Vortex pinning by inhomogeneities in type-II superconductors

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Abstract

The methods of formal matched asymptotics are used to examine the motion of a curvilinear vortex in an inhomogeneous type-II superconducting material in the limit as the vortex core radius tends to zero. The resulting law of motion indicates that the logarithm of the equilibrium density of the superconducting electrons acts as a pinning potential for the vortex, so that vortices will be attracted by impurities in the superconducting material.

Key words. superconductivity, vortex, pinning, impurity, Ginzburg-Landau

AMS subject classifications. 82D55

1 Introduction

Type-II superconductors are characterised by the appearance of thin filaments of normally conducting (normal) material in the superconducting matrix, each carrying a quantized amount of magnetic flux, and circled by a supercurrent vortex. These vortices will move in response to an electric current, whether the current is applied externally, generated by an applied magnetic field or another vortex, or self-induced. In applications it is crucial to understand this motion, since motion causes dissipation of energy and leads to an electrical resistance.

In practice attempts are made to pin vortices by introducing impurities into the material. Impurities act to lower the local equilibrium density of superconducting electrons a which lowers the energy penalty associated with the vortex core; thus it is energetically more favourable for a vortex to lie through an impurity.

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One approach to the modelling of vortex pinning is to include a "pinning potential" in the law of motion of individual vortices, which acts to draw the vortices towards the potential wells, which are the pinning sites.

In this paper we aim to study more carefully the effects on vortex motion of variations in the equilibrium density of superconducting electrons. Our starting point will be the time-dependent Ginzburg-Landau theory written down by Schmid and Gor'kov & Eliashberg [10, 19], modified to allow the equilibrium density of superconducting electrons to be a function of position. In their dimensional form these equations contain two important lengthscales, namely the penetration depth, λ, which is the typical lengthscale for variations in magnetic field, and the coherence length, ξ, which is the typical lengthscale for variations in the number density of superconducting electrons, and is therefore the vortex core radius. The ratio of these lengthscales, $\kappa = \lambda/\xi$, is known as the Ginzburg-Landau parameter. For type-II superconducting materials this parameter must be greater than $1/\sqrt{2}$. Typically it is much larger than this, taking values around 5-30 for low-temperature superconducting materials and up to 100 for high-temperature superconducting materials. Here we will be interested in the limit in which $\kappa \to \infty$. Thus, when lengths are nondimensionalized with repsect to λ, the nondimensional vortex core radius $\epsilon = 1/\kappa \to 0$.

In numerical simulations of these equations vortices are found to be attracted to the places with the lowest equilibrium densities of superconducting electrons [2].

The result of our asymptotic analysis will be a law of motion for well-separated vortices. We will find that the variation in equilibrium density does indeed act as a pinning potential, with the velocity of a vortex being given by

$$\left(\frac{\beta}{\log \kappa}\right) \mathbf{v} = C\mathbf{n} - \nabla \log a \quad a > 0, \tag{1}$$

where C is the curvature of the vortex, $\mathbf{n}$ is its unit normal, and β is an order-one function of the normal conductivity.

A law of motion for vortices with radii of curvature much greater than the penetration depth λ in materials for which a =constant, was first derived by Gor'kov and Kopnin [11]. Their analysis has been put into a more formal mathematical framework by Peres and Rubinstein [17], using techniques developed by Neu [16]. Recent extensions of this work include the derivation of the law of motion for vortices with small radii of curvature [3, 8], a derivation of the law of motion for vortices in anisotropic materials [4, 9] and a derivation of the law of motion when the time-dependent Ginzburg-Landau model has a complex relaxation parameter for ψ, which gives rise to vortex motions with non-zero Hall effect [7, 12]. In pricipal it is possible to extend the result we obtain (1) to both anisotropic superconductors and complex relaxation parameters by combining the analysis contained herein with that in either of [4, 9] or [12, 7].

2 Derivation of the law of motion

The starting point for our analysis is the time-dependent Ginzburg-Landau theory. For a more complete introduction to this theory, the reader is referred to [1] and [6]. Here we merely state the dimensionless equations as

$$\frac{\partial \Psi}{\partial t} + i\kappa\Psi\Phi + \kappa^2\Psi(|\Psi|^2 - a(\mathbf{x})) - (\nabla - i\kappa\mathbf{A})^2\Psi = 0, \tag{2}$$

$$-\nabla \wedge (\nabla \wedge \mathbf{A}) = \sigma(\mathbf{x})\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\Phi\right) + \frac{i}{2\kappa}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + |\Psi|^2\mathbf{A}, \tag{3}$$

Here Ψ, the complex superconducting order parameter, is such that $|\Psi|^2$ is proportional to the number density of superconducting electrons, and $\mathbf{A}$ and Φ, respectively the magnetic vector potential and electric scalar potential, are defines as follows:

$$\mathbf{H} = \nabla \wedge \mathbf{A}, \tag{4}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi, \tag{5}$$

where $\mathbf{H}$ is the magnetic field and $\mathbf{E}$ is the electric field. The vector potential $\mathbf{A}$ is unique up to the addition of a gradient; once $\mathbf{A}$ is given, Φ is unique up to the addition of a function of t. Here σ is the (dimensioness) conductivity of the normal electrons in the material and a is the (dimensionless) equilibrium density of superconducting electrons, both of which are allowed to vary spatially; κ is the Ginzburg-Landau parameter, and $*$ represents complex conjugation.

Lengths in (2)-(3) have been nondimensionalized with λ (so that both a and σ are supposed to vary on this lengthscale), while time has been nondimensionalized with the timescale associated with the relaxation of Ψ (so that the coefficient of $\partial\Psi/\partial t$ in (2) is unity). For constant a this model was formulated independently by Schmid [19] and Gor'kov & Eliashberg [10]. In the steady state the variable a model was written down by Likharev [15]; see also [2]. From a microscopic point of view, a position dependent coupling was first considered by Larkin & Ovchinnikov [14], which results in a position dependent a when the methods of Gor'kov & Eliashberg are used to derive Ginzburg-Landau-type equations.

Equations (2)-(3) are gauge invariant in the sense that they are invariant under transformations of the form

$$\mathbf{A} \to \mathbf{A} + \nabla w, \qquad \Phi \to \Phi - \frac{\partial w}{\partial t}, \qquad \Psi \to \Psi e^{i\kappa w}. \tag{6}$$

We may take advantage of this invariance to write the equations in terms of real variables by introducing the new potentials

$$\mathbf{Q} = \mathbf{A} - \frac{1}{\kappa}\nabla\chi, \tag{7}$$

$$\Theta = \Phi + \frac{1}{\kappa}\frac{\partial \chi}{\partial t}, \tag{8}$$

where $\Psi = fe^{i\chi}$ with f and χ real, giving

$$-\frac{\partial f}{\partial t} + \nabla^2 f = \frac{1}{\epsilon^2}\left(f^3 - a(\mathbf{x})f + f|\mathbf{Q}|^2\right), \tag{9}$$

$$f^2\Theta + \nabla \cdot \left(f^2\mathbf{Q}\right) = 0, \tag{10}$$

$$
\begin{aligned}
-\nabla \wedge \mathbf{H} &= f^2\mathbf{Q} + \sigma(\mathbf{x})\epsilon^2 \left(\frac{\partial \mathbf{Q}}{\partial t} + \nabla\Theta\right), && (11)\\
\mathbf{H} &= \nabla \wedge \mathbf{Q}, && (12)
\end{aligned}
$$

where $\epsilon = 1/\kappa$. We also find it convenient when performing the asymptotic analysis to take the divergence of (11) to give

$$
\nabla \cdot (f^2\mathbf{Q}) + \sigma\epsilon^2 \left(\frac{\partial(\nabla\cdot\mathbf{Q})}{\partial t} + \nabla^2\Theta\right) + \epsilon^2\nabla\sigma \cdot \left(\frac{\partial \mathbf{Q}}{\partial t} + \nabla\Theta\right) = 0. \qquad (13)
$$

A vortex solution of (9)-(12) corresponds to a curve on which $f = 0$, and about which χ varies by $2n\pi$, where $n \in \mathbf{Z}^+$ is known as the vortex number and is the number of flux quanta carried by the vortex. Here we consider only the case $n = 1$, which is thought to be the only stable vortex solution, but an exactly similar analysis holds for arbitrary n.

The definition (7) of $\mathbf{Q}$ implies that

$$
\mathbf{Q} \sim -\frac{1}{\kappa r}\mathbf{e}_\theta, \qquad \text{as} \qquad r \to 0, \qquad (14)
$$

where r and θ are local polar coordinates centered on the vortex.

We consider the limit $\epsilon \to 0$ for a system of curvilinear vortices which are separated by distances of order one. Our methodology is as follows. We begin by formulating the leading-order outer problem, away from the vortex cores. We then perform a local analysis about each core to derive the leading-order inner solution. The outer limit of this inner solution provides a matching condition for the outer solution, which must have the correct singular behaviour as the vortex is approached. We then need to solve the leading-order outer problem subject to this matching condition, and determine the next term in its expansion as the vortex is approached. This term then gives a matching condition on the first-order inner problem. Finally, by applying the Fredholm alternative to the first-order inner problem we can derive a solvability condition, which will give the law of motion of the vortex.

2.1 The far-field equations

In the constant a case examined in [3] it was found that the magnetic field generated by a single vortex was $O(\epsilon)$ in this scaling. The variations in a do not affect this result, which motivates the following outer expansions, denoted by the subscript o, away from vortex cores:

$$
\begin{aligned}
f_o &= f_o^{(0)} + \cdots,\\
\mathbf{Q}_o &= \epsilon\mathbf{Q}_o^{(0)} + \cdots,\\
\mathbf{H}_o &= \epsilon\mathbf{H}_o^{(0)} + \cdots,\\
\Theta_o &= \epsilon\Theta_o^{(0)} + \cdots.
\end{aligned}
$$

Substituting the above expansion into (9)-(7) we obtain the leading-order outer problem

$$
f_o = \sqrt{a(\mathbf{x})} + O(\epsilon^2) \qquad (15)
$$

$$
\begin{aligned}
\nabla \wedge \mathbf{H}_o^{(0)} &= -a(\mathbf{x})\mathbf{Q}_o^{(0)}, && (16)\\
\nabla \wedge \mathbf{Q}_o^{(0)} &= \mathbf{H}_o^{(0)} && (17)\\
\Theta_o^{(0)} &= -\frac{1}{a(\mathbf{x})}\nabla \cdot \left(a(\mathbf{x})\mathbf{Q}_o^{(0)}\right) = 0. && (18)
\end{aligned}
$$

To determine the singularity in the solution of (15)-(18) as the vortex is approached we need to consider the inner solution in the vicinity of one of the vortex cores.

2.2 The leading-order inner solution

Let the vortex lie along the curve Γ, given by $\mathbf{x} = (x, y, z) = \mathbf{q}(s, t)$. We define a local coordinate system (s, ρ, θ, T) by

$$
\begin{aligned}
\mathbf{x} &= \mathbf{q}\,(s,t) + \rho\cos\theta\,\mathbf{n}\,(s,t) + \rho\sin\theta\,\mathbf{b}\,(s,t)\,, && (19)\\
t &= T, && (20)
\end{aligned}
$$

where s is arclength, $\mathbf{n}\,(s,t)$ the unit normal and $\mathbf{b}\,(s,t)$ the unit binormal to the curve Γ. We define the inner coordinates by introducing the stretched variable R, defined by $\rho = \epsilon R$. The coordinate system so defined is not orthogonal away from the line $\mathbf{q}(s,t)$ when the torsion of the vortex is non-zero. However it is approximately orthogonal and the following expressions for ∇g, $\nabla \cdot \mathbf{B}$, $\nabla^2 g$, $\nabla \wedge \mathbf{B}$ and $\partial/\partial t$ in the inner region are obtained in [18]:

$$
\nabla g = \frac{\partial g}{\partial s}\mathbf{e_s} + \frac{1}{\epsilon}\frac{\partial g}{\partial R}\mathbf{e_R} + \frac{1}{\epsilon R}\frac{\partial g}{\partial \theta}\mathbf{e}_\theta + O(\epsilon), \qquad (21)
$$

$$
\nabla \cdot \mathbf{B} = \frac{1}{\epsilon R}\left(\frac{\partial\,(RB_R)}{\partial R} + \frac{\partial B_\theta}{\partial \theta}\right) - C\,(B_R\cos\theta - B_\theta\sin\theta) + \frac{\partial B_s}{\partial s} + \tau\frac{\partial B_s}{\partial \theta} + O(\epsilon), \qquad (22)
$$

$$
\nabla^2 g = \frac{1}{\epsilon^2 R}\left(\frac{\partial}{\partial R}\left(R\frac{\partial g}{\partial R}\right) + \frac{1}{R}\frac{\partial^2 g}{\partial \theta^2}\right) + \frac{C}{\epsilon}\left(\frac{1}{R}\frac{\partial g}{\partial \theta}\sin\theta - \frac{\partial g}{\partial R}\cos\theta\right) + O(1), \qquad (23)
$$

$$
\begin{aligned}
\nabla \wedge \mathbf{B} = {} & \frac{1}{\epsilon}\left[\frac{1}{R}\left(\frac{\partial}{\partial R}(RB_\theta) - \frac{\partial B_R}{\partial \theta}\right)\mathbf{e_s} + \frac{1}{R}\frac{\partial B_s}{\partial \theta}\mathbf{e_R} - \frac{\partial B_s}{\partial R}\mathbf{e}_\theta\right] \\
& + \left[\tau\left(\frac{\partial B_R}{\partial \theta} - \frac{\partial}{\partial R}(RB_\theta)\right) + C\cos\theta B_s + \frac{\partial B_R}{\partial s}\right]\mathbf{e}_\theta \\
& + \tau R\frac{\partial B_s}{\partial R}\mathbf{e_s} + \left(C\sin\theta B_s - \frac{\partial B_\theta}{\partial s}\right)\mathbf{e_R} + O(\epsilon), \qquad (24)
\end{aligned}
$$

$$
\frac{\partial}{\partial t} = -\frac{1}{\epsilon}\left((\mathbf{v}\cdot\mathbf{e_R})\frac{\partial}{\partial R} + (\mathbf{v}\cdot\mathbf{e}_\theta)\frac{1}{R}\frac{\partial}{\partial \theta}\right) + O(1), \qquad (25)
$$

where C is the curvature, τ is the torsion and $\mathbf{v}$ is the velocity of the vortex line (for definitions of the curvature and torsion see for example [13]).

Since in the inner variables the vortex lies along the line $R = 0$, we have the following boundary conditions at the origin:

$$
\begin{aligned}
\chi &\sim \theta && \text{as} \quad R \to 0,\\
f &= 0 && \text{on} \quad R = 0,
\end{aligned}
$$

which, as we remarked previously, imply the following behaviour for $\mathbf{Q}$ from (7):

$$\mathbf{Q} \sim -\frac{\mathbf{e}_\theta}{R} \quad \text{as} \quad R \to 0. \tag{26}$$

In the light of (26) we make the following ansatz for the leading-order behaviour of the inner solution, which we denote by the subscript i:

$$\begin{aligned} \mathbf{Q}_i &\sim Q_{i,\theta}^{(0)}(R,s)\mathbf{e}_\theta + \cdots, &(27)\\ f_i &\sim f_i^{(0)}(R,s) + \cdots. &(28) \end{aligned}$$

Substituting (27) and (28) into (9)-(12), using the expansion for the vector operators in the inner region given in (21)-(24), we obtain the leading-order inner equations

$$\begin{aligned} \mathbf{Q}_i^{(0)} &= -\frac{1}{R}\mathbf{e}_\theta, &(29)\\ \frac{\partial^2 f_i^{(0)}}{\partial R^2} + \frac{1}{R}\frac{\partial f_i^{(0)}}{\partial R} - \frac{f_i^{(0)}}{R^2} &= {f_i^{(0)}}^3 - a_0(s) f_i^{(0)}. &(30) \end{aligned}$$

Here we have expanded a in terms of the inner coordinates

$$a = a_0(s) + \epsilon A_1(s) R\cos\theta + \epsilon A_2(s) R \sin\theta + O(\epsilon^2), \tag{31}$$

where

$$A_1(s) = \left(\mathbf{n}(s)\cdot \nabla a|_{\mathbf{x}=\mathbf{q}(s)}\right), \qquad A_2(s) = \left(\mathbf{b}(s)\cdot \nabla a|_{\mathbf{x}=\mathbf{q}(s)}\right), \tag{32}$$

and $a_0(s) = a(\boldsymbol{q}(s))$. To match with the outer solution (15) we must have

$$f_i^{(0)} \to \sqrt{a_0(s)} \quad \text{as} \quad R \to \infty. \tag{33}$$

Writing

$$f_i^{(0)} = \sqrt{a_0(s)}\,\overline{f}(\sqrt{a_0(s)}R), \tag{34}$$

we find $\overline{f}$ satisfies

$$\begin{aligned} \overline{f}'' + \frac{\overline{f}'}{\zeta} - \frac{\overline{f}}{\zeta^2} &= \overline{f}^3 - \overline{f}, &(35)\\ \overline{f}(0) &= 0, &(36)\\ \overline{f}(\zeta) &\to 1 \quad \text{as} \quad \zeta \to \infty, &(37) \end{aligned}$$

where $\zeta = \sqrt{a_0(s)}R$ and $'$ represents $d/d\zeta$. Hence $\overline{f}$ is simply the solution corresponding to an isotropic vortex; the existence and uniqueness of a solution has been shown in [5].

This completes the leading-order inner solution for f and $\mathbf{Q}$. We now return to the outer problem.

2.3 The singularities induced by a vortex in the far-field equations

Matching (29) to the leading order outer solution implies

$$\mathbf{Q}_o^{(0)} \sim -\frac{\mathbf{e}_\theta}{\rho} \quad \text{as} \quad \rho \to 0. \tag{38}$$

Combining this with equation (16) gives the asymptotic behaviour of $\mathbf{H}_o^{(0)}$ as

$$\mathbf{H}_o^{(0)} \sim -\log \rho \mathbf{e_s} \quad \text{as} \quad \rho \to 0. \tag{39}$$

The procedure we adopted in [3] was to eliminate $\mathbf{Q}_o^{(0)}$ from (16) and (17), and observe that imposing the asymptotic behaviour (39) was equivalent to inserting a δ-function on the right hand side of the resulting equation. The Greens function could then be used to give a Biot-Savart integral for the solution, which could be asymptotically expanded as $\rho \to 0$ and matched with the inner solution.

Here, as in [4], such an approach is not amenable because of the coupling of the components of $\mathbf{H}_0^{(0)}$ through $a(\mathbf{x})$. However, by expanding the outer equations in terms of the inner variables, we can determine at least the singular terms in the inner limit of the outer expansion, which will be enough to deduce the law of motion for the vortex.

We write $\mathbf{H}_o^{(0)}$ and $\mathbf{Q}_o^{(0)}$ in terms of inner variables, substitute into (16)-(18), and expand in ϵ using the expansions of the vector operators given in equations (21)-(24). We find $\mathbf{Q}_o^{(0)}$ should be expanded as follows:

$$\mathbf{Q}_o^{(0)} = -\frac{\mathbf{e}_\theta}{R} + \log \epsilon \mathbf{q_1} + \mathbf{q_2} + \cdots. \tag{40}$$

Substituting into (17) we find

$$\frac{1}{R}\left(\frac{\partial}{\partial R}(Rq_{1,\theta}) - \frac{\partial q_{1,R}}{\partial \theta}\right) = \frac{\partial q_{1,s}}{\partial \theta} = \frac{\partial q_{1,s}}{\partial R} = 0, \tag{41}$$

$$\frac{1}{R}\left(\frac{\partial}{\partial R}(Rq_{2,\theta}) - \frac{\partial q_{2,R}}{\partial \theta}\right) = \frac{\partial q_{2,s}}{\partial \theta} = \frac{\partial q_{2,s}}{\partial R} = 0, \tag{42}$$

and may thus define scalar potentials for $\mathbf{q_1}$ and $\mathbf{q_2}$

$$\begin{aligned} q_{1,R} &= \frac{\partial \mu_1}{\partial R}, \quad q_{1,\theta} = \frac{1}{R}\frac{\partial \mu_1}{\partial \theta}, \quad q_{1,s} = q_{1,s}(s), \\ q_{2,R} &= \frac{\partial \mu_2}{\partial R}, \quad q_{2,\theta} = \frac{1}{R}\frac{\partial \mu_2}{\partial \theta}, \quad q_{2,s} = q_{2,s}(s). \end{aligned} \tag{43}$$

Substituting (40), (43) into (18) we obtain the following relations for μ_1 and μ_2

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \mu_1}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \mu_1}{\partial \theta^2} = 0, \tag{44}$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \mu_2}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \mu_2}{\partial \theta^2} - \frac{C\sin\theta}{R} + \frac{A_1 \sin\theta}{a_0 R} - \frac{A_2\cos\theta}{a_0 R} = 0. \tag{45}$$

Looking at the latter of these two equations we see that

$$\mu_2 = \frac{C}{2} R \log R \sin\theta - \frac{A_1}{2a_0} R \log R \sin\theta + \frac{A_2}{2a_0} R \log R \cos\theta + \tilde{\mu}_2, \tag{46}$$

where $\tilde{\mu}_2$ satisfies

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \tilde{\mu}_2}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \tilde{\mu}_2}{\partial \theta^2} \;=\; 0,$$

and leads to no further singular terms in $\mathbf{Q}_o^{(0)}$. We can now determine μ_1 from the condition that $\mathbf{Q}_o^{(0)}$ is a function of ϵ only through $\rho = \epsilon R$. The result is

$$\begin{aligned} \mathbf{Q}_o^{(0)} \;\sim\; & -\frac{\mathbf{e}_\theta}{\rho} + \frac{C}{2}\log\rho\,\mathbf{b}(s) - \frac{A_1}{2a_0}\log\rho\,\mathbf{b}(s) \\ & + \frac{A_2}{2a_0}\log\rho\,\mathbf{n}(s) + \cdots \quad \text{as} \quad \rho \to 0. \end{aligned} \tag{47}$$

We have thus been able to calculate two terms in the inner expansion of the first term in the outer expansion of $\mathbf{Q}$. We now need to match this with one term in the outer expansion of two terms in the inner expansion of $\mathbf{Q}$.

2.4 The first-order inner solution

The asymptotic behaviour of $\mathbf{Q}_o^{(0)}$ given in (47) indicates that the inner expansion should proceed as follows:

$$f_i \;=\; \sqrt{a_0}\,\overline{f}(\sqrt{a_0}R) + \epsilon\log\epsilon\, f_i^{(1)} + \cdots \tag{48}$$

$$\mathbf{Q}_i \;=\; -\frac{1}{R}\mathbf{e}_\theta + \epsilon\log\epsilon\,\mathbf{Q}_i^{(1)} + \cdots, \tag{49}$$

$$\mathbf{H}_i \;=\; \epsilon\log\epsilon\,\mathbf{H}_i^{(0)} + \cdots, \tag{50}$$

$$\Theta_i \;=\; \log\epsilon\,\Theta_i^{(0)} + \cdots, \tag{51}$$

$$\mathbf{v}_i \;=\; \mathbf{v}^{(0)}\log\epsilon + \cdots. \tag{52}$$

Expanding equations (9), (10) and (13) in inner coordinates and substituting in the expansions (48)-(52) we find at $O(\log\epsilon/\epsilon)$

$$\begin{aligned} (\mathbf{v}^{(0)}\cdot\mathbf{e}_{\mathbf{R}})a_0\overline{f}'(\sqrt{a_0}R) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f_i^{(1)}}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f_i^{(1)}}{\partial\theta^2} = & \\ 3a_0\overline{f}^2 f_i^{(1)} - a_0 f_i^{(1)} - \frac{2\sqrt{a_0}\overline{f}Q_{i,\theta}^{(1)}}{R} + \frac{f_i^{(1)}}{R^2}, & \end{aligned} \tag{53}$$

$$\overline{f}^2\Theta_i^{(0)} - \frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{2\overline{f}f_i^{(1)}}{\sqrt{a_0}R}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\overline{f}^2 Q_{i,R}^{(1)}\right) + \frac{1}{R}\frac{\partial}{\partial\theta}\left(\overline{f}^2 Q_{i,\theta}^{(1)}\right) \;=\; 0, \tag{54}$$

$$-\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{2\sqrt{a_0}\overline{f}f_i^{(1)}}{R}\right)+\frac{1}{R}\frac{\partial}{\partial R}\left(Ra_0\overline{f}^2Q_{i,R}^{(1)}\right)+\frac{1}{R}\frac{\partial}{\partial\theta}\left(a_0\overline{f}^2Q_{i,\theta}^{(1)}\right)$$

$$+\sigma_0\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Theta_i^{(0)}}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\Theta_i^{(0)}}{\partial\theta^2}\right)\quad=\quad 0, \tag{55}$$

where we have expanded σ in terms of inner variables

$$\begin{aligned}\sigma=\sigma_0(s)\quad&+\quad\epsilon\left(\mathbf{n}(s)\cdot\nabla\sigma|_{\mathbf{x}=\mathbf{q}(s)}\right)R\cos\theta\\&+\quad\epsilon\left(\mathbf{b}(s)\cdot\nabla\sigma|_{\mathbf{x}=\mathbf{q}(s)}\right)R\sin\theta+O(\epsilon^2),\end{aligned}\tag{56}$$

and where $\sigma_0(s)=\sigma(\boldsymbol{q}(s))$. From (54) and (55) we find

$$\overline{f}^2\Theta_i^{(0)}=\frac{\sigma_0}{a_0}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Theta_i^{(0)}}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\Theta_i^{(0)}}{\partial\theta^2}\right). \tag{57}$$

Substituting (49), (50) into (12) results in the following equations:

$$\frac{1}{R}\left(\frac{\partial}{\partial R}\left(RQ_{i,\theta}^{(1)}\right)-\frac{\partial Q_{i,R}^{(1)}}{\partial\theta}\right)=0,$$

$$\frac{\partial Q_{i,s}^{(1)}}{\partial\theta}=\frac{\partial Q_{i,s}^{(1)}}{\partial R}=0.$$

Again we define a scalar potential

$$Q_{i,R}^{(1)}=\frac{\partial\psi}{\partial R},\quad Q_{i,\theta}^{(1)}=\frac{1}{R}\frac{\partial\psi}{\partial\theta},\quad Q_{i,s}^{(1)}=Q_{i,s}^{(1)}(s).$$

Matching this second term in the inner expansion to the leading order outer solution we see that

$$\left.\begin{aligned}R\Theta_i^{(0)}\to 0\\ \psi\sim\frac{C}{2}R\sin\theta-\frac{A_1}{2a_0}R\sin\theta+\frac{A_2}{2a_0}R\cos\theta\end{aligned}\right\}\quad\text{as } R\to\infty,$$

whilst at the origin we require

$$\left.\begin{aligned}\Theta_i^{(0)}\sim-\frac{1}{R}\left(\mathbf{v^{(0)}}\cdot\mathbf{e}_\theta\right)\\ \nabla\psi\quad\text{bounded}\end{aligned}\right\}\quad\text{as } R\to 0.$$

Motivated by these boundary conditions we seek a solution to (53)-(57) of the form

$$f_i^{(1)}=g_1(\sqrt{a_0}R,s)\cos\theta+g_2(\sqrt{a_0}R,s)\cos(\theta-\alpha), \tag{58}$$

$$\psi_1=\frac{1}{\sqrt{a_0}}\phi_1(\sqrt{a_0}R,s)\sin\theta+\frac{1}{\sqrt{a_0}}\phi_2(\sqrt{a_0}R,s)\sin(\theta-\alpha), \tag{59}$$

$$\Theta_i^{(0)}=\sqrt{a_0}U_0\eta(\sqrt{a_0}R,s)\sin\theta+\sqrt{a_0}U_1\eta(\sqrt{a_0}R,s)\sin(\theta-\alpha), \tag{60}$$

$$\mathbf{v^{(0)}}=U_0\mathbf{n}+U_1\left(\cos\alpha\mathbf{n}+\sin\alpha\mathbf{b}\right), \tag{61}$$

where

$$\cos\alpha = \frac{A_1}{\sqrt{A_1^2+A_2^2}}, \qquad \sin\alpha = \frac{A_2}{\sqrt{A_1^2+A_2^2}}.$$

Substituting (58)-(61) into (53), (55) and (57) we find the following coupled system:

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial g_1}{\partial\zeta}\right) - 3\overline{f}^2 g_1 + g_1 = -U_0\frac{d\overline{f}}{d\zeta} + \frac{2g_1}{\zeta^2} - \frac{2\overline{f}}{\zeta^2}\phi_1, \tag{62}$$

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\overline{f}^2\frac{\partial\phi_1}{\partial\zeta}\right) + \sigma_0 U_0\frac{\partial}{\partial\zeta}\left(\frac{\partial\eta}{\partial\zeta}+\frac{\eta}{\zeta}\right) - \frac{\overline{f}^2\phi_1}{\zeta^2} + \frac{2\overline{f}g_1}{\zeta^2} = 0, \tag{63}$$

$$\frac{\partial^2\eta}{\partial\zeta^2} + \frac{1}{\zeta}\frac{\partial\eta}{\partial\zeta} - \frac{\eta}{\zeta^2} - \frac{\overline{f}^2\eta}{\sigma_0} = 0, \tag{64}$$

with boundary conditions

$$\left.\begin{array}{l} \phi_1 \to \dfrac{C\zeta}{2} \\ \zeta g_1 \to 0 \\ \zeta\eta \to 0 \end{array}\right\} \text{ as } \zeta\to\infty, \qquad \left.\begin{array}{c} \phi_1 \to 0 \\ g_1 \text{ bounded} \\ \eta \sim \dfrac{1}{\zeta} \end{array}\right\} \text{ as } \zeta \to 0.$$

The system of equations we obtain for (ϕ_2, g_2) are identical to (62) and (63), with ϕ_1 replaced by ϕ_2, g_1 by g_2 and U_0 by U_1 but now satisfy the boundary conditions

$$\left.\begin{array}{l} \phi_2 \to -\dfrac{\sqrt{A_1^2+A_2^2}}{2a_0}\zeta \\ \zeta g_2 \to 0 \end{array}\right\} \text{ as } \zeta\to\infty, \qquad \left.\begin{array}{c} \phi_2 \to 0 \\ g_2 \text{ bounded} \end{array}\right\} \text{ as } \zeta \to 0.$$

The same coupled system has been treated in [3] and using the analysis therein it can be shown that a solution exists only if

$$U_0 = -\frac{C}{\beta(s)}, \qquad U_1 = \frac{2}{\beta(s)}\frac{\sqrt{A_1^2(s)+A_2^2(s)}}{2a_0(s)}, \tag{65}$$

where

$$\beta(s) = \int_0^\infty \zeta\left(\frac{d\overline{f}}{d\zeta}\right)^2 d\zeta + \int_0^\infty \overline{f}^2\eta\, d\zeta; \tag{66}$$

β is a function of the normal conductivity σ_0 through η (see equation (64)). We see from (61) that the velocity of the vortex, to leading order, is given by

$$\mathbf{v} = \left(\frac{C\mathbf{n}(s)}{\beta(s)} - \frac{\nabla\log a|_{\mathbf{x}=\mathbf{q}(s)}}{\beta(s)}\right)\log(1/\epsilon). \tag{67}$$

3 Conclusion

We have derived a law of motion (67) for well-separated vortices through an asymptotic analysis of the time-dependent Ginzburg-Landau model of superconductivity as the Ginzburg-Landau parameter κ tends to infinity. We found that when the equilibrium number density of superconducting electrons a, where $a > 0$, is allowed to vary spatially, then its logarithm acts as a pinning potential in this law of motion, attracting vortices to the minimum of a. We have also allowed the normal conductivity σ of the material to vary spatially, which results in a spatially varying mobility β. We note that if we set $\sigma =$constant then β is equal to the constant value for homogeneous materials derived in [3].

The strength of this pinning potential depends on the lengthscale over which a varies. For ease of exposition we have considered the case in which the lengthscale of this variation is the penetration depth, λ, which is the natural lengthscale for the variation of the magnetic field. In this case the strength of the pinning potential is $O(\log \kappa)$. An exactly similar analysis holds when the lengthscale for the variation of a is δ, so long as $\delta \gg \xi$, the vortex core radius; the strength of the pinning potential is then $O((\lambda/\delta)\log \kappa)$.

Competing with this pinning potential in the law of motion are the driving terms due to the electric current density, which may be thought of as creating a Lorentz force on the vortex. These include a self-induced component due to vortex curvature (the first term in (67)), as well as components due to other vortices and background currents. The self-induced term is $O(\log \kappa)$ when the radius of curvature of the vortex is $O(\lambda)$. In the analysis above we chose the vortices to be separated by distances of order λ, in which case the associated current density $\mathbf{J} = (\mathrm{curl})^2\mathbf{A}$ is $O(1/\kappa)$; the corresponding contribution to the law of motion is $O(1)$, and is therefore dominated by the self-induced motion in this case, and does not appear in the leading order balance (67). However, it is shown in [3] that for more closely separated vortices higher current densities may result, in which case the term

$$\frac{2\kappa}{a\beta}\mathbf{J} \wedge \mathbf{t}(s), \tag{68}$$

is added to the law of motion, where $\mathbf{t}(s)$ is the tangent to the vortex line.

Thus, in general, the law of motion is

$$\mathbf{v} = \frac{\log \kappa C\mathbf{n}}{\beta} - \frac{\log \kappa}{\beta}\nabla \log a + \frac{2\kappa}{a\beta}\mathbf{J} \wedge \mathbf{t}. \tag{69}$$

Which of these terms dominates the law of motion depends on the relative magnitudes of $\log \kappa$, $(\lambda/\delta)\log \kappa$, and $|\kappa \mathbf{J}|$. In particular we note that the pinning potential due to variations in a is effective at pinning vortices until the current density in the material reaches a magnitude $|\mathbf{J}| \sim (\lambda/\delta)(\log \kappa)/\kappa$.

Finally we note that in our analysis the equilibrium density of superconducting electrons was always strictly positive. It remains an interesting open question to study the effects of allowing the coefficient of f in equation (9) to be positive, so that the equilibrium density of superconducting electrons is zero; our law of motion suggests that the pinning force will be an order of magnitude in κ stronger in this case.

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