Swarming Models with Repulsive-Attractive Effects

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Lecture 5, Ravello 2013

Outline

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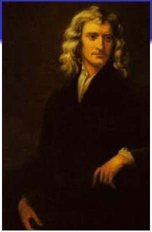
2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

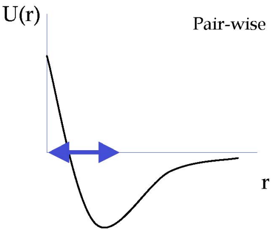
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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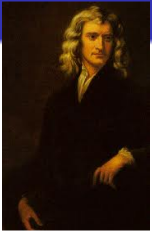

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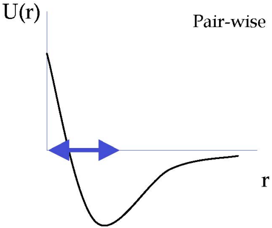

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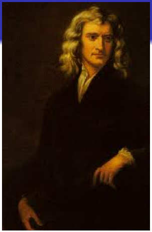

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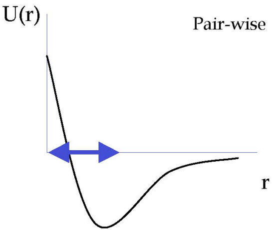

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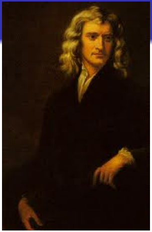

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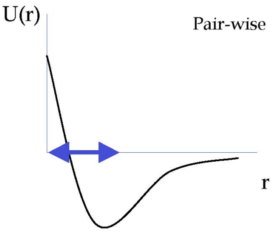
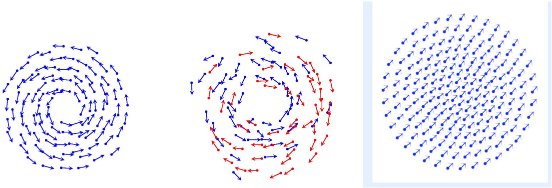


Model with an asymptotic speed

Typical patterns: milling, double milling or flocking:



Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\,(v_j - v_i)\,, \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Asymptotic flocking: $\gamma < 1/2$. (Cucker, Smale; Japan J. Math 2007).

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The Bertozzi-D'Orsogna model:

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = (\alpha - \beta|v_j|^2)v_j + \frac{1}{N}\sum_{\substack{l=1 \\ l\neq j}}^{N} \nabla U(x_l - x_j) \end{cases}, \quad j = 1, \ldots, N,$$

with $\alpha, \beta > 0$. Particular case $U(x) = k(|x|)$ with

$$k(r) = \frac{r^a}{a} - \frac{r^b}{b}, \qquad a > b > 0.$$

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = \frac{1}{N}\sum_{l=1}^{N} H(x_j - x_l)(v_l - v_j) + \frac{1}{N}\sum_{\substack{l=1 \\ l\neq j}}^{N} \nabla U(x_l - x_j) \end{cases}, \quad j = 1, \ldots, N$$

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Asymptotic solutions

Definition

• We call a *flock ring*, the solution such that $\{x_j\}_{j=1}^N$ are equally distributed on a circle with a certain radius, R and $\{v_j\}_{j=1}^N = u_0$, with $|u_0| = \sqrt{\alpha/\beta}$.

• We call a *mill ring*, the solution such that $\{x_j\}_{j=1}^N$ are equally distributed on a circle with a certain radius, R and $\{v_j\}_{j=1}^N = \sqrt{\alpha/\beta}\, x_j^\perp/|x_j|$ with $x_j^\perp$ the orthogonal vector.

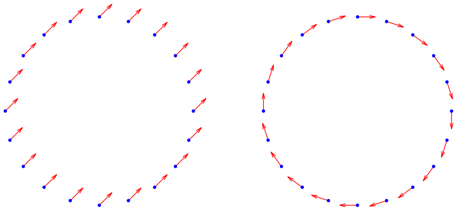

Figure: Flock and mill ring solutions.

Outline

Change of Variables

- Change of variables to the comoving frame:

$$\begin{cases} y_j = x_j(t) - u_0 t \\ z_j = v_j(t) - u_0 \end{cases} ,j = 1, \dots, N,$$

Then the system reads

$$\begin{cases} \dot{y}_j = z_j \\ \dot{z}_j = (\alpha - \beta |z_j|^2)(z_j + u_0) + \frac{1}{N} \sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla U(y_l - y_j) \end{cases} ,j = 1, \dots, N.$$

Write the stationary ring $(y_j^0, z_j^0) = (Re^{i\theta_j}, 0)$ where $\theta_j = \frac{2\pi j}{N}$, for $j = 1, \dots, N$.
A general flock spatial profile will be denoted by $(\hat{x}_j, 0)$.

- Consider the following type of perturbations:

$$\tilde{y}_j(t) = \hat{x}_j + h_j(t), \quad \text{with} \quad |h_j| \ll 1.$$

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Analysis of the stability of flock rings (I)

- Write the matrix of the linearized system for these perturbations

$$L = \begin{pmatrix} 0_{2N} & \mathrm{Id}_{2N} \\ \mathbf{M} & -2\beta\mathcal{U}_0 \end{pmatrix},$$

where **M** is symmetric and represents the $2N \times 2N$ Jacobian that results from linearizing the first order model, $M = (G_{ij})$ with G_{ij} being the 2×2-blocks defined as

$$G_{ij} = \begin{cases} -\sum_{j \neq i} \mathrm{Hess}\, U(\hat{x}_i - \hat{x}_j) & \text{for } i = j \\ \mathrm{Hess}\, U(\hat{x}_i - \hat{x}_j) & \text{for } i \neq j \end{cases},$$

with Hess U denoting the Hessian matrix of the interaction potential U.

$\mathcal{U}_0$ is the diagonal matrix with N blocks of the 2×2 matrix $u_0 u_0^T$ along the diagonal. Assume that $u_0 = e_1 = (1, 0)$ by rotational symmetry.

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Analysis of the stability of flock rings (II)

Symmetries & Linear Instability

Due to translational invariance and rotational invariance of the velocity configuration, zero is always an eigenvalue of the linearized matrix L.

Moreover, there is always a generalized eigenvector associated to the zero eigenvalue generated from the eigenvector due to rotational invariance of the velocity configuration.

Therefore, a flock solution is always linearly unstable.

Instability Result - Spectral Equivalence

The linearized second order system around the flock solution has an eigenvalue with positive real part if and only if the linearized first order system around the flock solution has a positive eigenvalue.

(Albi, Balagué, C., von Brecht; submitted)

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Eigenvalue Analysis

Eigenvalue problem:

$$\lambda \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathbf{M} & -2\beta U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} = L \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix},$$

Normalization $\mathbf{x}^*\mathbf{x} = 1$ of eigenvectors.

Substituting the first equation $\lambda \mathbf{x} = \mathbf{v}$ into the second equation yields

$$\lambda^2 \mathbf{x} + 2\beta\lambda U\mathbf{x} - \mathbf{M}\mathbf{x} = 0.$$

Let $|\mathbf{x}|_2$ denote the semi-norm on $\mathbb{C}^{2N}$ defined according to

$$|\mathbf{x}|_2^2 := \sum_{i=1}^{N} |\langle x_i, e_1\rangle|^2,$$

and let $E^N \cong \mathbb{C}^N$ denote the subspace

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Premultiplying by $\mathbf{x}^*$

$$\lambda = -\beta |\mathbf{x}|_2^2 \pm \sqrt{\beta^2 |\mathbf{x}|_2^4 + \mathbf{x}^* \mathbf{M} \mathbf{x}}.$$

As $\mathbf{M}$ is symmetric, we may write its $2N$ real eigenvalues and corresponding normalized ($\mathbf{x}^*\mathbf{x} = 1$) eigenvectors as

$$\mu_{2N} \leq \mu_{2N-1} \leq \cdots \leq \mu_2 \leq \mu_1 \qquad \mathbf{M}\mathbf{x}_i = \mu_i \mathbf{x}_i.$$

Zero Real Part Eigenvalues

Let λ denote an eigenvalue of L. Then $\Re(\lambda) = 0$ and $\Im(\lambda) \neq 0$ if and only if $\lambda = \pm i\sqrt{-\mu_l}$ for some l with $\mu_l < 0$ and $\mathbf{x}_l \in E^N$. The eigenspace consists only of eigenvectors.

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Eigenvalue Analysis

Generalized eigenvector: there exists an eigenvector $(\mathbf{x}, \lambda\mathbf{x})$ with $\mathbf{x} \in E^N$ so that the system of equations

$$\begin{pmatrix} -\lambda \mathrm{Id} & \mathrm{Id} \\ \mathbf{M} & -2\beta U - \lambda \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \lambda \mathbf{x} \end{pmatrix} \tag{1}$$

has a non-trivial solution.

Substituting the first equation $\mathbf{w} = \lambda\mathbf{u} + \mathbf{x}$ into the second equation, then pre-multiplying by $\mathbf{x}^*$

$$\mathbf{M}\mathbf{u} - 2\beta U\mathbf{w} = 2\lambda\mathbf{x} + \lambda^2\mathbf{u}$$
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Generalized eigenvector: there exists an eigenvector $(\mathbf{x}, \lambda\mathbf{x})$ with $\mathbf{x} \in E^N$ so that the system of equations

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Generalized Eigenvector equations: $\mathbf{w} = \mathbf{x}$ and

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If $(\mathbf{x}, \mathbf{0})$ generates a second generalized eigenvector then the system of equations

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Eigenvalue Analysis

Zero Real Part Eigenvalues

Let $\beta > 0$. Then

$$a_L(0) = \dim(\ker(\mathbf{M}) \cap E^N) + \dim(\ker(\mathbf{M})).$$

and

$$\det(L - \lambda \mathrm{Id}) = \lambda^{a_{\mathbf{M},\perp}(0) + a_{\mathbf{M}}(0)} \Pi_{j=1}^{l} (\lambda^2 - \mu_{i_j}) p_\beta(\lambda) ,$$

where $i_1 < i_2 < \cdots < i_l \leq 2N$ denote those (possibly non-existent) indices where $\mu_{i_j} < 0$ has an eigenvector $\mathbf{x}_{i_j} \in E^N$. The roots of the polynomial $p_\beta(\lambda)$ all have non-zero real part.

Suppose first that $\mu_1 \leq 0$. Then $\mathbf{x}^*\mathbf{M}\mathbf{x} \leq 0$ for any $\mathbf{x}$, whence all eigenvalues λ of L have non-positive real part due to

$$\lambda = -\beta |\mathbf{x}|_2^2 \pm \sqrt{\beta^2 |\mathbf{x}|_2^4 + \mathbf{x}^*\mathbf{M}\mathbf{x}}.$$

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Conversely, suppose $\mu_1 > 0$ and let $\mathcal{A}$ denote the set

$$\mathcal{A} := \left\{ \beta \in [0, \infty) : \max_{\lambda \in \sigma(L)} \Re(\lambda) > 0 \right\}.$$

Note that $0 \in \mathcal{A}$.

$\mathcal{A}$ is relatively open: by continuous dependence of the eigenvalues of L on β.

$\mathcal{A}$ is relatively closed: let $\beta_l \in \mathcal{A}$ and $\beta_l \to \beta_0 \in (0, \infty)$. Up to extraction of subsequences, it follows that there exists a corresponding sequence λ_l of eigenvalues with $\Re(\lambda_l) > 0$ converging to some λ_0 with $\Re(\lambda_0) \geq 0$.

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Eigenvalue Analysis

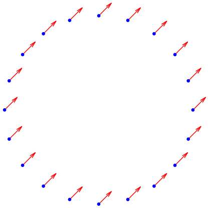
Linear Instability

As an artifact of translation invariance in the first order model, the vector defined by $\mathbf{e}_2 := (0, 1, \ldots, 0, 1)^T \in \mathbb{R}^{2N}$ always defines an eigenvector of $\mathbf{M}$ with eigenvalue zero. Due to the fact that $\mathbf{e}_2 \in E^N$, our results before imply that $(\mathbf{e}_2, \mathbf{e}_2)$ furnishes a generalized eigenvector with eigenvalue zero.

Outline

Eigenvalue Analysis: Ring Flocks

Ring Flock:



m-Mode Fourier Perturbations

A flock ring to the 2nd order model is spectrally stable if and only if the ring solution to the first order model is spectrally stable with respect to all *m*-mode perturbations.

Fourier mode Perturbations: $Re^{i\theta_j}(1+h_j)$ for $h_j = \xi_+ e^{im\theta_j} + \xi_- e^{-im\theta_j}$.

Eigenvalue Analysis: Ring Flocks

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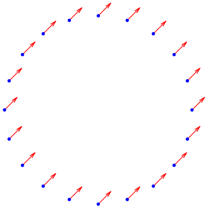

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The analysis in (Kolokonikov, Sun, Uminsky, Bertozzi; Physical Review E 2011) and (Bertozzi, von Brecht, Sun, Kolokolnikov, Uminsky; Comm. Math. Sci. 2012) shows that the stability under those perturbations reduces to a study of the decoupled set of 2×2 eigenvalue problems

$$\lambda \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \underbrace{\begin{pmatrix} I_1(m) & I_2(m) \\ I_2(m) & I_1(-m) \end{pmatrix}}_{M} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad 1 \le m \le N.$$

$$I_1(m) := 4 \sum_{p=1}^{N/2} G_1\left(\frac{\pi p}{N}\right) \sin^2\left(\frac{(m+1)\pi p}{N}\right)$$

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and for power-law potentials $k(r) = r^a/a - r^b/b$ the functions $G_i(\phi)$ are given by

$$G_1(\phi) := \frac{1}{2N}\left[-a(2R|\sin\phi|)^{a-2} + b(2R|\sin\phi|)^{b-2}\right],$$

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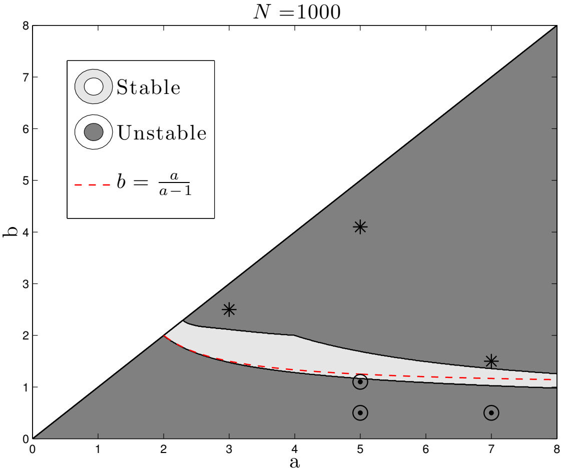

Figure: Stability areas for flock ring solutions for $N = 1000$.

Clustering Instability

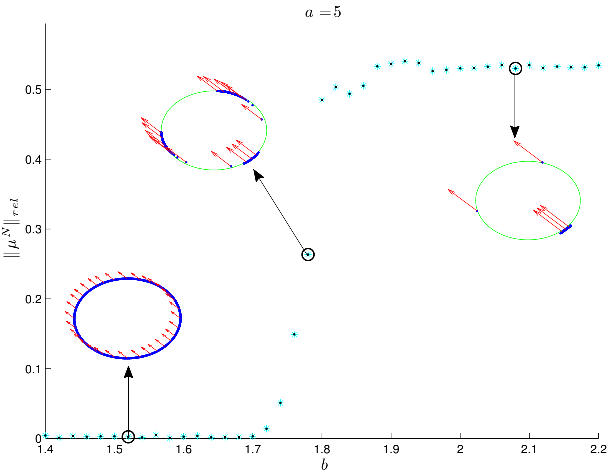

Figure: Bifurcation diagram for cluster formation at $T_f = 500$, with $N = 1000$ particles, $a = 5$, $|u_0| = 2.5$.

Fattening Instability

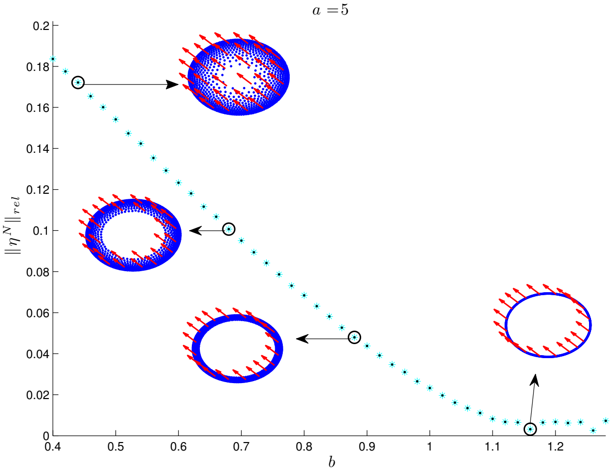

Figure: Bifurcation diagram for fattening instability at $T_f = 500$ with N=1000 particles, $a = 5$, $|u_0| = 2.5$.

Particle Simulations: Perturbation of flocks

Flock Rings: Cucker-Smale

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = \dfrac{1}{N} \displaystyle\sum_{l=1}^{N} H(x_j - x_l)(v_l - v_j) + \dfrac{1}{N} \sum_{\substack{l=1 \\ l \neq j}}^{N} \nabla U(x_l - x_j) \end{cases}, \quad j = 1, \dots, N$$

with $H(x) = g(|x|)$ given by

$$g(r) = \frac{1}{(1 + r^2)^\gamma}, \qquad \gamma > 0.$$

Spectral Equivalence

The linearized second order system around the flock ring solution has an eigenvalue with positive real part if and only if the linearized first order system around the ring solution has a positive eigenvalue. Moreover, the flock ring solution is unstable for m-mode perturbations for the second order model if and only if the ring solution is unstable for m-mode perturbations for the first order model.

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Outline

Stability: New change of variables

- Original coordinates: flock transversal profile
- New coordinates: relative to $m(t) = \frac{1}{N}\sum_i v_i(t)$.

→ all flocks are stationary, $4N+2$-dimensional dynamics $z \overset{t}{\mapsto} \mathcal{F}(z)$

- Reduce dynamics to mean-velocity consistent states, by choosing a invariant base B: $\mathcal{F}_B^B := \mathcal{F}|_{\text{span}\,B} \to \text{span}\,B$.

→ Study the linearisation $z \approx z_F + F_B^B(z - z_f)$

$$\begin{pmatrix} 0_{2N\times 2N} & \begin{matrix} I_{2N-2} \\ -1_{N-1}^T \otimes I_2 \end{matrix} & 0_{2N\times 2} \\ [G(\hat{x})] & -I_{N-1}\otimes 2\beta(m\otimes m^T) & 0_{2N-2\times 2} \\ 0_{2\times 2N} & 0_{2\times 2N-2} & -2\beta(m\otimes m^T) \end{pmatrix}.$$

$$\dot{x}_1 = \dots$$
$$\vdots$$
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$$\vdots$$
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flock solution:

$$z_F = (\hat{x} + v_0 t, v_0)^T,\ |v_0| = \sqrt{\alpha/\beta}.$$

Stability: New change of variables

- Original coordinates: flock transversal profile
- New coordinates: relative to mean velocity $m(t) = \frac{1}{N}\sum_i v_i(t)$.

→ all flocks are stationary, $4N+2$-dimensional dynamics $z \overset{t}{\mapsto} \mathcal{F}(z)$

- Reduce dynamics to mean-velocity consistent states, by choosing a invariant base B: $\mathcal{F}_B^B := \mathcal{F}|_{\operatorname{span} B} \to \operatorname{span} B$.

→ Study the linearisation $z \approx z_F + F_B^B(z - z_f)$

$$\begin{pmatrix} 0_{2N\times 2N} & \begin{matrix} I_{2N-2} \\ -1_{N-1}^T \otimes I_2 \end{matrix} & 0_{2N\times 2} \\ [G(\hat{x})] & -I_{N-1}\otimes 2\beta(m\otimes m^T) & 0_{2N-2\times 2} \\ 0_{2\times 2N} & 0_{2\times 2N-2} & -2\beta(m\otimes m^T) \end{pmatrix}.$$

$$\dot{x}_1 - m = \dots$$
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Result

Suppose the first-order aggregation system

$$\frac{dx_i}{dt} = -\sum_{i \neq j} \nabla U(x_i - x_j)\,,$$

is linearly stable except for translational and rotational invariance at a stationary profile $\hat{x}$.

Then the transformed second-order system behaves well:

- F_B^B has no generalised eigenvector for eigenvalue zero.
- $\dim(\text{eig}(F_B^B, 0)) = 4$ with 4 eigenvectors that all represent linearised flow within the set of stationary flock solutions.

  2 $\rightsquigarrow$ translation in space, 1 $\rightsquigarrow$ rotation in space, 1 $\rightsquigarrow$ rotation in mean velocity
- All non-zero eigenvalues of F_B^B have negative real-part.

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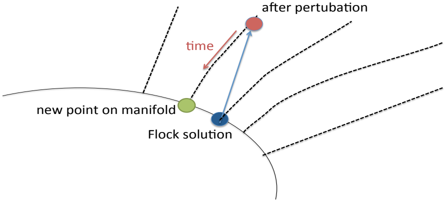
Stability Theorem

This is sufficient to establish that the family of flock solutions

$$Z_F = \left\{ (x^*, 0, m) \,,\, x^* = T_x R[\phi]\hat{x}, |m| = \sqrt{\alpha/\beta} \right\}$$

is a normally hyperbolic invariant manifold with a purely stable tangent-bundle splitting and exponentially decaying local stability (T_x translation, $R[\phi]$ rotation).

(C., Huang, Martin; preprint)



Outline

What about mills?

Let us consider the transformation

$$\begin{cases} y_j(t) = O(t)x_j(t) \\ z_j(t) = O(t)v_j(t) \end{cases} \quad , \quad j = 1, \ldots, N$$

where $O(t)$ is the rotation matrix defined as

$$O(t) = e^{St}, \quad S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad \text{and} \quad \dot{O}(t) = Se^{St}.$$

Fourier mode Perturbations:

$$\begin{pmatrix} \xi'_+ \\ \xi'_- \\ \eta'_+ \\ \eta'_- \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega i\alpha + \omega^2 + I_1(m) & -\omega i\alpha + I_2(m) & -\alpha - 2\omega i & \alpha \\ \omega i\alpha + I_2(m) & \omega i\alpha + \omega^2 + I_1(-m) & \alpha & -\alpha + 2\omega i \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \\ \eta_+ \\ \eta_- \end{pmatrix}$$

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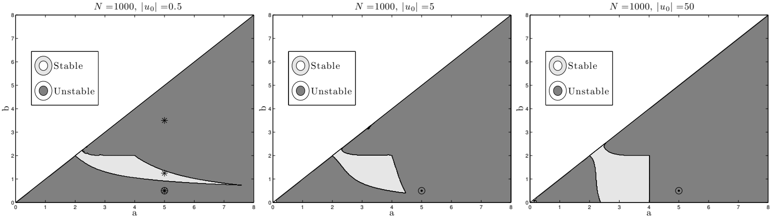

Figure: Stability region for $N = 1000$ and different values of $|u_0|$.

What about mills?

Conclusions

- The stability and instability of flocks for the second order model is implied from the analysis of the first order model.
- References:
    1. Albi-Balagué-C.-VonBrecht (preprint arxiv 2013).
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