ON THE NONUNIVERSALITY OF THE ERROR FUNCTION IN THE SMOOTHING OF STOKES DISCONTINUITIES

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Abstract. The nonuniversality of the error function in the smoothing of Stokes lines is demonstrated by means of an example with smoothing function $\int_{-\infty}^{\phi} e^{-u^{2m}+g(u)}\,du$, where m is any integer greater than 2 and g is any polynomial of degree less than or equal to $2m-1$.

Key words. Stokes phenomenon

1. Introduction. The role of Stokes' phenomenon in describing the asymptotic behaviour of an analytic function as its argument tends to an isolated singularity has been studied intensively in recent years. The basic picture is that an asymptotic expansion uniform in $\mathrm{ph}(z)$ can only be constructed if the coefficients of certain subdominant exponentials jump across certain lines in the Argand diagram known as Stokes lines. The detailed structure in the vicinity of the Stokes line was first considered by Berry [1], who truncated the (divergent) dominant series at its smallest term (that is, it was *optimally truncated*) and studied the behaviour of the remainder. In this way he was able to show (at least for certain classes of functions) that the change in the coefficient of the subdominant exponential was in fact rapid but continuous, and moreover was described universally in terms of an error function of the local coordinate across the Stokes line. This smoothing was interpreted in terms of matched asymptotic expansions in [5], and we will work within that framework in the present paper.

More recently Berry [2] has considered certain "superfactorial" series, and found that these also have an error function smoothing of the Stokes discontinuity. This led to the natural question as to whether *all* Stokes lines are described locally by error functions [3]. Here we answer this question in the negative by demonstrating an example of a Stokes transition with smoothing function of the form

$$\int_{-\infty}^{\phi} e^{-u^{2m}}\,du \qquad (1)$$

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for each integer $m \geq 2$. In the discussion we will see that these smoothing functions are not the only ones possible; in fact any polynomial of degree less than or equal to $2m-1$ may be added to $-u^{2m}$ in (1).

2. Example. Consider the following delay equation:

$$f(z) = f(z - 1/k) + \frac{e^{k^m z^{2m}}}{\sqrt{k}}, \tag{2}$$

where m is an integer greater than 2. Here k will be our (real) large parameter, while z will be complex and of unit modulus. We note that we could equally well fix z to be real, and look for Stokes lines in the complex k-plane. However, in the spirit of matched asymptotic expansions, we prefer to separate the asymptotically large parameter k from the independent variable z.

There are two lengthscales in equation (2). The homogeneous equation has solutions which are functions of kz, while the inhomogeneous term is a function of $\sqrt{k}z$. Such a situation also arises when considering the Stokes lines of $J_k(kz)$ as $k \to \infty$ for example, where J_k is the Bessel function, and in that case results in the Stokes lines being curved. We note also the work of Berry and Howls [4], who considered the smoothing of Stokes phenomenon in a problem with two parameters as one of the parameters was varied, and found the smoothing function to be an error function. For reasons to be discussed in the next section we do not believe the presence of two lengthscales is necessary in general for non-error function transitions at Stokes lines, but it is necessary in this particular example.

Applying a suitable Fourier transform to (2) yields the following integral representation of a solution:

$$f(z) = \frac{1}{2\pi\sqrt{k}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{k^m (z + \rho e^{i\pi/(2m)}/k)^{2m}} \frac{e^{is\rho}}{1 - e^{ise^{-i\pi/(2m)}}} \, d\rho \, ds. \tag{3}$$

It is not difficult to show that the ρ integration gives a function of order $e^{-c|s|^{2m/(2m-1)}}$ as $|s| \to \infty$ for some constant c with positive real part, so that the double integral exists. The asymptotics of (3) as $k \to \infty$ are nontrivial, but we expect that

$$f \sim \frac{e^{k^m z^{2m}}}{\sqrt{k}} \qquad \text{as } k \to \infty, \text{ for } z \text{ real and positive.} \tag{4}$$

Now, from (2), we find

$$f \sim \sum_{n=0}^{\infty} f_n \tag{5}$$

where

$$f_n = \frac{e^{(kz-n)^{2m}/k^m}}{\sqrt{k}}. \tag{6}$$

The expansion (5), (6) is asymptotic providing $\mathrm{Re}(z^{2m-1}) > 0$, that is, in the sector $-\pi/(4m-2) < \mathrm{ph}(z) < \pi/(4m-2)$. Clearly the series is also divergent, since $f_n \to \infty$ as $n \to \infty$. We aim to show that there is a Stokes line at ph $(z) = 0$, and that the transition there is not an error function.

The point of optimal truncation of the expansion (5), (6) is given by

$$\mathrm{ph}(kz-n)^{2m} \sim \pi,$$

i.e.

$$n \sim kr\cos\theta - \frac{kr\sin\theta}{\tan(\pi/(2m))}, \tag{7}$$

where $z = re^{i\theta}$. Let us write

$$f = \sum_{n=0}^{N-1} f_n + R_N,$$

where $N = kr\cos\theta - kr\sin\theta/\tan(\pi/(2m)) + \alpha$, and α is order one. Then

$$R_N(z) - R_N(z-1/k) = \frac{1}{\sqrt{k}}\exp\left[\frac{(kr\sin\theta(1/\tan(\pi/(2m))+i)+\alpha)^{2m}}{k^m}\right]. \tag{8}$$

Following [5] we seek the value of θ for which the modulus of the right-hand side of (8) is maximal, and we find that this is given by $\theta = 0$. We introduce a local coordinate ϕ by setting $\theta = \phi/(r\sqrt{k})$. Then to leading order the right-hand side of (8) is given by

$$\frac{1}{\sqrt{k}}\exp\left[-\left(\mathrm{cosec}(\pi/(2m))\right)^{2m}\phi^{2m}\right].$$

Now

$$R_N(z-1/k) \sim R_N(z) - \frac{1}{k}\frac{dR_N}{dz}(z) + \frac{1}{2k^2}\frac{d^2R_N}{dz^2}(z) - \cdots,$$

while

$$\frac{d}{dz} = \frac{-ie^{-i\theta}}{r}\frac{d}{d\theta} \sim -i\sqrt{k}\frac{d}{d\phi}.$$

Hence the dominant term on the left-hand side of (8) is given by

$$-\frac{i}{\sqrt{k}}\frac{dR_N}{d\phi}.$$

Thus, as $k \to \infty$ equation (8) for R_N becomes

$$\frac{dR_N}{d\phi} = i \exp\left[-\left(\operatorname{cosec}(\pi/(2m))\right)^{2m} \phi^{2m}\right]. \tag{9}$$

Hence

$$R_N = A + i \int_{-\infty}^{\phi} \exp\left[-\left(\operatorname{cosec}(\pi/(2m))\right)^{2m} u^{2m}\right] du. \tag{10}$$

Remark. It is worth commenting here that the change in the optimal truncation point due to variations in the phase of z is significant in the present example, unlike in the standard error function transition. We have

$$N \sim kr - \frac{\sqrt{k}\phi}{\tan(\pi/(2m))} + \alpha - \frac{\phi^2}{2r} + \cdots.$$

Thus variations in ϕ affect N at order $\sqrt{k}$, whereas for the standard transition they affect N only by an order one amount.

We note that this also means that $dR_N/d\phi$ will contain δ-functions as new terms in the optimally truncated series are added when ϕ varies, but that these appear at order $1/\sqrt{k}$ below the leading order behaviour of R_N. These can be removed working in a local coordinate system in which N is independent of the coordinate across the Stokes line, by making the transformation

$$\begin{aligned} \rho &= r - \frac{\phi}{\sqrt{k}\tan(\pi/(2m))}, \\ \psi &= \phi. \end{aligned}$$

Then N is a function of ρ only. Since

$$\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \psi} - \frac{1}{\sqrt{k}\tan(\pi/(2m))}\frac{\partial}{\partial \rho} \sim \frac{\partial}{\partial \psi}$$

the smoothing result is the same.

3. Discussion. For each integer $m \geq 2$ we have provided an example of a Stokes line with smoothing given by (10). Thus we have demonstrated the nonuniversality of the error function in the smoothing of Stokes discontinuities, answering a question of Berry.

We have generalised the error function transition into a class of transitions governed by similar integrals, but in fact much more general transitions can be constructed. If we replace the right-hand side of (2) with the function

$$\frac{e^{(k^m z^{2m} + g(z\sqrt{k}))}}{\sqrt{k}},$$

where g is any polynomial of degree less than or equal to $2m-1$, then we find that the smoothing function is given by

$$R_N = A + i\int_{-\infty}^{\phi} \exp\left[-\left(\operatorname{cosec}(\pi/(2m))\right)^{2m} u^{2m} + g\left(\left(i+\cot\frac{\pi}{2m}\right)u\right)\right]\,du.$$

For example, choosing $m=2$ and $g(z\sqrt{k}) = kz^2$ gives the smoothing function

$$R_N = A + i\int_{-\infty}^{\phi} e^{-4u^4+2iu^2}\,du.$$

Let us examine the reason for this freedom in the smoothing function. In the notation of Berry [2], suppose our expansion is given by

$$\begin{aligned} F(z) &= \sum_{n=0}^{N} e^{b(n,z)} + \sum_{n=N+1}^{\infty} e^{b(n,z)}, \\ &= F_T(z) + R_N(z), \end{aligned}$$

as $z \to \infty$. Berry proceeds by replacing the sum in R_N by an integral:

$$R_N(z) \approx \int_N^{\infty} e^{b(n,z)}\,dn. \tag{11}$$

The Stokes transition arises when the endpoint N passes through the saddle point of the integrand, which occurs when N takes its optimal value $N_0(z)$, say. Thus the Stokes transition is intimately related to the method of steepest descents with an end point near a saddle point.

Let us consider the steepest descents problem (11) as $z \to \infty$. Berry proceeds by expanding

$$b(n,z) \sim b(N_0(z),z) + \frac{(n-N_0(z))^2}{2} b_{nn}(N_0(z),z) + \cdots. \tag{12}$$

The quadratic term in (12) gives the error function transition. However, since $b(n,z)$ is not necessarily separable into a product of a function of z and a function of n (as would be the case with a Fourier inversion kernel for example), the remaining terms in this expansion may not be negligible. The required approximation to b is achieved by setting $n = N_0(z)+p(z)u$ and expanding $b(N_0(z)+p(z)u, z)$ as $z \to \infty$, where $p(z)$ is chosen so that the first term in u is order one (p determines the Stokes line width). It is quite possible that this limit is not quadratic in u; any term or combination of terms in the Taylor expansion (12) may appear at leading order. For example, with

$$b(n,z) = \frac{(n-z)^4}{z^2} + \frac{(n-z)^2}{z}$$

we have $N_0(z) = z$, $p(z) = \sqrt{z}$, $n = z + \sqrt{z}u$, and $b(n, z) \sim u^4 + u^2$ as $z \to \infty$. Indeed, all terms in the Taylor expansion may appear: with

$$b(n, z) = g\left(\frac{n - N_0(z)}{p(z)}\right)$$

the limit is $g(u)$.

Let us briefly examine the "stability" of these non-error function transitions. Suppose $b(n, z)$ is a polynomial in $(n-z)^2$ of degree M whose coefficients are functions of z only:

$$b(n, z) = \sum_{i=1}^{M} \alpha_i(z)(n - z)^{2i}. \tag{13}$$

Then, with $n = z + p(z)u$,

$$b(n, z) = \sum_{i=1}^{M} \alpha_i(z)p(z)^{2i}u^{2i}, \tag{14}$$

where $p(z)$ is chosen so that the leading term in u is order one.

If this results in one term in the polynomial being dominant at leading order (as will be the case generically), then a transition of the form (10) will result. It if results in two or more terms in the polynomial being in balance at leading order then a more complicated transition will result. The former group may be described as the stable transitions, since they will survive under a small perturbation of the coefficients α_i, whereas the latter group will not. Thus transitions of the family (10) seem the ones most likely to occur in practice.

Given the rich structure of possible smoothings, and the existence of other stable transitions, one is led naturally to the question of why the error function is so common. One reason is the frequency with which series with a factorial over a power divergence occur in practice. Such series arise when using Watson's lemma and in the asymptotic approximation of solutions to singularly perturbed differential equations. All such series result in an error function smoothing of Stokes lines.

Finally we comment that the need for two lengthscales in the examples presented was due to the desire to generate the asymptotic series from the delay equation (2). We expect the asymptotic series

$$f_n = \frac{e^{(kz-n)^{2m}/(kz)^m}}{\sqrt{z}}$$

and

$$f_n = \frac{e^{(kz-n)^{2m}/n^m}}{\sqrt{z}}$$

to have identical smoothings, even though there is only one lengthscale, but we have been unable to find a differential or difference equation whose asymptotic solution generates these series. The methods of [2], which can make sense of a divergent series without the need for an equation or integral to generate it, would be applicable in these cases. Thus although the question of Berry referred to problems involving only one lengthscale, we do not believe that the presence of the second lengthscale has any real bearing on the nature of the smoothing function.

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