Swarming Models with Repulsive-Attractive Effects

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Lecture 2, Ravello 2013

Outline

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Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \left[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\gamma} f(y,w,t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \right]$$

Orientation, Attraction and Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right].$$

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Definition of the distance

Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu \qquad \text{for all } \varphi \in C_o(\mathbb{R}^d)\,.$$

Random variables:

Say that X is a random variable with law given by μ, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(x) \, d\mu = \int_{\Omega} (\varphi \circ X) \, dP = \mathbb{E}\left[\varphi(X)\right].$$

Kantorovich-Rubinstein-Wasserstein Distance $p = 1, 2$:

$$W_p^p(\mu, \nu) = \inf_{(X,Y)} \left\{\mathbb{E}\left[|X - Y|^p\right]\right\}$$

where (X, Y) are couples of random variables with μ and ν as respective laws.

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Well-posedness in probability measures[1]

Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with P^t the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions f and g with initial data f_0 and g_0, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \leq \alpha(t)\, W_1(f_0, g_0)$$

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Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j\neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

is the solution corresponding to initial atomic measures.

- Convergence of approximations of measures by particles due to the stability at any given time T as an alternative derivation of the kinetic models.

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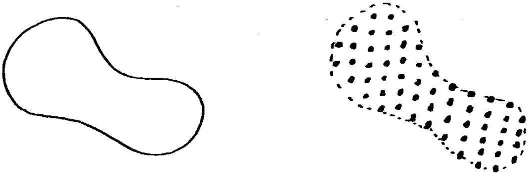
Mean-Field Limit

Just take as many particles as needed in order to have

$$W_1(f_t, f_t^N) \leq \alpha(t)\, W_1(f_0, f_0^N) \to 0 \qquad \text{as } N \to \infty$$

by sampling the initial data in a suitable way.

The sequences of particle solutions becomes a Cauchy sequence with the distance W_1 converging to the solution of the kinetic equation.



Hauray-Jabin 2011: mean field limit for Vlasov with potentials such that $|\nabla U| \leq r^{-\alpha}$, with $\alpha < 1$ with initial data for Vlasov in $L^1 \cap L^\infty$.

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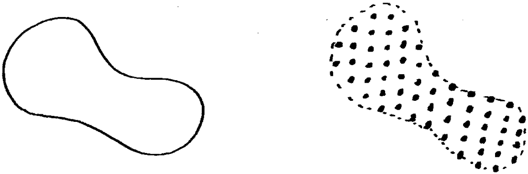

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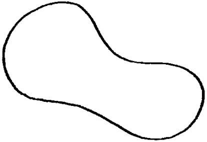

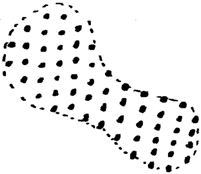

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Outline

Proof of the Theorem

Conditions on E:

1. E is continuous on $[0,T] \times \mathbb{R}^d$,
2. For some $C > 0$,
$$|E(t,x)| \le C_E(1+|x|), \quad \text{for all } t,x \in [0,T] \times \mathbb{R}^d, \text{ and}$$
3. E is locally Lipschitz with respect to x, i.e., for any compact set $K \subseteq \mathbb{R}^d$ there is some $L_K > 0$ such that
$$|E(t,x) - E(t,y)| \le L_K|x-y|, \qquad t \in [0,T], \quad x,y \in K.$$

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Proof of the Theorem

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta |v|^2) v f) = 0,$$

which is a linear first-order equation. The associated characteristic system of ODE's is

$$\frac{d}{dt} X = V,$$
$$\frac{d}{dt} V = E(t, X) + V(\alpha - \beta |V|^2).$$

Flow Map:

Given $(X_0, V_0) \in \mathbb{R}^d \times \mathbb{R}^d$ there exists a unique solution (X, V) to the ODE system in $C^1([0,T]; \mathbb{R}^d \times \mathbb{R}^d)$ satisfying $X(0) = X_0$ and $V(0) = V_0$. In addition, there exists a constant C which depends only on T, $|X_0|$, $|V_0|$, α, β and the constant C_E, such that

$$|(X(t), V(t))| \leq |(X_0, V_0)| \, e^{Ct} \qquad \text{for all } t \in [0, T].$$

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We can thus consider the flow at time $t \in [0, T)$ of ODE's equations

$$\mathcal{T}_E^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d.$$

Again by basic results in ode's, the map $(t, x, v) \mapsto \mathcal{T}_E^t(x, v) = (X, V)$ with (X, V) the solution at time t to the ODE system with initial data (x, v), is jointly continuous in (t, x, v).

For a measure $f_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, the function

$$f : [0, T) \to \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d), \quad t \mapsto f_t := \mathcal{T}_E^t \# f_0$$

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Estimates on characteristics:

- Taking characteristics with initial data inside a fixed ball then there exists $R > 0$ depending on T, in which the whole trajectories are inside a possibly larger ball of radius R for all times $t \in [0, T]$.
- For some constant C which depends only on α, β, R and $\mathrm{Lip}_R(E^i)$, for all P^0 in B_R

$$\left|\mathcal{T}^t_{E^1}(P^0) - \mathcal{T}^t_{E^2}(P^0)\right| \leq \frac{e^{Ct}-1}{C} \sup_{s\in[0,T)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)}.$$

- For some constant C as before

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$$\left|\mathcal{T}^t_{E^1}(P^0) - \mathcal{T}^t_{E^2}(P^0)\right| \leq \frac{e^{Ct}-1}{C} \sup_{s\in[0,T)} \left\|E^1_s - E^2_s\right\|_{L^\infty(B_R)} .$$

- For some constant C as before

$$\left|\mathcal{T}^t_E(P_1) - \mathcal{T}^t_E(P_2)\right| \leq |P_1 - P_2|\, e^{C\int_0^t (\mathrm{Lip}_R(E_s)+1)\,ds}, \quad t \in [0, T].$$

Proof of the Theorem

Error on transported measures through different flows:

Let $\mathcal{T}_1, \mathcal{T}_2 : \mathbb{R}^d \to \mathbb{R}^d$ be two Borel measurable functions. Also, take $f \in \mathcal{P}_1(\mathbb{R}^d)$. Then,

$$W_1(\mathcal{T}_1\#f, \mathcal{T}_2\#f) \le \|\mathcal{T}_1 - \mathcal{T}_2\|_{L^\infty(\mathrm{supp} f)} .$$

Continuity in time for solutions of the linear transport:

$$W_1(\mathcal{T}_E^s\#f, \mathcal{T}_E^t\#f) \le C\,|t-s|\,, \qquad \text{for any } t, s \in [0, T].$$

Error on transported measures through different initial data:

Take a locally Lipschitz map $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ and $f, g \in \mathcal{P}_1(\mathbb{R}^d)$, both with compact support contained in the ball B_R. Then,

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$$\begin{aligned}
W_1(f_t, g_t) &= W_1(\mathcal{T}_f^t \# f_0, \mathcal{T}_g^t \# g_0) \\
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&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} ds + L_t W_1(f_0, g_0) \\
&\leq C_3 \mathrm{Lip}_{2R}(\nabla U) \int_0^t e^{C_4(t-s)} W_1(f_s, g_s)\, ds + e^{C_1 t} W_1(f_0, g_0).
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Outline

Rigorous Statement of the Mean-Field Limit

Aggregation Equation:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \text{ with } u(t,x) := -\nabla U * \rho, & t > 0, \quad x \in \mathbb{R}^d, \\ \rho(0,x) := \rho_0(x), & x \in \mathbb{R}^d, \end{cases}$$

Particle Approximation and Empirical distribution $\mu_N(t)$:

$$\begin{cases} \dot{X}_i(t) = -\sum_{j \neq i} m_j \nabla U(X_i(t) - X_j(t)), \\ X_i(0) = X_i^0, \qquad i = 1, \ldots, N. \end{cases}$$

$$\mu_N(t) = \sum_{i=1}^N m_i \delta_{X_i(t)}, \quad \sum_{i=1}^N m_i = \int_{\mathbb{R}^d} \rho_0(x) dx = 1,$$

with $m_i > 0$, $i = 1, \ldots, N$. We set $\nabla U(0) = 0$ even if singular at the origin.

The convergence:

"$\mu_N^0 \rightharpoonup \rho^0$ weakly-$*$ as measures $\Longrightarrow \mu_N(t) \rightharpoonup \rho(t)$ weakly-$*$ as measures for small time or for every time?"

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Rigorous Statement of the Mean-Field Limit

Quantities to Control: the W_∞-distance between $\rho(t)$ and $\mu_N(t)$, and the minimum inter-particle distance:

$$\eta(t) := W_\infty(\mu_N(t), \rho(t)), \quad \eta_m(t) := \min_{1 \le i \ne j \le N} \left(|X_i(t) - X_j(t)| \right),$$

with $\eta^0 := \eta(0)$ and $\eta_m^0 := \eta_m(0)$.

Assumptions on the potential U: it is C^2 except at the origin, where it might have a singularity. We set $U(0) = 0$ by definition, and

$$|\nabla U(x)| \le \frac{C}{|x|^\alpha}, \quad \text{and} \quad |D^2 U(x)| \le \frac{C}{|x|^{1+\alpha}}, \quad \forall\, x \in \mathbb{R}^d \setminus \{0\},$$

for $-1 \le \alpha < d - 1$.

Note that due to the assumptions on U, we can always find $1 < p < \infty$ such that $(\alpha + 1)p' < d$, and thus ∇U belongs to $\mathcal{W}^{1,p'}_{loc}(\mathbb{R}^d)$.

Weak Solutions: $\rho \in L^\infty(0, T; (L^1 \cap L^p)(\mathbb{R}^d)) \cap \mathcal{C}([0, T], \mathcal{P}_1(\mathbb{R}^d))$, with initial data $\rho^0 \in (\mathcal{P}_1 \cap L^p)(\mathbb{R}^d)$.

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Main Result.-

Let ρ be a solution to the aggregation equation up to time $T > 0$, such that $\rho \in L^\infty(0,T;(L^1 \cap L^p)(\mathbb{R}^d)) \cap \mathcal{C}([0,T],\mathcal{P}_1(\mathbb{R}^d))$, with initial data $\rho^0 \in (\mathcal{P}_1 \cap L^p)(\mathbb{R}^d)$, $0 \le \alpha < -1 + d/p'$, and $1 < p \le \infty$. Furthermore, we assume μ_N^0 converges to ρ^0 for the distance d_∞ as the number of particles N goes to infinity,

$$W_\infty(\mu_N^0, \rho^0) \to 0 \quad \text{as} \quad N \to \infty,$$

and that the initial quantities η^0, η_m^0 satisfy

$$\lim_{N\to\infty} \frac{(\eta^0)^{d/p'}}{(\eta_m^0)^{1+\alpha}} = 0.$$

Under the previous assumptions on the potential, for N large enough the associated particle system is well-defined up to time T, in the sense that there is no collision between particles before that time, and moreover

$$\mu_N(t) \rightharpoonup \rho(t) \quad \text{weakly-}* \text{ as measures as} \quad N \to \infty, \quad \text{for all} \quad t \in [0,T].$$

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Outline

Strategy of the Proof

- In Step A, we estimate the growth of the W_∞ Wasserstein distance between the continuum and the discrete solutions η that involves η itself and η_m in the form:

$$\frac{d\eta}{dt} \le C\eta\|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right).$$

- In Step B, we estimate the decay of the minimum inter-particle distance η_m, which also involves the terms η and η_m in the form:

$$\frac{d\eta_m}{dt} \ge -C\eta_m\|\rho\| \left(1 + \eta^{d/p'} \eta_m^{-(1+\alpha)}\right).$$

- In Step C, under the assumption of a well prepared initial approximation, we combine the estimates above to conclude the desired result.

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Step A: Well-defined characteristics

The assumptions on the potential lead to

$$|\nabla U(x) - \nabla U(y)| \leq \frac{2|x-y|}{\min(|x|,|y|)^{\alpha+1}}\,.$$

Given the velocity fields $u(x,t) = -\nabla U * \rho$ and "$u_N := -\nabla U * \mu_N$". We define the flows:

$$\begin{cases} \dfrac{d}{dt}(\Psi(t;s,x)) = u(t;s,\Psi(t;s,x)), \\ \Psi(s;s,x) = x, \end{cases}$$

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Step A: Well-defined Flows

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\begin{aligned}
|u(t,x)-u(t,y)| &\le \int_{\mathbb{R}^d} |\nabla U(x-z)-\nabla U(y-z)|\rho(t,z)\,dz \\
&\le 2|x-y| \int_{\mathbb{R}^d} \frac{1}{\min(|x-z|,|y-z|)^{\alpha+1}}\rho(t,z)\,dz \\
&\le 4|x-y| \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}}\rho(t,z)\,dz\,.
\end{aligned}
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Now, splitting the last integral into the near- and far-field sets $\mathcal{A} := \{z : |x-z| \ge 1\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$ and estimating the two terms, we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}}\rho(t,z)\,dz &\le \|\rho(t)\|_1 + \left(\int_{\mathcal{B}} \frac{1}{|x-y|^{(1+\alpha)p'}}\,dy\right)^{1/p'} \|\rho(t)\|_p \\
&\le C\|\rho\|\,,
\end{aligned}
$$

for all $x \in \mathbb{R}^d$ due to the assumption $(1+\alpha)p' < d$.

Step A: Well-defined Flows

$$
\begin{aligned}
|u(t,x) - u(t,y)| &\le \int_{\mathbb{R}^d} |\nabla U(x-z) - \nabla U(y-z)| \rho(t,z)\, dz \\
&\le 2|x-y| \int_{\mathbb{R}^d} \frac{1}{\min(|x-z|,|y-z|)^{\alpha+1}} \rho(t,z)\, dz \\
&\le 4|x-y| \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-z|^{\alpha+1}} \rho(t,z)\, dz\,.
\end{aligned}
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Step A: Estimate of the evolution of W_∞

Fixed $0 \le t_0 < \min(T, T_0^N)$ and choose an optimal transport map for W_∞ denoted by $\mathcal{T}^0$ between $\rho(t_0)$ and $\mu_N(t_0)$; $\mu_N(t_0) = \mathcal{T}^0 \# \rho(t_0)$.

The solution of the aggregation equation is given by $\rho(t) = \Psi(t; t_0, \cdot\,) \# \rho(t_0)$ and obviously $\mu_N(t) = \Psi_N(t; t_0, \cdot\,) \# \mu_N(t_0)$ for $t \ge t_0$. We also notice that for $t \ge t_0$

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By Definition of the W_∞ Wasserstein distance, we get

$$\eta(t) = W_\infty(\mu_N(t), \rho(t)) \le \|\Psi(t; t_0, \cdot) - \Psi_N(t; t_0, \cdot) \circ \mathcal{T}^0\|_\infty .$$

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$$\frac{d}{dt}\left(\Psi_N(t; t_0, \mathcal{T}^0(x)) - \Psi(t; t_0, x)\right)\Big|_{t=t_0} = u_N(t_0, \mathcal{T}^0(x)) - u(t_0, x).$$

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and thus $\dfrac{d^+\eta}{dt} \le C \sup_{x\in\mathbb{R}^d} \displaystyle\int_{\mathbb{R}^d} |\nabla U(\mathcal{T}(x) - \mathcal{T}(y)) - \nabla U(x-y)|\rho(y)dy.$

We decompose the integral on $\mathbb{R}^d$ into the near- and the far-field parts as $\mathcal{A} := \{z : |x-z| \ge 4\eta\}$ and $\mathcal{B} := \mathbb{R}^d - \mathcal{A}$, to get

$$\mathcal{I}_1 \le \int_{\mathcal{A}} \frac{2\left(|x - \mathcal{T}(x)| + |y - \mathcal{T}(y)|\right)}{\min(|x-y|, |\mathcal{T}(x) - \mathcal{T}(y)|)^{\alpha+1}} \rho(y) dy \le C\eta \|\rho\|.$$

and

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Step B: Estimate of the evolution of η_m

Pick i, j so that $|X_i - X_j| = \eta_m$.

$$\frac{d}{dt}|X_i - X_j| \geq -|u_N(X_i) - u_N(X_j)|$$
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Step B: Estimate of the evolution of W_∞

For the integral over $\mathcal{B}$, we use that as soon as $X_i \neq \mathcal{T}(y)$, then

$$|\nabla U(X_i - \mathcal{T}(y))| \leq \frac{1}{|X_j - \mathcal{T}(y)|^\alpha} \leq \frac{1}{\eta_m^\alpha},$$

and $\nabla U(X_i - \mathcal{T}(y)) = 0$ otherwise, and similarly for X_j.

A simple Hölder computation implies that

$$\int_{\mathcal{B}} |\nabla U(X_i - \mathcal{T}(y)) - \nabla U(X_j - \mathcal{T}(y))| \, \rho(y) dy \leq C \eta^{d/p'} \eta_m^{-\alpha} \|\rho\|.$$

Putting together we finally conclude the estimate in Step B.

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$$\int_{\mathcal{B}} |\nabla U(X_i - \mathcal{T}(y)) - \nabla U(X_j - \mathcal{T}(y))| \, \rho(y) dy \leq C\eta^{d/p'} \eta_m^{-\alpha} \|\rho\|.$$

Putting together we finally conclude the estimate in Step B.

Step B: Estimate of the evolution of W_∞

For the integral over $\mathcal{B}$, we use that as soon as $X_i \neq \mathcal{T}(y)$, then

$$|\nabla U(X_i - \mathcal{T}(y))| \leq \frac{1}{|X_j - \mathcal{T}(y)|^\alpha} \leq \frac{1}{\eta_m^\alpha},$$

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Step C: Closing the Argument

$$\begin{cases} \dfrac{d^+\eta}{dt} & \le C\eta\|\rho\|\left(1+\eta^{d/p'}\eta_m^{-(1+\alpha)}\right), \\ \dfrac{d\eta_m}{dt} & \ge -C\eta_m\|\rho\|\left(1+\eta^{d/p'}\eta_m^{-(1+\alpha)}\right), \end{cases}$$

for $t \in [0, \min(T, T_0^N))$.

For this, we set

$$f(t) := \frac{\eta(t)}{\eta^0}, \quad g(t) := \frac{\eta_m(t)}{\eta_m^0} \quad \text{and} \quad \xi_N := (\eta^0)^{d/p'}(\eta_m^0)^{-(1+\alpha)}.$$

Note that ξ_N depends on the number of particles N. It yields

$$\frac{d^+f}{dt} \le C\|\rho\| f\left(1+\xi_N f^{d/p'} g^{-(1+\alpha)}\right),$$
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Then, there exists a positive constant $T_*^N \le T_0^N$ for sufficiently large N such that

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Our assumption for the initial data finally implies

$$\liminf_{N\to\infty} T_*^N \ge \lim_{N\to\infty} -\frac{\log(\xi_N)}{2(d/p' + (1+\alpha))\|\rho\|} = \infty\,,$$

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Outline

Stochastic Particle System

General Interacting Particle System with Noise:

N interacting $\mathbb{R}^{2d}$-valued processes $(X_t^i, V_t^i)_{t\geq 0}$ with $1 \leq i \leq N$ solution of

$$\begin{cases} dX_t^i = V_t^i dt, \\ dV_t^i = \sqrt{2} dB_t^i - F(X_t^i, V_t^i) dt - \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} H(X_t^i - X_t^j, V_t^i - V_t^j) dt, \end{cases}$$

with independent and commonly distributed initial data (X_0^i, V_0^i) with $1 \leq i \leq N$.

Empirical Measure:

$$\hat{f}_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_t^i, V_t^i)}$$

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Stochastic Particle System Associated to PDE:

N interacting processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ solutions of the kinetic McKean-Vlasov type equation on $\mathbb{R}^{2d}$:

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The stochastic processes are independent and identically distributed according to

$$\partial_t f_t + v \cdot \nabla_x f_t = \Delta_v f_t + \nabla_v \cdot ((F + H * f_t) f_t), \quad t > 0, x, v \in \mathbb{R}^d.$$

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Coupling Method 2

Conjecture: The N interacting processes $(X_t^i, V_t^i)_{t\geq 0}$ behave as $N \to \infty$ like the processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ associated to the PDE.

More precisely, the objective is to estimate the convergence as $N \to \infty$ of

$$\mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)$$

Consequences

1. $f_t^{(1)}$ of any of the particles X_t^i at time t converges to f_t as N goes to infinity:

$$W_2^2(f_t^{(1)}, f_t) \leq \mathbb{E}\left[|X_t^i - \overline{X}_t^i|^2 + |V_t^i - \overline{V}_t^i|^2\right] \leq \varepsilon(N)\,.$$

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Coupling Method 3

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2. Propagation of chaos: The law $f_t^{(k)}$ of any k particles (X_t^i, V_t^i) converges to the tensor product $f_t^{\otimes k}$ as N goes to infinity:

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3. Convergence of the empirical measure $\hat{f}_t^N$ to f_t: if φ is a Lipschitz map on $\mathbb{R}^{2d}$, then

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Main Result

Previous Results: If the functions involved F and H are globally Lipschitz then there are classical results by Snitzman and Meleard, implying that

$$\varepsilon(N) = O\left(\frac{1}{N}\right)$$

The typical F and H in our Cucker-Smale and D'Orsogna etal model are not globally Lipschitz.

Hypotheses:

Assume that F and H with $H(-x,-v) = -H(x,v)$, satisfy

$$-(v-w)\cdot(F(x,v)-F(x,w)) \le A\,|v-w|^2$$
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Main Result 2

Properties of the Stochastic Processes and PDE:

Assume that the particle system and the processes have global solutions on $[0, T]$ with initial data (X_0^i, V_0^i) such that the uniform moment condition holds:

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with $f_t = law(\overline{X}_t^i, \overline{V}_t^i)$ and some $p' > p$.

Result:

For all $0 < \epsilon < 1$ there exists a constant C such that

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Outline

Step 0.- Fluctuations:

Fluctuations: $x_t^i := X_t^i - \overline{X}_t^i$, $v_t^i := V_t^i - \overline{V}_t^i$, $i = 1, \ldots, N$.

Coupling: the Brownian motions $(B_t^i)_{t\geq 0}$ are equal for the stochastic interacting particle system and for the processes

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Step 0.- Fluctuations:

Fluctuations: $x_t^i := X_t^i - \overline{X}_t^i$, $v_t^i := V_t^i - \overline{V}_t^i$, $i = 1, \dots, N$.
Coupling: the Brownian motions $(B_t^i)_{t \geq 0}$ are equal for the stochastic interacting particle system and for the processes

$$
\begin{aligned}
dx^i &= v^i \, dt \,, \\
dv^i &= -\left(F(X^i, V^i) - F(\overline{X}^i, \overline{V}^i)\right) dt \\
&\quad - \frac{1}{N} \sum_{j=1}^{N} \left(H(X^i - X^j, V^i - V^j) - (H * f_t)(\overline{X}^i, \overline{V}^i)\right) dt.
\end{aligned}
$$

Consider the quantity

$$
\alpha(t) = \mathbb{E}\left[|x^i|^2 + |v^i|^2\right]
$$

independent of the label i by symmetry.

Step 0.- Fluctuations:

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|x^i|^2\right] = \mathbb{E}\left[x^i \cdot v^i\right] \leq \frac{1}{2}\alpha(t)$$

and

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|v^i|^2\right] = -\mathbb{E}\left[v^i \cdot \left(F(X^i,V^i) - F(\overline{X}^i,\overline{V}^i)\right)\right]$$
$$-\frac{1}{N}\mathbb{E}\left[\sum_{j=1}^N v^i \cdot \left(H(X^i - X^j, V^i - V^j) - H * f_t(\overline{X}^i,\overline{V}^i)\right)\right] =: I_1 + I_2.$$

Step 1.- Localization Estimate for I_1:

Using the hypotheses on F:

$$I_1 \le A\,\mathbb{E}\left[|v^i|^2\right] + L\,\mathbb{E}\left[|v^i|\min\{|x^i|,1\}\,(1+|\overline{V}^i|^p)\right] := I_{11} + L\,I_{12}\,.$$

Localizing in V + Markov's inequality:

$$I_{12} \le (1+R^p)\alpha(t) + \frac{1}{2}\left(\mathbb{E}\left[|\overline{V}^i|^{4p}\right]\right)^{1/2}\left(e^{-aR^p}\,\mathbb{E}\left[e^{a|\overline{V}_t^i|^p}\right]\right)^{1/2}$$

Final Estimate: given $T>0$, there exists $C>0$ such that

$$I_1 \le C(1+r)\,\alpha(t) + C\,e^{-r}$$

holds for all $r>0$ and all $0 \le t \le T$.

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Step 2.- Localization Estimate for I_2:

$$
\begin{aligned}
I_2 = & -\frac{1}{N}\mathbb{E}\left[\sum_{j=1}^{N} v^i \cdot \Big(H(X^i - X^j, V^i - V^j) - H(\overline{X}^i - \overline{X}^j, \overline{V}^i - \overline{V}^j) \Big)\right] \\
& -\frac{1}{N}\mathbb{E}\left[v^i \cdot \Big(H(0,0) - (H * f_t)(\overline{X}^i, \overline{V}^i) \Big)\right] \\
& -\frac{1}{N}\mathbb{E}\left[\sum_{j\neq i}^{N} v^i \cdot \Big(H(\overline{X}^i - \overline{X}^j, \overline{V}^i - \overline{V}^j) - (H * f_t)(\overline{X}^i, \overline{V}^i) \Big)\right] \\
=: & \, I_{21} + I_{22} + I_{23}.
\end{aligned}
$$

Localization in I_{21} and I_{22} + Argument of Law of Large Numbers (Snitzman):

$$
I_{23} \leq \frac{1}{N}\left(\mathbb{E}\left[|v^1|^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left|\sum_{j=2}^{N} Y^j\right|^2\right]\right)^{1/2} \leq \frac{C}{\sqrt{N}}\sqrt{\alpha(t)}
$$

where $Y^j := H(\overline{X}^1 - \overline{X}^j, \overline{V}^1 - \overline{V}^j) - (H * f_t)(\overline{X}^1, \overline{V}^1)$ for $j \geq 2$.

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Step 3.- Conclusion:

First Result: given $T > 0$, there exists $C > 0$ such that

$$\alpha'(t) \leq C\,(1+r)\,\alpha(t) + Ce^{-r} + \frac{C}{\sqrt{N}}\sqrt{\alpha(t)} \leq C\,(1+r)\,\alpha(t) + Ce^{-r} + \frac{C}{N}$$

for all $t \in [0,T]$, all $N \geq 1$ and all $r > 0$. This implies due to changes of variables:

$$u' \leq -u\log u + \frac{1}{N}$$

implying that $\alpha(t) \leq CN^{-e^{-Ct}}$.

Second Result: A better localization implies that given $T > 0$, there exists $C > 0$ such that

$$\alpha'(t) \leq C(1+r)\,\alpha(t) + C\,e^{-r^{p'}/p} + \frac{C}{N}$$

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Conclusions & Open Problems

- Simple modelling of the three main mechanisms leads to complicated patterns. More information from particular species should be included to make more realistic models (Helmelrijk & collaborators, ...)
- Mean-field limit under reasonable conditions leads to rigorous derivation of the mesoscopic/kinetic models with/without noise.
- References:

1. C.-D'Orsogna-Panferov (KRM 2008).
2. C.-Fornasier-Toscani-Vecil (Birkhäuser 2011).
3. C.-Klar-Martin-Tiwari (M3AS 2010).
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