EXPONENTIAL ASYMPTOTICS AND STOKES LINES IN NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

S.J. CHAPMAN*, J.R. KING†, AND K.L. ADAMS†‡

Key words. Stokes phenomenon, nonlinear, exponential asymptotics

Abstract. A technique for calculating exponentially small terms beyond all orders in singularly perturbed ordinary differential equations is presented. The approach is based on the application of a WKBJ-type ansatz to the late terms in the naive asymptotic expansion and the identification of Stokes lines, and is closely related to the well-known Stokes line smoothing phenomenon in linear ordinary differential equations. The method is illustrated by application to examples.

1. Introduction. We will be concerned with singularly perturbed nonlinear ordinary differential equations on the real line. In particular, for the cases of concern here a naive perturbation expansion can be made to satisfy the boundary conditions at each algebraic order, but it is possible to show that there is either no solution, or a solution only when a parameter is a member of a discrete set of eigenvalues. The reason for this is the switching on of exponentially small terms across Stokes lines. These terms are exponentially subdominant to the algebraic series at the point where they are switched on, though they may dominate it at infinity if the real axis crosses an "anti-Stokes" line [1].

The classic example of this type of problem is the geometric model for crystal growth considered by Kruskal and Segur [17]. The relevant equation is

$$\epsilon^2\phi''' + \phi' = \cos\phi, \tag{1.1}$$

where $' \equiv d/ds$, with boundary conditions

$$\phi \to -\pi/2 \text{ as } s \to -\infty, \qquad \phi \to \pi/2 \text{ as } s \to \infty. \tag{1.2}$$

A naive expansion

$$\phi \sim \sum_{n=0}^{\infty} \epsilon^{2n}\phi_n \tag{1.3}$$

can be shown to satisfy the boundary conditions at every order [17]. However, if we linearise about $-\pi/2$ as $s \to -\infty$, with $\phi \sim -\pi/2 + f$, we have

$$\epsilon^2 f''' + f' = f, \tag{1.4}$$

*Mathematical Institute, 24-29 St. Giles', Oxford OX1 3LB. Supported by a Royal Society Research Fellowship.

†Dept. of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD.

‡Supported by an EPSRC studentship.

so that

$$f \sim \alpha e^{s} + \beta e^{-s/2+is/\epsilon} + \gamma e^{-s/2-is/\epsilon} \quad \text{as } \epsilon \to 0, \tag{1.5}$$

for constants α, β and γ. Since only the α term decays as $s \to -\infty$ we require $\beta = \gamma = 0$, and imposing $\phi \to -\pi/2$ as $s \to -\infty$ therefore represents two boundary conditions. Similarly, the condition as $s \to \infty$ can be shown to represent two boundary conditions. This gives four boundary conditions on a third order equation (in fact it is worse than this, since the problem is translationally invariant), and we would not expect there to be a solution unless, as Kruskal and Segur put it, a miracle happens; in fact, it has been shown in [3] that no solution to the problem (1.1), (1.2) exists. In [17] it is shown that if a solution did exist it would necessarily have to be antisymmetric. Replacing (1.2) with the conditions

$$\phi \to -\pi/2 \qquad \text{as } s \to -\infty, \tag{1.6}$$

$$\phi(0) = 0, \tag{1.7}$$

(for which the solution to (1.1) is correctly specified) a solution to the problem (1.1), (1.2) will thus exist only if $\phi''(0)$ turns out to be equal to zero for the solution to (1.1) with (1.6)-(1.7). In fact it can be shown [17] that under the conditions (1.6), (1.7),

$$\phi''(0) \sim \frac{Ce^{-\pi/(2\epsilon)}}{\epsilon^{5/2}} \quad \text{as } \epsilon \to 0, \tag{1.8}$$

where C is a nonzero constant. This explains the inability of the algebraic expansion (1.3) to detect that there is no solution: the second derivative of ϕ at the origin is exponentially small in ϵ.

Kruskal and Segur developed a method for attacking such problems [17]. The method involves (i) finding the singularities of the leading order term in the asymptotic expansion (1.3) in the complex plane, observing that the expansion breaks down in the vicinity of these singularities, (ii) rescaling and performing an inner analysis in their vicinity, (iii) finding the exponentially small terms in the far field of the inner solution (iv) matching these with the possible exponentially small corrections to the algebraic expansion, and (v) evaluating these terms on the imaginary axis to check the symmetry condition. Because the inner problem exhibits Stokes phenomenon in its far field, the subdominant exponentials are only present in certain sectors when matching back into the outer expansion (that is, the expansion away from singularities).

Here we present an alternative method for finding the subdominant exponentials. Our method is based on making the exponentials visible by optimally truncating the algebraic asymptotic series, that is, by truncating it at its least term. An analysis of the remainder then shows that the exponentials are smoothly switched on across Stokes lines in the outer problem, exactly in the manner discovered by Berry for the linear case [5]; Berry's approach was later made rigorous by McLeod [18]. Here we use the framework of matched asymptotic expansions described in [19]. The procedure is as follows:

(i) Expand the solution naively as an algebraic asymptotic series.

(ii) Apply a WKBJ-type ansatz to the equation for ϕ_n to find the behaviour of ϕ_n as $n \to \infty$.

(iii) Optimally truncate the algebraic series and observe the subdominant exponential being switched on across a Stokes line.

Once the behaviour of ϕ_n as $n \to \infty$ has been determined the analysis proceeds as in the linear case described in [19].

Our method allows us to keep track of the domain in which the exponentially small terms are present. We will find that sometimes the Stokes lines in the outer problem are curved, so that it is important to know where they lie, and in particular whether they intersect the real line. Moreover, optimally truncating the outer expansion provides justification for including the exponentially small corrections away from the singularities and Stokes lines, as well as determining their form, this often being required even when using the Kruskal-Segur method.

2. Example. We demonstrate our procedure by means of examples. We first consider the equation

$$\epsilon^2 \phi''' + \phi' = F(\phi), \tag{2.1}$$

$$\phi \to \phi_- \qquad \text{as } s \to -\infty, \tag{2.2}$$

$$\phi \to \phi_+ \qquad \text{as } s \to \infty, \tag{2.3}$$

where $F(\phi_-) = F(\phi_+) = 0$. We take ϕ_- and ϕ_+ to be such that there is a heteroclinic orbit connecting them when $\epsilon = 0$. We assume the function F to be even and to be real for real ϕ; its precise form is not important for our exposition, but we note that the crystal growth and anisotropic crystal growth problems considered by Kruskal & Segur and the Kuromoto-Sivashinsky equation are all particular cases.

We begin by expanding in powers of ϵ to give

$$\phi \sim \sum_{n=0}^{\infty} \epsilon^{2n}\phi_n, \tag{2.4}$$

with

$$\phi_0' = F(\phi_0), \tag{2.5}$$

$$\phi_0 \to \phi_- \qquad \text{as } s \to -\infty, \tag{2.6}$$

$$\phi_0 \to \phi_+ \qquad \text{as } s \to \infty, \tag{2.7}$$

$$\phi_{n-1}''' + \phi_n' = F'(\phi_0)\phi_n + \cdots, \tag{2.8}$$

$$\phi_n \to 0 \qquad \text{as } s \to \pm\infty, \qquad n \geq 1. \tag{2.9}$$

If we are to truncate the series (2.4) optimally we need to know the behaviour of ϕ_n as $n \to \infty$. We note from equation (2.8) that the set of singular points of ϕ_n will be the same as that of ϕ_0 for all n. However, since each time we calculate a new term in the asymptotic expansion we differentiate the previous term three times and integrate once, if ϕ_{n-1} has a singularity of strength p, then ϕ_n will have a singularity of strength $p+2$. Thus whenever the leading order solution has a singularity (in the complex plane), the expansion (2.4) will be divergent in the form of factorial/power.

Suppose the leading order solution has singularities at the points $s = \sigma_i$, $i = 1, \cdots, 2m$, $m \neq 0$, with $\sigma_{2j} = \bar{\sigma}_{2j-1}$, where the overbar denotes complex conjugation. Motivated by the discussion above, we seek a solution of (2.8) as $n \to \infty$ of the form

$$\phi_n \sim \Lambda \frac{\Gamma(2n+\gamma+1)(-1)^n}{u^{2n+\beta}}, \tag{2.10}$$

where u, β and γ may be functions of s but are independent of n, and Λ is a constant. Substituting this ansatz into equation (2.8) we find that the leading order behaviour as $n \to \infty$ is given by

$$(u')^3 - u' = 0. \tag{2.11}$$

Hence $u' = \pm 1$; since replacing u by $-u$ simply changes the constant Λ we may take $u' = 1$ without loss of generality. Since the singular points of ϕ_n correspond to the singular points of ϕ_0, u will be zero at $s = \sigma_i$, for some i, giving $u = s - \sigma_i$. In general, the behaviour of ϕ_n will be a sum of expressions of the form (2.10), one for each σ_i. However, for real s this sum will be dominated by the singularities which are closest to the real axis, since this leads to the smallest value of $|u|$. We let σ denote the nearest singularity in the upper half plane.

To determine γ and β we need to go to the next order in (2.8) as $n \to \infty$. With $u = s - \sigma$ we find (2.8) requires

$$(2n+\beta-2)(2n+\beta-1) - 2\gamma' \log(n) \frac{(2n+\beta-2)(2n+\beta-1)}{2n+\beta}(s-\sigma) + 2\log(s-\sigma)\beta' \frac{(2n+\beta-2)(2n+\beta-1)}{2n+\beta}(s-\sigma) - (2n+\gamma)(2n+\gamma-1) = \frac{(2n+\gamma)(2n+\gamma-1)}{2n+\beta}(s-\sigma)F'(\phi_0) + \cdots. \tag{2.12}$$

The largest terms are order $n \log n$, and give $\gamma' = 0$, so that γ is independent of s. Equating the coefficients of n in (2.12) gives

$$2\beta - 2\gamma - 2 + 2(s-\sigma)\log(s-\sigma)\beta' = (s-\sigma)F'(\phi_0), \tag{2.13}$$

with solution

$$\beta = \gamma + 1 + \frac{1}{2\log(s-\sigma)} \int^s F'(\phi_0)\, ds, \tag{2.14}$$

i.e.

$$\beta = \gamma + 1 + \frac{\log \phi_0'}{2\log(s-\sigma)} + \frac{C}{\log(s-\sigma)}, \tag{2.15}$$

for some constant C. We emphasise that β in (2.10) therefore does depend on s, whereas γ turns out not to. Since

$$x^{C/\log x} = e^C,$$

changing C simply changes the constant Λ, so without loss of generality we may take $C = 0$. Suppose $\phi_0 \sim B(s-\sigma)^{-\beta_0}$ as $s \to \sigma$, for some constant B. Then as $s \to \sigma$ we have $(s-\sigma)F'(\phi_0) \to -(\beta_0 + 1)$, $\beta \to \beta_0$, and (2.15) gives

$$\gamma = \frac{3\beta_0 - 1}{2}. \tag{2.16}$$

This completes the first step in our procedure, namely determining the large n behaviour of ϕ_n: we have

$$\phi_n \sim \Lambda \frac{\Gamma(2n+\gamma+1)(-1)^n}{(s-\sigma)^{2n+\gamma+1}(\phi_0')^{1/2}}, \tag{2.17}$$

where γ is given by (2.16). The only unknown in (2.17) is the constant Λ; procedures for calculating this will be illustrated shortly.

Following Dingle [13] we expect there to be a Stokes line wherever ϕ_n and ϕ_{n+1} have the same phase as $n \to \infty$, i.e. when $-u^2$ is real and positive.

The next step is to optimally truncate the series (2.4). Let R_N be the remainder after N terms:

$$\phi = \sum_{n=0}^{N-1} \epsilon^{2n} \phi_n + R_N. \tag{2.18}$$

Then R_N satisfies

$$\begin{aligned} \epsilon^2 R_N''' + R_N' - F'(\phi_0) R_N &= -\epsilon^{2N} \phi_{N-1}''' + \cdots \\ &\sim \epsilon^{2N} \phi_N' \quad \text{as } \epsilon \to 0, N \to \infty. \end{aligned} \tag{2.19}$$

The homogeneous version of this equation has solutions with the following asymptotic behaviour as $\epsilon \to 0$:

$$R_N \sim \frac{e^{-is/\epsilon}}{(\phi_0')^{1/2}}, \quad R_N \sim \frac{e^{is/\epsilon}}{(\phi_0')^{1/2}}, \quad R_N \sim \phi_0'. \tag{2.20}$$

We aim to show that a multiple of the first is switched on across the Stokes line $\arg(s - \sigma) = -i\pi/2$.

Let $s - \sigma = re^{i\theta}$, where r and θ are real. The remainder R_N is minimal when $N \sim r/(2\epsilon)$. Therefore we let $N = r/(2\epsilon) + \alpha$ where α is bounded as $\epsilon \to 0$. Then, as $\epsilon \to 0$, $N \to \infty$

$$\begin{aligned} \epsilon^{2N} \phi_N' &\sim \Lambda \frac{\Gamma(2N + \gamma + 1)(-1)^{N+1}(2N + \beta)\epsilon^{2N}}{(s - \sigma)^{2N+\beta+1}} \\ &\sim \frac{\Lambda\sqrt{2\pi} r^{\gamma-\beta+1/2}}{\epsilon^{\gamma+3/2}} e^{-i\pi(r/\epsilon+2\alpha+2)/2} e^{-r/\epsilon} e^{-i\theta(r/\epsilon+2\alpha+\beta+1)}. \end{aligned} \tag{2.21}$$

Thus the remainder is roughly of order $e^{-r/\epsilon}$. Defining A by

$$R_N = \frac{Ae^{-i(s-\sigma)/\epsilon}}{(\phi_0')^{1/2}}, \tag{2.22}$$

then, since

$$\frac{d}{ds} = -\frac{ie^{-i\theta}}{r} \frac{d}{d\theta}, \tag{2.23}$$

equation (2.19) implies

$$\frac{2ie^{-i\theta}}{r} \exp(-ire^{i\theta}/\epsilon) \frac{dA}{d\theta} \sim \frac{\Lambda\sqrt{2\pi}}{\sqrt{r}\epsilon^{\gamma+3/2}} e^{-i\pi(r/\epsilon+2\alpha+2)/2} e^{-r/\epsilon} e^{-i\theta(r/\epsilon+2\alpha+\gamma+2)}. \tag{2.24}$$

We see that $dA/d\theta$ is exponentially small in ϵ except on $\theta = -\pi/2$, where it is algebraic in ϵ. Thus the main change in A occurs near this line, which is a Stokes line. To examine the Stokes switching we introduce local coordinates by setting $\theta = -\pi/2 + \delta\bar{\theta}$, with $\delta \ll 1$; the precise scaling of δ will be determined shortly. We then find that equation (2.24) gives

$$-\frac{2}{r\delta} e^{r\delta^2\bar{\theta}^2/(2\epsilon)} \frac{dA}{d\bar{\theta}} \sim \frac{\Lambda\sqrt{2\pi}}{\sqrt{r}\epsilon^{\gamma+3/2}} e^{i\pi\gamma/2}. \tag{2.25}$$

Thus we see that the correct scalings are $\delta = \epsilon^{1/2}$, $A = \hat{A}/\epsilon^{\gamma+1}$, giving

$$\frac{d\hat{A}}{d\bar{\theta}} \sim -\frac{\Lambda\sqrt{\pi}\sqrt{r}}{\sqrt{2}} e^{i\pi\gamma/2} e^{-r\bar{\theta}^2/2}. \tag{2.26}$$

Hence

$$\hat{A} \sim a - \frac{\Lambda\sqrt{\pi}}{\sqrt{2}} e^{i\pi\gamma/2} \int_{-\infty}^{\bar{\theta}\sqrt{r}} e^{-\bar{\theta}^2/2}\, d\bar{\theta}. \tag{2.27}$$

As $\bar{\theta} \to -\infty$, $\hat{A} \to a$, while as $\bar{\theta} \to \infty$, $\hat{A} \to a - \Lambda\pi e^{i\pi\gamma/2}$. This implies that for the outer solution (i.e. for $|\theta + \pi/2| = O(1)$) there is a jump in the coefficient A as we cross the Stokes line given by

$$A(\theta = -\pi/2^-) - A(\theta = -\pi/2^+) = -\frac{\Lambda\pi e^{i\pi\gamma/2}}{\epsilon^{\gamma+1}}. \tag{2.28}$$

We now consider the implications of this jump for three particular forms of F.

(i) Kuramoto-Sivashinsky. The Kuramoto-Sivashinsky equation, as analysed in [14, 16], corresponds to equation (2.1) with

$$F(\phi) = 1 - \phi^2. \tag{2.29}$$

The leading order solution is given by

$$\phi_0 = \tanh s, \tag{2.30}$$

which has simple poles at $s = \pm i\pi/2$ giving $\beta_0 = 1$, $\gamma = 1$. Hence there is a Stokes line down the imaginary axis from $s = i\pi/2$ across which (see (2.22) and (2.28))

$$-\frac{\Lambda\pi i \cosh s\, e^{-\pi/(2\epsilon)} e^{-is/\epsilon}}{\epsilon^2}$$

is switched on. Similarly, there is a Stokes line up the imaginary axis from $s = -i\pi/2$ across which

$$\frac{\bar{\Lambda}\pi i \cosh s\, e^{-\pi/(2\epsilon)} e^{is/\epsilon}}{\epsilon^2}$$

is switched on (see Fig. 2.1). On the real axis, the two exponentials are comparable, so that across $s = 0$

$$\frac{-2R\pi \cosh s\, e^{-\pi/(2\epsilon)} \sin(s/\epsilon - \Phi)}{\epsilon^2}$$

is switched on, where here and henceforth $\Lambda = Re^{i\Phi}$, with R and Φ real.

Since this function is growing as $s \to \infty$ we see that the naive expansion (2.4) ceases to be valid when s is of order $1/\epsilon$. We conclude that there is no solution of the boundary value problem which is approximated by (2.4) for all s.

Finally, we note that because the leading order solution has an infinite sequence of poles up and down the imaginary axis, we would expect there to be an infinite number of Stokes smoothings across the Stokes line, in a manner somewhat analogous to that occurring in the gamma function $\Gamma(z)$ on the Stokes lines $\arg(z) = \pm\pi/2$ [6, 7, 20]. However, the prefactor in each successive term switched on is exponentially subdominant to the previous term, and therefore only the first is important in determining the question of solvability.

(ii) Crystal growth. The crystal growth problem studied by Kruskal & Segur corresponds to equation (2.1) with

$$F(\phi) = \cos\phi. \tag{2.31}$$

The leading order solution is given by

$$\phi_0 = -\frac{\pi}{2} + 2\tan^{-1} e^s, \tag{2.32}$$

which has logarithmic singularities at $s = \pm i\pi/2$. Hence $\beta_0 = 0$, $\gamma = -1/2$. There is a Stokes line down the imaginary axis from $s = i\pi/2$ across which

$$\frac{-\Lambda\pi e^{-i\pi/4}(\cosh s)^{1/2} e^{-\pi/(2\epsilon)} e^{-is/\epsilon}}{\epsilon^{1/2}}$$

is switched on, and a Stokes line up the imaginary axis from $s = -i\pi/2$ across which

$$\frac{-\bar{\Lambda}\pi e^{i\pi/4}(\cosh s)^{1/2} e^{-\pi/(2\epsilon)} e^{is/\epsilon}}{\epsilon^{1/2}}$$

is switched on. On the real axis the two exponentials are comparable, so that across $s = 0$

$$\frac{-2R\pi(\cosh s)^{1/2} e^{-\pi/(2\epsilon)} \cos(s/\epsilon + \pi/4 - \Phi)}{\epsilon^{1/2}} \tag{2.33}$$

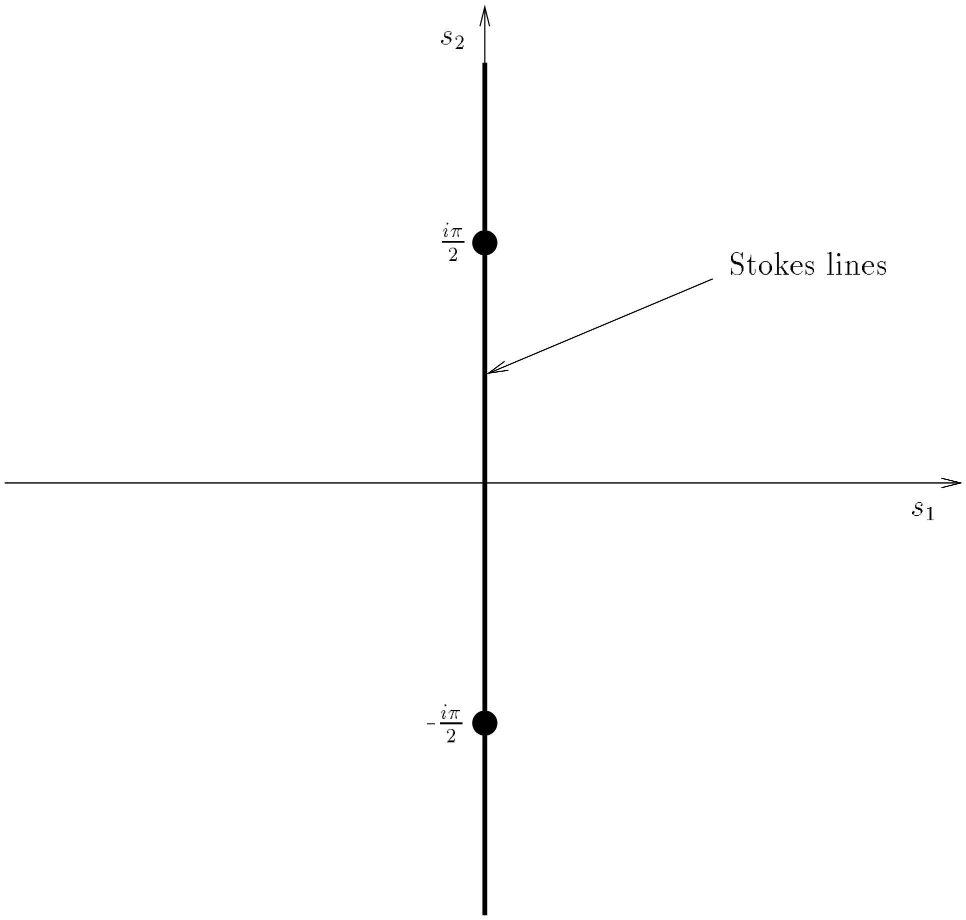

FIG. 2.1. *Stokes lines in the Kuramoto-Sivashinsky and crystal growth problems;* $s = s_1 + is_2$, s_1, s_2 *real.*

is switched on.

As in the previous example, this function is growing as $s \to \infty$ and thus the naive expansion (2.4) ceases to be valid when s is of order $1/\epsilon$. We conclude again that there is no solution of the boundary value problem which is approximated by (2.4) for all s.

For completeness let us now impose the conditions (1.6), (1.7) and determine the resulting value of $\phi''(0)$. From (2.33) we see that

$$\phi(0) \sim -\frac{R\pi e^{-\pi/(2\epsilon)}\cos(\pi/4 - \Phi)}{\epsilon^{1/2}}$$

(note the factor of a half due to the fact that we are evaluating exactly at the middle of the Stokes line; see (2.27)). Hence, in order to satisfy (1.7), we need to add a multiple of ϕ_0', which is a solution of the homogeneous version of (2.19). Since $\phi_0'(0) = 1$, we see that we need to add

$$\frac{R\pi e^{-\pi/(2\epsilon)}\phi_0'(s)\cos(\pi/4 - \Phi)}{\epsilon^{1/2}}$$

to the optimally truncated expansion (2.18). This term does not contribute to the leading order expression for $\phi''(0)$, and so from (2.33) we now see that the solution to (1.1) satisfying (1.6), (1.7) satisfies

$$\phi''(0) \sim \frac{R\pi e^{-\pi/(2\epsilon)}\cos(\pi/4 - \Phi)}{\epsilon^{5/2}}. \tag{2.34}$$

(iii) Anisotropic crystal growth. The anisotropic crystal growth problem considered by Kruskal & Segur corresponds to equation (2.1) with

$$F(\phi) = \frac{\cos\phi}{1 + \alpha\cos 4\phi}, \qquad 0 < \alpha < 1. \tag{2.35}$$

The leading order solution is given implicitly by

$$s = (1+\alpha)\log(\sec\phi_0 + \tan\phi_0) - \frac{8}{3}\alpha(\sin\phi_0)^3, \tag{2.36}$$

which has square-root singularities at the roots of $1 + \alpha\cos(4\phi_0) = 0$. There are four roots nearest the real axis, one in each quadrant. Denoting the root in the upper left-hand quadrant by σ, the others are given by $-\sigma$, $\bar{\sigma}$ and $-\bar{\sigma}$. We have $\beta_0 = -1/2$, giving $\gamma = -5/4$. There are Stokes lines parallel to the imaginary axis between $s = \sigma$ and $s = \bar{\sigma}$, and between $s = -\bar{\sigma}$ and $s = -\sigma$ (see Fig. 2.2). On the real axis we find that

$$-\frac{2\pi R\epsilon^{1/4}e^{-\Im(\sigma)/\epsilon}}{(\phi_0')^{1/2}}\cos\left(\frac{s - \Re(\sigma)}{\epsilon} + \frac{5\pi}{8} - \Phi\right)$$

is switched on at $s = \Re(\sigma)$, and that

$$-\frac{2\pi R\epsilon^{1/4}e^{-\Im(\sigma)/\epsilon}}{(\phi_0')^{1/2}}\cos\left(\frac{s + \Re(\sigma)}{\epsilon} - \frac{5\pi}{8} + \Phi\right)$$

is switched on at $s = -\Re(\sigma)$.

For there to be a solution to the boundary value problem the two functions switched on must exactly cancel for $s > -\Re(\sigma)$, i.e.

$$\cos\left(\frac{\Re(\sigma)}{\epsilon} - \frac{5\pi}{8} + \Phi\right) = 0. \tag{2.37}$$

Note that this is exactly the condition that $\phi(0) = \phi''(0) = 0$ to leading order.

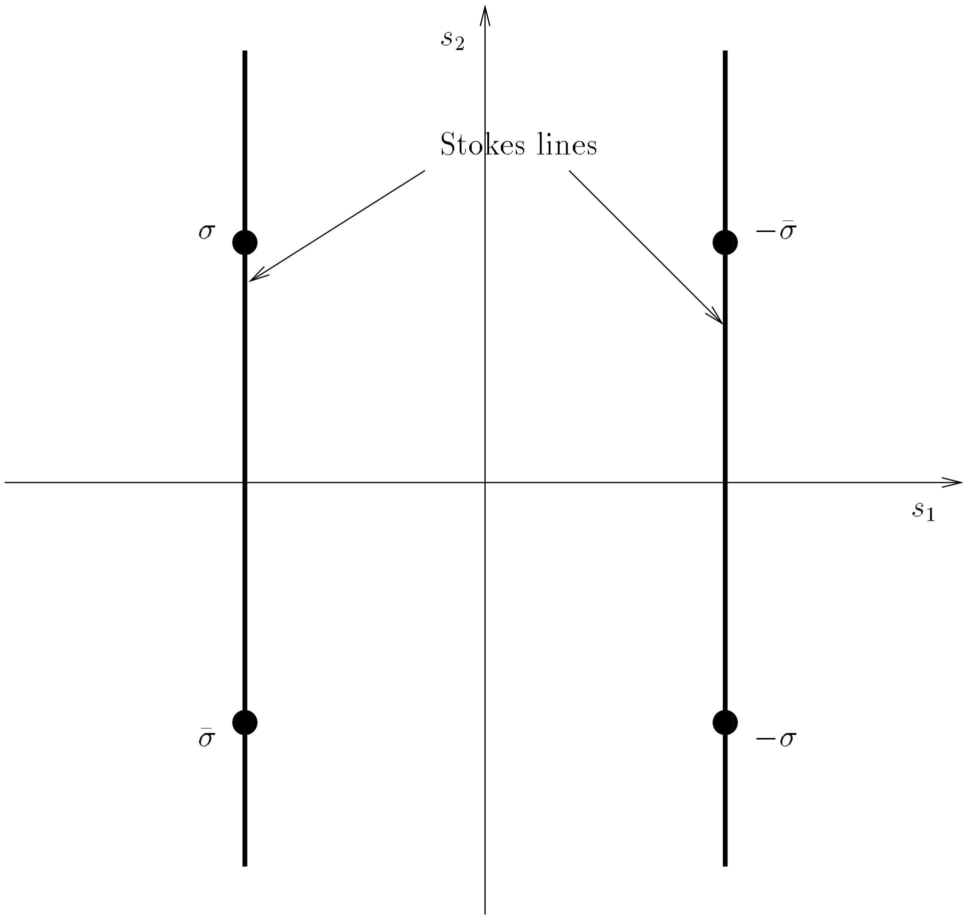

FIG. 2.2. *Stokes lines in the anisotropic crystal growth problem;* $s = s_1 + is_2$, s_1, s_2 *real.*

3. Calculation of the constant Λ. We now illustrate the procedures for calculating the required Stokes multipliers; all that is required is the determination of the most singular term in ϕ_n at each of the nearest singularities.

(i) Kuramoto-Sivashinsky. Following Grimshaw and Hooper [14] we adopt the rescaled form (with appropriate redefinition of ϵ)

$$\epsilon^2\phi''' + (1 - 4\epsilon^2)\phi' = 1 - \phi^2; \tag{3.1}$$

for (3.1) we have

$$\phi \sim -1 + Ce^{2s} \qquad \text{as } s \to -\infty, \tag{3.2}$$

for some real constant C, so that ϕ is periodic with its period πi not depending on ϵ. The original form (2.1) has the same most singular terms in each ϕ_n and this rescaling is not essential; however it makes some aspects of our discussion more straightforward. We have

$$\phi_0(s) = \tanh(s) = \sum_{k=0}^{\infty} \left(\frac{1}{(2k+1)i\pi/2 + s} - \frac{1}{(2k+1)i\pi/2 - s} \right), \tag{3.3}$$

and it can be shown that

$$\phi_n(s) = \tanh(s)\Omega_n(\mathrm{sech}^2(s)), \tag{3.4}$$

where $\Omega_n(x)$ is an nth degree polynomial in x, with $\Omega_n(0) = 0$ for $n \geq 1$. The expression (3.3) has a simple pole at $s = (2k-1)i\pi/2$ for each integer k, and we can deduce that

$$\phi_n(s) = \sum_{k=0}^{\infty}\sum_{j=0}^{n} a_{n,j} \left(\frac{1}{((2k+1)i\pi/2 + s)^{2j+1}} - \frac{1}{((2k+1)i\pi/2 - s)^{2j+1}} \right) \tag{3.5}$$

for some real constants $a_{n,j}$. As $n \to \infty$ the expansion (3.5) should be consistent with (2.17), and to determine the constant Λ we simply need to compare these two expressions for a particular value of s. We choose the simplest such value, that is, we compare as $s \to \sigma$, where $\sigma = i\pi/2$. In this case it is clear that (3.5) is dominated by the terms $A_n \equiv a_{n,n}$, so that

$$\phi_n(s) \sim \frac{A_n}{(s-\sigma)^{2n+1}} \qquad \text{as } s \to \sigma, n \to \infty. \tag{3.6}$$

We thus require to calculate A_n, which is given by

$$A_0 = 1, \qquad A_1 = -6, \tag{3.7}$$

and the nonlinear recurrence relation

$$A_n = -(2n+1)2nA_{n-1} + \frac{1}{2n-1}\sum_{m=1}^{n-1} A_{n-m}A_m, \qquad m \geq 2. \tag{3.8}$$

We thus have

$$A_n \sim (-1)^n (2n+1)!\, i\Lambda \tag{3.9}$$

as $n \to \infty$ ((2.17) with $\gamma = 1$) and Λ can be determined by iterating forward to large n using (3.8). We find that $\Lambda \approx -1.25i$, Λ being purely imaginary. Methods of speeding up the convergence to give more precise values of Λ are given by Byatt-Smith [9].

We note that, while a numerical solution of the recurrence relation (3.8) is required to obtain the value of Λ, it is clear from (3.8) that $\Lambda \neq 0$, which is often all that is required. This is the case in all the problems we consider.

(ii) Crystal Growth. For the rescaled form

$$\epsilon^2\phi''' + (1-\epsilon^2)\phi' = \cos\phi, \tag{3.10}$$

we have

$$\phi \sim -\pi/2 + Ce^s \qquad \text{as } s \to -\infty \tag{3.11}$$

and the period of the solution $(2\pi i)$ is again independent of ϵ. Here we have

$$\begin{aligned}\phi_0(s) &= -\pi/2 + 2\tan^{-1} e^s \\ &= i\sum_{k=0}^{\infty} \log\left(\frac{((2k+1/2)\pi i + s)((2k+3/2)\pi i - s)}{((2k+1/2)\pi i - s)((2k+3/2)\pi i + s)}\right),\end{aligned} \tag{3.12}$$

and

$$\phi_n(s) = \tanh(s)\mathrm{sech}(s)\Omega_{n-1}(\mathrm{sech}^2(s)) \qquad n \geq 1, \tag{3.13}$$

where $\Omega_n(x)$ is a (different) nth degree polynomial; more importantly $\phi_n(s)$ can also be written in the form

$$\begin{aligned}\phi_n(s) = i\sum_{k=0}^{\infty}\sum_{j=0}^{n} a_{n,j}\Bigg(&\frac{1}{((2k+1/2)\pi i + s)^{2j}} + \frac{1}{((2k+3/2)\pi i - s)^{2j}} \\ &- \frac{1}{((2k+1/2)\pi i - s)^{2j}} - \frac{1}{((2k+3/2)\pi i + s)^{2j}}\Bigg) \qquad n \geq 1,\end{aligned}$$

and, as $s \to \sigma$, with $\sigma = i\pi/2$,

$$\phi_n(s) \sim -\frac{iA_n}{(s-\sigma)^{2n}} \qquad \text{as } n \to \infty, \tag{3.14}$$

where $A_n = a_{n,n}$. It is readily shown that A_n can be determined as the (divergent) series solution

$$\psi \sim \log(-\zeta) + \sum_{n=1}^{\infty} \frac{A_n}{\zeta^{2n}} \tag{3.15}$$

to the equation

$$\frac{d^3\psi}{d\zeta^3} + \frac{d\psi}{d\zeta} = -e^{-\psi}. \tag{3.16}$$

Note that this may be derived as a leading order inner problem (cf. Kruskal and Segur [17]) by the scalings $s = \pi i/2 + \epsilon\zeta$, $\phi = i\log(2/\epsilon) - \pi/2 - i\psi$, and (3.15) can then be viewed as a matching condition between inner and outer. However, here we simply view (3.15)-(3.16) as a prescription for determining the A_n, without needing to solve an inner problem in the complex domain. While it is not difficult to write down the nonlinear difference equation for A_n, it is not very instructive to do so.

We have $\beta_0 = 0$, $\gamma = -1/2$; comparing (3.14) with (2.17) gives

$$\Lambda = \lim_{n\to\infty} \frac{-iA_n(-1)^n e^{-i\pi/4}}{\Gamma(2n+1/2)}. \tag{3.17}$$

Writing $\Lambda = Re^{i\pi/4}$, it is found by iterating the nonlinear recurrence relation numerically that $R \approx 1.35$.

(iii) Anisotropic Crystal Growth. For

$$\epsilon^2\phi''' + \phi' = \frac{\cos\phi}{1+\alpha\cos(4\phi)} \tag{3.18}$$

we treat the singularity in the expansion at $s = \sigma(\alpha)$, with $\phi_0 = \theta(\alpha)$ where

$$1 + \alpha\cos(4\theta(\alpha)) = 0. \tag{3.19}$$

Following Kruskal and Segur [17] we define

$$K(\alpha) = -\frac{\cos\theta}{4\alpha\sin(4\theta)} \tag{3.20}$$

and (2.36) satisfies

$$\phi_0 \sim \theta(\alpha) + (2K(\alpha))^{1/2}(s-\sigma)^{1/2} \text{ as } s \to \sigma. \tag{3.21}$$

Because of the square root branch cut, the details of the terms ϕ_n are much harder to calculate for $n = O(1)$ than for the previous examples, but the behaviour for $n \to \infty$ (and hence the optimal truncation) is still straightforward. It is easily seen that

$$\phi_n \sim (2K(\alpha))^{1/2} A_n (s-\sigma)^{-(2n-1/2)} \text{ as } s \to \sigma, \tag{3.22}$$

for some constants A_n (independent of α), so that $\beta_0 = -1/2$, $\gamma = -5/4$. The A_n are given by the recurrence relation

$$\begin{aligned}(2n-1)A_n = &-(2n-1/2)(2n-3/2)(2n-5/2)A_{n-1} \\ &- \sum_{m=1}^{n-1}(2m-1/2)(A_m + (2m-3/2)(2m-5/2)A_{m-1})A_{n-m},\end{aligned} \tag{3.23}$$

with $A_0 = 1$. We have

$$\Lambda = 2^{1/4}K(\alpha)^{3/4} \lim_{n\to\infty} \frac{(-1)^n A_n}{\Gamma(2n-1/4)} \tag{3.24}$$

and solving (3.23) numerically to large n gives

$$\frac{\Lambda}{2^{1/4}K(\alpha)^{3/4}} \approx -0.351. \tag{3.25}$$

Hence the condition (2.37) for a solution to exist becomes

$$\Re\left[K(\alpha)^{3/4}\exp(-5i\pi/8)\exp(i\Re(\sigma)/\epsilon)\right] = 0. \tag{3.26}$$

We note that since the phase of Λ can be calculated explicitly we are able to give an explicit analytic condition for a solution to exist, whereas the condition of Kruskal & Segur [17] contains a constant determined by numerical integration.

4. Another example. In Section 2 we were seeking heteroclinic orbits of the equation (2.1). Here we illustrate our procedure further by considering the problem of finding homoclinic orbits of the fourth order equation

$$\epsilon^2\phi'''' + \phi'' = F(\phi), \tag{4.1}$$

$$\phi \to \hat{\phi} \qquad \text{as } s \to \pm\infty, \tag{4.2}$$

where $F(\hat{\phi}) = 0$. We take $\hat{\phi}$ to be such that there is a homoclinic orbit when $\epsilon = 0$. Expanding in powers of ϵ as in (2.4) we find

$$\phi_0'' = F(\phi_0), \tag{4.3}$$

$$\phi_0 \to \hat{\phi} \qquad \text{as } s \to \pm\infty, \tag{4.4}$$

$$\phi_{n-1}'''' + \phi_n'' = F'(\phi_0)\phi_n + \cdots, \tag{4.5}$$

$$\phi_n \to 0 \qquad \text{as } s \to \pm\infty, \qquad n \geq 1. \tag{4.6}$$

With the ansatz

$$\phi_n \sim \Lambda\frac{\Gamma(2n+\gamma+1)(-1)^n}{u^{2n+\beta}}, \tag{4.7}$$

we find $u = s - \sigma$, $\gamma =$ constant, and

$$\beta = \gamma + 1 + \frac{C}{\log(s-\sigma)}, \tag{4.8}$$

where σ is again the closest singularity in ϕ_0 to the real axis. As before we have $C = 0$ without loss of generality, so that β is constant in this case, and $\gamma = \beta_0 - 1$. We expect there to be a Stokes line where the phases of ϕ_n and ϕ_{n+1} are equal as $n \to \infty$, i.e. where $-u^2$ is real and positive.

Truncating the expansion for ϕ after N terms:

$$\phi = \sum_{n=0}^{N-1} \epsilon^{2n}\phi_n + R_N, \tag{4.9}$$

we have

$$\epsilon^2 R_N'''' + R_N'' - F'(\phi_0)R_N \sim -\epsilon^{2N}\phi_{N-1}'''' \tag{4.10}$$

$$\sim \epsilon^{2N}\phi_N''. \tag{4.11}$$

The homogeneous version of this equation as solutions with the following asymptotic behaviour as $\epsilon \to 0$:

$$R_N \sim e^{-is/\epsilon}, \quad R_N \sim e^{is/\epsilon}, \quad R_N \sim \phi_0', \quad R_N \sim \phi_0' \int \frac{1}{(\phi_0')^2}\, ds. \tag{4.12}$$

The optimal truncation point is again $N = r/(2\epsilon) + \alpha$. Letting

$$R_N = Ae^{-i(s-\sigma)/\epsilon} \tag{4.13}$$

we find

$$\frac{dA}{d\theta} \sim \frac{\Lambda\sqrt{\pi}}{\sqrt{2}} \frac{\sqrt{r}}{\epsilon^{\gamma+3/2}} e^{i\theta} \exp(ire^{i\theta}/\epsilon) e^{-r/\epsilon} e^{-i\theta(r/\epsilon+2\alpha+\gamma+3)} e^{-i\pi/2(r/\epsilon+2\alpha)} \tag{4.14}$$

We see that $dA/d\theta$ is exponentially small in ϵ except at the Stokes line $\theta = -\pi/2$. Rescaling by setting $\theta = -\pi/2 + \delta\bar{\theta}$ we find

$$\frac{dA}{d\bar{\theta}} \sim \frac{\Lambda\sqrt{\pi}}{\sqrt{2}} \frac{\sqrt{r}}{\epsilon^{\gamma+3/2}} \delta e^{i\pi/2(\gamma+2)} \exp(-r\bar{\theta}^2\delta^2/(2\epsilon)). \tag{4.15}$$

As before, the correct scalings are $\delta = \epsilon^{1/2}$, $A = \hat{A}/\epsilon^{\gamma+1}$, giving

$$\frac{d\hat{A}}{d\bar{\theta}} \sim \frac{\Lambda\sqrt{\pi r}}{\sqrt{2}} e^{i\pi/2(\gamma+2)} \exp(-r\bar{\theta}^2/2). \tag{4.16}$$

Thus the jump in A across the Stokes line is given by

$$A(\theta = -\pi/2^-) - A(\theta = -\pi/2^+) = -\frac{\Lambda\pi e^{i\pi\gamma/2}}{\epsilon^{\gamma+1}}. \tag{4.17}$$

Example. The KdV equation with surface tension studied by [2, 9, 15, 21] corresponds to (4.1) with

$$F(\phi) = \phi - \phi^2, \tag{4.18}$$

$$\phi \to 0 \qquad \text{as } s \to \pm\infty. \tag{4.19}$$

The leading order solution is given by

$$\phi_0 = \frac{3}{2}\mathrm{sech}^2(s/2), \tag{4.20}$$

which has double poles at $s = \pm i\pi$, giving $\beta_0 = 2$, $\gamma = 1$. Hence there are Stokes lines down the imaginary axis from $s = i\pi$ and up the imaginary axis from $s = -i\pi$. On the real axis

$$-\frac{2R\pi e^{-\pi/\epsilon}}{\epsilon^2}\sin(s/\epsilon - \Phi) \tag{4.21}$$

is switched on, where $\Lambda = Re^{i\Phi}$. Since $\phi_0 \sim 6e^{-s}$ as $s \to \infty$ the exponentially small term switched on at $s = 0$ ceases to be dominated by the regular expansion for s of order $1/\epsilon$. Thus there is no solution to (4.1) which is approximated by a naive expansion (2.4) for all s.

5. Examples in which $u' \neq 1$. Thus far all our examples have resulted in the simple equation $u' = 1$. Here we consider the extra complication introduced when the equation for u' is nontrivial. As before, we find it easiest to explain the method by reference to examples, and we have chosen two. The first problem arises in the theory of fluid resonance in tanks, and has been considered using the Kruskal-Segur method in [8, 10]; see also [4]. The second problem arises when considering the reconnection of fluid vortices, and is described in [12]. It is the most complicated of the problems we consider, and as far as we know has not been attacked by any other method.

Fluid resonance in tanks. Consider the following equation and boundary conditions:

$$\epsilon^2\phi'' = \phi^2 - s^2 - 1, \tag{5.1}$$

$$\phi \sim -|s| - \frac{1}{2|s|} \qquad \text{as } s \to \pm\infty. \tag{5.2}$$

Expanding in powers of ϵ we find

$$\phi \sim \sum_{n=0}^{\infty} \epsilon^{2n}\phi_n, \tag{5.3}$$

with

$$\phi_0 = -\sqrt{1+s^2}, \tag{5.4}$$

$$\phi''_{n-1} = -2\sqrt{1+s^2}\phi_n + 2\phi_1\phi_{n-1} + \cdots. \tag{5.5}$$

We seek a solution of the form

$$\phi_n \sim \Lambda \frac{\Gamma(2n+\gamma+1)}{u^{2n+\beta}} \tag{5.6}$$

as $n \to \infty$. Substituting this ansatz into (5.5) we find that at leading order

$$(u')^2 = -2\sqrt{1+s^2} \tag{5.7}$$

giving

$$u = i\sqrt{2}\int_i^s (1+p^2)^{1/4}\, dp. \tag{5.8}$$

The easiest way to proceed now is to change coordinates by writing $\phi = \phi(u)$. Then

$$\epsilon^2 (u')^2 \frac{d^2\phi}{du^2} + \epsilon^2 u'' \frac{d\phi}{du} = \phi^2 - s(u)^2 - 1, \tag{5.9}$$

and

$$\frac{d^2\phi_{n-1}}{du^2} + \frac{u''}{(u')^2}\frac{d\phi_{n-1}}{du} = \phi_n + \cdots. \tag{5.10}$$

Substituting the ansatz (5.6) in (5.10) and equating coefficients at the next order gives $\gamma =$ constant and

$$\beta = \frac{1}{\log u}\left[\int^u \frac{\gamma+1}{u} + \frac{u''}{2(u')^2}\, du\right] \tag{5.11}$$

$$= \gamma + 1 + \frac{\log u'}{2\log u} + \frac{C}{\log u}. \tag{5.12}$$

As before we have $C = 0$ without loss of generality. As $u \to 0$, $u \sim c(s-i)^{5/4}$, where $c = e^{5\pi i/8} 2^{11/4}/5$, and hence

$$\frac{\log u'}{2\log u} \sim \frac{1}{10}. \tag{5.13}$$

Since $\phi_0 \sim -\sqrt{2i}\sqrt{s-i}$ as $s \to i$ we have $\beta_0 = -2/5$ and hence $\gamma = -3/2$. Thus as $n \to \infty$

$$\phi_n \sim \frac{\Lambda\Gamma(2n+\gamma+1)}{u^{2n+\gamma+1}(u')^{1/2}}, \tag{5.14}$$

and we expect there to be a Stokes line where u is real and positive, that is, down the imaginary axis from $s = i$.

As usual, we truncate the series after N terms:

$$\phi = \sum_{n=0}^{N-1} \epsilon^{2n}\phi_n + R_N, \tag{5.15}$$

giving

$$\epsilon^2 \frac{d^2 R_N}{du^2} + \epsilon^2 \frac{u''}{(u')^2} \frac{dR_N}{du} - R_N \sim -\epsilon^{2N} \phi_N. \tag{5.16}$$

The homogeneous version of this equation has solutions with the following asymptotic behaviour as $\epsilon \to 0$:

$$R_N \sim \frac{e^{-u/\epsilon}}{(u')^{1/2}}, \quad R_N \sim \frac{e^{u/\epsilon}}{(u')^{1/2}}. \tag{5.17}$$

Letting

$$R_N = \frac{Ae^{-u/\epsilon}}{(u')^{1/2}}$$

we find

$$-2\epsilon \frac{dA}{du} + \epsilon^2 \left(\frac{d^2 A}{du^2} + \cdots \right) \sim -\phi_N + \cdots. \tag{5.18}$$

Setting $u = re^{i\theta}$, optimal truncation occurs when $n = r/(2\epsilon) + \alpha$, with α bounded as $\epsilon \to 0$. We find

$$-\frac{ie^{-i\theta}\epsilon}{r} \frac{dA}{d\theta} \exp(-re^{i\theta}/\epsilon) \sim \frac{\sqrt{2\pi}\Lambda}{\sqrt{r}\epsilon^{\gamma+1/2}} e^{-r/\epsilon} e^{-i\theta(r/\epsilon+2\alpha+\gamma+1)}. \tag{5.19}$$

Hence $dA/d\theta$ is exponentially small except at the Stokes line $\theta = 0$. We rescale by setting $\theta = \delta\bar{\theta}$ to give

$$-\frac{i\epsilon}{r\delta} \frac{dA}{d\bar{\theta}} \sim \frac{\sqrt{2\pi}\Lambda}{\sqrt{r}\epsilon^{\gamma+1/2}} e^{-r\delta^2\bar{\theta}^2/(2\epsilon)} \tag{5.20}$$

As before, the correct scalings are $\delta = \sqrt{\epsilon}$, $A = \hat{A}/\epsilon^{\gamma+1}$, giving

$$\frac{d\hat{A}}{d\bar{\theta}} \sim i\sqrt{2\pi}\Lambda\sqrt{r} e^{-r\bar{\theta}^2/2}. \tag{5.21}$$

Hence

$$\hat{A} \sim a + i\Lambda\sqrt{2\pi} \int_{-\infty}^{\bar{\theta}\sqrt{r}} e^{-\bar{\theta}^2/2}\, d\bar{\theta}. \tag{5.22}$$

Matching with the outer solution this means that there is a jump in the coefficient A as we cross the Stokes line given by

$$A(\theta = 0^-) - A(\theta = 0^+) = \frac{i2\pi\Lambda}{\epsilon^{\gamma+1}} = i2\pi\Lambda\epsilon^{1/2}. \tag{5.23}$$

Thus there is a Stokes line down the imaginary axis from $s = i$ across which

$$\frac{2^{3/4}\pi e^{i\pi/4}\Lambda\epsilon^{1/2}}{(1+s^2)^{1/8}} \exp\left(-\frac{i\sqrt{2}}{\epsilon} \int_i^s (1+p^2)^{1/4}\, dp \right)$$

is switched on. Similarly, there is a Stokes line up the imaginary axis from $s = -i$ across which

$$\frac{2^{3/4}\pi e^{-i\pi/4}\bar{\Lambda}\epsilon^{1/2}}{(1+s^2)^{1/8}}\exp\left(\frac{i\sqrt{2}}{\epsilon}\int_{-i}^{s}(1+p^2)^{1/4}\,dp\right)$$

is switched on. On the real axis we find that the two exponentials are comparable, and at $s = 0$

$$\frac{2^{7/4}\pi R\epsilon^{1/2}}{(1+s^2)^{1/8}}\exp\left(-\frac{\sqrt{2}}{\epsilon}\int_0^1(1-x^2)^{1/4}\,dx\right)\cos\left(\frac{\pi}{4}-\frac{\sqrt{2}}{\epsilon}\int_0^s(1+p^2)^{1/4}\,dp+\Phi\right) \tag{5.24}$$

is switched on, where $\Lambda = Re^{i\Phi}$. Evaluating ϕ at $s = 0$ we find

$$\phi(0) \sim 2^{3/4}\pi R\epsilon^{1/2}\exp\left(-\frac{\sqrt{2}}{\epsilon}\int_0^1(1-x^2)^{1/4}\,dx\right)\cos\left(\frac{\pi}{4}+\Phi\right). \tag{5.25}$$

Note again the factor of a half since we are evaluating exactly at the middle of the Stokes line. Since the function (5.24) turned on across the Stokes line is $O(1/s^{1/4})$ as $s \to \infty$ we see that there is no solution ϕ satisfying the boundary conditions (5.2).

As before, to evaluate the constant Λ we consider the expression for ϕ_n as $s \to i$. It is not difficult to show that

$$\phi_n \sim \frac{-2^{1/2}e^{i\pi/4}e^{-5\pi i n/4}A_n}{(s-i)^{5n/2-1/2}} \quad \text{as } s \to i, \tag{5.26}$$

where the A_n satisfy

$$A_n = \frac{(5n-6)(5n-4)A_{n-1}}{8\sqrt{2}} - \sum_{k=1}^{n-1}\frac{A_k A_{n-k}}{2}, \tag{5.27}$$

$$A_0 = 1. \tag{5.28}$$

Comparing (5.26) with (5.14) we find

$$\Lambda = -\frac{5^{2/5}}{2^{3/5}}\lim_{n\to\infty} 2^{3n/2}\left(\frac{4}{5}\right)^{2n}\frac{A_n}{\Gamma(2n-1/2)}. \tag{5.29}$$

Solving (5.27), (5.28) numerically up to large n we find

$$\Lambda \approx 0.247. \tag{5.30}$$

Vortex reconnection. We consider the following problem:

$$4\epsilon^2\phi''' + \frac{4\phi'}{\phi^2} + s^2\phi' - s\phi = 0, \tag{5.31}$$

$$\phi \sim \pm s + O(1/s^3) \quad \text{as } s \to \pm\infty. \tag{5.32}$$

Linearising about the leading order behaviour at infinity we find the following possible behaviour:

$$\phi \sim s + a_0 s + \frac{a_1}{s^2} e^{is^2/(4\epsilon)} + \frac{a_2}{s^2} e^{-is^2/(4\epsilon)}. \tag{5.33}$$

We seek solutions without rapid ocillations at infinity: the boundary condition (5.32) forces $a_0 = a_1 = a_2 = 0$. Since we therefore have three conditions at infinity, and three conditions at minus infinity, we have six boundary conditions on a third order equation. Alternatively, we can replace the boundary conditions at minus infinity with the symmetry condition $\phi'(0) = 0$.

Expanding in powers of ϵ as in (5.3) we find

$$\phi_0 = \frac{1}{\sqrt{2}} \sqrt{(s^2 + \sqrt{s^4 + 8})}, \tag{5.34}$$

$$4\phi'''_{n-1} + \frac{4\phi'_n}{\phi_0^2} + s^2 \phi'_n - s\phi_n = \frac{4\phi_1 \phi'_{n-1}}{\phi_0^3} + \cdots. \tag{5.35}$$

With the ansatz

$$\phi_n \sim \frac{\Lambda(-1)^n \Gamma(2n + \gamma + 1)}{u^{2n+\beta}}, \tag{5.36}$$

we find

$$(u')^2 = \frac{1}{\phi_0^2} + \frac{s^2}{4}, \tag{5.37}$$

giving

$$u = \int_\sigma^s \frac{(x^4 + 8)^{1/4}}{2}\, dx, \tag{5.38}$$

where σ is the fourth root of -8 in the upper left-hand quadrant. Changing coordinates to write $\phi = \phi(u)$ as before gives

$$4\epsilon^2 \left[(u')^3 \frac{d^3\phi}{du^3} + 3u'u'' \frac{d^2\phi}{du^2} + u''' \frac{d\phi}{du} \right] + \frac{4u'}{\phi^2} \frac{d\phi}{du} + s(u)^2 u' \frac{d\phi}{du} - s(u)\phi = 0. \tag{5.39}$$

and

$$\begin{aligned} \frac{d^3\phi_{n-1}}{du^3} &+ 3\frac{u''}{(u')^2} \frac{d^2\phi_{n-1}}{du^2} + \frac{u'''}{(u')^3} \frac{d\phi_{n-1}}{du} \\ &+ \frac{d\phi_n}{du} - \frac{s(u)}{4(u')^3}\phi_n - \frac{2}{\phi_0^3 (u')^2} \frac{d\phi_0}{du}\phi_n = \frac{\phi_1}{(u')^2 \phi_0^3} \frac{d\phi_{n-1}}{du} + \cdots. \end{aligned} \tag{5.40}$$

Substituting the ansatz (5.36) into equation (5.40) and equating coefficients at the next order gives $\gamma =$ constant and

$$\beta + u \log u \frac{d\beta}{du} = \gamma + 1 + \frac{u'' u}{2(u')^2} + \frac{3s(u)u}{8(u')^3}, \tag{5.41}$$

with solution

$$\beta = \gamma + 1 + \frac{\log u'}{2\log u} + \frac{3\sinh^{-1}(s^2/\sqrt{8})}{4\log u} + \frac{C}{\log u}. \tag{5.42}$$

As before, $C = 0$ without loss of generality. As $u \to 0$, $u \sim -i(2\sqrt{2}/5)\sigma^{3/4}(s-\sigma)^{5/4}$, and $\phi_0 \sim \sigma/\sqrt{2} - (\sigma/2)^{1/2}(s-\sigma)^{1/2}$. Hence $\beta_0 = -2/5$ and

$$\gamma = \beta_0 - \frac{11}{10} = -\frac{3}{2}. \tag{5.43}$$

Thus

$$\phi_n \sim \frac{\Lambda(-1)^n\Gamma(2n+\gamma+1)}{u^{2n+\gamma+1}(u')^{1/2}}\exp\left(-\frac{3}{4}\sinh^{-1}\left(\frac{s^2}{\sqrt{8}}\right)\right). \tag{5.44}$$

Truncating after N terms,

$$\phi \sim \sum_{n=0}^{N-1}\epsilon^{2n}\phi_n + R_N, \tag{5.45}$$

gives

$$\begin{aligned}\epsilon^2\frac{d^3R_N}{du^3} + 3\epsilon^2\frac{u''}{(u')^2}\frac{d^2R_N}{du^2} + \epsilon^2\frac{u'''}{(u')^3}\frac{dR_N}{du}& \\ + \frac{dR_N}{du} - \frac{s(u)}{4(u')^3}R_N - \frac{2}{\phi_0^3(u')^2}\frac{d\phi_0}{du}R_N &\sim -\epsilon^{2N}\frac{d^3\phi_{N-1}}{du^3}.\end{aligned} \tag{5.46}$$

The homogeneous version of this equation has solutions with the following asymptotic behaviour:

$$R_N \sim \frac{e^{-iu/\epsilon}\exp(-3\sinh^{-1}(s^2/\sqrt{8})/4)}{(u')^{1/2}}, \tag{5.47}$$

$$R_N \sim \frac{e^{iu/\epsilon}\exp(-3\sinh^{-1}(s^2/\sqrt{8})/4)}{(u')^{1/2}}, \tag{5.48}$$

$$R_N \sim \frac{\exp(3\sinh^{-1}(s^2/\sqrt{8})/2)}{(u')^2}. \tag{5.49}$$

We find the optimal truncation point is $N = r/(2\epsilon) + \alpha$ as before. Setting

$$R_N = A\frac{e^{-iu/\epsilon}\exp(-3(\sinh^{-1}(s^2/\sqrt{8}))/4)}{(u')^{1/2}} \tag{5.50}$$

we find

$$\frac{2ie^{-i\theta}}{r}\frac{dA}{d\theta}\exp(-ire^{i\theta}/\epsilon) \sim -\frac{\sqrt{2\pi}\Lambda}{\sqrt{r}\epsilon^{\gamma+3/2}}e^{-i\pi/2(r+2\alpha)}e^{-r/\epsilon}e^{-i\theta(r/\epsilon+2\alpha+\gamma+2)}. \tag{5.51}$$

Hence $dA/d\theta$ is exponentially small except on the Stokes line $\theta = -\pi/2$. We rescale by setting $\theta = -\pi/2 + \delta\bar{\theta}$. The correct scalings are $\delta = \sqrt{\epsilon}$, $A = \hat{A}/\epsilon^{\gamma+1}$ as before, giving

$$\frac{d\hat{A}}{d\bar{\theta}} \sim \frac{\sqrt{\pi}\Lambda e^{i\pi/2(\gamma+2)}\sqrt{r}e^{-r\theta^2/2}}{\sqrt{2}}. \tag{5.52}$$

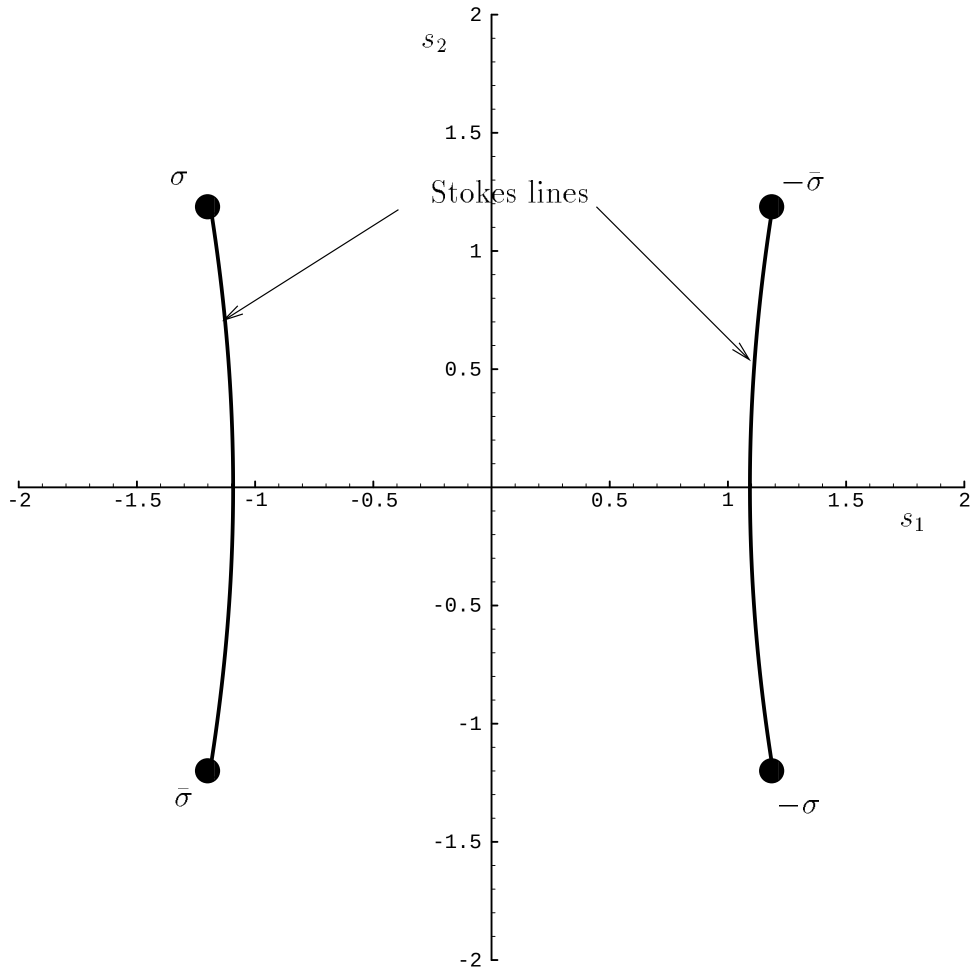

FIG. 5.1. *Stokes lines in the vortex reconnection problem;* $s = s_1 + is_2$, s_1, s_2 *real.*

Thus there is a jump in A across the Stokes line given by

$$A(\theta = -\pi/2^-) - A(\theta = -\pi/2^+) = -\frac{\Lambda \pi e^{i\pi\gamma/2}}{\epsilon^{\gamma+1}} = -\Lambda \epsilon^{1/2} \pi e^{-3i\pi/4}. \tag{5.53}$$

The Stokes lines occur when $-u^2$ is real and positive. As in the anisotropic crystal growth problem, there are Stokes lines linking the two singularities in the left half-plane, and Stokes lines linking the two singularities in the right half-plane. However, in the present problem the Stokes lines are curved, which will generically be the case when u' is not constant. They are shown in Figure 5.1.

Where the left-hand Stokes line cuts the real axis

$$(5.54)\quad -\frac{R2^{1/2}\pi\epsilon^{1/2}}{(s^4+8)^{1/8}}\exp\left(-\frac{3}{4}\sinh^{-1}\left(\frac{s^2}{\sqrt{8}}\right)\right)\times$$
$$\exp\left(-\frac{1}{2\sqrt{2}\,\epsilon}\int_0^{8^{1/4}}(8-y^4)^{1/4}\,dy\right)\times$$
$$\cos\left(\Phi-\frac{3\pi}{4}-\frac{1}{2\sqrt{2}\,\epsilon}\int_0^{8^{1/4}}(8-y^4)^{1/4}\,dy-\frac{1}{2\epsilon}\int_0^s(x^4+8)^{1/4}\,dx\right)$$

is switched on. Similarly, where the right-hand Stokes line cuts the real axis

$$(5.55)\quad \frac{R2^{1/2}\pi\epsilon^{1/2}}{(s^4+8)^{1/8}}\exp\left(-\frac{3}{4}\sinh^{-1}\left(\frac{s^2}{\sqrt{8}}\right)\right)\times$$
$$\exp\left(-\frac{1}{2\sqrt{2}\,\epsilon}\int_0^{8^{1/4}}(8-y^4)^{1/4}\,dy\right)\times$$
$$\cos\left(-\Phi+\frac{3\pi}{4}+\frac{1}{2\sqrt{2}\,\epsilon}\int_0^{8^{1/4}}(8-y^4)^{1/4}\,dy-\frac{1}{2\epsilon}\int_0^s(x^4+8)^{1/4}\,dx\right)$$

is switched on. As in the anisotropic crystal growth problem, there will be a solution only if these two contributions cancel as $s\to\infty$. Hence the condition for a solution is

$$(5.56)\qquad \sin\left(\Phi-\frac{3\pi}{4}-\frac{1}{2\sqrt{2}\,\epsilon}\int_0^{8^{1/4}}(8-y^4)^{1/4}\,dy\right)=0.$$

This is exactly the condition that $\phi'(0)=0$ to leading order. Note that even though we only have one symmetry condition in this case (unlike the two conditions $\phi(0)=\phi''(0)=0$ in the crystal growth problem) we cannot add a multiple of the nonoscillatory homogeneous solution (5.49) to the truncated asymptotic expansion in order to fix $\phi'(0)=0$, since this solution fails to satisfy the boundary condition at infinity.

To calculate the constant Λ we consider the behaviour of ϕ_n as $s\to\sigma$ as before. It is not difficult to show that

$$(5.57)\qquad \phi_n\sim-\frac{\sigma^{5n/2+1/2}A_n}{\sqrt{2}(s-\sigma)^{5n/2-1/2}},\qquad \text{as } s\to\sigma,$$

where the A_n satisfy the recurrence relation

$$(5.58)\qquad 4(5n/2-1)A_n=-(5n/2-3)(5n/2-2)(5n/2-1)A_{n-1}-4\sum_{m=1}^{n-1}(5m/2-1/2)A_mA_{n-m},$$

$$(5.59)\qquad A_0=1.$$

Comparing (5.44) and (5.57) we have

$$\Lambda = -\exp\left(\frac{3}{4}\sinh^{-1}\left(\frac{\sigma^2}{\sqrt{8}}\right)\right)\frac{\sigma^{3\gamma/4+13/8}e^{-i\pi/2(\gamma+3/2)}8^{\gamma/2+1/4}}{5^{\gamma+1}}\times \lim_{n\to\infty}\left(\frac{64}{25}\right)^n\frac{(-1)^n A_n}{\Gamma(2n+\gamma+1)}, \tag{5.60}$$

$$= -\frac{5^{1/2}}{8^{3/8}}\lim_{n\to\infty}\left(\frac{64}{25}\right)^n\frac{(-1)^n A_n}{\Gamma(2n-1/2)}. \tag{5.61}$$

Solving (5.58), (5.59) numerically up to large n gives

$$\Lambda \approx 0.202. \tag{5.62}$$

Hence the condition (5.56) for a solution to exist becomes

$$\sin\left(-\frac{3\pi}{4} - \frac{1}{2\sqrt{2}\,\epsilon}\int_0^{8^{1/4}}(8-y^4)^{1/4}\,dy\right) = 0, \tag{5.63}$$

i.e.

$$\epsilon = \epsilon_n \sim \frac{1}{2\sqrt{2}\,\pi(n+1/4)}\int_0^{8^{1/4}}(8-y^4)^{1/4}\,dy, \tag{5.64}$$

$$= \frac{\Gamma(1/4)\Gamma(5/4)}{2\pi^{3/2}(n+1/4)} \qquad n \in \mathbb{Z}. \tag{5.65}$$

Of course, since we have been performing asymptotics as $\epsilon \to 0$, this formula is valid in the limit of large n. With $n = 0$ the asymptotic value $\epsilon_0 \approx 1.180$ is 57% larger than the true value $\epsilon_0 \approx 0.749$, obtained by direct numerical solution of the eigenvalue problem. However, even with $n = 1$ the asymptotic value $\epsilon_1 \approx 0.236$ is only 10% larger than the true value $\epsilon_1 \approx 0.215$.

6. Conclusion. We have developed a method of calculating the exponentially small terms "beyond all orders" in the asymptotic expansion of the solution of singularly perturbed nonlinear ordinary differential equations. The method is based on applying a WKBJ-type ansatz to the late terms in the asymptotic expansion and identifying Stokes lines, and is closely related to the well-known Stokes line smoothing phenomenon in linear ordinary differential equations.

The method is also closely related to that of Kruskal & Segur, who perform a local analysis near the singularities of the leading order solution using matched asymptotics to identify the subdominant exponentials in the outer expansion. These exponentials can be identified as solutions of the homogeneous linearised equation, and the candidate Stokes lines obtained by equating phases of pairs of exponentials. Which of these candidates are actually Stokes lines, and the exponentials present in each region, can then be obtained by matching with the inner region.

Here we have found the Stokes lines and subdominant exponentials present by optimally truncating the algebraic expansion and observing the Stokes switching directly. A complete determination of the outer solution involves finding the constant Λ, for which we use the inner limit of the outer expansion. In the problems we have considered we have been able to determine the phase of Λ exactly, leading to explicit analytical conditions for the existence of solutions. For many problems the precise value of the modulus of Λ is unimportant, so long as it is nonzero.

One point our approach makes transparent is that the most important singularity is not necessarily the closest singularity to the real axis; the most important singularity is that which leads to the smallest value of $|\Im(u)|$ on the real axis (with the ansatz (2.10)). This will be the nearest singularity to the real axis when $u = s - \sigma$, as in the crystal growth problem, but for nontrivial u it may not be.

As we have noted, the method we use and the Kruskal-Segur method are closely related (and necessarily lead to the same results), but we believe that the extra tools we have presented will be valuable, particularly for more complicated problems such as partial differential equations with free boundaries.

Our approach may also be extended to nonlinear delay equations, for which there are infinitely many solutions of the linearised equation which may be switched on across Stokes lines[1]. In this case the polynomial equation for u' is of infinite order, but can typically be summed.

Acknowledgement. We would like to thank the referees for their valuable comments.

REFERENCES

[1] K.L. Adams, Ph. D. Thesis, University of Nottingham (1997).
[2] T.R. Akylas & T.-S. Yang, *On short-scale oscillatory tails of long-wave disturbances*, Stud. Appl. Math. **94** (1995), pp. 1–20.
[3] C.J. Amick & J.B. McLeod, *A singular perturbation problem in needle crystals*, Arch. Rat. Mech. Anal., **109** (1990), pp. 139–171.
[4] C.J. Amick & J.F. Toland, *A differential equation in the theory of resonant oscillations of water waves*, Proc. Roy. Soc. Edin., **114A** (1990), pp. 15–26.
[5] M.V. Berry, *Waves near Stokes lines*, Proc. R. Soc. Lond. A**427** (1990), pp. 265–280.

[1]Note, however, that not all delay equations have factorial over power divergence, with the corresponding error function smoothing of Stokes lines, so that some care needs to be taken [11].

[6] M.V. Berry, *Infinitely many Stokes smoothings in the Gamma-function*, Proc. Roy. Soc. Lond. A**434** (1991), pp. 465–472.
[7] W.G.C. Boyd, *Gamma-function asymptotics by an extension of the method of steepest descents*, Proc. Roy. Soc. Lond. A**447** (1994), pp. 609–630.
[8] J.G. Byatt-Smith, *The asymptotic solution of a connection problem of a second-order ordinary differential equation*, Stud. Appl. Math. **80** (1989), pp. 109–135.
[9] J.G. Byatt-Smith, *On the existence of homoclinic and heteroclinic orbits for differential equations with a small parameter*, Euro. J. Appl. Math. **2** (1991), pp. 133–158.
[10] J.G. Byatt-Smith & A.M. Davie, *Exponentially small oscillations in the solution of an ordinary differential equation*, Asymptotics beyond All Orders, NATO ASI Series B: Physics Vol. 284 (1991), pp. 223–240.
[11] S.J. Chapman, *On the nonuniversality of the error function in the smoothing of Stokes discontinuities*, Proc. R. Soc. Lond. **452**, pp. 2225–2230 (1996).
[12] S.J. Chapman, B.J. Hunton & J.R. Ockendon, *Vortices and boundaries*, Quart. Appl. Math., (1997) to appear.
[13] R.B. Dingle, *Asymptotic Expansions: their Derivation and Interpretation*, Academic Press (1973).
[14] R. Grimshaw & A.P. Hooper, *The non-existence of a certain class of travelling wave solutions of the Kuramoto-Sivashinsky equation*, Physica D, **50** (1991), pp. 231–238.
[15] R. Grimshaw & N. Joshi, *Weakly nonlocal solitary waves in a singularly perturbed Korteweg-de Vries equation*, SIAM J. Appl. Math., **55** (1995), pp. 124–135.
[16] J. Jones, W.C. Troy & A.D. MacGillivary, *Steady solutions of the Kuramoto-Sivashinsky equation for small wave speed*, J. Diff. Eqn. **96** (1992), pp. 28–55.
[17] M.D. Kruskal & H. Segur, *Asymptotics beyond all orders in a model of crystal growth*, Stud. Appl. Math., **85** (1991), pp. 129-181.
[18] J.B. McLeod, *Smoothing of Stokes discontinuities*, Proc. Roy. Soc. Lond., A**437** (1992), pp. 343–354.
[19] A.B. Olde Daalhuis, S.J. Chapman, J.R. King, J.R. Ockendon & R.H. Tew, *Stokes phenomenon and matched asymptotic expansions*, SIAM J. Appl. Math., **55** (1995), pp. 1469–1483.
[20] R.B. Paris & A.D. Wood, *Exponentially-improved asymptotics for the Gamma function*, J. Comp. Appl. Math. **41** (1992), pp. 135–143.
[21] Y. Pomeau, A. Ramani & B. Grammaticos, *Structural stability of the Korteweg-de Vries solitons under a singular perturbation*, Physica D **31** (1988), pp. 127–134.