of local minimizers of the interaction energy can be classified in terms of the repulsion strength of the potential near zero.
- If the strength of the repulsion is stronger than or equal to Newtonian, they are bounded uniformly.
- Compactly supported global minimizers exist under the reasonable condition that it costs less energy to be near the origin than to be at infinity.
- References:

1. Balagué-C.-Laurent-Raoul (Physica D & ARMA 2013).
2. Cañizo-C.-Patacchini (to appear in ARMA 2015).
3. C.-Delgadino-Mellet (preprint 2014).
4. C.-Figalli-Patacchini (in preparation).

Conclusions

- The dimensionality of the support of local minimizers of the interaction energy can be classified in terms of the repulsion strength of the potential near zero.
- If the strength of the repulsion is stronger than or equal to Newtonian, they are bounded uniformly.
- Compactly supported global minimizers exist under the reasonable condition that it costs less energy to be near the origin than to be at infinity.
- References:

1. Balagué-C.-Laurent-Raoul (Physica D & ARMA 2013).
2. Cañizo-C.-Patacchini (to appear in ARMA 2015).
3. C.-Delgadino-Mellet (preprint 2014).
4. C.-Figalli-Patacchini (in preparation).

Conclusions

- The dimensionality of the support of local minimizers of the interaction energy can be classified in terms of the repulsion strength of the potential near zero.
- If the strength of the repulsion is stronger than or equal to Newtonian, they are bounded uniformly.
- Compactly supported global minimizers exist under the reasonable condition that it costs less energy to be near the origin than to be at infinity.
- References:

1. Balagué-C.-Laurent-Raoul (Physica D & ARMA 2013).
2. Cañizo-C.-Patacchini (to appear in ARMA 2015).
3. C.-Delgadino-Mellet (preprint 2014).
4. C.-Figalli-Patacchini (in preparation).