Swarming Models with Repulsive-Attractive Effects

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Lecture 4, Ravello 2013

Outline

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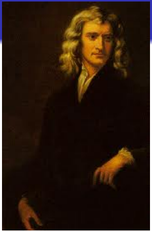
2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

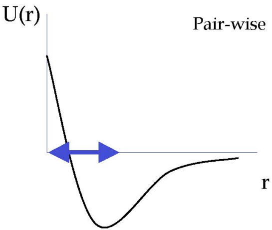
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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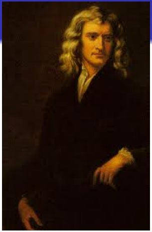

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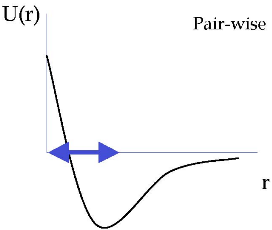

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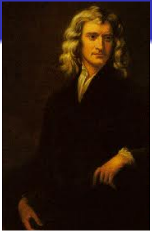

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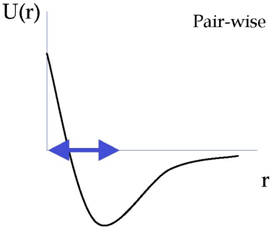

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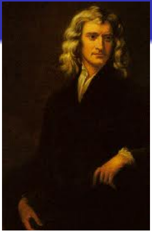

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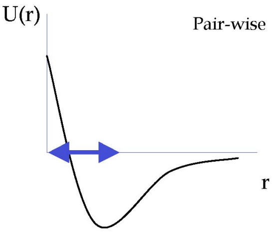
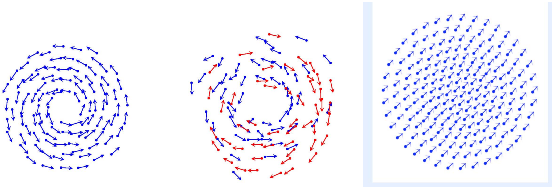


Model with an asymptotic speed

Typical patterns: milling, double milling or flocking:



Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\,(v_j - v_i)\,, \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}\,.$$

Asymptotic flocking: $\gamma < 1/2$. (Cucker, Smale; Japan J. Math 2007).

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Macroscopic equations

Monokinetic Solutions

Assuming that there is a deterministic velocity for each position and time, $f(x,v,t) = \rho(x,t)\,\delta(v - u(x,t))$ is a distributional solution if and only if,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) = 0, \\ \rho\,\dfrac{\partial u}{\partial t} + \rho\,(u \cdot \nabla_x)u = \rho\,(\alpha - \beta|u|^2)u - \rho\,(\nabla_x U \star \rho). \end{cases}$$

Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

Flocking

Travelling wave case, $u =$ const such that $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$, then $\rho(x,t) = \tilde{\rho}(x - ut)$, and the density is determined by

$$\tilde{\rho}\left(\nabla_{\mathbf{x}} U \star \tilde{\rho}\right) = 0,$$

from which

$$U \star \tilde{\rho} = C, \quad \tilde{\rho} \neq 0,$$

in the support of $\tilde{\rho}$ if the support has not empty interior.

Complete set of solutions depending on regularity of the potential and stability are open problems.

Particular example (Delta Rings): $U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$ with $a > b \geq 2 - d$, then there is a unique explicit radius R_{ab} such that the uniform distribution on the sphere of radius R_{ab} is a flocking solution.

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Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

Milling

we set **u** in a rotatory state,

$$u = \pm\sqrt{\frac{\alpha}{\beta}}\frac{x^\perp}{|x|},$$

where $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$, and look for $\rho = \rho(|x|)$ radial, then

$$U \star \rho = D + \frac{\alpha}{\beta}\log|x|, \quad \text{whenever } \rho \neq 0.$$

Complete set of solutions depending on regularity of the potential and stability are open problems.

Particular example (Delta Rings): $U(x) = \frac{|x|^a}{a} - \frac{|x|^b}{b}$ with $a > b \geq 2 - d$, then there is a unique explicit radius $\hat{R}_{ab}$ such that the uniform distribution on the sphere of radius $\hat{R}_{ab}$ is a mill solution.

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Particular solutions

Superposition of Monokinetic Solutions: Double Mills

$f(x,v,t) = \rho_1(x,t)\,\delta(v - u_1(x,t)) + \rho_2(x,t)\,\delta(v - u_2(x,t))$ is a distributional solution if and only if

$$\begin{cases} \dfrac{\partial(\rho_1+\rho_2)}{\partial t} + \mathrm{div}_x(\rho_1 u_1 + \rho_2 u_2) = 0. \\ \displaystyle\sum_{i=1}^{2} \rho_i \left[\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla_x) u_i - (\alpha - \beta |u_i|^2) u_i \right] = -(\nabla_x U \star \rho)\,\rho. \end{cases}$$

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$$\rho = \frac{1}{2}\delta_{\hat{R}_{ab}}\delta(v - u(x)) + \frac{1}{2}\delta_{\hat{R}_{ab}}\delta(v + u(x))$$

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Mill solutions: Quasi-Morse Potentials

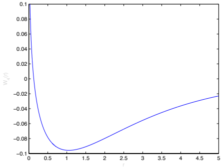
Quasi-Morse Potential: Let V denote the radially symmetric fundamental solution of the n-dimensional *screened Poisson equation*

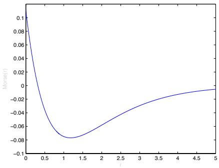
$$\Delta u - k^2 u = \delta_0, \qquad k > 0\,.$$

Let $C, l, \lambda > 0$. The *n-dimensional Quasi-Morse potential* is defined as

$$U(r) := \lambda \left(V(r) - C\, V\left(\frac{r}{l}\right)\right)\,.$$

Biologically reasonable: $l < 1$, $Cl^{n-2} > 1$.





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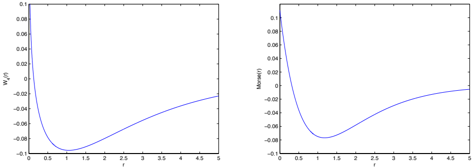


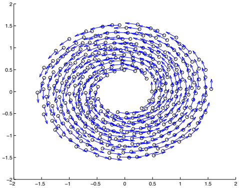
Mill solutions: Quasi-Morse Potentials 2

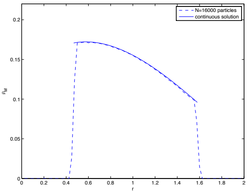
Explicit Solvability

Solve $(U \star \rho)(r) = s(r)$ on $\mathrm{supp}(\rho)$ with $\mathrm{supp}(\rho) = B(0, R_F)$, $s(r) = D$ for flocks, or $\mathrm{supp}(\rho) = B(R_m, R_M)$, $s(r) = D + \frac{\alpha}{\beta}\log(r)$ for mills respectively:

flock	$A > 0$	$\rho_F = \mu_1 J_0(ar) + \mu_2$
	$A = 0$	$\rho_F = \mu_1 r^2 + \mu_2$
	$A < 0$	$\rho_F = \mu_1 I_0(ar) + \mu_2$
mill	$A > 0$	$\rho_M = \rho_{\mathrm{inhom}} + \mu_1 J_0(ar) + \mu_2 Y_0(ar) + \mu_3$
	$A = 0$	$\rho_M = \frac{\alpha}{\beta} \frac{k^4}{4\lambda l^2 (1-C)} r^2 (\log(r) - 1) + \mu_1 r^2 + \mu_2 \log(r) + \mu_3$
	$A < 0$	$\rho_M = \rho_{\mathrm{inhom}} + \mu_1 I_0(-ar) + \mu_2 \cdot K_0(ar) + \mu_3$

with $A = k^2 \frac{Cl^d - 1}{l^2 - Cl^d}$, $a^2 = |A|$, and ρ has to satisfy $\rho > 0$, $\int \rho \, dx = 1$.





Mill solutions: Quasi-Morse Potentials 3

Applying the operators $\Delta - k^2\,\mathrm{Id}$ and $\Delta - \frac{k^2}{\ell^2}\,\mathrm{Id}$ to both sides of $(U \star \rho)(r) = s(r)$, the density ρ now satisfies

$$\Delta\rho + A\rho = \frac{k^4}{\ell^2 - C\ell^n} D, \qquad \text{on } \operatorname{supp}\rho\,.$$

In radial coordinates, this equation reads

$$\frac{1}{r^{d-1}}\frac{d}{dr} r^{d-1}\frac{d\rho}{dr} \pm a^2\rho = \frac{k^4}{\ell^2 - C\ell^d} D, \qquad a = \sqrt{|A|}.$$

One can show that

$$(U \star \rho)(r) = D + \lambda_1 r^{1-\frac{d}{2}} I_{\frac{d}{2}-1}(kr/\ell) + \lambda_2 r^{1-\frac{d}{2}} I_{\frac{d}{2}-1}(kr)$$
$$+ \lambda_3 r^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(kr/\ell) + \lambda_4 r^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(kr), \qquad 0 \le r \le R,$$

By boundedness $\lambda_3 = \lambda_4 = 0$ and there is a linear relation between λ_1, λ_2 and μ_1, μ_2.

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Mill solutions: Quasi-Morse Potentials 4

	λ_1	λ_2
$A > 0$	$-C\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_+(\ell)K_{\frac{n}{2}}(kR/\ell)$	$\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_+(1)K_{\frac{n}{2}}(kR)$
$A = 0$	$-CR^{\frac{n}{2}}\ell^{n-1}B_0(\ell)K_{\frac{n}{2}}(kR/\ell)$	$R^{\frac{n}{2}}\ell^{n-1}B_0(1)K_{\frac{n}{2}}(kR)$
$A < 0$	$-C\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_-(\ell)K_{\frac{n}{2}}(kR/\ell)$	$\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_-(1)K_{\frac{n}{2}}(kR)$

There exists a flock profile only if the homogeneous equations for $\mu = (\mu_1, \mu_2)^t$

$$M\mu = \begin{pmatrix} \tilde{B}(\ell) & 1 \\ \tilde{B}(1) & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are satisfied. These two homogeneous equations, together with the total unit mass constraint for the non-negative density ρ, determine the three characterizing parameters (μ_1, μ_2, R_F) of the flock profile.

Mill solutions: Quasi-Morse Potentials 4

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$A=0$	$-CR^{\frac{n}{2}}\ell^{n-1}B_0(\ell)K_{\frac{n}{2}}(kR/\ell)$	$R^{\frac{n}{2}}\ell^{n-1}B_0(1)K_{\frac{n}{2}}(kR)$
$A<0$	$-C\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_-(\ell)K_{\frac{n}{2}}(kR/\ell)$	$\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_-(1)K_{\frac{n}{2}}(kR)$

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$$M\boldsymbol{\mu}=\begin{pmatrix}\tilde{B}(\ell) & 1\\ \tilde{B}(1) & 1\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

are satisfied. These two homogeneous equations, together with the total unit mass constraint for the non-negative density ρ, determine the three characterizing parameters (μ_1,μ_2,R_F) of the flock profile.

Mill solutions: Quasi-Morse Potentials 4

	λ_1	λ_2
$A > 0$	$-C\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_+(\ell)K_{\frac{n}{2}}(kR/\ell)$	$\frac{R^{\frac{n}{2}}}{k}\ell^{n-1}B_+(1)K_{\frac{n}{2}}(kR)$
$A = 0$	$-CR^{\frac{n}{2}}\ell^{n-1}B_0(\ell)K_{\frac{n}{2}}(kR/\ell)$	$R^{\frac{n}{2}}\ell^{n-1}B_0(1)K_{\frac{n}{2}}(kR)$
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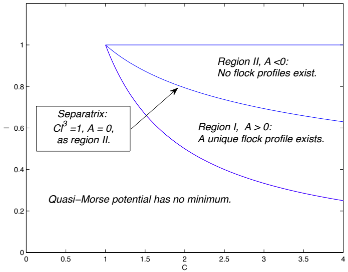
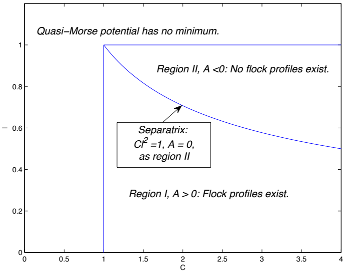
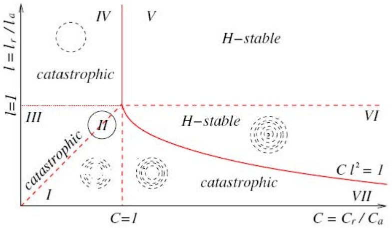
Mill solutions: Quasi-Morse Potentials 5

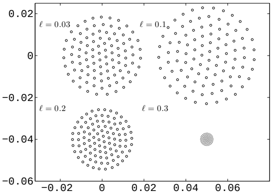




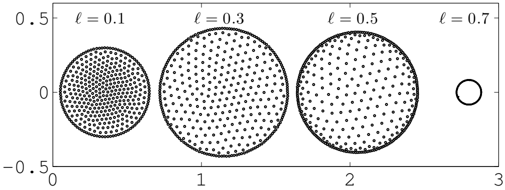


Mill solutions: Quasi-Morse Potentials 6

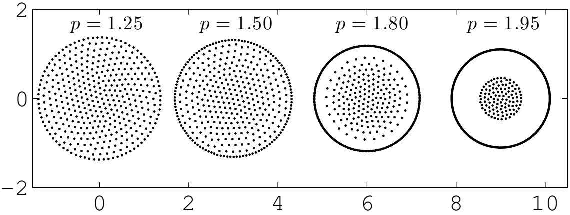
$$V(r) = -e^{-\frac{r^p}{p}}, \qquad p > 0.$$



(a) $p = 1/2, C = 0.6$



(b) $p = 3/2, C = 0.6$



(c) Different p's with $C = 10/9, \ell = 3/4$

Outline

Short Relaxation towards Cruising Speed

Scaled Vlasov equation in $d = 2, 3$ dimensions:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + a^\varepsilon(t,x) \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon} \mathrm{div}_v \{ f^\varepsilon (\alpha - \beta |v|^2) v \} = 0, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$$

with $a^\varepsilon(t,\cdot) = -\nabla_x U \star \rho^\varepsilon(t,\cdot) - H \star f^\varepsilon(t,\cdot)$.

This asymptotic limit enforces that particles move at cruising speed $\sqrt{\alpha/\beta}$. If one formally does the expansion

$$f^\varepsilon = f + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \ldots$$

we get

$$\mathrm{div}_v \{ f(\alpha - \beta |v|^2) v \} = 0$$

$$\partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v(fa(t,x)) + \mathrm{div}_v \{ f^{(1)} (\alpha - \beta |v|^2) v \} = 0 \,,$$

up to first order.

To eliminate the higher order term we use the invariants of the flow associated to the field $(\alpha - \beta |v|^2) v \cdot \nabla_v$, functions of x and $v/|v|$.

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Vicsek Model as Asymptotic Limit

Fixed Speed Models

Assume that $U \in C_b^2(\mathbb{R}^d)$, $H(x,v) = h(x)v$ with $h \in C_b^1(\mathbb{R}^d)$ nonnegative, $f^{\text{in}} \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, $\text{supp}(f^{\text{in}}) \subset \{(x,v) : |x| \leq L_0, r_0 \leq |v| \leq R_0\}$.

Then for all $\delta > 0$, the sequence $(f^\varepsilon)_\varepsilon$ converges towards the measure solution $f(t,x,\omega)$ on $(x,\omega) \in \mathbb{R}^d \times \sqrt{\alpha/\beta}\mathbb{S}$ of the problem

$$\partial_t f + \text{div}_x(f\omega) - \text{div}_\omega \left\{ f \left(I - \frac{1}{r^2}(\omega \otimes \omega) \right) (\nabla_x U \star \rho + H \star f) \right\} = 0$$

with initial data $f(0) = \langle f^{\text{in}} \rangle$.

Remarks:

- Adding noise we get from $\Delta_v f$ to the Laplace-Beltrami operator on the sphere $\Delta_\omega f$.
- This shows that the fixed speed limit of the Cucker-Smale's model is the Vicsek's model.

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Conclusions

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- Flock profiles are among Local minimizers of the interaction energy. Explicit compactly supported flocks can be found for some biologically relevant potentials (Quasi-Morse).
- References:

1. Balagué-C.-Laurent-Raoul (Physica D 2013 & ARMA 2013).
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