Swarming Models with Repulsive-Attractive Effects

J. A. Carrillo

Imperial College London

Lecture 3, Ravello 2013

Outline

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Reduction - 1st Order Friction Model:

Edelshtein-Keshet, Mogilner (JMB 2000): Assume the variations of the velocity and speed are much smaller than spatial variations, then from Newton's equation:

$$m\frac{d^2x_i}{d^2t} + \alpha\frac{dx_i}{dt} + \sum_{j\neq i} \nabla U(|x_i - x_j|) = 0$$

so finally, we obtain

$$\frac{dx_i}{dt} = -\sum_{j\neq i} \nabla U(x_i - x_j) \qquad \text{in the continuum setting} \Rrightarrow \qquad \begin{cases} \frac{\partial \rho}{\partial t} + \text{div}\,(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

"repelling/attracting field": $-\nabla U : \mathbb{R}^d \to \mathbb{R}^d$

For which interaction repulsive/attractive potentials do we get convergence towards some nontrivial steady states/patterns?

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Formal Gradient Flow

Basic Properties

1. **Conservation of the center of mass.**
2. **Liapunov Functional:** Gradient flow of

$$\mathcal{F}[\rho] = \frac{1}{2} \iint U(x-y)\, \rho(x)\, \rho(y)\, dxdy$$

with respect to the Wasserstein distance W_2.
(C., McCann, Villani; RMI 2003, ARMA 2006).

The macroscopic equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\nabla\left[\frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right]\right).$$

with entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^2} \rho(t,x)\left|\nabla \frac{\delta \mathcal{F}}{\delta \rho}(t,x)\right|^2 dx.$$

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Sliding down in a Energy Landscape

Finite Dimensional Gradient flows

A *gradient flow* in $\mathbb{R}^d$ defined by an energy $\mathcal{F}$ is given by

$$\frac{dx_t}{dt} = -\nabla\mathcal{F}(x_t)\,.$$

It is the continuous version of the *steepest descent* on the energy landscape determined by $\mathcal{F}$ given by the implicit Euler scheme: given a time step Δt and an approximation to the solution at time $t_k = k\Delta t$, we find the approximation at time t_{k+1} by solving

$$x_{k+1} = x_k - \Delta t \nabla\mathcal{F}(x_{k+1})\,.$$

which is equivalent under convexity conditions to the following variational problem: Solve

$$x_{k+1} = \arg\min_{x\in\mathbb{R}^d} \left\{ \frac{1}{2\,\Delta t}|x - x_k|^2 + \mathcal{F}(x) \right\}$$

with $|\cdot|$ the euclidean norm.

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JKO scheme (Jordan-Kinderlehrer-Otto)

A discrete time approximation of the PDE is obtained by solving a sequences of variational problem.

- Choose a time step Δt.
- Solve

$$\rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2^a(\mathbb{R}^d)} \left\{ \frac{1}{2\,\Delta t} W_2^2(\rho, \rho_k) + \mathcal{F}(\rho) \right\}$$

- As $\Delta t \to 0$ it converges to the solution of a weak form of

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 \\ v = -\nabla U * \rho \end{cases}$$

The convergence for smooth C^1 potentials U with at most quadratic growth at infinity given in "Gradient Flow in Metric Spaces" book by Ambrosio, Gigli, Savaré, 2005.

Is this theory with initial data measures really necessary?

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How to deal with concentrations?

Let $U(x) = k(|x|)$ be a radial fully attractive potential with its only possible singularity located at zero such that

$$\text{No-Osgood condition: } \boxed{\int_0^L \frac{dr}{k'(r)} < +\infty,}$$

Then we are doomed to deal with concentrations in finite time.

- Given an initial condition in $L^1 \cap L^\infty$ (Bertozzi, C., Laurent; Nonlinearity 2009) or in $L^1 \cap L^p$ for $p > d/(d-1)$ (Bertozzi, Laurent, Rosado; CPAM 2011), the solutions blow up in finite time.
- Assume additionally that U is λ-convex: $U(x) - \frac{\lambda}{2}|x|^2$ is convex, then one can construct a well-posedness theory for measures going over the blow-up time in a unique way (C., Di Francesco, Figalli, Laurent, Slepcev; Duke MJ 2012). This assumption restrict the possible singularities at the origin, $U(x) \simeq |x|^\alpha$ locally at 0, with $1 \leq \alpha < 2$, for instance.
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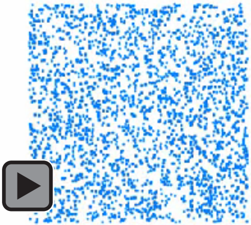
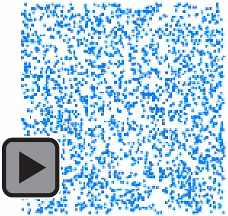
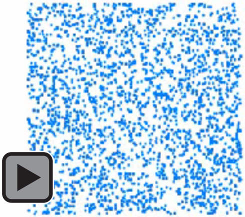
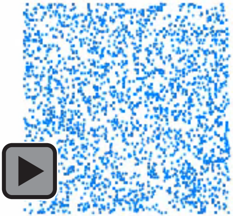
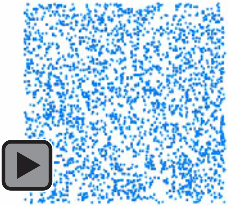
Nontrivial patterns? - Particle Simulations





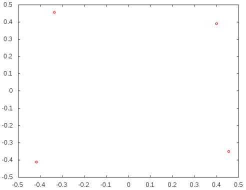




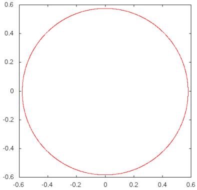


Summary: Particle Simulations $d = 2$

Potential $a = 4$, $b = 2.1$



Potential $a = 4$, $b = 1.85$



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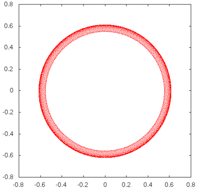

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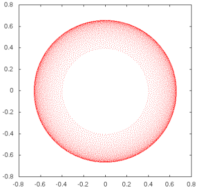
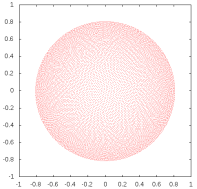


Potential $a = 4$, $b = 0.05$



$$\dot{X}_i = -\sum_{j \neq i} m_j \nabla U(X_i - X_j)$$

$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$$

$$2 - d \leq b < a$$

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Radial Setting

The velocity field generated by a spherical shell of radius η is given by:

$$\omega(r,\eta) = -\frac{1}{\sigma_N}\int_{\partial B(0,1)} \nabla U(re_1 - \eta y)\cdot e_1\, d\sigma(y),$$

Under some conditions on the potential U, the function $\omega \in C^1(\mathbb{R}^2_+)$.

The equation $\frac{\partial\rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x)\left[\nabla U * \rho\right](t,x)\right)$ written in radial coordinates is

$$\partial_t\hat\mu + \partial_r(\hat\mu\hat v) = 0$$

$$\hat v(t,r) = \int_0^{+\infty}\omega(r,\eta)d\hat\mu_t(\eta)\,.$$

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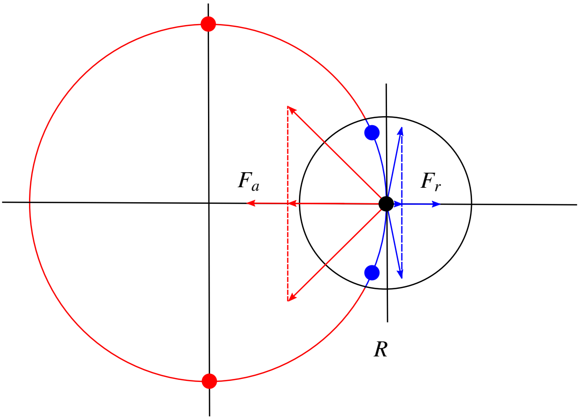
Spherical shell

A spherical shell (a uniform distribution on the sphere) of radius R (δ_R from now on) is a stationary state for the aggregation equation if

$$\omega(R,R) = 0.$$



Conditions for radial ins/stability

Suppose that δ_R is a stationary state, $\omega(R,R) = 0$.

- **Instability:** Suppose that one of the following cases is satisfied
  - $\omega \in C^1(\mathbb{R}^2_+)$ and $\partial_1\omega(R,R) > 0$.
  - $\omega \in C(\mathbb{R}^2) \cap C^1(\mathbb{R}^2 \setminus \mathcal{D})$ and

$$\lim_{\substack{(r_1,r_2)\notin\mathcal{D} \\ (r_1,r_2)\to(R,R)}} \partial_1\omega(r_1,r_2) = +\infty.$$

  Conclusion: it is not possible for an L^p radially symmetric solution to converge toward a δ_R when $t \to \infty$.

- **Stability:** If the following conditions hold
  - $\partial_1\omega(R,R) < 0$, (fattening instability)
  - $\partial_1\omega(R,R) + \partial_2\omega(R,R) < 0$. (shifting instability)

  Then if the radial initial data is close enough to δ_R there is convergence. This is a local result of stability.

Conditions for radial ins/stability

Suppose that δ_R is a stationary state, $\omega(R,R) = 0$.

- **Instability:** Suppose that one of the following cases is satisfied
  - $\omega \in C^1(\mathbb{R}^2_+)$ and $\partial_1\omega(R,R) > 0$.
  - $\omega \in C(\mathbb{R}^2) \cap C^1(\mathbb{R}^2 \setminus \mathcal{D})$ and

$$\lim_{\substack{(r_1,r_2)\notin\mathcal{D} \\ (r_1,r_2)\to(R,R)}} \partial_1\omega(r_1,r_2) = +\infty.$$

  Conclusion: it is not possible for an L^p radially symmetric solution to converge toward a δ_R when $t \to \infty$.

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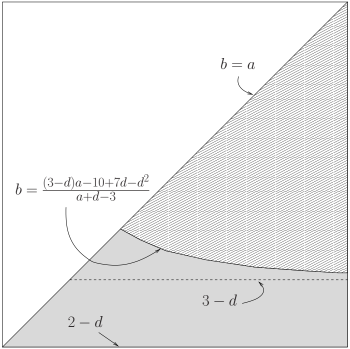
Power-Law Case

$$U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b} \qquad 2 - d < b < a$$

Theorem: Ins/Stability of Delta Rings with respect to radial perturbations.

- There is a computable value of R such that the uniform distribution on the sphere of radius R, δ_R is an steady state.
- If the velocity field generated by δ_R is strictly increasing at R then it is unstable.
- If the velocity field generated by δ_R is strictly decreasing at R then it is locally asymptotically stable.

(Balagué, C., Laurent, Raoul; Physica D 2013)



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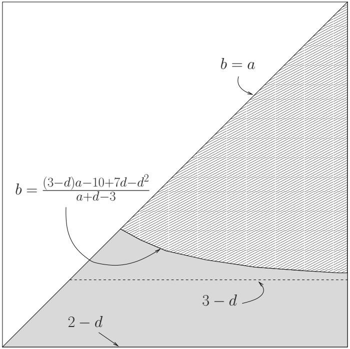

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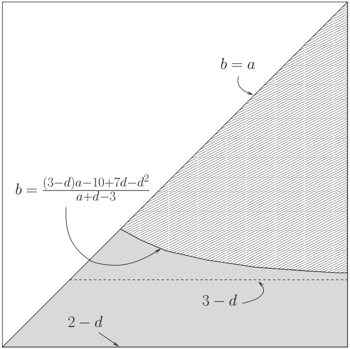

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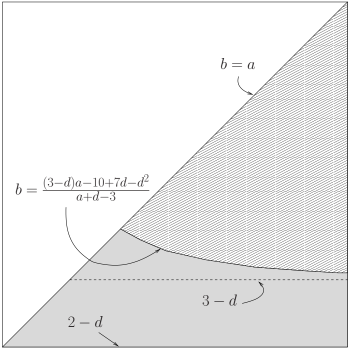
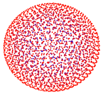
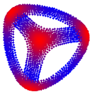


Outline

Dimensionality of the support

Some simulations with power law potentials of the form

$$W(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}, \qquad 2 - d < b < a$$

dim=3	dim=2	dim=2	dim=0
			
$b = -0.5$ $a = 5$	$b = 0.5$ $a = 23$	$b = 1.25$ $a = 15$	$b = 2.5$ $a = 5$

Local minimizers in 3D for different parameters when $b > -1$ increases. The computations were done with $n = 2,500$ particles.

Mild versus Strong Repulsive potentials

Support is essentially 0-dimensional for $b > 2$.

Let $U \in C^2(\mathbb{R}^N)$ be a radially symmetric potential which behaves like $-|x|^b/b$ in a neighborhood of the origin with $b > 2$.

Then a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ cannot have a s-dimensional component for any $1 \leq s \leq d$.

Dimension of the Support depends on $2 - d < b < 2$.

Assume that μ is a local minimizer of the interaction energy $\mathcal{F}$ with respect to W_∞ such that U is radial with $U(x) \sim -|x|^b$ near zero and $2 - d < b < 2$. If μ contains s-Haussdorff dimensional connected components in its support, then $s > 2 - b$.

(Balagué, C., Laurent, Raoul; ARMA 2013)

Strategy: Pure variational approach: by contradiction we build better competitors. For the first result, by locally sending part of the mass to a Dirac Delta.

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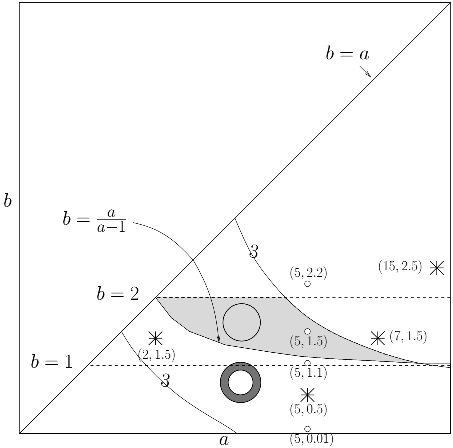
Relevant example: Power-law Potential in 2d



$$\frac{dx_i}{dt} = -\sum_{j \neq i}^{N} \nabla U(x_i - x_j)$$

The spectral gap of the linearized stability of the Delta ring depending on N disappears as $N \to \infty$.

(Kolokonikov, Sun, Uminsky, Bertozzi; Physical Review E 2011)
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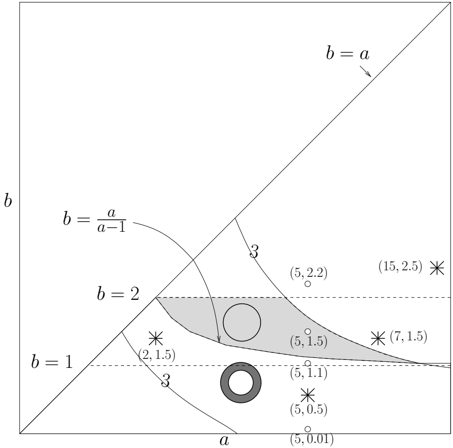

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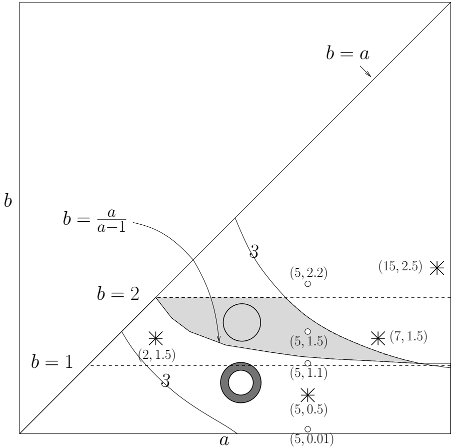

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