Trace and Transfer Maps in the Algebraic K-theory of Spaces

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Abstract.
We prove that the composition of the A-theory transfer with the trace map to stable homotopy is weakly homotopic to the Becker-Gottlieb transfer. This shows that the Waldhausen splitting $\mathrm{A}(*) \simeq \mathrm{Q}(S^0) \times \mathrm{Wh}(*)$ of A-theory into stable homotopy and the Whitehead space is natural with respect to transfer maps.

Introduction

In their paper "A Parametrized Index Theorem for the Algebraic K-Theory Euler Class" [9], Dwyer, Weiss, and Williams show that the interaction of the algebraic K-theory transfer and the Becker-Gottlieb transfer detects subtle information about a fibration. They prove that if a fibration $E \to B$ with homotopy finite fibres is fibre homotopy equivalent to a bundle of smooth manifolds then the diagram

$$\begin{tikzcd} & \mathrm{A}(E) \\ B \arrow[ur] \arrow[r] & \mathrm{Q}(E_+) \arrow[u] \end{tikzcd}$$

Diagram 1

commutes up to homotopy, where $B \to \mathrm{A}(E)$ is the algebraic K-theory (A-theory) transfer, $B \to \mathrm{Q}(E_+)$ is the Becker-Gottlieb transfer, and $\mathrm{Q}(E_+) \to \mathrm{A}(E)$ is the inclusion.

Stable homotopy and algebraic K-theory are also related by the trace map $\mathrm{A}(E) \to \mathrm{Q}(E_+)$ and one can consider the commutativity of the diagram

$$\begin{tikzcd} & \mathrm{A}(E) \arrow[d] \\ B \arrow[ur] \arrow[r] & \mathrm{Q}(E_+). \end{tikzcd}$$

Diagram 2

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Algebraic K-theory contains strictly more information than stable homotopy and so the commutativity of this diagram is a less subtle question than that of the commutativity of the first diagram. Indeed, one expects this diagram to commute for any fibration with homotopy finite fibres. The difficulty proving this arises because existing descriptions of the trace map $\mathrm{A}(E) \to \mathrm{Q}(E_+)$ (unlike those of the inclusion $\mathrm{Q}(E_+) \to \mathrm{A}(E)$) are complicated and generally incompatible with the natural definitions of the two transfer maps. (See [13, 14, 5, 4] for versions of the trace map.)

Here we consider a modified version of the above commutativity question. Namely, does the diagram

$$\begin{tikzcd}
 & \mathrm{A}(E) \arrow[r] & \mathrm{A}(*) \arrow[d] \\
B \arrow[ur] \arrow[r] & \mathrm{Q}(E_+) \arrow[r] & \mathrm{Q}(S^0)
\end{tikzcd}$$

Diagram 3

commute up to weak homotopy? We do so using a description of the trace map in terms of Euclidean neighborhood retracts (ENRs) that permits comparison both with the standard definition of the A-theory transfer and with a simple geometric description of the Becker-Gottlieb transfer. We prove that Diagram 3 does indeed commute up to weak homotopy provided the fibration has compact fibres and a total space that is a Euclidean neighborhood retract over its base space. These conditions are satisfied, for example, if the fibration is locally trivial and the fibre is a finite CW complex, or if both the total space and the base are finite CW complexes. Diagram 4 illustrates an expanded version of Diagram 3 and provides a visual outline of the paper.

We can reformulate the homotopy commutativity of Diagram 3 as follows: the Waldhausen splitting $\mathrm{A}(*) \simeq \mathrm{Q}(S^0) \times \mathrm{Wh}(*)$ of A-theory into stable homotopy and the Whitehead space is natural with respect to transfer maps. Indeed, the Whitehead transfer $B \to \mathrm{Wh}(E)$ is most simply defined as the composite $B \to \mathrm{A}(E) \to \mathrm{Wh}(E)$; the splitting of the A-theory transfer

$$B \to \mathrm{A}(E) \to \mathrm{A}(*) \xrightarrow{\simeq} \mathrm{Q}(S^0) \times \mathrm{Wh}(*)$$

is therefore weakly homotopic to the product of the stable homotopy and Whitehead transfers

$$B \to \mathrm{Q}(E_+) \times \mathrm{Wh}(E) \to \mathrm{Q}(S^0) \times \mathrm{Wh}(*).$$

We expect that the same ENR method that we use to prove the commutativity of Diagram 3 will prove the weak homotopy commutativity of Diagram 2; the ENR description of the trace map $\mathrm{A}(E) \to \mathrm{Q}(E_+)$ is complicated only by some fibrewise technicalities. But more importantly and along somewhat different lines, we believe that the trace map can be defined categorically; that is, we anticipate a simple category (with cofibrations and weak equivalences) whose K-theory has the homotopy type of $\mathrm{Q}(S^0)$ and which admits a functor (preserving cofibrations and weak equivalences) from the category of retractive spaces $\mathrm{R}^{\mathrm{f}}(*)$. If such a category exists, this description of the trace map will dramatically simplify the contents of this paper and indeed any comparison of algebraic K-theory and stable homotopy.

The ENR approach to the trace map was suggested to us independently by Graeme Segal and Tom Goodwillie. Lydakis [11] has also given such a description, though with different aims. Our trace map is very similar to his in spirit but in places rather different in detail.

Notes on Diagram 4:

The space in the lower left is the base space $B = |B.|$ of the fibration, the space in the upper right is A-theory $\mathrm{A}(*) = \Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{hf}}(*)|$, and the space in the lower right is stable homotopy $\mathrm{Q}(S^0) \simeq |\mathrm{Q}.(S^0)|$. The upper left path from $|B.|$ to $\Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{hf}}(*)|$ is the standard construction of the A-theory transfer (Section 1). The righthand vertical column is a model for the trace map (Section 2). We prove that the wrong way maps are homotopy equivalences, but in fact we suspect all the maps in this column except τ are homotopy equivalences. The map $|B.| \xrightarrow{\beta} |\mathrm{Q}.(S^0)|$ is the Becker-Gottlieb transfer (projected from $\mathrm{Q}(E_+)$ to $\mathrm{Q}(S^0)$); the map $|B.| \xrightarrow{\beta_f} |\mathcal{F}.|$ may be regarded as an alternate Becker-Gottlieb transfer (Section 3). The map $|B.| \xrightarrow{\theta_b} \Omega|\mathrm{s}.\mathcal{E}.|$ may be regarded as an alternate A-theory transfer; we will compare it to the composition θ^t.

$$
\begin{tikzcd}
\widetilde{\mathrm{A}}_B(E) \arrow[d, equal] & \widetilde{\mathrm{A}}(E) \arrow[d, equal] & \mathrm{A}(E) \arrow[d, equal] & \mathrm{A}(*) \arrow[d, equal] \\
\operatorname{hocolim} |\mathrm{N}.w\mathrm{R}^{\mathrm{hf}}(E_x)| \arrow[r] & |\mathrm{N}.w\mathrm{R}^{\mathrm{hf}}(E)| \arrow[r] & \Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{hf}}(E)| \arrow[r] & \Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{hf}}(*)| \\
\operatorname{hocolim} |\mathrm{N}.(\mathrm{simp}(B.)/x)| \arrow[u] \arrow[d, "\sim"'] & & & \Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{f}}(*)| \arrow[u, "\sim"'] \arrow[d] \\
|\mathrm{N}.\,\mathrm{simp}(B.)| \arrow[d, "\sim"'] & & & \Omega|\mathrm{N}.w\mathrm{S}.\mathcal{E}| \arrow[d] \\
B = |B.| \arrow[uu, dashed, bend left, "\lambda"] \arrow[rrr, dashed, bend left=60, "\overline{\theta}^t"] \arrow[drrr, "\theta_b"] \arrow[ddddrrr, "\beta_f"] \arrow[dddddddddrrr, "\beta"] & & & \Omega|\mathrm{N}.w\mathrm{S}.\mathcal{E}.| \\
& & & \Omega|\mathrm{s}.\mathcal{E}.| \arrow[u, "\sim"'] \\
& & & \Omega|\mathrm{s}.\widetilde{\mathcal{E}}.| \arrow[u, "\sim"'] \arrow[d, "\tau"'] \\
& & & \Omega|\mathrm{t}.\mathcal{F}.| \\
& & & |\mathcal{F}.| \arrow[u, "\sim"'] \\
& & & |\overline{\mathcal{F}}.| \arrow[u, "\sim"'] \\
& & & |\widetilde{\mathcal{F}}.| \arrow[u, "\sim"'] \arrow[d] \\
& & & |\mathcal{F}.^{\mathbb{R}}| \arrow[d, "\nu"'] \\
& & & |\mathcal{V}.| \\
& & & \mathrm{Q}(S^0) = |\mathrm{Q}.(S^0)| \arrow[u, "\sim"']
\end{tikzcd}
$$

Diagram 4

1. The A-theory Transfer

We briefly recall the definitions of A-theory [15] and the A-theory transfer [9]. All categories will be discrete.

We take the A-theory of X to be the K-theory of the category $\mathrm{R}^{\mathrm{hf}}(X)$ of homotopy-finite-CW retractive spaces over X. The category $\mathrm{R}(X)$ of retractive spaces over X has objects spaces Y with maps $Y \underset{s}{\overset{r}{\rightleftharpoons}} X$ such that s is a section of the retraction r; a morphism of retractive spaces is a map over and under X. The category of homotopy-finite-CW retractive spaces is the full subcategory consisting of those objects (Y, r, s) that admit a homotopy equivalence (in $\mathrm{R}(X)$) to a space admitting a finite CW structure relative to X. This category is a category with cofibrations and weak equivalences, where we take the cofibrations to be maps satisfying the homotopy extension property and weak equivalences to be weak homotopy equivalences. We will also have occasion to consider the category $\mathrm{R}^{\mathrm{f}}(X)$ of finite-CW retractive spaces over X. These are retractive spaces equipped with the structure of a finite CW complex relative to X; morphisms are cellular maps.

Given a category $\mathcal{C}$ with cofibrations and weak equivalences, we form the simplicial category $\mathrm{S}.\mathcal{C}$ of filtered objects of $\mathcal{C}$. Let $[k]$ denote the category $0 \to 1 \to 2 \to \ldots \to k$. We have the arrow category $\mathrm{Ar}[k] = [k]^{[1]}$. The objects of $\mathrm{S}_k\mathcal{C}$ are functors $F : \mathrm{Ar}[k] \to \mathcal{C}$ satisfying the following properties. Let F_{ij} denote the image of the arrow $i \to j$ in $\mathcal{C}$. We require that $F_{ii} = *$, where $*$ is the basepoint of $\mathcal{C}$, for all i, and that the sequence $F_{ij} \to F_{ik} \to F_{jk}$ is a cofibration sequence. Morphisms in $\mathrm{S}.\mathcal{C}$ are natural transformations.

The K-theory of $\mathcal{C}$ is by definition $\Omega|\mathrm{N}.w\mathrm{S}.\mathcal{C}|$, the loop space on the realization $|-|$ of the nerve N. of the subcategory of weak equivalences w of the category of filtered objects S. of $\mathcal{C}$. In particular, $\mathrm{A}(X) = \Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{hf}}(X)|$.

Let $E \to B$ be a (Hurewicz) fibration with homotopy finite fibres. Given a point $p \in B$, let E_p be the fibre over p. We can associate to p an object in $\mathrm{R}^{\mathrm{hf}}(E)$, namely $E_p \amalg E$ with the retraction to E given by the disjoint union of the inclusion and the identity. Heuristically, the A-theory transfer is induced by this map.

Precisely, the A-theory transfer $B \to \mathrm{A}(E)$ is defined, up to homotopy, as follows. Assume that for some simplicial set $B.$ we have $B = |B.|$; in general we can replace B by the (homotopy equivalent) realization of its set of singular simplices. Let $\mathrm{simp}(B.)$ be the category of simplices of $B.$ – the objects of $\mathrm{simp}(B.)$ are the simplices of $B.$ and a morphism $y \to x$ from $y \in B_n$ to $x \in B_m$ is a monotone map $f : \{0, \ldots, n\} \to \{0, \ldots, m\}$ such that $f^*(x) = y$. There is a homotopy equivalence

$$\mathrm{N.\,simp}(B.) \xrightarrow{\sim} B.$$

called "Kan's last vertex map". This map is given on an n-diagram in $\mathrm{N}_n\,\mathrm{simp}(B.)$ by

$$(x_0 \overset{g_0}{\to} x_1 \overset{g_1}{\to} \dots \overset{g_{n-1}}{\to} x_n) \mapsto g^*(x_n) \quad \text{where} \quad g : \{0,\dots,n\} \to \{0,\dots,n\}$$
$$\text{with} \quad g(i) = g_{n-1}\dots g_{i+1}g_i(|x_i|)$$

We have another homotopy equivalence

$$\underset{x\in \mathrm{simp}(B.)}{\mathrm{hocolim}} |\mathrm{N.}(\mathrm{simp}(B.)/x)| \overset{\sim}{\to} |\mathrm{N.}\,\mathrm{simp}(B.)|$$

This map is induced by the trivial natural transformation $|\mathrm{N.}(\mathrm{simp}(B.)/x)| \to *$.

Let $\widetilde{\mathrm{A}}(E) = |\mathrm{N.}w\mathrm{R}^{\mathrm{hf}}(E)|$ denote the "pre-group-complete A-theory". There is an "exact sequence group completion" map

$$\widetilde{\mathrm{A}}(E) = |\mathrm{N.}w\mathrm{R}^{\mathrm{hf}}(E)| \to \Omega|\mathrm{N.}w\mathrm{S.R}^{\mathrm{hf}}(E)| = \mathrm{A}(E).$$

We also have the "fibrewise pre-group-complete A-theory" $\widetilde{\mathrm{A}}_B(E) = \underset{x\in \mathrm{simp}(B.)}{\mathrm{hocolim}}\, \widetilde{\mathrm{A}}(E_x)$, where E_x denotes the fibre of $E \to B$ over x. Inclusion induces an assembly-type map

$$\widetilde{\mathrm{A}}_B(E) \to \widetilde{\mathrm{A}}(E).$$

Now we need only construct a map

$$\underset{x\in \mathrm{simp}(B.)}{\mathrm{hocolim}} |\mathrm{N.}(\mathrm{simp}(B.)/x)| \to \mathrm{hocolim}\, |\mathrm{N.}w\mathrm{R}^{\mathrm{hf}}(E_x)| = \mathrm{hocolim}\, \widetilde{\mathrm{A}}(E_x) = \widetilde{\mathrm{A}}_B(E).$$

This map is induced by a natural transformation T from $x \mapsto (\mathrm{simp}(B.)/x)$ to $x \mapsto w\mathrm{R}^{\mathrm{hf}}(E_x)$; these are both functors $(\mathrm{simp}(B.)) \to \mathcal{CAT}$. To each $x \in \mathrm{simp}(B.)$ we need to associate a functor $T_x : (\mathrm{simp}(B.)/x) \to w\mathrm{R}^{\mathrm{hf}}(E_x)$. Let y be an object in $(\mathrm{simp}(B.)/x)$. Set $T_x(y) = E_y \amalg E_x$; here the retraction map $E_y \amalg E_x \to E_x$ is induced by the inclusion $E_y \to E_x$ and the identity $E_x \to E_x$. Morphisms are also induced by the inclusion maps and are all obviously weak equivalences.

We have constructed up to homotopy the transfer $B \to \mathrm{A}(E) = \Omega|\mathrm{N.}w\mathrm{S.R}^{\mathrm{hf}}(E)|$. We compose this with the A-theory map induced by projecting E to a point, namely

$$\Omega|\mathrm{N.}w\mathrm{S.R}^{\mathrm{hf}}(E)| \to \Omega|\mathrm{N.}w\mathrm{S.R}^{\mathrm{hf}}(*)|,$$

and refer also to the composite $B \to \mathrm{A}(E) \to \mathrm{A}(*)$ as the A-theory transfer. Finally we note that we can substitute for the category of homotopy finite retractive spaces the category of finite retractive spaces; that is, the map

$$\Omega|\mathrm{N.}w\mathrm{S.R}^{\mathrm{hf}}(*)| \overset{\sim}{\leftarrow} \Omega|\mathrm{N.}w\mathrm{S.R}^{\mathrm{f}}(*)|$$

is a homotopy equivalence [15].

2. The A-theory Trace

We present a definition of the trace map $\mathrm{A}(*) \to \mathrm{Q}(S^0)$ using Euclidean neighborhood retracts (ENRs) that will make transparent the relationship between the A-theory transfer and the Becker-Gottlieb transfer. As mentioned in the introduction, this trace map is quite similar to Lydakis' and we refer to his paper [11] where convenient. The main references for the definitions and properties of ENRs are Dold's papers [7, 8].

We think heuristically of the trace map as follows. To a pointed finite CW complex X, that is an object of $\mathrm{R}^{\mathrm{f}}(*)$, we want to associate a map $S^n \to S^n$, for some n. Suppose X is embedded in $\mathbb{R}^n$ and $r : U \to X$ is a neighborhood retraction. Then there is a subneighborhood $V_\epsilon \subset U$ so that the map $\nu : V_\epsilon \to D^n_\epsilon$, given by $\nu(x) = x - r(x)$, is proper. (Here D^n_ϵ denotes the open ϵ-ball in $\mathbb{R}^n$.) Thus after one point compactification, the maps $\mathbb{R}^n \hookleftarrow V_\epsilon \xrightarrow{\nu} D^n_\epsilon$ yield a map $\xi(X) : S^n \to S^n$. This is simply a version of the Pontryagin-Thom construction. Note that the degree of $\xi(X)$ is the euler characteristic of X. If $X = Y_+$, that is if the basepoint of X is disjoint, then the trace of X is morally $\xi(Y)$; thus the degree of the trace is the reduced euler characteristic.

Now given any self map $f : X \to X$, let F be the fixed points of f. Then there is a neighborhood V_ϵ of F in $\mathbb{R}^n$ so that the map $\nu_f : V_\epsilon \to D^n_\epsilon$, given by $\nu_f(x) = x - fr(x)$, is proper. One point compactification produces a map $\xi_f(X) : S^n \to S^n$. The degree of $\xi_f(X)$ is a homotopy invariant of f and only depends on f in a neighborhood of the fixed set F. In particular, given any space X in $\mathrm{R}^{\mathrm{f}}(*)$, let $f : X \to X$ be a map homotopic to the identity that retracts a neighborhood of the basepoint to the basepoint. Then F splits as $F' \amalg *$. Because of this splitting we can form the map $\xi_f(X; F')$ associated only to a neighborhood of the fixed points F', and we take this map to represent the trace of X.

In section 2.1 we produce maps from A-theory $\mathrm{A}(*)$ to the K-theory of appropriate categories of ENRs. This will allow us to use embeddings and neighborhood retractions, as just described. In section 2.2, we first discuss a simplicial set of "fixed point problems" that encodes the information (namely the self map and fixed point set) needed to perform the above variant of the Pontryagin-Thom construction. Second, we formalize conditions on the self map that ensure, among other things, that the basepoint splits off the fixed point set. Finally we associate a fixed point problem to an ENR with an appropriate self map. In section 2.3 we construct a map from the set of fixed point problems to stable homotopy.

2.1. A-theory via Euclidean Neighborhood Retracts

In section 1 we described the A-theory transfer $B \to \mathrm{A}(*) = \Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{f}}(*)|$. Again, we identify $\mathrm{R}^{\mathrm{f}}(*)$ with the category of pointed finite CW complexes. Finite CW complexes are compact Euclidean neighborhood retracts, and we now define maps from $\mathrm{A}(*)$ to the K-theory of appropriate categories of ENRs. Recall that a space E is by definition a Euclidean neighborhood retract if it admits an embedding $\phi : E \to \mathbb{R}^n$ in some Euclidean space and a retraction $r : U \to \phi(E)$ of some neighborhood U of the image of ϕ.

Definition 2.1 ([11]). The category $\mathcal{E}$ of pointed ENRs has objects compact ENRs with basepoint, and morphisms based maps. It can be considered a category with cofibrations and weak equivalences where the cofibrations are inclusions and the weak equivalences are weak homotopy equivalences.

The inclusion functor $\mathrm{R}^{\mathrm{f}}(*) \to \mathcal{E}$ induces a map in K-theory

$$\Omega|\mathrm{N}.w\mathrm{S}.\mathrm{R}^{\mathrm{f}}(*)| \to \Omega|\mathrm{N}.w\mathrm{S}.\mathcal{E}|.$$

In order to define the trace map most simply, we would like to "ignore the morphisms" in the definition of the K-theory of $\mathcal{E}$; that is, we want to work not with the nerve of the category of weak equivalences of filtered ENRs, but with the simplicial set of filtered ENRs. We can do this if we introduce another simplicial direction and consider ENRs parametrized by a simplex.

Let B be a compact space. Recall that a space $E \to B$ over B is an "ENR over B", denoted ENR_B, if there is an embedding $\phi : E \to B \times \mathbb{R}^n$, for some n, and a neighborhood retraction $r : U \to \phi(E)$ such that both ϕ and r commute with the obvious maps to B.

Definition 2.2 ([11]). We define the simplicial category $\mathcal{E}.$ of parameterized pointed ENRs. The objects $\mathcal{E}_k$ in degree k are compact spaces E that are ENR_{Δ^k} and have a section $s : \Delta^k \to E$ of the projection $E \to \Delta^k$. The morphisms are maps over and under Δ^k. The face and degeneracy maps are given by restriction and pullback.

(Similarly, for any B, we have the category $\mathcal{E}_B$ of parameterized pointed ENRs over B. For the remainder of the paper, all maps of spaces over B, for any B, will be maps over B.) The inclusion as the zero simplices $\mathcal{E} \to \mathcal{E}.$ induces a map in K-theory

$$\Omega|\mathrm{N}.w\mathrm{S}.\mathcal{E}| \to \Omega|\mathrm{N}.w\mathrm{S}.\mathcal{E}.|.$$

We denote by $\mathrm{s}.\mathcal{E}.$ the (bi)simplicial set of objects of the category $\mathrm{S}.\mathcal{E}.$ of filtered parametrized pointed ENRs. We refer to $\mathrm{s}.\mathcal{E}.$ simply as the simplicial set of filtered ENRs. We now note that we can dispense with the morphisms.

Proposition 2.3 ([11], §4). *The inclusion as the zero-simplices of the nerve* $\mathrm{s}.\mathcal{E}. \to \mathrm{N}.w\mathrm{S}.\mathcal{E}.$ *induces a homotopy equivalence on realization.*

We end this subsection by recalling a few properties of ENR_Bs. The first proposition explains the main reason we work with *compact* ENR_Bs.

Proposition 2.4 ([7]). *If* $E \xrightarrow{\pi} B$ *is an* ENR_B *with* E *compact, then* π *is a (Hurewicz) fibration.*

The next proposition helpfully formulates the main extension property of ENRs. It is an immediate consequence of the Tietze extension theorem.

Proposition 2.5 ([11],§3). *Let* Y *be a closed subspace of a normal space* X *over* B *and* E *any* ENR_B. *Any map* $Y \to E$ *over* B *extends over* B *to a neighborhood of* Y *in* X.

Corollary 2.6 ([11],§3). *Let* $E \subset C \subset G$ *be such that* E *and* G *are* ENR_B *and* C *is a closed subset of* G. *Any retraction* $C \to E$ *extends to a retraction of a neighborhood of* E *in* G.

This proposition implies another, using the fact that any two extensions are homotopic.

Proposition 2.7 ([11],§3). *Let* $E \subset G$ *be such that* E *and* G *are* ENR_B, *with* E *closed in* G. *Then* E *is a neighborhood deformation retract over* B *in* G.

Corollary 2.8. *Let* $E \subset G$ *be an inclusion of* ENR_B*s with* E *closed in* G. *Then* $E \subset G$ *is a cofibration, that is satisfies the homotopy extension property.*

The corollary follows because, given a neighborhood deformation retraction of E in G, one can construct a retraction $G \times I \to E \times I \cup G \times \{0\}$.

2.2. The Fixed-Point Trace Map

As mentioned above, there is a simplicial set $\mathcal{F}.$ of "fixed point problems" admitting a map, via a Pontryagin-Thom construction, to $\mathrm{Q}.(S^0)$.

Definition 2.9. The simplicial set $\mathcal{F}.$ of fixed point problems has as its set of k-simplices $\mathcal{F}_k$ triples (E, F, f) where E is a (not necessarily compact) ENR_k, F is a compact subset of E, and f is the germ at F of a map $E \to E$ whose fixed points are precisely F. That is, more precisely, f is a map $U \to E$ for some neighborhood U of F with fixed points F, and we introduce the equivalence relation that $f \sim g$ if f and g agree on some neighborhood of F.

There is a simplicial map $\mathcal{E}. \to \mathcal{F}.$ taking an ENR E to the fixed point problem of the identity (E, E, id_E). Morally, this induces an unreduced version of the trace map. Properly, the trace has as its domain K-theory, or filtered ENRs – we will construct a reduced trace map from filtered ENRs to "filtered fixed point problems". The natural notion of a filtered fixed point problem is a splitting of the fixed points of a self map into disjoint components.

Definition 2.10. We define the simplicial set $\mathrm{t}.\mathcal{F}.$ of filtered fixed point problems. An element of $\mathrm{t}_k\mathcal{F}_n$ is an element (E, F, f) of $\mathcal{F}_n$ together with a splitting into disjoint pieces $F = \coprod_{i=1}^k F_i$ where we allow the F_i to be empty. The face and degeneracy maps are as expected, with

$$\begin{aligned}
&\partial_0 : \coprod_{i=1}^{k} F_i \mapsto \coprod_{i=2}^{k} F_i,\\
&\partial_k : \coprod_{i=1}^{k} F_i \mapsto \coprod_{i=1}^{k-1} F_i, \text{and}\\
&\partial_j : \coprod_{i=1}^{k} F_i \mapsto (\coprod_{i=1}^{j-1} F_i) \amalg (F_j \cup F_{j+1}) \amalg (\coprod_{i=j+2}^{k} F_i), \quad \text{for} \quad 0 < j < k.
\end{aligned}$$

The fixed points of the identity map of a filtered ENR are not split into disjoint pieces, but the identity map is always homotopic to a map with fixed points split in a manner compatible with the filtration. Moreover, the choice of such a homotopy is contractible. We now make precise this enlargement of $\mathrm{s}.\mathcal{E}.$ to the simplicial set $\widetilde{\mathrm{s}.\mathcal{E}}.$ of filtered ENRs with filtering homotopy.

Recall that an element E of $s_k\mathcal{E}_n$ is given by a functor (satisfying various properties) from $\mathrm{Ar}[k] = [k]^{[1]}$ to $\mathcal{E}_n = \mathcal{E}_{\Delta^n}$. In particular the images of the arrows $0 \to i$ determine a sequence $E_{00} \subset \ldots \subset E_{0i} \subset \ldots \subset E_{0k}$. For simplicity, we do not distinguish between the functor E and the filtered sequence $\{E_{0i}\}$, which we denote $\{E_i\}$. We think of the E_{ij}, $i > 0$, as determined by taking quotients.

Definition 2.11. We call a homotopy $h : E_k \times I \to E_k$ a *filtering homotopy* for $E \in \mathrm{s}_k\mathcal{E}_n$ if:

1. h preserves the filtration, ie $h(E_i \times I) \subset E_i$ for all i,
2. h is a homotopy from the identity, ie $h|(E_k \times \{0\})$ is the identity,
3. h is a homotopy to a map contracting a neighborhood of each level of the filtration, ie for all i there exists a neighborhood U_i of E_i in E_k such that $h(U_i \times \{1\}) \subset E_i$, and
4. h fixes E_0, ie $h|(E_0 \times \{t\}) = \mathrm{id}_{E_0}$ for all $t \in I$.

(One may equally well define a filtering homotopy for a filtration of any ENR_B; that is, nothing in the definition depends on having ENR_{Δ^n}.)

Definition 2.12. The simplicial set $\widetilde{\mathrm{s.}\mathcal{E}}.$ of filtered parameterized ENRs with filtering homotopy is, in degree (k, n), equal to the set of pairs of elements (E, h) of a filtered ENR $E \in \mathrm{s}_k\mathcal{E}_n$ and a filtering homotopy h for E.

Proposition 2.13. *The forgetful map* $\widetilde{\mathrm{s.}\mathcal{E}}. \to \mathrm{s.}\mathcal{E}.$ *induces a homotopy equivalence on realization.*

Proof. We prove that for all k the map $\widetilde{\mathrm{s}_k\mathcal{E}}. \to \mathrm{s}_k\mathcal{E}.$ induces a homotopy equivalence on realization. The result follows because the realization of a simplicial map of simplicial spaces is a homotopy equivalence provided the map is a homotopy equivalence in each simplicial dimension.

The simplicial sets $\widetilde{\mathrm{s}_k\mathcal{E}}.$ and $\mathrm{s}_k\mathcal{E}.$ are both Kan. We show that the relative simplicial homotopy group vanishes. Let $E = E_\Delta = (E^0_\Delta \subset \ldots \subset E^k_\Delta)$ be an element of $\mathrm{s}_k\mathcal{E}_n$. (We now write the degree of the filtration as a superscript for notational convenience.) Denote by E_∂ the restriction of E to $\partial\Delta$. Let h_∂ be a filtering homotopy for E_∂. We will construct an extension of h_∂ to a filtering homotopy over all of Δ.

Proceed by induction on the filtration. By property 4 of a filtering homotopy, we may define h on $E^0_\Delta \times I$ to be the identity. Suppose h is defined on $E^i_\Delta \times I$, satisfying all the properties of a filtering homotopy for E^i_Δ. We must construct a map $\overline{h} : E^{i+1}_\Delta \times I \to E^{i+1}_\Delta$ such that

a. $\overline{h}|(E^i_\Delta \times I)$ agrees with h,

b. $\overline{h}|(E^{i+1}_\partial \times I)$ agrees with h_∂,

c. $\overline{h}|(E^{i+1}_\Delta \times \{0\})$ is the identity, and

d. $\overline{h}|(E^{i+1}_\Delta \times \{1\})$ contracts a neighborhood of E^i_Δ into itself.

We begin by ensuring condition (d). By assumption, there is a neighborhood U^i_∂ of E^i_∂ in E^{i+1}_∂ such that $h^{i+1}_\partial \times \{1\}$ contracts U^i_∂ into E^i_∂. Choose a closed subneighborhood $N^i_\partial \subset U^i_\partial$ of E^i_∂. We have a map

$$h^i_\Delta \times \{1\} \cup h^{i+1}_\partial \times \{1\} : E^i_\Delta \cup N^i_\partial \to E^i_\Delta$$

which by proposition 2.5 extends to a map $N^i_\Delta \to E^i_\Delta$ where N^i_Δ is a (closed) neighborhood of E^i_Δ in E^{i+1}_Δ. (We have used here and will in the following use without note that all the 'E's are compact Hausdorff spaces and are therefore normal; similarly, using normality, we freely take closed subneighborhoods of existing open neighborhoods. Also, we use 'extends' loosely in that we mean only that the new map agrees with the old wherever both are defined.)

Next we extend in a fashion determined by condition (a). Gluing together the existing maps, we have a map

$$E^i_\Delta \times I \cup E^{i+1}_\Delta \times \{0\} \cup E^{i+1}_\partial \times I \cup N^i_\Delta \times \{1\} \to E^{i+1}_\Delta.$$

By proposition 2.5 this extends to a map $N^i_\Delta \times I \to E^{i+1}_\Delta$ for some new neighborhood N^i_Δ of E^i_Δ.

Finally we extend over the remainder of $E^{i+1}_\Delta \times I$ respecting (b) and (c). Recall the homotopy extension-lifting property: given a cofibration $A \to X$ and a fibration $E \to B$, a homotopy $X \times I \to B$ with an existing lift $A \times I \to E$ over A, and a compatible initial lift $X \times \{0\} \to E$ of X, there exists a lift $X \times I \to E$ making all relevant diagrams commute. (See, for example, [10] and [6].) Here we have $A = E^{i+1}_\partial \cup N^i_\Delta$, $X = E^{i+1}_\Delta$, $E = E^{i+1}_\Delta$, and $B = \Delta$. We may assume that N^i_Δ is itself ENR_B and then use 2.8 to see that $E^{i+1}_\partial \cup N^i_\Delta \to E^{i+1}_\Delta$ is a cofibration. The resulting homotopy h is a filtering homotopy for E_Δ agreeing with the given homotopy on E_∂. This completes the proof. □

We can now define the trace map from a filtered ENR with filtering homotopy to a filtered fixed point problem. A typical element of $\widetilde{\mathrm{s}_k\mathcal{E}_n}$ is a pair (E, h), where $E = \{E_0 \subset \ldots \subset E_k\}$ is a $k+1$-stage filtered ENR_{Δ^n} and $h : E \times I \to E$ is a homotopy, satisfying various conditions. The fixed point set of $h_1 : E \times \{1\} \to E$ is split into $k+1$ disjoint pieces $F_0, F_1, \ldots, F_k$, with $F_i \subset E_i$. By assumption $F_0 = \Delta^n = E_0$. The map is given by

$$\begin{aligned} \widetilde{\mathrm{s}_k\mathcal{E}_n} &\xrightarrow{\tau} \mathrm{t}_k\mathcal{F}_n \\ (E, h) &\mapsto (E_k, F_1 \amalg F_2 \amalg \ldots \amalg F_k, h_1). \end{aligned}$$

That we disregard the fixed points F_0 corresponds to the fact that we are defining a reduced trace map.

Next we check that adding the filtration to the fixed point problems did not affect the homotopy type. Note that we can identify the fixed point problems $\mathcal{F}.$ with $\mathrm{t}_1\mathcal{F}.$, the one-stage filtered fixed point problems.

Proposition 2.14. *The inclusion of* $\mathcal{F}.$ *as the 1-simplices of* $\mathrm{t}.\mathcal{F}.$ *induces a homotopy equivalence* $|\mathcal{F}.| \xrightarrow{\sim} \Omega|\mathrm{t}.\mathcal{F}.|$.

Proof. Proposition 1.5 of [12] shows that $|\mathcal{F}.| \to \Omega|\mathrm{t}.\mathcal{F}.|$ is a homotopy equivalence if:

1. $|\mathrm{t}_0\mathcal{F}.|$ is contractible;

2. the map $|\mathrm{t}_k\mathcal{F}.| \xrightarrow{\xi} |\mathrm{t}_1\mathcal{F}.| \times \ldots \times |\mathrm{t}_1\mathcal{F}.|$ (k copies) is a homotopy equivalence, where ξ is induced by the k face maps $\{0, 1\} \to \{0, \ldots, k\}$ of the form $0 \mapsto i, 1 \mapsto i+1$; and

3. the components $\pi_0(|\mathrm{t}_1\mathcal{F}.|)$ form a group with composition given by

$$|\mathrm{t}_1\mathcal{F}.| \times |\mathrm{t}_1\mathcal{F}.| \xleftarrow{\simeq} |\mathrm{t}_2\mathcal{F}.| \xrightarrow{\mu} |\mathrm{t}_1\mathcal{F}.|,$$

where μ is induced by the face map $0 \mapsto 0, 1 \mapsto 2$.

(We are using the fact that the realization of a simplicial set has a numerable covering by contractible sets.)

1. Note that $\mathrm{t}_0\mathcal{F}_n$ is the set of ENR_{Δ^n}. These elements need not be compact and can be empty. Given any $E_\partial \xrightarrow{p} \partial\Delta^n$ in $\mathrm{ENR}_{\partial\Delta^n}$, construct an E' in ENR_{Δ^n} with $E'_\partial = E_\partial$ as follows. Identify Δ^n with $\Delta^n \cup (\partial\Delta^n \times I)$, where the boundary of Δ^n is identified with $\partial\Delta^n \times \{1\}$. Let E' be $E \times [0,1)$ with projection $E \times [0,1) \xrightarrow{p'} \Delta^n \cup (\partial\Delta^n \times I)$ by $p'(e,t) = (p(e),t) \in \partial\Delta^n \times I$. E' is ENR_{Δ^n} as desired.

2. The map

$$\begin{array}{rcl} |\mathrm{t}_k\mathcal{F}.| & \xrightarrow{\xi} & |\mathrm{t}_1\mathcal{F}.| \times \ldots \times |\mathrm{t}_1\mathcal{F}.| \\ (E, F_1 \amalg \ldots \amalg F_k, f) & \mapsto & (E, F_1, f) \times \ldots \times (E, F_k, f) \end{array}$$

has a homotopy inverse

$$\begin{array}{rcl} |\mathrm{t}_1\mathcal{F}.| \times \ldots \times |\mathrm{t}_1\mathcal{F}.| & \xrightarrow{\zeta} & |\mathrm{t}_k\mathcal{F}.| \\ (E_1, F_1, f_1) \times \ldots \times (E_k, F_k, f_k) & \mapsto & (E_1 \amalg \ldots \amalg E_k, F_1 \amalg \ldots \amalg F_k, f_1 \amalg \ldots \amalg f_k). \end{array}$$

The composition $\xi \circ \zeta$ is given by

$$\begin{array}{l} (E_1, F_1, f_1) \times \ldots \times (E_k, F_k, f_k) \mapsto \\ (E_1 \amalg \ldots \amalg E_k, F_1, f_1 \amalg \ldots \amalg f_k) \times \ldots \times (E_1 \amalg \ldots \amalg E_k, F_k, f_1 \amalg \ldots \amalg f_k). \end{array}$$

Using the construction from item 1 above to get rid of ENR components with no fixed points, this map is homotopic to the identity.

On the other hand, the composition $\zeta \circ \xi$ is given by

$$\alpha = (E, F_1 \amalg \ldots \amalg F_k, f) \mapsto (E \amalg \ldots \amalg E, F_1 \amalg \ldots \amalg F_k, f \amalg \ldots \amalg f) = \zeta \circ \xi(\alpha).$$

We will construct, in a canonical fashion, an element of $\mathrm{t}_k\mathcal{F}_{\Delta^n \times I}$ cobounding α and $\zeta \circ \xi(\alpha)$, thereby showing $\zeta \circ \xi$ is homotopic to the identity in $|\mathrm{t}_k\mathcal{F}.|$. When we have an element of $\mathrm{t}_k\mathcal{F}_{\Delta^n \times I}$ cobounding two elements β, β' of $\mathrm{t}_k\mathcal{F}_n$, we will say loosely that β and β' are homotopic, writing $\beta \sim \beta'$.

Consider $E \amalg \ldots \amalg E$ as contained in $E \times \mathbb{R}$ with the i^{th} copy of E identified with $E \times \{i\}$. We have a chain of homotopies of filtered fixed point problems

$$\begin{gathered}
(E \times \{1\} \amalg \ldots \amalg E \times \{k\}, F_1 \times \{1\} \amalg \ldots \amalg F_k \times \{k\}, f \times \{1\} \amalg \ldots \amalg f \times \{k\}) \\
\wr \\
(E \times (1-\epsilon, 1+\epsilon) \amalg \ldots \amalg E \times (k-\epsilon, k+\epsilon), F_1 \times \{1\} \amalg \ldots \amalg F_k \times \{k\}, f') \\
\wr \\
(E \times (1-\epsilon, k+\epsilon), F_1 \times \{1\} \amalg \ldots \amalg F_k \times \{k\}, f') \\
\wr \\
(E \times (1-\epsilon, k+\epsilon), F_1 \times \{1\} \amalg F_2 \times \{1\} \amalg \ldots \amalg F_k \times \{1\}, f'') \\
\wr \\
(E \times \{1\}, F_1 \times \{1\} \amalg \ldots \amalg F_k \times \{1\}, f \times \{1\})
\end{gathered}$$

where, in the above, $f'(e,t) = (f(e), i)$ on $E \times (i-\epsilon, i+\epsilon)$ and $f''(e,t) = (f(e), 1)$ in a neighborhood of $E \times \{1\}$. The last filtered fixed point problem is simply α, as needed.

3. To show that the components of $|\mathcal{F}.|$ form a group, we construct an explicit homotopy inverse. Let (E, F, f) be in $\mathcal{F}_n$. Then

$$(E, F, f) \sim (E \times (-1,1), F \times \{0\}, f_r)$$

where

$$\begin{aligned}
f_r(e,t) = (f(e), \phi(t)) \quad &\text{with} \;\; \phi(t) > t \quad \text{for} \quad t < 0 \\
&\text{and} \;\; \phi(t) < t \quad \text{for} \quad t > 0.
\end{aligned}$$

The homotopy inverse will be

$$(E \times (2,4), F \times \{3\}, f_q)$$

where

$$\begin{aligned}
f_q(e,t) = (f(e), \phi'(t)) \quad &\text{with} \;\; \phi'(t) < t \quad \text{for} \quad t < 3 \\
&\text{and} \;\; \phi'(t) > t \quad \text{for} \quad t > 3.
\end{aligned}$$

Now

$$(E \times ((-1,1) \cup (2,4)), F \times (\{1\} \cup \{3\}), f_r \cup f_q) \sim (E \times (-1,4), F \times (\{1\} \cup \{3\}), f_p)$$

where

$$\begin{aligned}
f_p(e,t) = (f(e), \psi(t)) \quad \text{with} \quad &\psi(t) > t \quad \text{for} \quad t < 0, \\
&\psi(t) < t \quad \text{for} \quad 0 < t < 3, \\
\text{and} \quad &\psi(t) > t \quad \text{for} \quad t > 3.
\end{aligned}$$

Finally, this is homotopic to a object with no fixed points, which is 0 in homotopy. □

2.3. Stable Homotopy via Fixed Point Problems

We now construct a chain of maps between the simplicial set of fixed point problems $\mathcal{F}.$ and the simplicial set of singular simplices of $\mathrm{Q}(S^0)$.

The first step is to enlarge the fixed point problems $\mathcal{F}.$ to include a particular embedding in some $\mathbb{R}^n$.

Definition 2.15. The simplicial set $\overline{\mathcal{F}.}^n$ has as its k-simplices quadruples (E, F, ϕ, f) where (E, F, f) is in $\mathcal{F}_k$ and $\phi : E \to \Delta^k \times \mathbb{R}^n$ is an embedding. The suspension map $\overline{\mathcal{F}.}^n \to \overline{\mathcal{F}.}^{n+1}$ is given by $(E, F, \phi, f) \mapsto (E, F, \phi \times \{0\}, f)$, and the simplicial set of embedded fixed point problems $\overline{\mathcal{F}.}$ is the colimit over n of $\overline{\mathcal{F}.}^n$.

Proposition 2.16. *The forgetful map $\overline{\mathcal{F}.} \to \mathcal{F}.$ induces a homotopy equivalence on realization.*

Proof. The simplicial sets are both Kan. We show that the relative simplicial homotopy group vanishes. Let (E, F, f) be in $\mathcal{F}_k$ and let $\phi : E_{\partial\Delta^k} \to \partial\Delta^k \times \mathbb{R}^n$ be an embedding. We extend this to an embedding of E_{Δ^k}. First extend (Proposition 2.5) ϕ to any map $\phi' : E_U \to U \times \mathbb{R}^n$ where U is a neighborhood of $\partial\Delta^k$ in Δ^k. Let $\psi : E_{\Delta^k} \to \Delta^k \times \mathbb{R}^m$ be any embedding. Choose a function $\gamma : \Delta^k \to [0, 1]$ that is 0 precisely on $\partial\Delta^k$ and is 1 outside U. Now $((1 - \gamma)\phi + \gamma\psi) : E_{\Delta^k} \to \Delta^k \times \mathbb{R}^{n+m}$ is an embedding extending ϕ, as desired. $\square$

Now we further enlarge the fixed point problems to include a retraction of a neighborhood of their embedding in Euclidean space.

Definition 2.17. The simplicial set $\widetilde{\mathcal{F}.}^n$ has as its k-simplices quintuples (E, F, ϕ, f, r) where (E, F, ϕ, f) is in $\overline{\mathcal{F}.}^n$ and r is the germ of a retraction onto $\phi(E)$ of a neighborhood of $\phi(E)$ in $\Delta^k \times \mathbb{R}^n$. The suspension map $\widetilde{\mathcal{F}.}^n \to \widetilde{\mathcal{F}.}^{n+1}$ is given by $(E, F, \phi, f, r) \mapsto (E, F, \phi \times \{0\}, f, r \times \{0\})$, and the simplicial set of embedded fixed point problems with retraction $\widetilde{\mathcal{F}.}$ is the colimit over n of $\widetilde{\mathcal{F}.}^n$.

Proposition 2.18. *The forgetful map $\widetilde{\mathcal{F}.} \to \overline{\mathcal{F}.}$ induces a homotopy equivalence on realization.*

Proof. The simplicial sets are both Kan. We show that the relative simplicial homotopy group vanishes. Let (E, F, ϕ, f) be in $\overline{\mathcal{F}_k}$ and let r be the germ of a neighborhood retraction of $\phi(E_{\partial\Delta^k}) \subset \partial\Delta^k \times \mathbb{R}^n$. By Corollary 2.6 this neighborhood retraction extends to some neighborhood retraction of $\phi(E) \subset \Delta^k \times \mathbb{R}^n$, as desired. $\square$

We now dispense with the ENR E and retain only the fixed points and the germ of a self map.

Definition 2.19. The simplicial set of Euclidean fixed point problems $\mathcal{F}^{\mathbb{R}}_{.}$ is the colimit over n of those fixed point problems of the form $(\Delta^k \times \mathbb{R}^n, F, f)$. Suspension, again, is given by $(\Delta^k \times \mathbb{R}^n, F, f) \mapsto (\Delta^k \times \mathbb{R}^{n+1}, F \times \{0\}, f \times \{0\})$.

There is a map $\widetilde{\mathcal{F}}. \to \mathcal{F}^{\mathbb{R}}_{.}$ taking (E, F, ϕ, f, r) to $(\Delta^k \times \mathbb{R}^n, \phi(F), fr)$.

Next we form the collapse map of the germ at the fixed point set, producing a germ at the vanishing set.

Definition 2.20. The simplicial set $\mathcal{V}.^n$ has as k-simplices triples $(\Delta^k \times \mathbb{R}^n, V, g)$ where g is the germ at $V \subset \Delta^k \times \mathbb{R}^n$ of a map to $\mathbb{R}^n$ that sends V to $0 \in \mathbb{R}^n$. The suspension $\mathcal{V}.^n \to \mathcal{V}.^{n+1}$ takes $(\Delta^k \times \mathbb{R}^n, V, g)$ to $(\Delta^k \times \mathbb{R}^{n+1}, V \times \{0\}, g \times \mathrm{id}_{\mathbb{R}})$. The simplicial set of vanishing problems $\mathcal{V}.$ is the colimit over n of $\mathcal{V}.^n$.

The collapse map is given by

$$\mathcal{F}^{\mathbb{R}}_{.} \xrightarrow{\nu} \mathcal{V}.$$
$$(\Delta^k \times \mathbb{R}^n, F, f) \mapsto (\Delta^k \times \mathbb{R}^n, F, \mathrm{id}_{\mathbb{R}^n} - f).$$

Finally, let $\mathrm{Q}.(S^0)$ denote the singular set of $\mathrm{Q}(S^0)$. That is, $\mathrm{Q}_k(S^0)$ is a colimit of simplicial sets with elements given by maps $\Delta^k \times S^n \to S^n$ such that for all $p \in \Delta^k$ the map $\{p\} \times S^n \to S^n$ is based; we also sometimes consider these elements as maps $\Delta^k \times S^n \to \Delta^k \times S^n$ over Δ^k. We identify S^n with $\mathbb{R}^n \cup \infty$ and take infinity to be the basepoint. There is a map $\mathrm{Q}.(S^0) \to \mathcal{V}.$ given by $(\Delta^k \times S^n \xrightarrow{\phi} S^n) \mapsto (\Delta^k \times \mathbb{R}^n, \phi^{-1}(0), [\phi])$, where $[\phi]$ is the germ of ϕ at $\phi^{-1}(0)$.

Proposition 2.21. *The restriction-germ map* $\mathrm{Q}.(S^0) \to \mathcal{V}.$ *induces a homotopy equivalence on realization.*

Proof. Again, both simplicial sets are Kan and we show that the relative simplicial homotopy groups vanish. Let $(\Delta^k \times \mathbb{R}^n, V, g)$ represent an element in $\mathcal{V}_k$. Let U denote the neighborhood of $V \subset \Delta^k \times \mathbb{R}^n$ on which g is defined. Let $\widetilde{g}_\partial$ be a lift to $\mathrm{Q}.(S^0)$ of g over $\partial\Delta^k$. That is, $\widetilde{g}_\partial$ is a map from $\partial\Delta^k \times S^n$ to $\partial\Delta^k \times S^n$ that agrees with g on U_∂.

We have therefore a partial map $h : \Delta^k \times S^n \to \Delta^k \times S^n$ defined on $\partial\Delta^k \times S^n \cup U \cup \Delta^k \times \{\infty\}$ agreeing with $\widetilde{g}_\partial$ and g. Using standard extension theorems we can construct a partial map $h' : \Delta^k \times S^n \to \Delta^k \times S^n$ defined on a neighborhood of $\partial\Delta^k \times S^n \cup V \cup \Delta^k \times \{\infty\}$ such that h' agrees with h where both are defined.

Denote the projections by $\pi : \Delta^k \times \mathbb{R}^n \to \Delta^k$ and $\rho : \Delta^k \times \mathbb{R}^n \to \mathbb{R}^n$. Given a function $\epsilon : \Delta^k \to \mathbb{R}^+ \cup \{\infty\}$ that sends $\partial\Delta^k$ to $\{\infty\}$, let

$$P_\epsilon = \{x \in \Delta^k \times \mathbb{R}^n \text{ s.t. } |\rho(x)| < \epsilon(\pi(x))\}.$$

This P_ϵ is a neighborhood of $\Delta \times \{0\}$. There exists such a function ϵ so that the map $h' : h'^{-1}(P_\epsilon) \to P_\epsilon$ is proper. Choose a homeomorphism $\alpha : P_\epsilon \to \Delta^k \times \mathbb{R}^n$ that is the identity in a neighborhood of $\Delta \times \{0\} \cup \partial\Delta^k \times \mathbb{R}^n$. Now define

$$h'' : \Delta^k \times S^n \to \Delta^k \times S^n \quad \text{by} \quad h''(x) = \begin{cases} \alpha h'(x) & \text{if } x \in h'^{-1}(P_\epsilon) \\ \infty & \text{otherwise.} \end{cases}$$

Here we again identify S^n with $R^n \cup \infty$. Note that h'' agrees with $\widetilde{g}_\partial$ on $\partial\Delta^k$ and with g on some neighborhood of V, as desired. $\square$

3. The Becker-Gottlieb Transfer and Commutativity

We recall the Becker-Gottlieb transfer $|B.| \xrightarrow{\beta} |\mathrm{Q}.(S^0)|$ from [1, 2, 3], define certain intermediate maps $|B.| \xrightarrow{\theta_b} \Omega|\mathrm{s}.\mathcal{E}.|$ and $|B.| \xrightarrow{\beta_f} |\mathcal{F}.|$, and compare the composition of the A-theory transfer and the A-theory trace with the Becker-Gottlieb transfer.

For a fibration $E \to B$ with ENR_B total space and compact fibres, we define the Becker-Gottlieb transfer $B \to \mathrm{Q}(S^0)$ as follows. (Though the transfer typically has range $\mathrm{Q}(E_+)$, we refer also to the composite $B \to \mathrm{Q}(E_+) \to \mathrm{Q}(S^0)$ as the Becker-Gottlieb transfer.) Let $\phi : E \to B \times \mathbb{R}^n$ be an embedding, and let $r : U \to \phi(E)$ be a neighborhood retraction of $\phi(E) \subset B \times \mathbb{R}^n$. Again denote the projections by $\pi : B \times \mathbb{R}^n \to B$ and $\rho : B \times \mathbb{R}^n \to \mathbb{R}^n$. Given a function $\epsilon : B \to \mathbb{R}^+$, let

$$V_\epsilon = \{x \in U \text{ s.t. } |\rho(x) - \rho(r(x))| < \epsilon(\pi(x))\}.$$

Choose such a function ϵ so that the map

$$\begin{aligned} V_\epsilon &\to D^n \\ x &\mapsto (\rho(x) - \rho(r(x)))/\epsilon(\pi(x)) \end{aligned}$$

is proper, where D^n is the open unit ball in $\mathbb{R}^n$. We now have maps

$$B \times \mathbb{R}^n \xleftarrow{\text{open}} V_\epsilon \xrightarrow{\text{proper}} D^n \cong \mathbb{R}^n$$

with the left map an open embedding and the right map proper. After one-point compactification we have a map $B_+ \wedge S^n \to S^n$ and in particular a map $B \to \mathrm{Q}(S^0)$. This is the Becker-Gottlieb transfer. It depends on the choices of ϕ, r, and ϵ but only up to homotopy. Doing the same construction simplex by simplex clearly gives a simplicial map $B. \xrightarrow{\beta} \mathrm{Q}.(S^0)$.

There is a map $B. \xrightarrow{\beta_f} \mathcal{F}.$ taking a simplex $\sigma \in B_k$ to $(E_\sigma, E_\sigma, \mathrm{id}) \in \mathcal{F}_k$. The choice of an embedding ϕ and a neighborhood retraction r determine a map $B. \to \widetilde{\mathcal{F}}.$ by $\sigma \mapsto (E_\sigma, E_\sigma, \phi, \mathrm{id}, [r])$, where $[r]$ is the germ of the retraction. The diagram

$$\begin{tikzcd}
 & \mathcal{F}. \\
B. \arrow[ur, "\beta_f"] \arrow[dr] & \overline{F.} \arrow[u] \\
 & \widetilde{\mathcal{F}}. \arrow[u]
\end{tikzcd}$$

clearly commutes. Similarly the diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{F}}. \arrow[d] \\
B. \arrow[ur] \arrow[ddr, "\beta"'] & \mathcal{F}^{\mathbb{R}}. \arrow[d] \\
 & \mathcal{V}. \\
 & \mathrm{Q}.(S^0) \arrow[u]
\end{tikzcd}$$

clearly commutes when we make the same choice of ϕ and r for the two diagonal maps. Thus for any choice of ϕ, r, and ϵ and any choice of homotopy inverses to the wrong way maps above, the diagram

$$\begin{tikzcd}
{|B.|} \arrow[r, "\beta_f"] \arrow[dr, "\beta"'] & {|\mathcal{F}.|} \arrow[d] \\
 & {|\mathrm{Q}.(S^0)|}
\end{tikzcd}$$

commutes up to homotopy.

We have another map $|B.| \xrightarrow{\theta_b} \Omega|\mathrm{s}.\mathcal{E}.|$ induced by the simplicial map $B. \to \mathrm{s}_1\mathcal{E}.$ taking $\sigma \in B_k$ to $(\Delta^k \subset \Delta^k \amalg E_\sigma) \in \mathrm{s}_1\mathcal{E}_k$. Moreover, this map lifts canonically to a map $B. \to \widetilde{\mathrm{s}_1\mathcal{E}}.$ taking σ to $(\Delta^k \subset \Delta^k \amalg E_\sigma, \mathrm{id} \times I)$. That is, because $\Delta^k \amalg E_\sigma$ is a disjoint union, the identity map is a filtering homotopy.

Now the diagram

$$\begin{tikzcd}
 & \widetilde{\mathrm{s}_1\mathcal{E}.} \arrow[d, "\tau"] \\
B. \arrow[ur] \arrow[dr, "\beta_f"'] & \mathrm{t}_1\mathcal{F}. \\
 & \mathcal{F}. \arrow[u]
\end{tikzcd}$$

clearly commutes. Thus for any choice of homotopy inverses the diagram

$$\begin{tikzcd}
{|B.|} \arrow[r, "\theta_b"] \arrow[dr, "\beta_f"'] & \Omega|\mathrm{s}.\mathcal{E}.| \arrow[d] \\
 & {|\mathcal{F}.|}
\end{tikzcd}$$

commutes up to homotopy.

We now examine the commutativity of the top rectangular section of Diagram 4. First we define simplicial replacements for the two homotopy colimits. Let $\mathrm{ho}_{\cdot\cdot}(B)$ denote a (bi)simplicial set whose elements in degree (k,l) are pairs (s,t) where s is a k-diagram in $\mathrm{simp}(B.)$ and t is an l-diagram in $(\mathrm{simp}(B.)/s_0)$; that is, an element of $\mathrm{ho}_{(k,l)}(B)$ is a $(k+l)$-diagram in $\mathrm{simp}(B.)$ of the form

$$t_l \to \ldots \to t_0 \rightsquigarrow s_0 \to \ldots \to s_k.$$

(The squiggled arrow distinguishes the end of the l-diagram from the beginning of the k-diagram.) The realization of $\mathrm{ho}_{\cdot\cdot}(B)$ is $\underset{x\in\mathrm{simp}(B.)}{\mathrm{hocolim}}|\mathrm{N}.(\mathrm{simp}(B.)/x)|$. Similarly let $\mathrm{ho}^{A}_{\cdot\cdot}(B)$ denote a simplicial set with elements in degree (k,l) given by pairs (s,T) where s is a k-diagram in $\mathrm{simp}(B.)$ and T is an l-diagram of weak equivalences in $\mathrm{R}^{\mathrm{hf}}(E_{s_0})$. The realization of $\mathrm{ho}^{A}_{\cdot\cdot}(B)$ is $\underset{x\in\mathrm{simp}(B.)}{\mathrm{hocolim}}|\mathrm{N}.w\mathrm{R}^{\mathrm{hf}}(E_x)|$. In terms of the simplicial replacements, the maps in Diagram 4 are given as follows:

$$\begin{aligned}
\mathrm{ho}_{\cdot\cdot}(B) &\to \mathrm{N}.\,\mathrm{simp}(B.)\\
(t_l \to \ldots \to t_0 \rightsquigarrow s_0 \to \ldots \to s_k) &\mapsto (s_0 \to \ldots \to s_k)
\end{aligned}$$

$$\begin{aligned}
\mathrm{ho}_{\cdot\cdot}(B) &\to \mathrm{ho}^{A}_{\cdot\cdot}(B)\\
(t_l \to \ldots \to t_0 \rightsquigarrow s_0 \to \ldots \to s_k) &\mapsto (s_0 \to \ldots \to s_k, E_{t_l} \to \ldots \to E_{t_0})
\end{aligned}$$

$$\begin{aligned}
\mathrm{ho}^{A}_{\cdot\cdot}(B) &\to \mathrm{N}.w\mathrm{R}^{\mathrm{hf}}(E)\\
(s_0 \to \ldots \to s_k, E_{t_l} \to \ldots \to E_{t_0}) &\mapsto (E \amalg E_{t_l} \to \ldots \to E \amalg E_{t_0})
\end{aligned}$$

Also recall Kan's last vertex map $\mathrm{N.simp}(B.) \to (B.)$; this map is given by

$$(s_0 \to s_1 \to \ldots \to s_k) \mapsto (\mathrm{m}(s_0)\,\mathrm{m}(s_1)\ldots \mathrm{m}(s_k))$$

where $\mathrm{m}(S)$ denotes the maximum vertex of S in the vertex ordering of $(B.)$ and $(v_0\, v_1 \ldots v_k)$ denotes the simplex with v_i as vertices. This differs only in notation from the description given in section 1.

We now construct a homotopy inverse to the composition

$$|\mathrm{ho..}(B)| \xrightarrow{\simeq} |\mathrm{N.simp}(B.)| \xrightarrow{\simeq} |B.|$$

Note that neither of the maps in this composition has a simplicial homotopy inverse. (For example, $\mathrm{N.simp}(B.)$ is the barycentric subdivision of $(B.)$ and there is no sensible simplicial map from a simplicial set to its barycentric subdivision.) For simplicity, we assume that $(B.)$ is finite dimensional, that is, that there exists a number N such that there are no nondegenerate simplices of $(B.)$ of dimension greater than N. When $(B.)$ is not finite dimensional our discussion applies to the finite skeleta of $(B.)$; we assume implicitly only that the map $|B.| \xrightarrow{\lambda} |\mathrm{ho..}(B)|$ that we construct on a finite dimensional skeleton can be extended to a homotopy inverse on all of $(B.)$. Let $s = (01\ldots n)$ be a simplex in (B_n) with vertices labeled, in order, $0, 1, \ldots, n$. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ be a point in the standard simplex Δ^n; that is, $\alpha_i \in [0,1]$ and $\sum \alpha_i = 1$. Thus (s, α) labels a point in $|B.|$. The order in $[0,1]$ of the $n+2$ numbers $(1/(N+2), \alpha_0, \alpha_1, \ldots, \alpha_n)$ determines a subdivision of Δ^n. The segments of this subdivision have the form

$$\Lambda = \{\alpha \text{ s.t. } \alpha_{i_0} > \ldots > \alpha_{i_k} > 1/(N+2) > \alpha_{i_{k+1}} > \ldots > \alpha_{i_n}\} \overset{\phi}{\cong} \Delta^k \times \Delta^{n-k}$$

where $(i_0, \ldots, i_n)$ is a fixed permutation of $(0, \ldots, n)$, and ϕ is a homeomorphism that rescales the indicated segment of Δ^n onto the standard product $\Delta^k \times \Delta^{n-k}$. (We assume that ϕ interpolates linearly between the obvious correspondence of vertices.) Given such an α, we now define the homotopy inverse by

$$\begin{aligned}
&|B.| \xrightarrow{\lambda} |\mathrm{ho..}(B)| \\
&(s,\alpha) \mapsto (((i_0) \to (i_0 i_1) \to \ldots \to (i_0 \ldots i_k) \rightsquigarrow \\
&\qquad\qquad (i_0 \ldots i_k) \to (i_0 \ldots i_{k+1}) \to \ldots \to (i_0 \ldots i_n)), \phi(\alpha)).
\end{aligned}$$

Here again a collection $(i_0 \ldots i_l)$ denotes the simplex of (B_l) with vertex set $\{i_0, \ldots, i_l\}$; we allow repetitions in the vertex set, and so the simplex may be degenerate.

Perhaps a picture is in order. The subdivision of Δ^2 looks like the following:

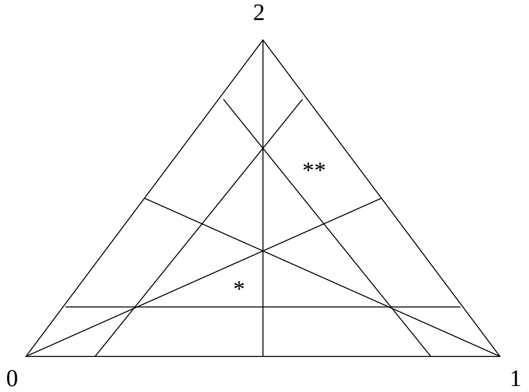

Figure 1. A subdivision of the 2-simplex

The segment labeled $(*)$ is mapped to $(0 \to 01 \to 012 \rightsquigarrow 012)$ in $\mathrm{ho}_{(0,2)}(B)$, while the segment $(**)$ is mapped to $(2 \to 12 \rightsquigarrow 12 \to 012)$ in $\mathrm{ho}_{(1,1)}(B)$. (As the vertex set is ordered, the simplex 12 is equivalent to 21, and 012 to 210.)

We now have two maps $\theta^t, \theta_b : |B.| \to \Omega|\mathrm{N}.w\mathrm{S}.\mathcal{E}.|$ given by going around the top and bottom respectively of the rectangle of Diagram 4. (Here we do not distinguish between the map $\theta_b : |B.| \to \Omega|\mathrm{s}.\mathcal{E}.|$ and the composition $\theta_b : |B.| \to \Omega|\mathrm{s}.\mathcal{E}.| \to \Omega|\mathrm{N}.w\mathrm{S}.\mathcal{E}.|$.) We think of both θ^t and θ_b as maps to $|\mathrm{N}.w\mathrm{S}_1\mathcal{E}.|$ and write them as such. Take $s = (01\ldots n)$ and $\alpha \in \Delta^n$ as above. The maps are given by

$$\theta^t(s,\alpha) = ((* \subset * \amalg E_{i_0}) \to \ldots \to (* \subset * \amalg E_{i_0 i_1 \ldots i_k}), \pi(\phi(\alpha)))$$
$$\theta^b(s,\alpha) = ((\Delta^n \subset \Delta^n \amalg E_{01\ldots n}), \alpha)$$

Here $\pi : \Delta^k \times \Delta^n \to \Delta^k$ denotes projection. Notice that the image of θ^t is in $\mathrm{N}_k w\mathrm{S}_1\mathcal{E}_0$, while the image of θ_b is in $\mathrm{N}_0 w\mathrm{S}_1\mathcal{E}_n$.

We will define a homotopy between these two maps, but first we need a preliminary construction. Suppose $(S_0, S_1, \ldots, S_l)$ is a series of sets of vertices of $(B.)$; we allow repetition both within and between the S_i. As usual, $E_{(S_i)}$ denotes the segment of the fibration $E \to B$ over the simplex (S_i); this space is $\mathrm{ENR}_{\Delta^{|S_i|}}$ and we imagine it as such even if (S_i) is degenerate. We now construct an ENR_{Δ^l} which we will denote

$$E_{S_0} — E_{S_1} — \ldots — E_{S_l}.$$

This ENR_{Δ^l} has vertices $E_{S_0}, E_{S_1}, \ldots, E_{S_l}$; that is, these are the inverse images of the vertices of Δ^l. The ENR_{Δ^l} is given by the composition

$$E_{S_0 \amalg \ldots \amalg S_l} \to \Delta^{|S_0| + \ldots + |S_l|} \xrightarrow{\mu} \Delta^l$$

where μ takes a vertex p to the vertex i if p corresponds to an element in S_i. Though it is somewhat surprising, this is an ENR_{Δ^l}.

Now recall that θ^t took a point

$$(01\dots n,\alpha)\in\Lambda=\{\alpha_{i_0}>\dots>\alpha_{i_k}>1/(N+2)>\alpha_{i_{k+1}}>\dots>\alpha_{i_n}\}$$

to a point in

$$(*\subset *\amalg E_{i_0})\to(*\subset *\amalg E_{i_0i_1})\to\dots\to(*\subset *\amalg E_{i_0i_1\dots i_k}),$$

which for simplicity we will now abbreviate to

$$E_{i_0}\to E_{i_0i_1}\to\dots\to E_{i_0i_1\dots i_k}.$$

(All constructions from now on are understood to apply also to the '$*\subset *$' segment of the ENR; similar abbreviation will be applied to 'ENR_{Δ^l}'s for any l.) Consider the element $\mathfrak{E}\in N_k w\mathrm{S}_1 E_k$ given by

$$\begin{tikzcd}
E_{i_0} \arrow[r, dash] \arrow[d] & E_{\mathrm{m}(i_0i_1)} \arrow[r, dash] \arrow[d] & E_{\mathrm{m}(i_0i_1i_2)} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & E_{\mathrm{m}(i_0i_1i_2\dots i_k)} \arrow[d] \\
E_{i_0i_1} \arrow[r, dash] \arrow[d] & E_{i_0i_1} \arrow[r, dash] \arrow[d] & E_{i_0i_1\cup\mathrm{m}(i_0i_1i_2)} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & E_{i_0i_1\cup\mathrm{m}(i_0i_1i_2\dots i_k)} \arrow[d] \\
E_{i_0i_1i_2} \arrow[r, dash] \arrow[d] & E_{i_0i_1i_2} \arrow[r, dash] \arrow[d] & E_{i_0i_1i_2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & E_{i_0i_1i_2\cup\mathrm{m}(i_0i_1i_2\dots i_k)} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\
E_{i_0i_1i_2\dots i_k} \arrow[r, dash] & E_{i_0i_1i_2\dots i_k} \arrow[r, dash] & E_{i_0i_1i_2\dots i_k} \arrow[r, dash] & \cdots \arrow[r, dash] & E_{i_0i_1i_2\dots i_k}
\end{tikzcd}$$

In the realization this element is a $\Delta^k\times\Delta^k$. Consider the two faces

$$\begin{aligned}\mathfrak{E}_1&=E_{i_0}\to E_{i_0i_1}\to\dots\to E_{i_0i_1\dots i_k}\\ \text{and }\ \mathfrak{E}_2&=E_{\mathrm{m}(i_0)}-E_{\mathrm{m}(i_0i_1)}-\dots-E_{\mathrm{m}(i_0i_1\dots i_k)}.\end{aligned}$$

These elements $\mathfrak{E}_1\in\mathrm{N}_k w\mathrm{S}_1\mathcal{E}_0$ and $\mathfrak{E}_2\in\mathrm{N}_0 w\mathrm{S}_1\mathcal{E}_k$ determine respectively maps $\mathfrak{E}_1,\mathfrak{E}_2:\Delta^k\to|N_k w\mathrm{S}_1 E_k|$. Furthermore, the element $\mathfrak{E}$ determines a linear homotopy between $\mathfrak{E}_1$ and $\mathfrak{E}_2$.

Notice that by definition the diagram

$$\begin{tikzcd}
\Lambda \arrow[r, hook] \arrow[d, "\phi"'] & \Delta^n \arrow[r, "\theta^t"] & {|\mathrm{N}.w\mathrm{S}_1\mathcal{E}.|} \\
\Delta^k\times\Delta^{n-k} \arrow[rr, "\pi"] & & \Delta^k \arrow[u, "\mathfrak{E}_1"']
\end{tikzcd}$$

commutes. Informally, the map θ^t restricted to Λ "just is" $\mathfrak{E}_1$, pointwise in Δ^{n-k}. The homotopies determined by the $\mathfrak{E}$ (for various Λ) fit together to give a homotopy

$$\Delta^n \times I \xrightarrow{\mathfrak{E}} |\mathrm{N}.w\mathrm{S}_1\mathcal{E}.|$$

from θ^t to a map

$$\begin{aligned} \Delta^n_{(s)} &\xrightarrow{\mathfrak{E}_2} |\mathrm{N}.w\mathrm{S}_1\mathcal{E}.| \\ (s,\alpha) &\mapsto (E_{\mathrm{m}(i_0)} \text{ — } \dots \text{ — } E_{\mathrm{m}(i_0\dots i_k)}, \pi(\phi(\alpha))). \end{aligned}$$

The simplex $(E_{\mathrm{m}(i_0)} \text{ — } \dots \text{ — } E_{\mathrm{m}(i_0\dots i_k)})$ is a possibly degenerate subface of $(E_0 \text{ — } \dots \text{ — } E_n) = E_{0\dots n}$. There is therefore an obvious linear homotopy between $\mathfrak{E}_2$ and θ_b. The composition of this homotopy with $\mathfrak{E}$ gives a homotopy between $\theta^t|\Delta^n_{(s)}$ and $\theta_b|\Delta^n_{(s)}$. The homotopies so constructed for two simplices s and t of $(B.)$ agree on the intersection $s \cap t$. We have thus defined a homotopy between θ^t and θ_b, as desired.

To summarize, we have shown that the diagram

$$\begin{tikzcd} & \mathrm{A}(*) \arrow[d] \\ B \arrow[ur] \arrow[r] & \Omega|\mathrm{s}.\mathcal{E}.| \end{tikzcd}$$

is homotopy commutative on finite skeleta of $B = |B.|$, and is therefore weakly homotopy commutative. This completes the proof of the following.

Theorem 3.1. *Let $E \to B$ be a fibration with compact fibres. Assume the total space E is a Euclidean neighborhood retract over B; (this is always true, for example, if the fibration is locally trivial and the fibre is a finite CW complex, or if both E and B are finite CW complexes). Then the composition of the A-theory transfer with the trace map to stable homotopy is weakly homotopic to the Becker-Gottlieb transfer. That is, the diagram*

$$\begin{tikzcd} & \mathrm{A}(*) \arrow[d] \\ B \arrow[ur] \arrow[r] & \mathrm{Q}(S^0) \end{tikzcd}$$

commutes up to weak homotopy.

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