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# Notices

## of the American Mathematical Society

January 2023

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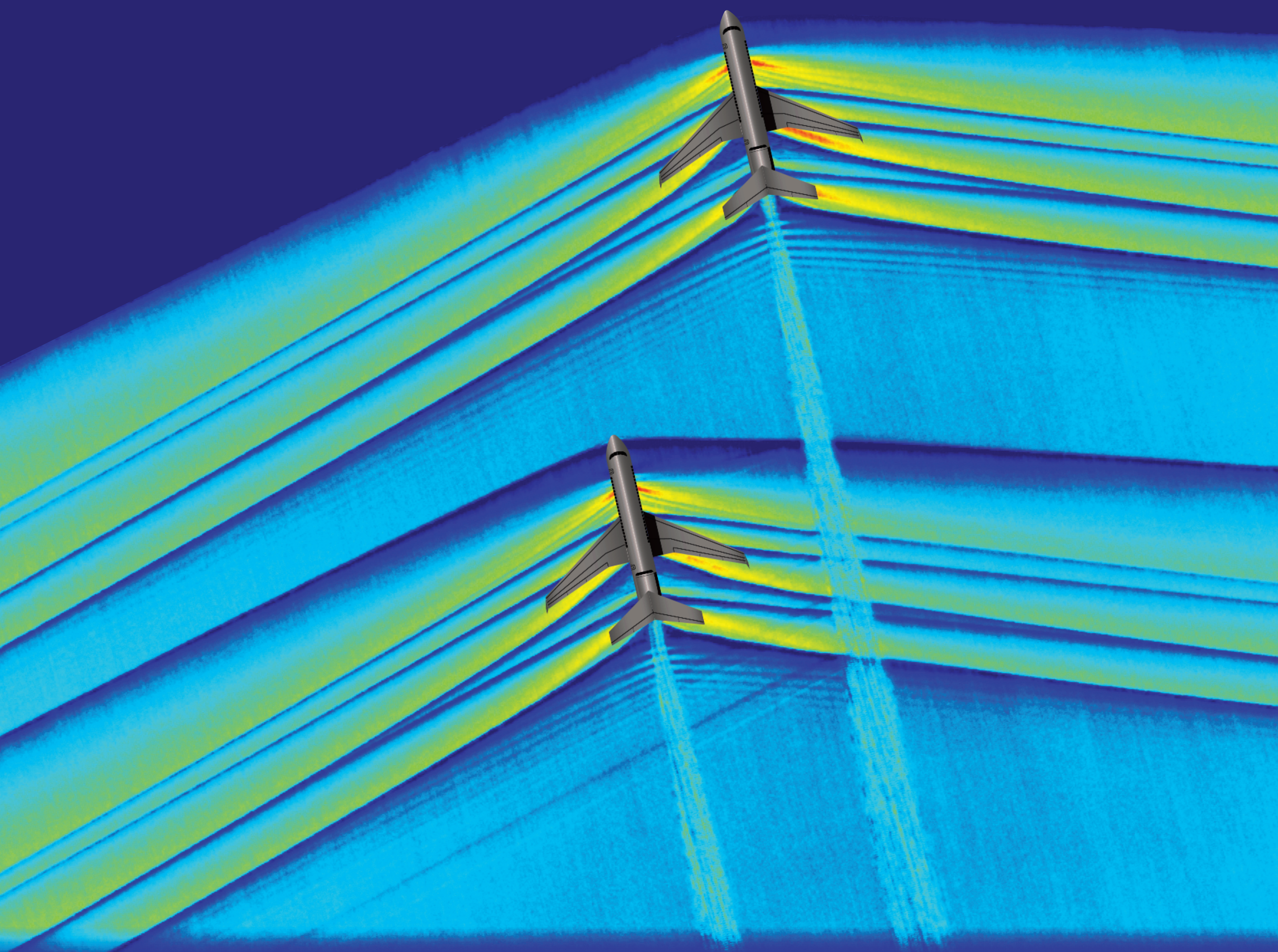

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*Advancing research. Creating connections.*

*The cover design is based on imagery from "Partial Differential Equations of Mixed Type—Analysis and Applications," page 8.*

AMERICAN MATHEMATICAL SOCIETY

# Current Events Bulletin

Friday, January 6, 2023
2:00–6:00 pm

Ballroom AB, Hynes Convention Center | Joint Mathematics Meetings, Boston, MA

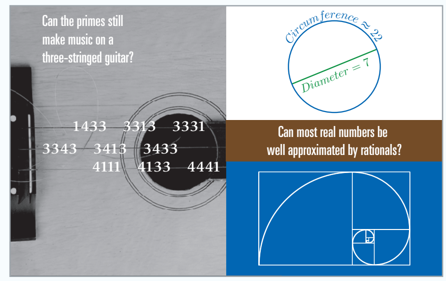
## 2:00 pm

**Andrew Granville**
*Université de Montréal*

*Supported by the Bose, Datta, Mukhopadhyay, and Sarkar Fund.*

### Missing digits, and good approximations

What wonder will be next in the ancient study of the sequence of primes?



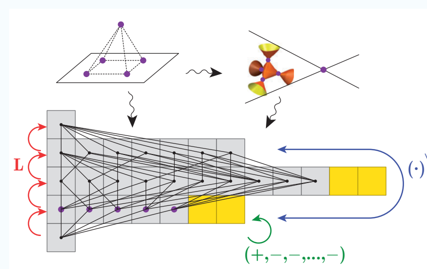

Courtesy of The American Mathematical Society. Guitar image courtesy of CaoChunhai / iStock / Getty Images.

## 3:00 pm

**Christopher Eur**
*Harvard University*

### An essence of independence: recent works of June Huh on combinatorics and Hodge theory

Matroids, an abstract setting for linear independence are a backbone of combinatorics. Now they have fused with a central part of algebraic geometry over the complex numbers.




Courtesy of The American Mathematical Society. Real points of the Cayley surface image courtesy of Salix alba/Wikimedia Commons.

## 4:00 pm

**Henry Cohn**
*Massachusetts Institute of Technology*

### From sphere packing to Fourier interpolation

What's with dimension 8 that makes it so special?

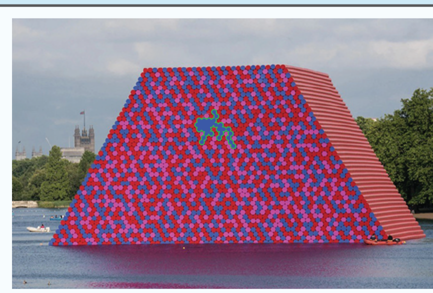
Courtesy of Baloncici / iStock / Getty Images Plus.

## 5:00 pm

**Martin Hairer**
*Imperial College London*

### A stroll around the critical Potts model

Phase transitions are all around us. Perhaps this is a phase transition in the theory of phase transitions!



Courtesy of The American Mathematical Society. Photography by Wolfgang Volz.

Organized by **David Eisenbud**, *University of California, Berkeley*

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jmm 2023 Joint Mathematics Meetings
Boston • January 4–7

Notices
of the American Mathematical Society

January 2023

*Cover Credit: The image used in the cover design appears in "Partial Differential Equations of Mixed Type—Analysis and Applications," p. 8, and is courtesy of NASA.*

## FEATURES

## FROM THE AMS SECRETARY



## INVITATIONS FROM THE AMS



ERRATUM. As noted by Serge Lvovski, the caption for Figure 5 in "Nikolay Konstantinov, 01.02.1932–07.03.2021, a Mathematical Educator Par Excellence" in the December 2022 *Notices* contained an error. Instead of "Nikolai Konstantinov, ?, Mikhail Tsfasman, Sergei Lando" it should read "Nikolai Konstantinov, Tatiana Galkina, Vladimir Pavlov, Sergei Lando," and instead of "Sergey Novikov, ?, Alexander Muzykantsky, ?, Vladimir Tikhomirov" it should read "Sergey Novikov, Andrei Pogrebkov, Alexander Muzykantsky, Andrei Kirillov, Vladimir Tikhomirov."

# Notices
of the American Mathematical Society




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The American Mathematical Society is committed to promoting and facilitating equity, diversity and inclusion throughout the mathematical sciences. For its own long-term prosperity as well as that of the public at large, our discipline must connect with and appropriately incorporate all sectors of society. We reaffirm the pledge in the AMS Mission Statement to "advance the status of the profession of mathematics, encouraging and facilitating full participation of all individuals," and urge all members to conduct their professional activities with this goal in mind. (*as adopted by the April 2019 Council*)


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# LETTERS TO THE EDITOR

## Letter to the Editor

*On August 23, 2022, this letter went to 2,176 department chairs and faculty of mathematics in the United States and was featured on AMS social media on September 1, 2022.*

Dear Colleague:

This week, we received a letter of concern from multiple faculty, including several of whom are directing REU programs. It seems that several of their students have encountered problems accessing the Math GRE Subject Test this fall. The exam is not offered in several states and even when it is offered, the spots are filling up quickly. If there is a test center within 125 miles (as the crow flies), students are expected to travel there to take the exam; otherwise, students can seek permission to have their home campus designated as an alternative test site. We even heard of one student who had arranged with ETS to have their school administer their test, but subsequently ETS added a new test center 110 miles away from campus so now the student has to pivot and figure out how to get to this new site.

We understand that for some graduate programs in mathematics, the GRE Subject Test provides useful information, but we are alarmed to learn that many of our students are finding scheduling and traveling to the exam to be a serious obstacle.

We want to bring this issue to your attention, and request that you be understanding in reviewing applications from students who may ultimately be unable to take the exam. We also ask that you please alert your faculty to this issue so they can take it into consideration when writing their letters of recommendation.

*Sincerely,*
*Ruth Charney, AMS President*
*Catherine Roberts, AMS Executive Director*

## Listening to I. M. Singer with I. M. Singer

The year was 1974, the city Vancouver, B. C. Canada. The occasion: ICM 1974. During a break, I went looking for a restroom. I opened a door to a small room and there sat a gentleman listening intently to a lecture on a small TV screen. I recognized the man. I had listened to the same talk live with him as the speaker. As I was about to duck away, he said "No, join me." Together, we watched and listened to his talk as if in prayer. At the end of the talk, we shook hands silently and went our separate ways. The speaker was I. M. Singer.

*Sincerely,*
*Frank Okoh*
*Department of Mathematics*
*Wayne State University*



*We invite readers to submit letters to the editor at notices-letters@ams.org.*

# A WORD FROM...

Trena L. Wilkerson, President of the
National Council of Teachers of Mathematics

*The opinions expressed here are not necessarily those of the* Notices *or the AMS.*

Photo courtesy of Baylor University

Who is a mathematician? Are we facilitating the development and support for individuals to explore and engage in the wonder, joy, and beauty of mathematics? The National Council of Teachers of Mathematics (NCTM) advocates for high-quality mathematics teaching and learning for each and every student. This aligns with the American Mathematical Society's mission of encouraging and promoting the transmission of mathematical understanding and skills, and fostering an awareness and appreciation of mathematics and its connections to other disciplines and everyday life. These are important directions for our mathematics community. If we are to expand opportunities for all students in mathematics we must be committed to actions that support all students in doing mathematics across PK–12, postsecondary, and beyond. We must work to develop individuals so that they see themselves as confident, capable, lifelong learners of mathematics and statistics. This is vital to our future as a democratic society and for our place in the world community. I want to focus on actions we must commit to if we are to prepare students to successfully engage and lead in their world.

In 2018 NCTM released *Catalyzing Change in High School Mathematics: Initiating Critical Conversations* with the purpose of challenging stakeholders in PK–12 education to engage in essential conversations to examine current beliefs, policies, and practices that have had significant negative consequences both for students and the mathematical community as a whole. These conversations then should lead to plans to provide opportunities for all students to engage in mathematics in significant depth to prepare them to not only meet the needs of society but to be leaders in our world. It was apparent that it would be important to also have these critical conversations across early childhood, elementary and middle grades so that as students transition to high school they are fully prepared. To that end in 2020 NCTM released *Catalyzing Change in Early Childhood and Elementary Mathematics* and *Catalyzing Change in Middle School Mathematics*. Across all three publications there are four proposed recommendations that span the grade levels so that all learners have a "successful life-long journey with mathematics" [2, p. 9].

1. **Broaden the Purposes of Learning Mathematics**. Each and every individual should develop deep mathematical understanding as confident and capable learners; understand and critique the world through mathematics; and experience the wonder, joy, and beauty of mathematics.
2. **Create Equitable Structures in Mathematics**. PK–12 mathematics should dismantle inequitable structures, including ability grouping and tracking students into qualitatively different learning experiences and dead-end course pathways, and challenge spaces of marginality and privilege.
3. **Implement Equitable Mathematics Instruction**. Mathematics instruction should be consistent with research-informed and equitable teaching practices that foster students' positive mathematical identities and strong sense of agency.
4. **Develop Deep Mathematical Understanding.** Early childhood settings and elementary schools should build a strong foundation of deep mathematical understanding within middle school and at least the first two years of high school offering a common shared pathway with all students having a continuous four-year mathematics

*Trena L. Wilkerson is the president of the National Council of Teachers of Mathematics. Her email address is* `Trena_Wilkerson@baylor.edu`.

pathway. These should be grounded in the use of mathematical practices and process to coherently develop deep mathematical understanding ensuring the highest quality mathematics education for each and every student [1–3].

As you read and reflect on these recommendations, you may wonder what tracking is and if there is room for appropriate acceleration in mathematics? *Catalyzing Change in High School Mathematics* [1] distinguishes between tracking and acceleration. "Tracking is the practice of placing students into qualitatively different course pathways or qualitatively different mathematical learning experiences" [3, p. 29]. Tracking often reinforces the notion that some are capable of doing mathematics and others are not and becomes a system for labeling students in terms of perceived mathematical ability or potential. This often results in those labeled has having lower ability or potential being placed in mathematics classes that do not prepare them for continued study of mathematics, thus limiting their opportunities. Each and every student has the ability to learn significant mathematics when provided appropriate learning opportunities with an emphasis on reasoning, sense-making, and problem solving. Opportunities to expand understanding and explore mathematics should be open to a wide range of students, and if there are structural barriers that inhibit access to students these should be addressed and removed.

But what of acceleration? In 2016, NCTM's position statement *Providing Opportunities for Students with Exceptional Promise*, stated that "Students with exceptional mathematical promise must be engaged in enriching learning opportunities during and outside the school day to allow them to pursue their interests, develop their talent, and maintain their passion for mathematics." Acceleration can enable students with exceptional mathematical promise and interest to move ahead in the curriculum. It can support students who show skills, insights, or interests to be challenged to go deeper into mathematics, but we need to be cautious that acceleration practices do not set up mathematics learning as a race with winners and losers.

Rather than perpetuating a system based on moving as quickly as possible through a set of courses, we should work to develop a system that allows every student the opportunity to think deeply about mathematics and values sense-making and application. We want to grow the varied fields of mathematics, which I am fairly certain are not at capacity! We want and need more mathematicians. To do this, we need to ensure that multiple opportunities are available to all students and that critical concepts are not skipped or addressed in a rushed manner. We do not want students to believe that mathematics is about memorizing a process and that they should move faster at all costs, as this could lead them to dislike mathematics and all that the system represents. I believe that working together to support a system that values understanding over speed, values an experience where most if not all leave seeing the importance of mathematics, and cultivates within each student a belief that they can do, understand, and apply mathematics, will help to sustain the field of mathematics and a society that values the work that mathematicians do.

It is essential that a deep understanding is developed across mathematical concepts. For some, calculus may be a goal in high school. According to a joint Calculus position statement of NCTM and MAA "A high school calculus course should not be the singular end goal of the PK–12 mathematics curriculum at the expense of providing a broad spectrum of mathematical preparation." Thus, it is imperative that students have additional opportunities through pathways that include areas such as statistics, mathematical modeling, or data science which are important for students to understand and critique their world. These critical conversations and collaborations called for in *Catalyzing Change* are needed across PK–16 to ensure that all students are prepared for their future.

What does this mean for postsecondary mathematics education? *Catalyzing Change in High School* [1, p. 92–93] identifies five beginning actions for postsecondary educators.

- Ensuring strong articulation and seamless pathways between the high school and the postsecondary mathematics curricula;
- Collaborating with school- and district-based mathematics educators;
- Working with in-service and preservice teachers to support research-informed and equitable instructional practices focused on essential concepts across content domains of number, algebra and functions, statistics and probability, and geometry and measurement (See *Catalyzing Change in High School* for specifics related to these concepts);
- Collaborating with school and district educators to develop additional mathematics pathways and populating courses with essential concepts across content domains of number, algebra and functions, statistics and probability, and geometry and measurement;
- Collaborating with school and district educators to challenge and dismantle system structures that impede students' access to and success in mathematics.

Consider Maryam Mirzakhani's journey into mathematics. You know her as the first female and first Iranian to win the Fields Medal, mathematics' highest award. Initially she did not see herself as a mathematician. She loved stories, reading novels, writing, and doodling her thoughts. In middle school her math teacher told her that she was not particularly talented in mathematics. The following year, she had a teacher who encouraged her and introduced her to geometry, which she saw as different from any mathematics she had known before. It changed the direction of her journey into mathematics. One teacher's encouragement made her aware of the beauty of mathematics and opened opportunities for her to engage in rich mathematics, and our world has been forever changed.

If Maryam had not been supported in seeing mathematics with many purposes, think what would have been lost to her and to us. Every student who comes through our door is capable of doing mathematics, of learning and expanding their understanding of their world through mathematics. Are we providing learning spaces to unearth the mathematician in each student and foster in all this love of mathematics that we have? I challenge us to have these needed conversations among and across our organizations, in our mathematics and education departments, with PK–16 educators, and all stakeholders. Where will you start?

### References

[1] National Council of Teachers of Mathematics, *Catalyzing Change in High School Mathematics: Initiating Critical Conversations*, NCTM, Reston, VA, 2018.

[2] National Council of Teachers of Mathematics, *Catalyzing Change in Early Childhood and Elementary Mathematics: Initiating Critical Conversations*, NCTM, Reston, VA, 2020.

[3] National Council of Teachers of Mathematics, *Catalyzing Change in Middle School Mathematics: Initiating Critical Conversations*, NCTM, Reston, VA, 2020.

# Partial Differential Equations of Mixed Type —Analysis and Applications

*Gui-Qiang G. Chen*

Partial differential equations (PDEs) are at the heart of many mathematical and scientific advances. While great progress has been made on the theory of PDEs of standard types during the last eight decades, the analysis of nonlinear PDEs of mixed type is still in its infancy. The aim of this expository paper is to show – through several longstanding fundamental problems in fluid mechanics, differential geometry, and other areas – that many nonlinear PDEs arising in these areas are no longer of standard types, but lie at the boundaries of the classification of PDEs or, indeed, go beyond the classification and are of mixed type. Some interrelated connections, historical perspectives, recent developments, and current trends in the analysis of nonlinear PDEs of mixed type are also presented.

## 1. Linear Partial Differential Equations of Mixed Type

Three of the basic types of PDEs are *elliptic*, *hyperbolic*, and *parabolic*, following the classification introduced by Jacques Salomon Hadamard in 1923 (see Figure 1).

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*Communicated by* Notices *Associate Editor Reza Malek-Madani.*



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The prototype of second-order elliptic equations is the *Laplace equation*:

$$\Delta_{\mathbf{x}} u := \sum_{j=1}^{n} \partial_{x_j x_j} u = 0 \qquad \text{for } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n. \quad (1.1)$$

This equation often describes physical equilibrium states whose solutions are also called harmonic functions or potential functions, where $\partial_{x_j x_j}$ is the second-order partial derivative in the $x_j$-variable, $j = 1, \dots, n$. The simplest representative of hyperbolic equations is the *wave equation*:

$$\partial_{tt} u - \Delta_{\mathbf{x}} u = 0 \qquad \text{for } (t, \mathbf{x}) \in \mathbb{R}^{n+1}, \qquad (1.2)$$

which governs the propagation of linear waves (such as acoustic waves and electromagnetic waves). The prototype of second-order parabolic equations is the *heat equation*:

$$\partial_t u - \Delta_{\mathbf{x}} u = 0 \qquad \text{for } (t, \mathbf{x}) \in \mathbb{R}^{n+1}, \qquad (1.3)$$

which often describes the dynamics of temperature and diffusion/stochastic processes.

**Figure 1.** Jacques Salomon Hadamard (December 8, 1865–October 17, 1963) first introduced the classification of PDEs in [16].

At first glance, the forms of the Laplace/heat equations and the wave equation look quite similar. In particular, any steady solution of the wave/heat equations is a solution of the Laplace equation, and a solution of the Laplace equation often determines an asymptotic state of the time-dependent solutions of the wave/heat equations. However, the properties of the solutions of the Laplace/heat equations and the wave equation are significantly different. One important difference is in terms of the *infinite versus finite speed of propagation* of the solution, while another pertains to the *gain versus loss of regularity* of the solution; see [14, 16] and the references cited therein. Since the solutions of elliptic/parabolic PDEs share many common features, we focus mainly on PDEs of mixed elliptic-hyperbolic type from now on.

The distinction between the elliptic and hyperbolic types can be seen more clearly from the classification of two-dimensional (2-D) constant-coefficient second-order PDEs:

$$a_{11}\partial_{x_1x_1}u + 2a_{12}\partial_{x_1x_2}u + a_{22}\partial_{x_2x_2}u = f(\mathbf{x}) \qquad (1.4)$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Let $\lambda_1 \le \lambda_2$ be the two constant eigenvalues of the $2 \times 2$ symmetric coefficient matrix $(a_{ij})_{2\times 2}$. Then Equation (1.4) is classified as *elliptic* if

$$\det(a_{ij}) > 0 \iff \lambda_1\lambda_2 > 0 \iff a_{12}^2 - a_{11}a_{22} < 0, \quad (1.5)$$

while it is classified as *hyperbolic* if

$$\det(a_{ij}) < 0 \iff \lambda_1\lambda_2 < 0 \iff a_{12}^2 - a_{11}a_{22} > 0. \quad (1.6)$$

Notice that the left-hand side of Equation (1.4) is analogous to the quadratic (homogeneous) form:

$$a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2$$

for conic sections. Thus, the classification of Equation (1.4) is consistent with the classification of conic sections and quadratic forms in algebraic geometry, based on the sign of the discriminant: $a_{12}^2 - a_{11}a_{22}$. The corresponding quadratic curves are ellipses (incl. circles), hyperbolas, and parabolas (see Figure 2).

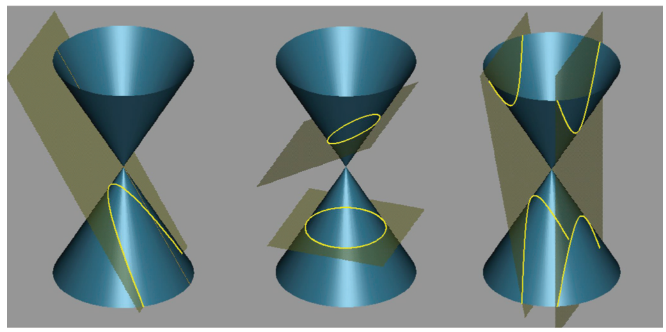

**Figure 2.** Types of conic sections: parabolas, ellipses, and hyperbolas.

This classification can also be seen by taking the Fourier transform on both sides of Equation (1.4):

$$(a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2)\hat{u}(\xi) = -\hat{f}(\xi) \qquad (1.7)$$

for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Here $\hat{w}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} w(\mathbf{x})e^{-i\mathbf{x}\cdot\xi}\,d\mathbf{x}$ is the Fourier transform of a function $w(\mathbf{x})$, such as $u(\mathbf{x})$ and $f(\mathbf{x})$ for (1.7). When Equation (1.4) is *elliptic*, the Fourier transform $\hat{u}(\xi)$ of solution $u(\mathbf{x})$ gains two orders of decay for the high Fourier frequencies (*i.e.*, $|\xi| \gg 1$) so that the solution gains the regularity of two orders from $f(\mathbf{x})$. When Equation (1.4) is *hyperbolic*, $\hat{u}(\xi)$ fails to gain two orders of decay for the high Fourier frequencies along the two characteristic directions in which $a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2 = 0$, even though it still gains two orders of decay for the high Fourier frequencies away from these two characteristic directions.

For the classification above, a general homogeneous constant-coefficient second-order PDE (*i.e.*, $f(\mathbf{x}) = 0$) with (1.5) or (1.6) can be transformed correspondingly into the Laplace equation (1.1) with $n = 2$, or the wave equation (1.2) with $n = 1$, via the corresponding coordinate transformations. This reveals the beauty of the classification theory that was first introduced by Hadamard in [16].

On the other hand, for general variable-coefficient second-order PDEs:

$$a_{11}(\mathbf{x})\partial_{x_1x_1}u + 2a_{12}(\mathbf{x})\partial_{x_1x_2}u + a_{22}(\mathbf{x})\partial_{x_2x_2}u = f(\mathbf{x}), \quad (1.8)$$

the situation is different. The classification depends upon the signature of the eigenvalues $\lambda_j(\mathbf{x}), j = 1, 2$, of the coefficient matrix $(a_{ij}(\mathbf{x}))$. In general, $\lambda_1(\mathbf{x})\lambda_2(\mathbf{x})$ may change its sign as a function of $\mathbf{x}$, which leads to the *mixed elliptic-hyperbolic type* of (1.8). Equation (1.8) is *elliptic* when $\lambda_1(\mathbf{x})\lambda_2(\mathbf{x}) > 0$ and *hyperbolic* when $\lambda_1(\mathbf{x})\lambda_2(\mathbf{x}) < 0$ with a transition boundary/region where $\lambda_1(\mathbf{x})\lambda_2(\mathbf{x}) = 0$.

Three of the classical prototypes for linear PDEs of mixed elliptic-hyperbolic type are as follows:

(i) *The Lavrentyev-Bitsadze equation*:

$$\partial_{x_1x_1}u + \mathrm{sign}(x_1)\partial_{x_2x_2}u = 0.$$

This equation exhibits a jump transition at $x_1 = 0$. It becomes the Laplace equation (1.1) in the half-plane $x_1 > 0$ and the wave equation (1.2) in the half-plane $x_1 < 0$, and changes its type from *elliptic* to *hyperbolic* via the jump-discontinuous coefficient $\mathrm{sign}(x_1)$.

(ii) *The Tricomi equation*: $\partial_{x_1x_1}u + x_1\partial_{x_2x_2}u = 0.$

This equation is of hyperbolic degeneracy at $x_1 = 0$. It is *elliptic* in the half-plane $x_1 > 0$ and *hyperbolic* in the half-plane $x_1 < 0$, and changes its type from *elliptic* to *hyperbolic* through the degenerate line $x_1 = 0$. This equation is of hyperbolic degeneracy in the domain $x_1 \le 0$, where the two characteristic families coincide *perpendicularly* to the line $x_1 = 0$. The degeneracy of the equation is determined by the classical elliptic or hyperbolic *Euler-Poisson-Darboux equation*:[1]

$$\partial_{\tau\tau}u \pm \partial_{x_2x_2}u + \frac{\beta}{\tau}\partial_\tau u = 0, \quad (1.9)$$

with $\beta = \frac{1}{3}$ for $\tau = \frac{2}{3}|x_1|^{\frac{3}{2}}$, and signs "$\pm$" corresponding to the half-planes $\pm x_1 > 0$ for $\mathbf{x}$ to lie in.

(iii) *The Keldysh equation*: $x_1\partial_{x_1x_1}u + \partial_{x_2x_2}u = 0.$

This equation is of parabolic degeneracy at $x_1 = 0$. It is *elliptic* in the half-plane $x_1 > 0$ and *hyperbolic* in the half-plane $x_1 < 0$, and changes its type from elliptic to hyperbolic through the degenerate line $x_1 = 0$. This equation is of parabolic degeneracy in the domain $x_1 \le 0$, in which the two characteristic families are quadratic parabolas lying in the half-plane $x_1 < 0$, and tangential at contact points to the degenerate line $x_1 = 0$. Its degeneracy is also determined by the classical elliptic or hyperbolic *Euler-Poisson-Darboux equation* (1.9) with $\beta = -\frac{1}{4}$ for $\tau = \frac{1}{2}|x_1|^{\frac{1}{2}}$.

For such a linear PDE, the transition boundary (*i.e.*, the boundary between the elliptic and hyperbolic domains) is known *a priori*. Thus, one traditional approach is to regard such a PDE as a degenerate elliptic or hyperbolic PDE in the corresponding domain, and then to analyze the solution behavior of these degenerate PDEs separately in the elliptic and hyperbolic domains with degeneracy on the transition boundary, determined, say, by the Euler-Poisson-Darboux type equations as (1.9). Another successful approach for dealing with such a PDE is the fundamental solution approach. With this approach, we first understand the behavior of the fundamental solution of the mixed-type PDE, especially its singularity, from which analytical/geometric properties of the solutions can then be revealed, since the fundamental solution is a generator of all of the solutions of the linear PDE. Great effort and progress have been made in the analysis of linear PDEs of mixed type by many leading mathematicians since the early 20th century (*cf.* [4, 6, 16, 18] and the references cited therein). Still, there are many important problems regarding linear PDEs of mixed type which require further understanding.

In the sections to come, we show, through several longstanding fundamental problems in fluid mechanics, differential geometry, and other areas, that many nonlinear PDEs arising in mathematics and science are no longer of standard type, but are in fact of mixed type. In contrast to the linear case, the transition boundary for a nonlinear PDE of mixed type is often *a priori* unknown, and the nonlinearity generates additional singularities in general. Thus, many classical methods and techniques for linear PDEs no longer work directly for nonlinear PDEs of mixed type. The lack of effective unified approaches is one of the main obstacles for tackling the elliptic/hyperbolic phases together for nonlinear PDEs of mixed type. Over the course of the last eight decades, the PDE research community has been largely partitioned according to the approaches taken to the analysis of different classes of PDEs (elliptic/hyperbolic/parabolic). However, advances in the analysis of nonlinear PDEs over the last several decades have made it increasingly clear that many difficult questions faced by the community lay at the boundaries of this classification or, indeed, go beyond this classification. In particular, many important nonlinear PDEs that arise in longstanding fundamental problems across diverse areas are of mixed type. As we will show in §2–§4, below, these problems include steady transonic flow problems and shock reflection/diffraction problems in gas dynamics, high-speed flow, and related areas (*cf.* [2, 3, 6, 12, 13, 15, 18–20]), and isometric embedding problems with optimal target dimensions and assigned regularity/curvatures in elasticity, geometric analysis, materials science, and other areas (*cf.* [11, 17]). The solution to these problems will advance our understanding of shock reflection/diffraction phenomena, transonic flows, properties/classifications of elastic/biological

[1] *J. Hadamard,* La Théorie des Équations aux Dérivées Partielles, *in French, Éditions Scientifiques, Peking; Gauthier-Villars Éditeur, Paris, 1964.*

surfaces/bodies/manifolds, and other scientific issues, and lead to significant developments of these areas and related mathematics. To achieve these goals, a deep understanding of the underlying nonlinear PDEs of mixed type (for instance, the solvability, the properties of solutions, *etc.*) is key.

## 2. Nonlinear PDEs of Mixed Type and Steady Transonic Flow Problems in Fluid Mechanics

In many applications, fluid flows are often regarded as time-independent; this is the case for some longstanding fundamental problems, such as that of transonic flows past multi-dimensional (M-D) obstacles (wedges/conic bodies, airfoils, *etc.*), or de Laval nozzles; see Figures 3–4. Furthermore, steady-state solutions are often global attractors as time-asymptotic equilibrium states, and serve as building blocks for constructing time-dependent solutions (*cf.* [6, 12, 13, 15]). The underlying nonlinear PDEs governing these fluid flows are generically of mixed type.

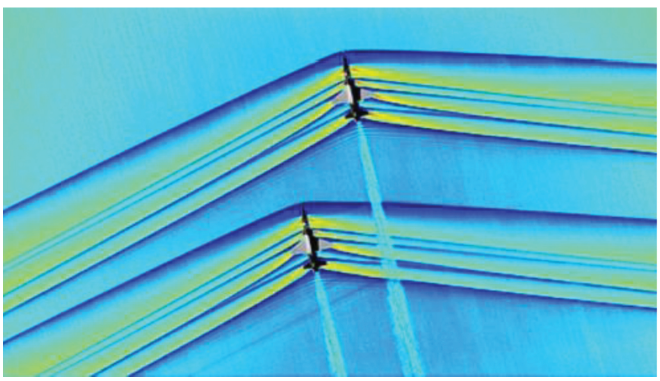

**Figure 3.** NASA's first Schlieren photo of shock waves interacting between two aircraft (taken in March 2019).

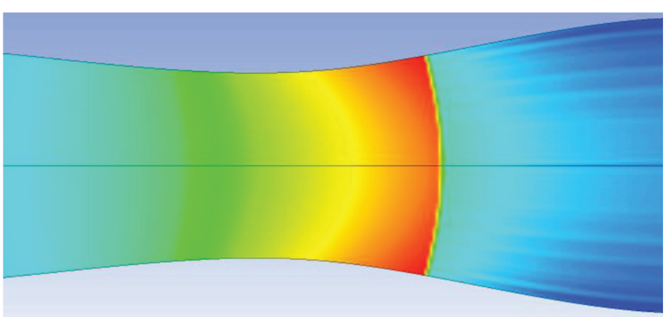

**Figure 4.** Diagram of a de Laval nozzle for the approximate flow velocity.

Our first example is steady potential fluid flows governed by the steady Euler equations of the conservation law of mass and Bernoulli's law:

$$\operatorname{div}(\rho\nabla\varphi) = 0, \ \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} = \frac{B_0}{\gamma-1} \qquad (2.1)$$

for $\mathbf{x} \in \mathbb{R}^n$ after scaling, where $\rho$ is the density, $\varphi$ is the velocity potential (*i.e.*, $v = \nabla\varphi$ is the velocity), $\gamma > 1$

**Figure 5.** Leonhard Euler (April 15, 1707–September 18, 1783) formulated the Euler equations for fluid mechanics; these are among the first PDEs to be written down.

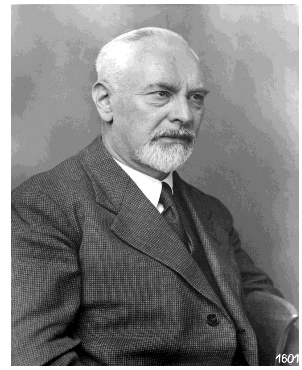

**Figure 6.** In 1936, Ludwig Prandtl (February 4, 1875–August 15, 1953) identified, via the shock polar analysis, two oblique shock configurations when a steady uniform supersonic gas flow passes a solid wedge.

is the adiabatic exponent for the ideal gas, $B_0/(\gamma-1)$ is the Bernoulli constant, and $\nabla$ is the gradient in $\mathbf{x}$. System (2.1), along with its time-dependent version (see (3.1) below), is one of the first PDEs to be written down by Euler (*cf.* Figure 5), and has been employed widely in aerodynamics and other areas in instances when the vorticity waves are weak in the fluid flow under consideration (*cf.* [3, 6, 12, 13, 15]). System (2.1) for the steady velocity potential $\varphi$ can be rewritten as

$$\operatorname{div}\big(\rho_B(|\nabla\varphi|)\nabla\varphi\big) = 0 \qquad (2.2)$$

with $\rho_B(q) = \big(B_0 - (\gamma-1)q^2/2\big)^{1/(\gamma-1)}$. Equation (2.2) is a *nonlinear conservation law of mixed elliptic-hyperbolic type*:

- strictly *elliptic* (subsonic) if $|\nabla\varphi| < c_* := \sqrt{2B_0/(\gamma+1)}$;
- strictly *hyperbolic* (supersonic) if $|\nabla\varphi| > c_*$.

The transition boundary here is $|\nabla\varphi| = c_*$ (sonic), a degenerate set of (2.2), which is *a priori* unknown, since it is determined by the solution itself.

Similarly, the time-independent full Euler flows are governed by the steady Euler equations:

$$\operatorname{div}(\rho v) = 0, \quad \operatorname{div}(\rho v\otimes v) + \nabla p = 0, \quad \operatorname{div}\big(\rho v(E + \frac{p}{\rho})\big) = 0, \qquad (2.3)$$

where $p$ is the pressure, $v$ is the velocity, and $E = \frac{1}{2}|v|^2 + e$ is the energy with $e = \frac{p}{(\gamma-1)\rho}$ as the internal energy determined by the thermodynamic constitutive equation of state. System (2.3) is a *system of conservation laws of mixed-composite hyperbolic-elliptic type*:

- strictly *hyperbolic* when $|v| > c$ (supersonic);

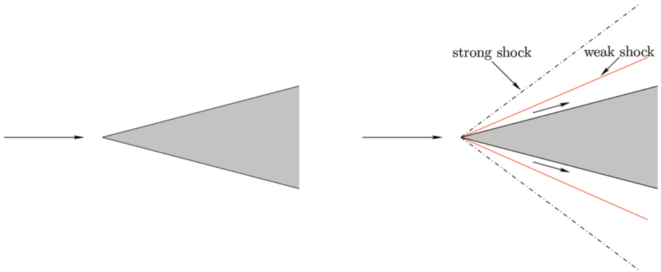


**Figure 7.** Two steady solutions with shocks around the solid wedge with an angle $\theta_\mathrm{w} \in (0, \theta_\mathrm{w}^\mathrm{s})$ or even $\theta_\mathrm{w} \in [\theta_\mathrm{w}^\mathrm{s}, \theta_\mathrm{w}^\mathrm{d})$.

- *mixed-composite elliptic-hyperbolic* (two of these are elliptic and the others are hyperbolic) when $|v| < c$ (subsonic),

where $c = \sqrt{\gamma p/\rho}$ is the sonic speed. The transition boundary between the supersonic/subsonic phase is $|v| = c$, a degenerate set of the solution of System (2.3), which is *a priori* unknown.

Many fundamental transonic flow problems in fluid mechanics involve these nonlinear PDEs of mixed type. One of these is a classical shock problem in which an upstream steady uniform supersonic gas flow passes a symmetric straight-sided solid wedge

$$W := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_2| < x_1 \tan\theta_\mathrm{w}, x_1 > 0\}, \quad (2.4)$$

whose (half-wedge) angle $\theta_\mathrm{w}$ is less than the detachment angle $\theta_\mathrm{w}^\mathrm{d}$ (*cf.* Figure 7).

Since this problem involves shocks, its global solution should be a weak solution of Equation (2.2) or System (2.3) in the distributional sense (which admits shocks)[2] in the domain under consideration (see [7]). For example, for Equation (2.2), a shock is a curve across which $\nabla\varphi$ is discontinuous. If $\Lambda^+$ and $\Lambda^-(:= \Lambda \setminus \overline{\Lambda^+})$ are two nonempty open subsets of a domain $\Lambda \subset \mathbb{R}^2$, and $\mathcal{S} := \partial\Lambda^+ \cap \Lambda$ is a $C^1$-curve across which $\nabla\varphi$ has a jump, then $\varphi \in C^1(\Lambda^\pm \cup \mathcal{S}) \cap C^2(\Lambda^\pm)$ is a global weak solution of (2.2) in $\Lambda$ if and only if $\varphi$ is in $W^{1,\infty}_{\mathrm{loc}}(\Lambda)$[3] and satisfies Equation (2.2) in $\Lambda^\pm$ and the Rankine-Hugoniot conditions on $\mathcal{S}$:

$$\begin{aligned} &\varphi_{\Lambda^+\cap\mathcal{S}} = \varphi_{\Lambda^-\cap\mathcal{S}}, \\ &\rho_B(|\nabla\varphi|^2)\nabla\varphi\cdot\nu|_{\Lambda^+\cap\mathcal{S}} = \rho_B(|\nabla\varphi|^2)\nabla\varphi\cdot\nu|_{\Lambda^-\cap\mathcal{S}}, \end{aligned} \quad (2.5)$$

where $\nu$ is the unit normal to $\mathcal{S}$ in the flow direction; *i.e.*, $\nabla\varphi\cdot\nu|_{\Lambda^\pm\cap\mathcal{S}} > 0$. A piecewise smooth solution with discontinuities satisfying (2.5) is called an *entropy solution* of (2.2) if it satisfies the following entropy condition: *The density $\rho$ increases in the flow direction of $\nabla\varphi_{\Lambda^+\cap\mathcal{S}}$ across any discontinuity*. Then such a discontinuity is called a *shock* (see [12]); see also Figure 8.[4]

**Figure 8.** Richard Courant (January 8, 1888–January 27, 1972) and Kurt Otto Friedrichs (September 28, 1901–December 31, 1982); their monumental book [12] has had a great impact upon the development of the M-D theory of shock waves and nonlinear PDEs of hyperbolic/mixed types.

For this problem, there are two configurations: the weak oblique shock reflection with supersonic/subsonic downstream flow (determined by the sonic angle $\theta_\mathrm{w}^\mathrm{s}$), and the strong oblique shock reflection with subsonic downstream flow; both of these satisfy the entropy condition, as was discovered by Prandtl (*cf.* Figure 6). The weak oblique shock is transonic with subsonic downstream flow for $\theta_\mathrm{w} \in (\theta_\mathrm{w}^\mathrm{s}, \theta_\mathrm{w}^\mathrm{d})$, while the weak oblique shock is supersonic with supersonic downstream flow for $\theta_\mathrm{w} \in (0, \theta_\mathrm{w}^\mathrm{s})$. However, the strong oblique shock is always transonic with subsonic downstream flow. The question of physical admissibility of one or both of the strong/weak shock reflection configurations was hotly debated for eight decades in the wake of Courant-Friedrichs [12] and von Neumann [20], and has only recently been better understood (*cf.* [7] and the references cited therein). There are two natural approaches to understanding this phenomenon: One is to examine whether these configurations are stable under steady perturbations, and the other is to determine whether these configurations are attainable as large-time asymptotic states (*i.e.*, the *Prandtl-Meyer problem*); both approaches involve the analysis of nonlinear PDEs (2.2) or (2.3) of mixed type.

Mathematically, the steady stability problem can be formulated as a free boundary problem with the perturbed shock-front:

$$\mathcal{S} = \{\mathbf{x} : x_2 = \sigma(x_1), x_1 \geq 0\} \quad (2.6)$$

with $\sigma(0) = 0$ and $\sigma(x_1) > 0$ for $x_1 > 0$ as a free boundary

[2] *P. D. Lax,* Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, *CBMS-RCSAM, No. 11, SIAM, Philadelphia, Pennsylvania, 1973.*

[3] *A $W^{k,p}$ function, for $1 \leq p \leq \infty$ and $k \geq 1$ integer, is a real-valued function such that itself and its (weak) derivatives up to order k are all $L^p$ functions.*

[4] *Author of the picture: Konrad Jacobs. Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach.*

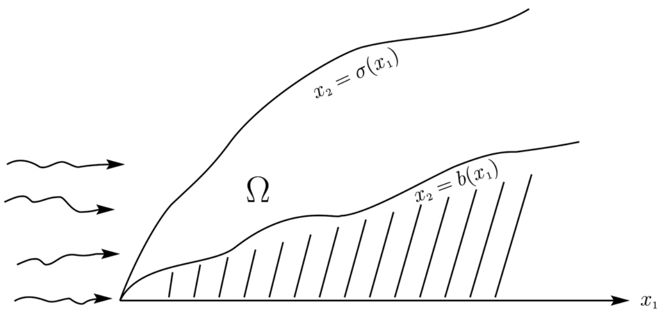


**Figure 9.** The leading steady shock $x_2 = \sigma(x_1)$ as a free boundary under the perturbation.

(with the Rankine-Hugoniot conditions, say (2.5), as free boundary conditions) to determine the domain behind $\mathcal{S}$:

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : b(x_1) < x_2 < \sigma(x_1), x_1 > 0\}, \qquad (2.7)$$

and the downstream flow in $\Omega$ for Equation (2.2) or System (2.3) of mixed elliptic-hyperbolic type, where $x_2 = b(x_1)$ is the perturbation of the flat wedge boundary $x_2 = x_1 \tan\theta_\mathrm{w}$. Such a global solution of the free boundary problem provides not only the global structural stability of the steady oblique shock, but also a more detailed structure of the solution.

Supersonic (*i.e.*, supersonic-supersonic) shocks correspond to the case when $\theta_\mathrm{w} \in (0, \theta_\mathrm{w}^\mathrm{s})$; these are shocks of weak strength. The local stability of such shocks was first established in the 1960s. The global stability and uniqueness of the supersonic oblique shocks for both Equation (2.2) and System (2.3) have been solved for more general perturbations of both the upstream steady flow and the wedge boundary, even in $BV$,[5] by purely hyperbolic methods and techniques (*cf.* [7] and the references cited therein).

For transonic (*i.e.*, supersonic-subsonic) shocks, it has been proved that the oblique shock of weak strength is always stable under general steady perturbations. However, the oblique shock of strong strength is stable only conditionally for a certain class of steady perturbations that require the exact match of the steady perturbations near the wedge-vertex and the downstream condition at infinity, which reveals one of the reasons why the strong oblique shock solutions have not been observed experimentally. In these stability problems for transonic shocks, the PDEs (or parts of the systems) are expected to be elliptic for global solutions in the domains determined by the corresponding free boundary problems; that is, we solve an expected elliptic free boundary problem. However, the earlier methods and approaches for elliptic free boundary problems do not directly apply to these problems, such as the variational methods, the Harnack inequality approach, and other elliptic methods/approaches. The main reason for this is that the type of equations needs to be controlled before we can apply these methods, and this requires some strong *a priori* estimates. To overcome these difficulties, the global structure of the problems is exploited, which allows us to derive certain properties of the solution so that the type of equations and the geometry of the problem can be controlled. With this, the free boundary problem, as described above, has been solved by an iteration procedure; see Chen-Feldman [7] and the references cited therein for more details.

When a subsonic flow passes through a *de Laval nozzle*, the flow may form a supersonic bubble with a transonic shock (see Figure 4); full understanding of how the geometry of the nozzle helps to create/stabilize/destabilize the transonic shock requires a deep understanding of the nonlinear PDEs of mixed type. Likewise, for the *Morawetz problem* for a steady subsonic flow past an airfoil, experimental results show that a supersonic bubble may be formed around the airfoil (see Figures 10–11), and the flow behavior is determined by the solution of a nonlinear PDE of mixed type.

Some fundamental problems for transonic flow posed in the 1950s–60s (*e.g.*, [3, 6, 12, 15, 19]) remain unsolved, though some progress has been made in recent years (*e.g.*, [6, 7, 13] and the references cited therein).

## 3. Nonlinear PDEs of Mixed Type and Shock Reflection/Diffraction Problems in Fluid Mechanics and Related Areas

In general, fluid flows are time-dependent. We now describe how some longstanding M-D time-dependent fundamental shock problems in fluid mechanics can naturally be formulated as problems for nonlinear PDEs of mixed type through a prototype: *the shock reflection-diffraction problem*.

When a planar shock separating two constant states (0) and (1), with constant velocities and densities $\rho_0 < \rho_1$ (state (0) is ahead or to the right of the shock, and state (1) is behind the shock), moves in the flow direction (*i.e.*, $v_1 > 0$) and hits a symmetric wedge (2.4) with (a half-wedge) angle $\theta_\mathrm{w}$ head-on at time $t = 0$, a reflection-diffraction process takes place for $t > 0$. A fundamental question that arises is which types of wave patterns of shock reflection-diffraction configurations may be formed around the wedge. The complexity of these configurations was first reported by Ernst Mach (*cf.* Figure 12), who observed two patterns of shock reflection-diffraction configurations: Regular reflection (two-shock configuration) and Mach reflection (three-shock/one-vortex-sheet configuration); these are shown in Figure 14, below.[6] The issue remained dormant until the 1940s, when John von Neumann [19, 20] (also *cf.* Figure 13) and other

[5] *A BV function is a real-valued function whose total variation is bounded.*

[6] *M. Van Dyke,* An Album of Fluid Motion, *The Parabolic Press, Stanford, 1982.*

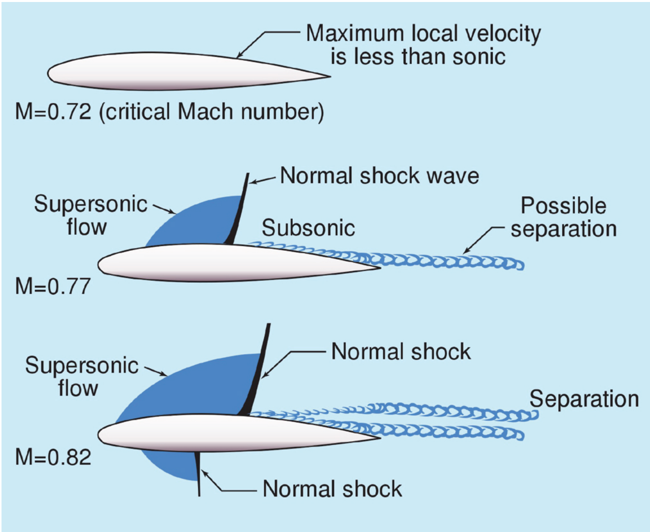


**Figure 10.** Transonic flow patterns on an airfoil showing flow patterns at and above the critical Mach number.

**Figure 11.** Aerodynamic condensation evidences of supersonic expansion fans around a transonic aircraft.

mathematical/experimental scientists (*cf.* [2, 6, 12, 15] and the references cited therein) began extensive research into all aspects of shock reflection-diffraction phenomena. It has been found that the situation is much more complicated than that which Mach originally observed; the shock reflection can be divided into more specific subpatterns, and various other patterns of shock reflection-diffraction configurations such as supersonic regular reflection, subsonic regular reflection, attached regular reflection, double Mach reflection, von Neumann reflection, and Guderley reflection may occur; see [2, 6, 12, 15] and the references cited therein (also see Figures 14–19, below). Then the fundamental scientific issues include:

(i) the structures of the shock reflection-diffraction configurations;

(ii) the transition criteria between the different patterns of the configurations;

**Figure 12.** Ernst Waldfried Josef Wenzel Mach (18 February 1838 – 19 February 1916), who first observed the complexity of shock reflection-diffraction configurations (1878).

**Figure 13.** John von Neumann (December 28, 1903–February 8, 1957), who proposed the sonic conjecture and the detachment conjecture for shock reflection-diffraction configurations.

(iii) the dependence of the patterns upon physical parameters such as the wedge angle $\theta_{\mathrm{w}}$, the incident-shock-wave Mach number (*i.e.*, the strength of the incident shock), and the adiabatic exponent $\gamma > 1$.

In particular, several transition criteria between the different patterns of shock reflection-diffraction configurations have been proposed; these include the *sonic conjecture* and the *detachment conjecture*, both put forward by von Neumann [19] (see also [2, 6]).

To present this more clearly, we now focus on the Euler equations for time-dependent compressible potential flow, which consist of the conservation law of mass and Bernoulli's law:

$$\partial_t\rho + \mathrm{div}(\rho\nabla\Phi) = 0, \quad \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1} \tag{3.1}$$

for $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^2$ after scaling, where $\Phi$ is the time-dependent velocity potential (*i.e.*, $v = \nabla\Phi$ is the velocity). Equivalently, System (3.1) can be reduced to the nonlinear wave equation of second-order:

$$\partial_t\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)\nabla_{\mathbf{x}}\Phi\big) = 0, \tag{3.2}$$

with $\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2)\big)^{\frac{1}{\gamma-1}}$, which is one of the original motivations for the extensive study of *nonlinear wave equations*.

Mathematically, the shock reflection-diffraction problem is a 2-D lateral Riemann problem for (3.1) or (3.2) in domain $\mathbb{R}^2 \setminus \overline{W}$ with $\rho_0, \rho_1, v_1 > 0$ satisfying

$$\rho_1 > \rho_0, \quad v_1 = (\rho_1 - \rho_0)\sqrt{\frac{2(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{\rho_1^2 - \rho_0^2}}. \tag{3.3}$$

**Problem 3.1** (Shock Reflection-Diffraction Problem). *Piecewise constant initial data, consisting of state* (0) *with*

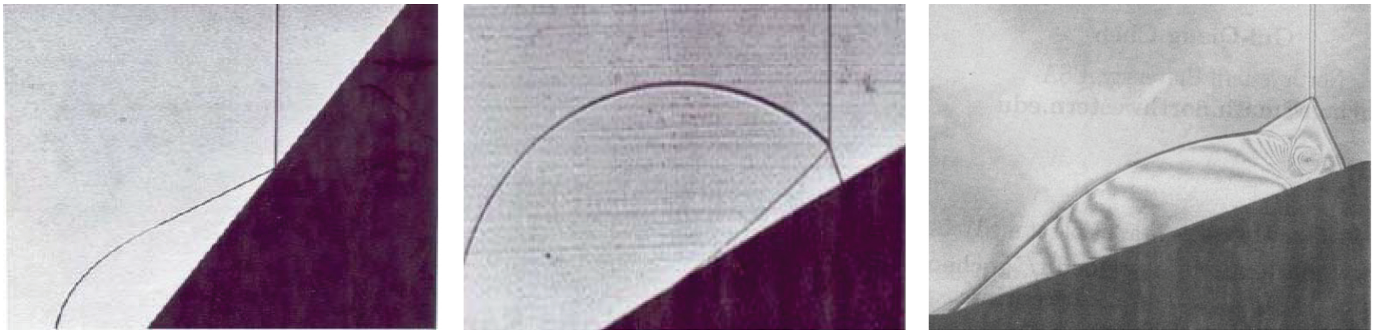

**Figure 14.** Three patterns of shock reflection-diffraction configurations.

*velocity* $\mathbf{v}_0 = (0,0)$ *and density* $\rho_0 > 0$ *on* $\{x_1 > 0\} \setminus \overline{W}$ *and state* (1) *with velocity* $\mathbf{v}_1 = (v_1, 0)$ *and density* $\rho_1 > 0$ *on* $\{x_1 < 0\}$ *connected by a shock at* $x_1 = 0$*, are prescribed at* $t = 0$*, satisfying* (3.3)*. Seek a solution of the Euler system* (3.1)*, or Equation* (3.2)*, for* $t \geq 0$*, subject to the initial data and the boundary condition* $\nabla\Phi \cdot \nu_{\mathrm{w}} = 0$ *on* $\partial W$*, where* $\nu_{\mathrm{w}}$ *is the unit outward normal to* $\partial W$*.*

Problem 3.1 is invariant under scaling: $(t, \mathbf{x}, \Phi) \to (\alpha t, \alpha \mathbf{x}, \frac{\Phi}{\alpha})$ for any $\alpha \neq 0$. Thus the problem admits self-similar solutions in the form:

$$\Phi(t, \mathbf{x}) = t\phi(\xi) \qquad \text{for } \xi = \frac{\mathbf{x}}{t}. \tag{3.4}$$

Then the pseudo-potential function $\varphi(\xi) = \phi(\xi) - \frac{1}{2}|\xi|^2$ satisfies the equation:

$$\operatorname{div}\big(\rho_B(|D\varphi|^2, \varphi)D\varphi\big) + 2\rho_B(|D\varphi|^2, \varphi) = 0, \tag{3.5}$$

with $\rho_B(|D\varphi|, \varphi) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\frac{1}{2}|D\varphi|^2 + \varphi)\big)^{\frac{1}{\gamma-1}}$, where the divergence div and gradient D are with respect to $\xi \in \mathbb{R}^2$. Define the pseudo-sonic speed $c = c(|D\varphi|, \varphi)$ by

$$c^2(|D\varphi|, \varphi) = \rho^{\gamma-1}(|D\varphi|^2, \varphi) = B_0 - (\gamma-1)\big(\frac{1}{2}|D\varphi|^2 + \varphi\big). \tag{3.6}$$

Equation (3.5) is of mixed elliptic-hyperbolic type:

- strictly *elliptic* if $|D\varphi| < c(|D\varphi|, \varphi)$ (pseudo-subsonic);
- strictly *hyperbolic* if $|D\varphi| > c(|D\varphi|, \varphi)$ (pseudo-supersonic).

The transition boundary between the pseudo-supersonic and pseudo-subsonic phases is $|D\varphi| = c(|D\varphi|, \varphi)$ (*i.e.*, $|D\varphi| = \sqrt{\frac{2}{\gamma+1}(B_0 - (\gamma-1)\varphi)}$), a degenerate set of the solution of Equation (3.5), which is *a priori* unknown and more delicate than that of Equation (2.2).

One class of solutions of (3.5) is that of *constant states*; these are solutions with constant velocity $\mathbf{v}_* \in \mathbb{R}^2$. Then the pseudo-potential of a constant state satisfies $D\varphi = \mathbf{v}_* - \xi$ so that

$$\varphi(\xi) = -\frac{1}{2}|\xi|^2 + \mathbf{v}_* \cdot \xi + C, \tag{3.7}$$

where $C$ is a constant. For this $\varphi$, the density $\rho$ and sonic speed $c = \rho^{(\gamma-1)/2}$ are positive constants, independent of $\xi$. Then, from (3.7), the ellipticity condition for the constant state is $|\xi - \mathbf{v}_*| < c$. Thus, for a constant state $\mathbf{v}_*$, Equation (3.5) is *elliptic* inside the *sonic circle*, with center $\mathbf{v}_*$ and radius $c$, and it is *hyperbolic* outside this circle. Moreover, if the density $\rho$ is a constant, then the solution is a constant state; that is, the corresponding pseudo-potential $\varphi$ is of form (3.7).

Problem 3.1 involves transonic shocks such that its global solution should be a weak solution of Equation (3.5) in the distributional sense within the domain in the $\xi$–coordinates (see [7]). If $\Lambda^+$ and $\Lambda^-(:= \Lambda \setminus \overline{\Lambda^+})$ are two nonempty open subsets of a domain $\Lambda \subset \mathbb{R}^2$, and $\mathcal{S} := \partial\Lambda^+ \cap \Lambda$ is a $C^1$-curve with a normal $\nu$ across which $D\varphi$ has a jump, then $\varphi \in C^1(\Lambda^\pm \cup \mathcal{S}) \cap C^2(\Lambda^\pm)$ is a global entropy solution of (3.5) in $\Lambda$ with $\mathcal{S}$ as a shock *if and only if* $\varphi$ is in $W^{1,\infty}_{\mathrm{loc}}(\Lambda)$ and satisfies Equation (3.5) in $\Lambda^\pm$, the Rankine-Hugoniot conditions on $\mathcal{S}$:

$$\varphi_{\Lambda^+\cap\mathcal{S}} = \varphi_{\Lambda^-\cap\mathcal{S}}, \tag{3.8}$$

$$\rho(|D\varphi|^2, \varphi)D\varphi \cdot \nu|_{\Lambda^+\cap\mathcal{S}} = \rho(|D\varphi|^2, \varphi)D\varphi \cdot \nu|_{\Lambda^-\cap\mathcal{S}}, \tag{3.9}$$

and the entropy condition stating that *the density* $\rho$ *increases in the pseudo-flow direction of* $D\varphi_{\Lambda^+\cap\mathcal{S}}$ *across any discontinuity.*

We now show how such solutions of the nonlinear PDE (3.5) of mixed elliptic-hyperbolic type in self-similar coordinates $\xi = \frac{\mathbf{x}}{t}$ can be constructed.

First, by the symmetry of the problem with respect to the $\xi_1$–axis, it suffices for us to focus only on the upper half-plane $\{\xi_2 > 0\}$, and to prescribe the following slip boundary condition: $D\varphi \cdot \nu_{\mathrm{sym}} = 0$ on the symmetry line $\Gamma_{\mathrm{sym}} := \{\xi_2 = 0\}$ for the interior unit normal $\nu_{\mathrm{sym}} = (0,1)$. Then Problem 3.1 can be reformulated as a boundary value problem in the unbounded domain:

$$\Lambda := \mathbb{R}^2_+ \setminus \{\xi : |\xi_2| \leq \xi_1 \tan\theta_{\mathrm{w}}, \xi_1 > 0\}$$

in the self-similar coordinates $\xi = (\xi_1, \xi_2)$, where $\mathbb{R}^2_+ := \mathbb{R}^2 \cap \{\xi_2 > 0\}$.

**Problem 3.2** (Boundary Value Problem). *Seek a solution* $\varphi$ *of Equation* (3.5) *in the self-similar domain* $\Lambda$ *with the slip*

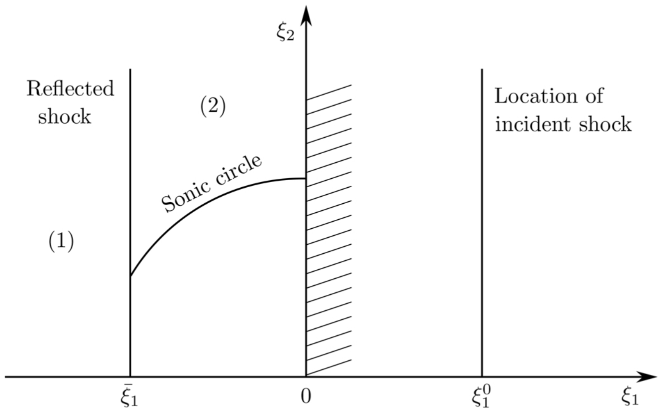


**Figure 15.** Normal reflection configuration.

*boundary condition*: $D\varphi \cdot \nu|_{\partial\Lambda} = 0$ *for the interior unit normal* $\nu$ *on* $\partial\Lambda$, *and the asymptotic boundary condition at infinity*:

$$\varphi \longrightarrow \bar{\varphi} = \begin{cases} \varphi_0 \text{ for } \xi_1 > \xi_1^0, \xi_2 > \xi_1 \tan\theta_{\mathrm{w}}, \\ \varphi_1 \text{ for } \xi_1 < \xi_1^0, \xi_2 > 0, \end{cases} \quad \text{when } |\xi| \to \infty,$$

*where* $\varphi_0 = -\frac{1}{2}|\xi|^2$ *and* $\varphi_1 = -\frac{1}{2}|\xi|^2 + v_1(\xi_1 - \xi_1^0)$ *with* $\xi_1^0 = \frac{\rho_1 v_1}{\rho_1 - \rho_0}$, *which is the location of the incident shock* $\mathcal{S}_0 = \{\xi_1 = \xi_1^0\} \cap \Lambda$ *determined by the Rankine-Hugoniot conditions* (3.8)–(3.9) *between states* (0) *and* (1) *on* $\mathcal{S}_0$.

The simplest case is when $\theta_{\mathrm{w}} = \frac{\pi}{2}$; this is called *normal reflection* (see Figure 15). In this case, the incident shock normally reflects from the flat wall to become the flat reflected shock $\xi_1 = \bar{\xi}_1 < 0$.

When $\theta_{\mathrm{w}} \in (0, \frac{\pi}{2})$, a necessary condition for the existence of a regular reflection solution, whose configurations are as shown in Figures 16–19, is the existence of the uniform state (2) with pseudo-potential $\varphi_2$ at $P_0$, determined by the three conditions at $P_0$:

$$D\varphi_2 \cdot \nu_{\mathrm{w}} = 0, \varphi_2 = \varphi_1, \rho(|D\varphi_2|^2, \varphi_2) D\varphi_2 \cdot \nu_{\mathcal{S}_1} = \rho_1 D\varphi_1 \cdot \nu_{\mathcal{S}_1} \tag{3.10}$$

for $\nu_{\mathcal{S}_1} = \frac{D(\varphi_1 - \varphi_2)}{|D(\varphi_1 - \varphi_2)|}$ across the flat shock $\mathcal{S}_1 = \{\varphi_1 = \varphi_2\}$ that separates state (2) from state (1) and satisfies the entropy condition: $\rho_2 > \rho_1$. These conditions lead to the system of algebraic equations (3.10) for the constant velocity $\mathbf{v}_2$ and the density $\rho_2$ of state (2). For any fixed densities $0 < \rho_0 < \rho_1$ of states (0) and (1), there exist a sonic angle $\theta_{\mathrm{w}}^{\mathrm{s}}$ and a detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}}$ satisfying that

$$0 < \theta_{\mathrm{w}}^{\mathrm{d}} < \theta_{\mathrm{w}}^{\mathrm{s}} < \frac{\pi}{2}$$

such that the algebraic system (3.10) has two solutions for each $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$ which become equal when $\theta_{\mathrm{w}} = \theta_{\mathrm{w}}^{\mathrm{d}}$. Thus, for each $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$, there exist two states (2), called weak and strong, with densities $0 < \rho_1 < \rho_2^{\mathrm{weak}} < \rho_2^{\mathrm{strong}}$ (the entropy condition). The weak state (2) is supersonic at the reflection point $P_0$ for $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$, sonic for $\theta_{\mathrm{w}} = \theta_{\mathrm{w}}^{\mathrm{s}}$, and subsonic for $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \hat{\theta}_{\mathrm{w}}^{\mathrm{s}})$ for some $\hat{\theta}_{\mathrm{w}}^{\mathrm{s}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \theta_{\mathrm{w}}^{\mathrm{s}}]$. The strong state (2) is always subsonic at $P_0$ for all $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$.

There had been a long debate to determine which of the two states (2) for $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$, the weak or the strong, is physical for the local theory; see [2, 7, 12]. Indeed, it has been shown in Chen-Feldman [5, 7] that the weak shock reflection-diffraction configuration tends to the unique normal reflection in Figure 15, but that the strong one does not, when the wedge angle $\theta_{\mathrm{w}}$ tends to $\frac{\pi}{2}$. The strength of the corresponding reflected shock near $P_0$ in the weak shock reflection-diffraction configuration is relatively weak, compared to the shock given by the strong state (2). From now on, for the given wedge angle $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \frac{\pi}{2})$, state (2) represents the unique weak state (2), and $\varphi_2$ is its pseudo-potential.

If the weak state (2) is supersonic, the speeds of propagation of the solution are finite, and state (2) is determined completely by the local information: state (1), state (0), and the location of point $P_0$. That is, any information from the reflection-diffraction domain, particularly the disturbance at corner $P_3$, cannot travel towards the reflection point $P_0$. However, if it is subsonic, the information can reach $P_0$ and interact with it, potentially altering the subsonic reflection-diffraction configuration. This argument motivated the following conjectures by von Neumann in [19] (see also [2, 6]):

**The von Neumann Sonic Conjecture**: *There exists a supersonic regular shock reflection-diffraction configuration when* $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{s}}, \frac{\pi}{2})$ *for* $\theta_{\mathrm{w}}^{\mathrm{s}} > \theta_{\mathrm{w}}^{\mathrm{d}}$. *That is, the supersonicity of the weak state* (2) *implies the existence of a supersonic regular reflection solution, as shown in* Figure 16.

Another conjecture is that the global regular shock reflection-diffraction configuration is still possible whenever the local regular reflection at the reflection point is possible; this is known as

**The von Neumann Detachment Conjecture**: *There exists a subsonic regular shock reflection-diffraction configuration for any wedge angle* $\theta_{\mathrm{w}} \in (\theta_{\mathrm{w}}^{\mathrm{d}}, \theta_{\mathrm{w}}^{\mathrm{s}})$. *That is, the existence of subsonic weak state* (2) *beyond the sonic angle implies the existence of a subsonic regular reflection solution, as shown in* Figure 17.

State (2) determines the straight shock $\mathcal{S}_1$ and the sonic arc $\Gamma_{\mathrm{sonic}} := P_1P_4$ when state (2) is supersonic at $P_0$, and the slope of $\Gamma_{\mathrm{shock}}$ at $P_0$ (arc $\Gamma_{\mathrm{sonic}}$ on the boundary of $\Omega$ becomes a corner point $P_0$) when state (2) is subsonic at $P_0$. Thus, the unknowns are the domain $\Omega$ (or equivalently, the curved part of the reflected-diffracted shock $\Gamma_{\mathrm{shock}}$) and the pseudo-potential $\varphi$ in $\Omega$. Then, from (3.8)–(3.9), in order to construct a solution of Problem 3.2 for the supersonic/subsonic regular shock reflection-diffraction configurations, it suffices to solve the following problem:

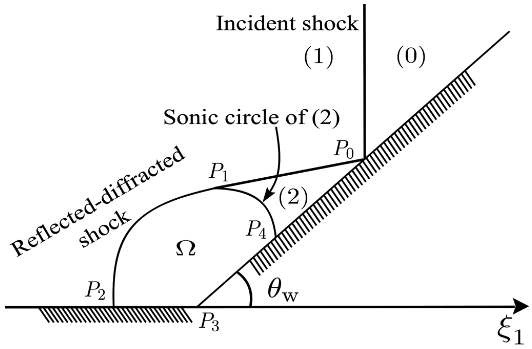


**Figure 16.** Supersonic regular reflection-diffraction configuration [6].

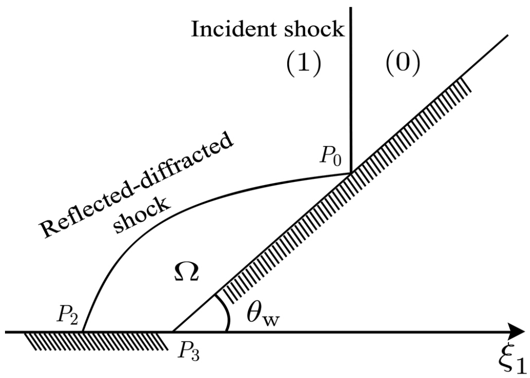


**Figure 17.** Subsonic regular reflection-diffraction configuration [6].

**Problem 3.3** (Free Boundary Problem). *For $\theta_w \in (\theta_w^d, \frac{\pi}{2})$, find a free boundary (curved reflected shock) $\Gamma_{shock} \subset \Lambda \cap \{\xi_1 < \xi_{1P_1}\}$ ($\Gamma_{shock} = P_1P_2$ on* Figure 16 *and $\Gamma_{shock} = P_0P_2$ on* Figure 17) *and a function $\varphi$ defined in the domain $\Omega$ as shown in* Figures 16–17 *such that*

(i) *Equation* (3.5) *is satisfied in $\Omega$, and the equation is strictly elliptic for $\varphi$ in $\overline{\Omega} \setminus \overline{\Gamma_{sonic}}$;*
(ii) *$\varphi = \varphi_1$ and $\rho D\varphi \cdot \nu_s = \rho_1 D\varphi_1 \cdot \nu_s$ on the free boundary $\Gamma_{shock}$;*
(iii) *$\varphi = \varphi_2$ and $D\varphi = D\varphi_2$ on $\Gamma_{sonic}$ in the supersonic case as shown in* Figure 16 *and at $P_0$ in the subsonic case as shown in* Figure 17;
(iv) *$D\varphi \cdot \nu_w = 0$ on $\Gamma_{wedge} = P_0P_3$, and $D\varphi \cdot \nu_{sym} = 0$ on $\Gamma_{sym}$,*

*where $\nu_s$ is the interior unit normal to $\Omega$ on $\Gamma_{shock}$.*

Indeed, if $\varphi$ is a solution of Problem 3.3, we extend $\varphi$ from $\Omega$ to $\Lambda$ to become a global entropy solution (see Figures 16–17) by defining that

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_w, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2, \\ \varphi_2 & \text{in region } P_0P_1P_4. \end{cases} \tag{3.11}$$

For the subsonic reflection case, domain $P_0P_1P_4$ is one point, and curve $P_0P_1P_2$ is $P_0P_2$. Then the global solutions involve two types of transonic (hyperbolic-elliptic) transition: One is from the *hyperbolic* to the *elliptic* phases via $\Gamma_{shock}$, and the other is from the *hyperbolic* to the *elliptic* phases via $\Gamma_{sonic}$.

The conditions in Problem 3.3(ii) are the Rankine-Hugoniot conditions (3.8)–(3.9) on $\Gamma_{shock}$ between $\varphi_{|\Omega}$ and $\varphi_1$. Since $\Gamma_{shock}$ is a free boundary and Equation (3.5) is strictly elliptic for $\varphi$ in $\overline{\Omega} \setminus \overline{\Gamma_{sonic}}$, then two conditions on $\Gamma_{shock}$ — the Dirichlet and oblique derivative conditions — are consistent with one-phase free boundary problems for nonlinear elliptic PDEs of second order.

In the supersonic case, the conditions in Problem 3.3(iii) are the Rankine-Hugoniot conditions on $\Gamma_{sonic}$ (weak discontinuity) between $\varphi_{|\Omega}$ and $\varphi_2$ so that, if $\varphi$ is a solution of Problem 3.3, its extension by (3.11) is a weak solution of Problem 3.2. Since $\Gamma_{sonic}$ is not a free boundary (its location is fixed), it is impossible in general to prescribe the two conditions given in Problem 3.3(iii) on $\Gamma_{sonic}$ for a second-order elliptic PDE. In the iteration problem, we prescribe the condition that $\varphi = \varphi_2$ on $\Gamma_{sonic}$, and then prove that $D\varphi = D\varphi_2$ on $\Gamma_{sonic}$ by exploiting the elliptic degeneracy on $\Gamma_{sonic}$.

We observe that there is an additional possibility to the regular shock reflection-diffraction configurations (beyond the conjectures by von Neumann [19]): For some wedge angle $\theta_w^a \in (\theta_w^d, \frac{\pi}{2})$, $\Gamma_{shock}$ may attach to the wedge vertex $P_3$, as observed by experimental results (*cf.* [6]); see Figs. 18–19. To describe the conditions of such an attachment, we use the explicit expressions of (3.3) to see that, for each $\rho_0$, there exists $\rho^c > \rho_0$ such that

$$v_1 \le c_1 \quad \text{if } \rho_1 \in (\rho_0, \rho^c]; \qquad v_1 > c_1 \quad \text{if } \rho_1 \in (\rho^c, \infty).$$

If $v_1 \le c_1$, we can rule out the solution with a shock attached to $P_3 = (0, 0)$. This is based on the fact that, if $v_1 \le c_1$, then $P_3$ lies within the sonic circle $\overline{B_{c_1}(\mathbf{v}_1)}$ of state (1), and $\Gamma_{shock}$ does not intersect with $\overline{B_{c_1}(\mathbf{v}_1)}$, as we show below. If $v_1 > c_1$, there would be a possibility that $\Gamma_{shock}$ could be attached to $P_3$, as the experiments show. Given these facts, the following results have been obtained:

**Theorem 3.4** (Chen-Feldman [5, 6]). *There are two cases:*

(i) *If $\rho_0$ and $\rho_1$ are such that $v_1 \le c_1$, then the supersonic/subsonic regular reflection solution exists for each (half-wedge) angle $\theta_w \in (\theta_w^d, \frac{\pi}{2})$. That is, for each $\theta_w \in (\theta_w^d, \frac{\pi}{2})$, there exists a solution $\varphi$ of* Problem 3.3 *such that*

$$\Phi(t, \mathbf{x}) = t\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t} \qquad \textit{for } \frac{\mathbf{x}}{t} \in \Lambda, t > 0,$$

*with $\rho(t, \mathbf{x}) = \big(\rho_0^{\gamma-1} - (\gamma-1)\big(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big)\big)^{\frac{1}{\gamma-1}}$, is a global weak solution of* Problem 3.1 *satisfying the entropy condition; that is, $\Phi(t, \mathbf{x})$ is an entropy solution.*

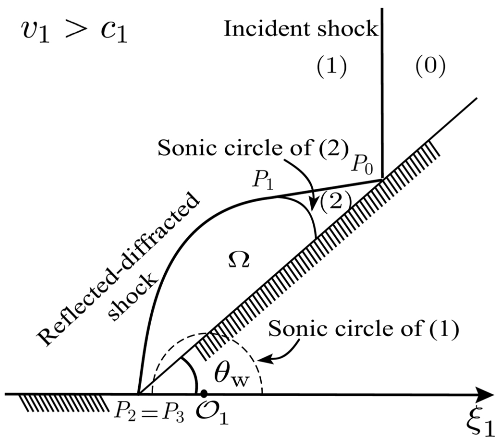


**Figure 18.** The attached supersonic regular reflection-diffraction configuration [6].

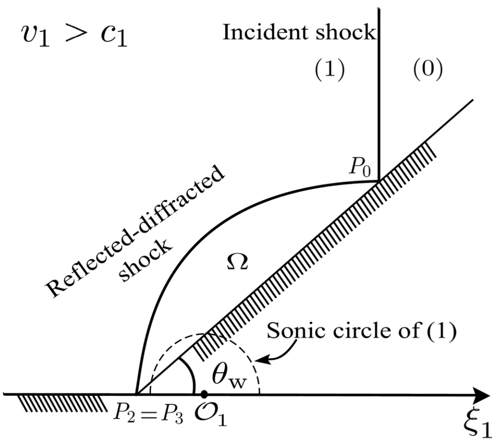


**Figure 19.** The attached subsonic regular reflection-diffraction configuration [6].

(ii) *If $\rho_0$ and $\rho_1$ are such that $v_1 > c_1$, then there exists $\theta_w^a \in [\theta_w^d, \frac{\pi}{2})$ so that the regular reflection solution exists for each angle $\theta_w \in (\theta_w^a, \frac{\pi}{2})$, and the solution is of the self-similar structure described in* (i), *above. Moreover, if $\theta_w^a > \theta_w^d$, then, for the wedge angle $\theta_w = \theta_w^a$, there exists an attached solution; that is, $\varphi$ is a solution of* Problem 3.3 *with $P_2 = P_3$.*

*The type of regular shock reflection-diffraction configurations* (*supersonic as in* Figure 16 *and* Figure 18, *or subsonic as in* Figure 17 *and* Figure 19) *is determined by the type of state* (2) *at $P_0$:*

(a) *For the supersonic/sonic reflection case, the reflected-diffracted shock $P_0P_2$ is $C^{2,\alpha}$–smooth for some $\alpha \in (0,1)$ and its curved part $\Gamma_{\text{sonic}}$ is $C^\infty$ away from $P_1$. The solution $\varphi$ is in $C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$, and is $C^{1,1}$ across $\Gamma_{\text{sonic}}$ which is optimal; that is, $\varphi$ is* not *$C^2$ across $\Gamma_{\text{sonic}}$.*

(b) *For the subsonic reflection case* (*as in* Figure 17 *and* Figure 19), *the reflected-diffracted shock $P_0P_2$ and solution $\varphi$ in $\Omega$ are in $C^{1,\alpha}$ near $P_0$ and $P_3$ for some $\alpha \in (0,1)$, and $C^\infty$ away from $\{P_0, P_3\}$.*

*Moreover, the regular reflection solution tends to the unique normal reflection* (*as in* Figure 15) *when the wedge angle $\theta_w$ tends to $\frac{\pi}{2}$. In addition, for both supersonic and subsonic reflection cases,*

$$\varphi_2 < \varphi < \varphi_1 \qquad \text{in } \Omega,$$

$$D(\varphi_1 - \varphi) \cdot \mathbf{e} \le 0 \qquad \text{in } \overline{\Omega} \text{ for all } \mathbf{e} \in \overline{Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1})},$$

*where $Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1}) := \{a\mathbf{e}_{\xi_2} + b\mathbf{e}_{\mathcal{S}_1} \,:\, a, b > 0\}$ with $\mathbf{e}_{\xi_2} = (0,1)$ and with $\mathbf{e}_{\mathcal{S}_1}$ as the tangent unit vector to $\mathcal{S}_1$.*

Theorem 3.4 was established by solving Problem 3.3. The first results on the existence of global solutions of the free boundary problem (Problem 3.3) were obtained for the wedge angles sufficiently close to $\frac{\pi}{2}$ in Chen-Feldman [5]. Later, in Chen-Feldman [6], these results were extended up to the detachment angle, as stated in Theorem 3.4. For this extension, the techniques developed in [5], notably the estimates near $\Gamma_{\text{sonic}}$, were the starting point.

To establish Theorem 3.4, a theory for free boundary problems for nonlinear PDEs of mixed elliptic-hyperbolic type has been developed, including new methods, techniques, and related ideas. Some features of these methods and techniques include:

(i) exploitation of the global structure of solutions to ensure that the nonlinear PDE (3.5) is elliptic for the regular reflection solution in $\Omega$ enclosed by the free boundary $\Gamma_{\text{shock}}$ and the fixed boundary for all wedge angles $\theta_w$ up to the detachment angle $\theta_w^d$ for all physical cases (see Figures 16–19);

(ii) optimal regularity estimates for the solutions of the *degenerate elliptic PDE* (3.5) both near $\Gamma_{\text{sonic}}$ and at corner $P_1$ between the free boundary $\Gamma_{\text{shock}}$ and the elliptic degenerate fixed boundary $\Gamma_{\text{sonic}}$ for the supersonic reflection case (see Figure 16 and Figure 18);

(iii) for fixed incident shock strength and $\gamma > 1$, the dependence of the structural transition of the global solution configurations on the wedge angle $\theta_w$ from the supersonic to subsonic reflection cases, *i.e.*, from the degenerate elliptic to the uniformly elliptic Equation (3.5) near a part of the boundary;

(iv) uniform *a priori* estimates required for all stages of the structural transition between the different configurations.

Based on the methods and techniques used to establish Theorem 3.4, further approaches and related techniques have been developed to prove that the steady weak oblique transonic shocks (discussed in §2) are attainable as large-time asymptotic states by constructing the global Prandtl-Meyer reflection configurations in self-similar coordinates in Bae-Chen-Feldman [1] and the references cited therein, and that all of the self-similar transonic shocks and related free boundaries in these problems are always convex in Chen-Feldman-Xiang [8].

These types of questions also arise in other shock reflection/diffraction problems, which can be formulated as free boundary problems for transonic shocks for nonlinear PDEs of mixed type. These problems have the following important attributes: They are physically fundamental and are supported by a wealth of experimental/numerical data indicating diverse patterns of complicated configurations (*cf.* Figures 14–19), and their solutions are building blocks and asymptotic attractors of general solutions of M-D hyperbolic conservation laws whose mathematical theory is also in its infancy (*cf.* [2, 6, 13, 15]).

Similarly, for the full Euler case, a self-similar solution is a solution of the form: $(\mathbf{V}, p, \rho)(t, \mathbf{x}) = (v - \xi, p, \rho)(\xi)$, $\xi = \mathbf{x}/t$, governed by

$$\begin{cases} \nabla \cdot (\rho \mathbf{V}) + n\rho = 0, \\ \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{V}) + \nabla p + (n+1)\rho \mathbf{V} = 0, \\ \nabla \cdot \left(\rho \mathbf{V}(E + \frac{p}{\rho})\right) + n\rho(E + \frac{p}{\rho}) = 0. \end{cases} \tag{3.12}$$

System (3.12) is a *system of conservation laws of mixed-composite elliptic-hyperbolic type*:

- strictly *hyperbolic* when $|\mathbf{V}| > c := \sqrt{\gamma p/\rho}$ (pseudo-supersonic);
- *mixed-composite elliptic-hyperbolic* (two of them are elliptic and the others are hyperbolic) when $|\mathbf{V}| < c := \sqrt{\gamma p/\rho}$ (pseudo-subsonic).

The transition boundary between the pseudo-supersonic and pseudo-subsonic phases is $|\mathbf{V}| = c$, a degenerate set of the solution of System (3.12), which is unknown *a priori*.

Similar fundamental mixed problems arise in other applications, where nonlinear PDEs of mixed type are the core parts of even more sophisticated systems; examples include the relativistic Euler equations, the Euler-Poisson equations, and the Euler-Maxwell equations.

## 4. Nonlinear PDEs of Mixed Type and Isometric Embedding Problems in Differential Geometry and Related Areas

Nonlinear PDEs of mixed type also arise naturally from many longstanding problems in differential geometry and related areas. In this section, we first show how the fundamental problem – the *isometric embedding problem* – in differential geometry can be formulated in terms of problems for nonlinear PDEs of mixed type, or even of no type.

The isometric embedding problem can be stated as follows: *Seek an embedding/immersion of an n-D (semi-) Riemannian manifold $(\mathcal{M}^n, g)$ with metric $g = (g_{ij}) > 0$ into an N-D (semi-) Euclidean space so that the metric, often along with assigned regularity/curvatures, is preserved.*

This problem has assumed a position of fundamental conceptual importance in differential geometry, thanks in part to the works of Darboux (1894), Weyl (1916), Janet (1926), and Cartan (1927). A classical question is whether a smooth Riemannian manifold $(\mathcal{M}^n, g)$ can be embedded into $\mathbb{R}^N$ with sufficiently large $N$; for more on this, see Nash (1956), Gromov (1986), and Günther (1989). A further fundamental issue is whether $(\mathcal{M}^n, g)$ can be embedded/immersed in $\mathbb{R}^{s_n}$ with the critical Janet dimension $s_n = \frac{n(n+1)}{2}$ and assigned regularity/curvatures. The solution to this issue promises to advance our understanding of the properties of (semi-)Riemannian manifolds and to provide frameworks/approaches for real applications, including the problems for realization/stability/rigidity/classification of isometric embeddings in many important application areas (*e.g.* elasticity, materials science, optimal design, thin shell/biological leaf growth, protein folding, cell/tissue organization, and manifold data analysis).

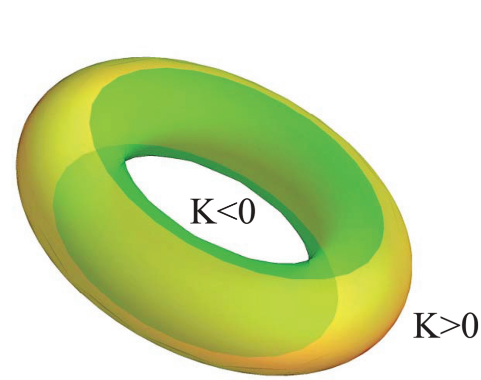


**Figure 20.** The Gauss curvature $K$ of a torus with mixed sign.

When $n = 2$, following Darboux,[7] the isometric embedding problem on a chart can be reduced to finding a function $u$ that solves the nonlinear Monge-Ampère equation (*cf.* [17]):

$$\det(\nabla^2 u) = |g|\left(1 - |\nabla u|_g^2\right)K, \tag{4.1}$$

with $|g| = \det(g)$, $|\nabla u|_g := \frac{1}{|g|}(g_{22}|\partial_{x_1} u|^2 - 2g_{12}\partial_{x_1}\partial_{x_2} u + g_{11}|\partial_{x_2} u|^2) < 1$ as required, and the Gauss curvature $K = K(g)$ of metric $g$. Equation (4.1) is *elliptic* if $K > 0$, *hyperbolic* if $K < 0$, and *degenerate* when $K = 0$. The sign change of $K$ is very common for surfaces and is necessary for many important cases; the simplest example of such a surface is the torus shown in Figure 20.

Nirenberg (1953) first solved the *Weyl problem*, establishing that any smooth metric $g$ on $\mathbb{S}^2$ can be globally embedded into $\mathbb{R}^3$ smoothly if the Gauss curvature $K > 0$. One could then ask whether any 2-D Riemannian surface is always embeddable into $\mathbb{R}^3$. The answer is *no* if $K \leq 0$. The embedding problem is still largely open for global results for general $K$, even though some local results have been obtained; see [17] and the references therein.

[7] *G. Darboux,* Leçons sur la Théorie des Surface, *Vol.* 3, *Gauthier-Villars, Paris, 1894.*

**Figure 21.** Johann Carl Friedrich Gauss (April 30, 1777–February 23, 1855) introduced the notion of Gauss (or Gaussian) curvature and the *Theorema Egregium*.

**Figure 22.** Jean-Gaston Darboux (August 14, 1842–February 23, 1917) indicated the connection between the isometric embedding and the nonlinear Monge-Ampère equation.

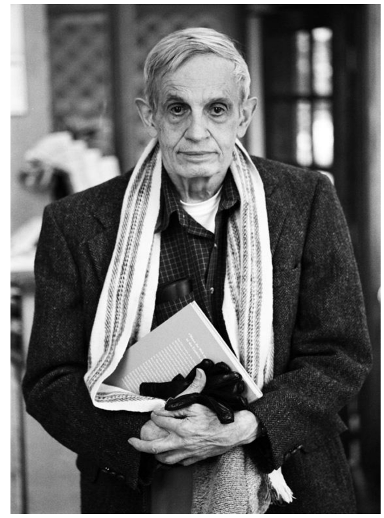

**Figure 23.** John Forbes Nash Jr. (June 13, 1928–May 23, 2015) established the Nash embedding theorems.

On the other hand, the fundamental theorem of surface theory states that *there exists a simply connected surface in* $\mathbb{R}^3$ *whose first and second fundamental forms are* $I = g_{ij}dx_idx_j$ *and* $I\!I = h_{ij}dx_idx_j$ *on a domain for* $i, j = 1, 2$, *provided that the coefficients* $\{h_{ij}\}$, *together with metric* $g = (g_{ij}) > 0$, *satisfy the Gauss-Codazzi equations*:

$$LN - M^2 = K, \tag{4.2}$$

$$\begin{cases} \partial_{x_1}N - \partial_{x_2}M = -\Gamma^1_{22}L + 2\Gamma^1_{12}M - \Gamma^1_{11}N, \\ \partial_{x_1}M - \partial_{x_2}L = \Gamma^2_{22}L - 2\Gamma^2_{12}M + \Gamma^2_{11}N, \end{cases} \tag{4.3}$$

where $L = h_{11}/\sqrt{|g|}, M = h_{12}/\sqrt{|g|}$, and $N = h_{22}/\sqrt{|g|}$, and $\Gamma^k_{ij}$ are the Christoffel symbols for $i, j, k = 1, 2$. This theorem still holds for immersion even when $\{h_{ij}\}$ is only in $L^p$ for $p > 2$.[8] Thus, given $(g_{ij}) > 0$, System (4.2)–(4.3) consists of three nonlinear PDEs for the unknowns $(L, M, N)$ determining $\{h_{ij}\}$, the knowledge of which gives the desired immersion. Then the problem can be reduced to the solvability of System (4.3) under constraint (4.2), which is of *mixed elliptic-hyperbolic type* determined by the sign of the Gauss curvature $K$. From the viewpoint of geometry, (4.2) is a constraint condition, while (4.3) involves compatibility conditions.

System (4.2)–(4.3) has features similar to those in gas dynamics in §2–§3. A natural question is whether or not this system can be written in a gas dynamic formulation to examine underlying interrelations and connections. Indeed, a novel observation in Chen-Slemrod-Wang [11] has indicated that this is indeed the case: The Codazzi system (4.3) can be formulated as the familiar nonlinear balance laws of momentum:

$$\begin{cases} \partial_{x_1}(\rho u^2 + p) + \partial_{x_2}(\rho uv) \\ \qquad = -\Gamma^1_{22}(\rho v^2 + p) - 2\Gamma^1_{12}\rho uv - \Gamma^1_{11}(\rho u^2 + p), \\ \partial_{x_1}(\rho uv) + \partial_{x_2}(\rho v^2 + p) \\ \qquad = -\Gamma^2_{22}(\rho v^2 + p) - 2\Gamma^2_{12}\rho uv - \Gamma^2_{11}(\rho u^2 + p), \end{cases} \tag{4.4}$$

and the Gauss equation (4.2) becomes the *Bernoulli* relation: $\rho = (q^2 + K)^{-\frac{1}{2}}$ if $p = -\frac{1}{\rho}$ is chosen as the Chaplygin pressure for $q = \sqrt{u^2 + v^2}$. In this case, define the sound speed as $c = \sqrt{p'(\rho)} = \frac{1}{\rho}$. Then

- $q < c$ and the *flow* is subsonic when $K > 0$;
- $q > c$ and the *flow* is supersonic when $K < 0$;
- $q = c$ and the *flow* is sonic when $K = 0$.

A weak compactness framework has been introduced and applied for establishing the existence and weak continuity/stability of isometric embeddings in $W^{2,p}, p \geq 2$, in [10, 11]; this has shown the high potential. In particular, the weak continuity/stability of the Gauss-Codazzi equations (4.2)–(4.3) and isometric immersions of (semi-)Riemannian manifolds, independent of local coordinates, have been established in [9, 10], even for the case $p = 2$.

For the higher-dimensional case, the Gauss-Codazzi equations for $h = \{h^a_{ij}\}$ are coupled with the Ricci equations for the coefficients $\varkappa = \{\varkappa^a_{lb}\}$ of the connection form on the normal bundle to become the Gauss-Codazzi-Ricci

[8] *S. Mardare,* The fundamental theorem of surface theory for surfaces with little regularity, *J. Elasticity* **73** *(2003), 251–290.*

equations in a local coordinate chart of the manifold:

$$h^a_{ji}h^a_{kl} - h^a_{ki}h^a_{jl} = R_{ijkl} \quad \text{(Gauss equations)}, \tag{4.5}$$

$$\partial_{x_k} h^a_{lj} - \partial_{x_l} h^a_{kj} = -\Gamma^m_{lj} h^a_{km} + \Gamma^m_{kj} h^a_{lm} - \left(\kappa^a_{kb} h^b_{lj} - \kappa^a_{lb} h^b_{kj}\right) \quad \text{(Codazzi equations)}, \tag{4.6}$$

$$\partial_{x_k} \kappa^a_{lb} - \partial_{x_l} \kappa^a_{kb} = g^{ij}\left(h^a_{li}h^b_{kj} - h^a_{ki}h^b_{lj}\right) + \kappa^a_{lc}\kappa^c_{kb} - \kappa^a_{kc}\kappa^c_{lb} \quad \text{(Ricci equations)}, \tag{4.7}$$

where $\kappa^a_{kb} = -\kappa^b_{ka}$ are the coefficients of the connection form on the normal bundle, $R_{ijkl}$ is the Riemann curvature tensor, the indices $a, b, c$ run from 1 to $N$, and $i, j, k, l, m, n$ run from 1 to $d \geq 3$. System (4.5)–(4.7) has no type, neither purely hyperbolic nor purely elliptic, for general Riemann curvature tensor $R_{ijkl}$. Nevertheless, the weak continuity of the nonlinear system (4.5)–(4.7) has been established.

**Theorem 4.1** (Chen-Slemrod-Wang [11]). *Let $(h^\varepsilon, \kappa^\varepsilon)$ be a sequence of solutions of the Gauss-Codazzi-Ricci system* (4.5)–(4.7)*, which is uniformly bounded in $L^p$ for $p > 2$. Then the weak limit vector field $(h, \kappa)$ of the sequence $(h^\varepsilon, \kappa^\varepsilon)$ in $L^p$ is still a solution of the Gauss-Codazzi-Ricci system* (4.5)–(4.7).

The proof of this is based on the following key observation in [11] for the div-curl structure of System (4.5)–(4.7): For fixed $i, j, k, l, a, b, c$,

$$\operatorname{div}(\overbrace{\underbrace{0, \cdots, 0, h^{a,\varepsilon}_{li}, 0, \cdots, -h^{a,\varepsilon}_{ki}}_{l}}^{k}, 0, \cdots, 0) = R^\varepsilon_1, \tag{4.8}$$

$$\operatorname{curl}(h^{b,\varepsilon}_{1j}, h^{b,\varepsilon}_{2j}, \cdots, h^{b,\varepsilon}_{dj}) = R^\varepsilon_2, \tag{4.9}$$

$$\operatorname{div}(\overbrace{\underbrace{0, \cdots, 0, \kappa^{a,\varepsilon}_{lc}, 0, \cdots, -\kappa^{a,\varepsilon}_{kc}}_{l}}^{k}, 0, \cdots, 0) = R^\varepsilon_3, \tag{4.10}$$

$$\operatorname{curl}(\kappa^{c,\varepsilon}_{1b}, \kappa^{c,\varepsilon}_{2b}, \cdots, \kappa^{c,\varepsilon}_{db}) = R^\varepsilon_4, \tag{4.11}$$

$$\operatorname{div}(\overbrace{\underbrace{0, \cdots, 0, h^{b,\varepsilon}_{lj}, 0, \cdots, -h^{b,\varepsilon}_{kj}}_{l}}^{k}, 0, \cdots, 0) = R^\varepsilon_5, \tag{4.12}$$

$$\operatorname{curl}(\kappa^{a,\varepsilon}_{1b}, \kappa^{a,\varepsilon}_{2b}, \cdots, \kappa^{a,\varepsilon}_{db}) = R^\varepsilon_6, \tag{4.13}$$

where $R_r, r = 1, \ldots, 6$, consist of the three types of nonlinear quadratic terms:

$$h^{a,\varepsilon}_{li}h^{b,\varepsilon}_{kj} - h^{a,\varepsilon}_{ki}h^{b,\varepsilon}_{lj}, \quad \kappa^{a,\varepsilon}_{lc}\kappa^{c,\varepsilon}_{kb} - \kappa^{a,\varepsilon}_{kc}\kappa^{c,\varepsilon}_{lb}, \quad \kappa^{a,\varepsilon}_{kb}h^{b,\varepsilon}_{lj} - \kappa^{a,\varepsilon}_{lb}h^{b,\varepsilon}_{kj},$$

as well as several linear terms involving $(h^\varepsilon, \kappa^\varepsilon)$, while the nonlinear quadratic terms are actually the scalar products of the vector fields given on the left-hand sides of (4.8)–(4.13). Therefore, this div-curl structure fits the following classical div-curl lemma divinely (Murat 1978, Tartar 1979): *Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be open and bounded. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that, for $\varepsilon > 0$, two fields $\mathbf{u}^\varepsilon \in L^p(\Omega; \mathbb{R}^d)$ and $\mathbf{v}^\varepsilon \in L^q(\Omega; \mathbb{R}^d)$ satisfy the conditions that*

(i) $\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}$ *weakly in* $L^p(\Omega; \mathbb{R}^d)$ *and* $\mathbf{v}^\varepsilon \rightharpoonup \mathbf{v}$ *weakly in* $L^q(\Omega; \mathbb{R}^d)$ *as* $\varepsilon \to 0$;

(ii) $\operatorname{div} \mathbf{u}^\varepsilon$ *are confined in a compact subset of* $W^{-1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$;

(iii) $\operatorname{curl} \mathbf{v}^\varepsilon$ *are confined in a compact subset of* $W^{-1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^{d\times d})$,

*where $W^{-1,p}(\Omega; \mathbb{R})$ is the dual space of $W^{1,q}(\Omega; \mathbb{R})$, and vice versa. Then the scalar product of $\mathbf{u}^\varepsilon$ and $\mathbf{v}^\varepsilon$ is weakly continuous: $\mathbf{u}^\varepsilon \cdot \mathbf{v}^\varepsilon \longrightarrow \mathbf{u} \cdot \mathbf{v}$ in the sense of distributions.*

With this div-curl lemma, the weak continuity result in Theorem 4.1 can be seen as follows: For the uniformly bounded sequence $(h^\varepsilon, \kappa^\varepsilon)$ in $L^p, p > 2$, $R^\varepsilon_r, r = 1, \ldots, 6$, are uniformly bounded in $L^{p/2}$, which implies that $R^\varepsilon_r, r = 1, \ldots, 6$, are compact in $W^{-1,q}_{\mathrm{loc}}$ for some $q \in (1, 2)$. On the other hand, System (4.8)–(4.13) implies that $R^\varepsilon_r, r = 1, \ldots, 6$, are uniformly bounded in $W^{-1,p}_{\mathrm{loc}}$ for $p > 2$. Then the interpolation compactness argument yields that

$$R^\varepsilon_r, r = 1, \ldots, 6, \text{ are confined in a compact set in } H^{-1}_{\mathrm{loc}}(\Omega).$$

With this, we can employ the div-curl lemma to conclude that

$$\begin{aligned}&(h^{a,\varepsilon}_{li}h^{b,\varepsilon}_{kj} - h^{a,\varepsilon}_{ki}h^{b,\varepsilon}_{lj}, \kappa^{a,\varepsilon}_{lc}\kappa^{c,\varepsilon}_{kb} - \kappa^{a,\varepsilon}_{kc}\kappa^{c,\varepsilon}_{lb}, \kappa^{a,\varepsilon}_{kb}h^{b,\varepsilon}_{lj} - \kappa^{a,\varepsilon}_{lb}h^{b,\varepsilon}_{kj}) \\ &\longrightarrow (h^a_{li}h^b_{kj} - h^a_{ki}h^b_{lj}, \kappa^a_{lc}\kappa^c_{kb} - \kappa^a_{kc}\kappa^c_{lb}, \kappa^a_{kb}h^b_{lj} - \kappa^a_{lb}h^b_{kj}),\end{aligned}$$

in the sense of distributions, as $\varepsilon \to 0$. Then Theorem 4.1 follows.

This local weak continuity result can be extended to the global weak continuity of the Gauss-Codazzi-Ricci system (4.5)–(4.7) as follows:

**Theorem 4.2** (Chen-Li [10]). *Let $(M, g)$ be a Riemannian manifold with $g \in W^{1,p}$ for $p > 2$. Let $(h^\varepsilon, \kappa^\varepsilon)$ be a sequence of solutions (i.e., the coefficients of the second fundamental form and the connection form on the normal bundle) in $L^p$ of the Gauss-Codazzi-Ricci system* (4.5)–(4.7) *in the distributional sense. Assume that, for any submanifold $K \Subset M$, there exists $C_K > 0$ independent of $\varepsilon$ such that*

$$\sup_{\varepsilon>0} \|(h^\varepsilon, \kappa^\varepsilon)\|_{L^p(K)} \leq C_K.$$

*Then, when $\varepsilon \to 0$, there exists a subsequence of $(h^\varepsilon, \kappa^\varepsilon)$ that converges weakly in $L^p$ to a pair $(h, \kappa)$ that is still a weak solution of the Gauss-Codazzi-Ricci system* (4.5)–(4.7).

The proof is based on a compensated compactness theorem in Banach spaces, which leads directly to a globally intrinsic div-curl lemma on Riemannian manifolds, developed in Chen-Li [10]. From the viewpoint of geometry, the $L^p$ bounded requirement on the connection form on the normal bundle $\kappa^\varepsilon$ is not intrinsic. Therefore, Theorem 4.2 has been reformulated as follows:

**Theorem 4.3** (Chen-Giron [9]). *Let $(M, g)$ be a Riemannian manifold with $g \in W^{1,p}$ for $p > 2$. Let $(h^\varepsilon, \kappa^\varepsilon)$ be a sequence of solutions (i.e., the coefficients of the second fundamental form* *and the connection form on the normal bundle) in $L^p$ of the Gauss-Codazzi-Ricci system* (4.5)–(4.7) *in the distributional sense. Assume that, for any submanifold $K \Subset M$, there exists $C_K > 0$ independent of $\varepsilon$ such that*

$$\sup_{\varepsilon>0} \|h^{\varepsilon}\|_{L^p(K)} \leq C_K.$$

*Then there exists a refined sequence $(\tilde{h}^{\varepsilon}, \tilde{\kappa}^{\varepsilon})$, each of which is still a weak solution of the Gauss-Codazzi-Ricci system* (4.5)–(4.7), *such that, when $\varepsilon \to 0$, $(\tilde{h}^{\varepsilon}, \tilde{\kappa}^{\varepsilon})$ converges weakly in $L^p$ to a pair $(h, \kappa)$ that is still a weak solution of the Gauss-Codazzi-Ricci system* (4.5)–(4.7).

As a direct corollary, the weak limit of isometrically immersed surfaces with lower regularity in $W^{2,p}$ is still an isometrically immersed surface in $\mathbb{R}^d$ governed by the Gauss-Codazzi-Ricci system (4.5)–(4.7) for any $R_{ijkl}$ (without sign/type restriction) with respect only to the coefficients of the second fundamental form. The weak continuity result in Theorem 4.3 is global and intrinsic, independent of local coordinates, without restriction on both the Riemann curvatures and the types of System (4.5)–(4.7). The key to the proof is to exploit the invariance for a choice of suitable gauge to control the full connection form and to develop a non-abelian div-curl lemma on Riemannian manifolds (see Chen-Giron [9]).

This approach and related observations have been motivated by the theory of polyconvexity in nonlinear elasticity,[9] intrinsic methods in elasticity and nonlinear Korn inequalities,[10] and Uhlenbeck compactness and Gauge theory,[11,12] among other ideas.

## 5. Further Connections, Unified Approaches, and Current Trends

In §2–§4, we have presented several important sets of nonlinear PDEs of mixed elliptic-hyperbolic type, or even of no type, in shock wave problems in fluid mechanics and isometric embedding problems in differential geometry and related areas. Such nonlinear PDEs of mixed type arise naturally in other problems in fluid mechanics, differential geometry/topology, nonlinear elasticity, materials science, mathematical physics, dynamical systems, and related areas.

We have shown in §2–§4 that free boundary methods, weak convergence methods, and related techniques are useful as unified approaches for dealing with the nonlinear mixed problems involving *both elliptic and hyperbolic phases*. Friedrichs's positive symmetric techniques have also demonstrated high potential in solving mixed-type problems.[13] Entropy methods and kinetic methods have been useful for solving nonlinear PDEs of hyperbolic or mixed hyperbolic-parabolic type. Variational approaches deserve to be further explored, especially for handling transonic flow problems, since the solutions of these problems are critical points of the corresponding functionals. Some approximate methods, such as viscosity methods, relaxation methods, shock capturing methods, stochastic methods, and related numerical methods should be further analyzed/developed, and numerical calculations/simulations should be performed to gain new ideas and motivations. These methods, along with energy estimate techniques, functional analytical methods, measure-theoretic techniques (*esp.* divergence-measure fields), and other methods, should be developed into even more powerful approaches, applicable to wider classes of nonlinear PDEs of mixed type. The underlying structures of the nonlinear PDEs of mixed type under consideration here have been one of the main motivating factors in developing new methods/techniques/ideas for unified approaches. As mentioned earlier, the analysis of nonlinear PDEs of mixed type is still in its early stages, and most nonlinear mixed-type problems are wide open and ripe for the development of new ideas, methods, and techniques.

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# Stochastic Separation Theorems: How Geometry May Help to Correct AI Errors


*Alexander Gorban, Bogdan Grechuk, and Ivan Tyukin*

## 1. Introduction

Recent years have seen explosive progress in data-driven artificial intelligence (AI) systems. Many decades of the development of mathematics underpinning statistical learning theory coupled with advancements in approximation theory, numerical analysis, technology, and computing gave rise to new-generation AI transforming our life. These systems show great promise in cancer diagnostics [MSG+20], they are a part of autonomous cars [22], automated face recognition and biometrics [KE21], image segmentation [SBKV+20], language processing and translation tools [DZS+22], and as such become our new reality. Availability of unprecedented volumes of data, citizens' expectations and participation are further driving this change.

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New reality, however, brings new challenges. Uncertainties and biases are inherent within any empirical data. They enter production pipelines of data-driven AI and ripple through them causing errors. AI instabilities and adversarial examples—errors due to minor changes in data or structure—have recently been found in many advanced data-driven AI models. Moreover, mounting evidence suggests that these errors are in fact expected in such systems [THG20] and may not always be cured by larger volumes of data or better training algorithms [BHV21] as long as the AI architecture remains fixed.

This leads to the following question: if errors are inevitable in data-driven AI then how do we deal with them once they occur?

One way to address this imminent challenge is to equip an AI with an "error filter" or "error corrector" [GT18]. The function of the AI corrector is to learn from errors "on-the-job," supplementing the AI's initial training. Dynamic addition of AI correctors continuously extends AI architecture, adapts to data uncertainty [GGG$^+$18], and enables AI to escape the stability barrier revealed in [BHV21]. When a new data arrives at AI input, the AI error corrector then decides if it is likely to cause an error, and if so, then reports. To do this, the filter uses some set $I$ of attributes, such as, for example, internal latent representations of the input in AI decision space. To each attribute $i \in I$, the system assigns some weight $w_i$. For each new input, the system computes numerical values $x_i$ of all attributes $i \in I$, and compares the weighted sum $\sum_{i\in I} w_i x_i$ with some threshold $t$ to decide whether to report the input as an error.

However, how does the filter determine the weights $w_i$ of all attributes? To do this, the filter is provided a training set of example inputs marked as correct and errors. Then the system tries to find weights $w_i$ such that, ideally, all data in the training set are classified correctly. Moreover the system tries to ensure that all (or a large proportion of) future "unseen" inputs would be processed correctly too. In other words, the system seeks to learn the weights from some training data, and the error filter itself is therefore an example of a machine learning (ML) system.

Geometrically, any input is described by the values $x_i$ of the attributes, and can therefore be represented as a point $x = (x_1, \ldots, x_n)$ in the $n$-dimensional Euclidean space, where $n = |I|$ is the number of attributes. Then the criterion $\sum_{i\in I} w_i x_i \geq t$ for an input being an error defines a half-space, whose boundary is the hyperplane $H$ defined by the equation $\sum_{i\in I} w_i x_i = t$. If we mark points corresponding to errors and correct AI behavior as red and blue, respectively, the machine learning task of error identification reduces to finding a hyperplane that separates the red points from the blue ones; see Figure 1.

Assume that such hyperplane $H$ exists and we have started to use the filter with the corresponding weights $w_i$.

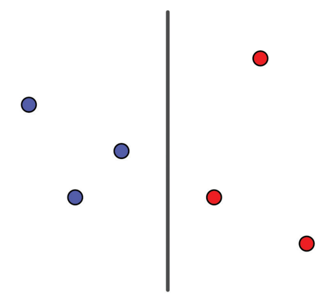

**Figure 1.** Separation of red and blue points by a hyperplane.

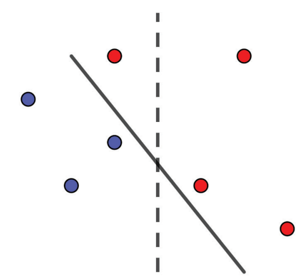

**Figure 2.** Retraining the system by recomputing a hyperplane.

Imagine, however, that a new input has arrived, which the filter classified as correct but the user marked as an error. In other words, the filter itself made an error. Of course, we would like the system to be able to learn from such errors and improve its performance in the future. An obvious way to do this is to add a new point to the training set and recompute the weights. This constitutes "retraining the system." Geometrically, this means that a new red point $X$ appears on the "wrong" side of the hyperplane, so that we try to find a different hyperplane that separates all points correctly; see Figure 2. Obviously, it is not always possible to find such a hyperplane; see Figure 3. Moreover, even if it is possible, it may require substantial time to recompute all weights every time the filter makes an error.

Alternatively, one may use the following error-correction method, suggested in [GMT19]: separate a new red point $X$ from the existing blue points by another hyperplane $H'$ given by an equation $\sum_{i\in I} w'_i x_i = t'$; see Figure 3. After this, classify any new input as error if either $\sum_{i\in I} w_i x_i \geq t$ or $\sum_{i\in I} w'_i x_i \geq t'$.

A careful reader may have already noticed a limitation of this approach that appears to be fundamental: why did we assume that a point can be separated from all other points by a hyperplane? Obviously, if that point belongs

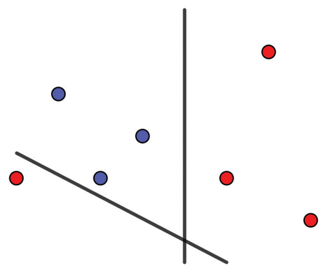

**Figure 3.** Separation of new red point by a different hyperplane.

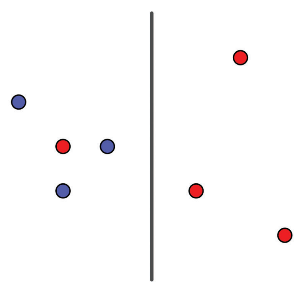

**Figure 4.** A red point not separable by a hyperplane.

to the convex hull[1] of other points, then a separating hyperplane does not exist and the method does not work; see Figure 4. For example, even if we have just 3 points $X_1, X_2, X_3$, then $X_3$ may lie in the interior of the line interval $X_1X_2$, and in this case it cannot be separated from $X_1, X_2$.

However, intuitively, the described case is in some sense "degenerate" and should not happen too often with real data. The best way to formalise this intuition is to use the language of probability theory, and ask what is the probability that the method would work for random data. This leads to a very nice problem that lies on the borderline of probability theory and geometry.

**Problem 1.** Given a set $K$ of $m$ random points in $\mathbb{R}^n$, what is the probability that each point $X \in K$ can be separated from all other points by a hyperplane? Equivalently, what is the probability that points in $K$ are in convex position (in sense that each point $X \in K$ is a vertex of the convex hull of $K$)?

## 2. Sylvester's Problem

Problem 1 has a long history and goes back to at least the question asked by Sylvester in 1864: given 4 random points $X, Y, Z, W$ on the plane, what is the probability $p$ that they form a convex quadrilateral?

To address this question, it is convenient to introduce the following random variables. Let $I_X$ be the random variable equal to 1 if point $X$ is inside the triangle $YZW$ and 0 otherwise. Let random variables $I_Y$, $I_Z$ and $I_W$ be defined similarly. Then random variable

$$I = I_X + I_Y + I_Z + I_W$$

counts the number of points that are inside the triangle formed by other points. Hence, $I = 0$ precisely if $X, Y, Z, W$ form a convex quadrilateral, and this happens with probability $p$. With probability $1-p$, $I = 1$. Thus, the expected value $\mathbb{E}[I] = 0 \cdot p + 1 \cdot (1-p) = 1-p$, and $p = 1 - \mathbb{E}[I]$.

This implies that to find $p$ it suffices to find $\mathbb{E}[I]$. From the linearity of the expectation, and assuming that $X, Y, Z, W$ are drawn independently from the same distribution,

$$\mathbb{E}[I] = \mathbb{E}[I_X] + \mathbb{E}[I_Y] + \mathbb{E}[I_Z] + \mathbb{E}[I_W] = 4\mathbb{E}[I_X].$$

Next, $I_X$ is a random variable that takes values 0 or 1, and

$$\mathbb{E}[I_X] = 0 \cdot (1 - p_X) + 1 \cdot p_X = p_X,$$

where $p_X$ is the probability that $X$ lies inside triangle $YZW$.

If points $X, Y, Z, W$ are selected independently and uniformly at random from the unit disk $\mathbb{D}$, then by the law of total expectation,

$$\mathbb{E}[I_X] = \mathbb{E}[\mathbb{E}[I_X | Y, Z, W]] = \mathbb{E}\left[\frac{A(YZW)}{A(\mathbb{D})}\right]$$

where $A$ denotes the area. Hence, the problem reduces to determining the expected area of the triangle $YZW$. In 1867, Woolhouse determined that

$$\mathbb{E}\left[\frac{A(YZW)}{A(\mathbb{D})}\right] = \frac{35}{48\pi^2},$$

hence

$$p = 1 - \mathbb{E}[I] = 1 - 4\mathbb{E}[I_X] = 1 - \frac{35}{12\pi^2} = 0.704\ldots.$$

Of course, random points can be selected inside regions different from a disk. Sylvester also asked the same question in the following modified form. Let $S$ be a convex body in the plane (that is, a compact convex set with nonempty interior) and choose four points from $S$ independently and uniformly at random. What is the probability $p(4, S)$ that these points are the vertices of a convex quadrilateral? Further, for what $S$ is this probability the smallest and the largest? The second question has been solved by Blaschke, who proved in 1917 that for all convex bodies $S$,

$$\frac{2}{3} = p(4, \mathbb{T}) \leq p(4, S) \leq p(4, \mathbb{D}) = 1 - \frac{35}{12\pi^2} = 0.704\ldots,$$

where $\mathbb{T}$ and $\mathbb{D}$ denotes a triangle and a disk in the plane, respectively.

[1] *Recall that the convex hull of set $K \subset \mathbb{R}^n$ is the intersection of all convex sets containing $K$.*

Sylvester's question can be asked for $m$ points: if they are selected uniformly at random in a convex body $S$ in the plane, what is the probability $p(m,S)$ that they form a convex $m$-gon?

In 1995, Valtr solved this problem exactly for a parallelogram $\mathbb{L}$, and proved that

$$p(m,\mathbb{L}) = \left(\frac{\binom{2m-2}{m-1}}{m!}\right)^2.$$

In 1996, Valtr also solved this problem for triangle $\mathbb{T}$, and showed that

$$p(m,\mathbb{T}) = \frac{2^m(3m-3)!}{(m-1)!^3\,(2m)!}.$$

Using Stirling's approximation for the factorial, it is straightforward to prove that

$$\lim_{m\to\infty}\left(m^2\sqrt[m]{p(m,\mathbb{T})}\right) = \frac{27}{2}e^2.$$

Because any convex body $S$ in the plane can be sandwiched between two triangles, this implies the existence of universal constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 \le m^2\sqrt[m]{p(m,S)} \le c_2$$

for all $m$ and all $S$. In fact, Bárány [Bár99] proved in 1999 that

$$\lim_{m\to\infty}\left(m^2\sqrt[m]{p(m,S)}\right) = c(S)$$

for some constant $c(S)$ that depends on $S$. For example, $c(\mathbb{L}) = 16e^2$ for parallelogram, $c(\mathbb{T}) = \frac{27}{2}e^2$ for triangle, and $c(\mathbb{D}) = 2\pi^2e^2$ for disk. In particular,

$$p(m,\mathbb{D}) \approx \left(\frac{2\pi^2e^2}{m^2}\right)^m.$$

approaches 0 as $m\to\infty$ with super-exponential speed.

Can we have the exact (non-asymptotic) formulas for $p(m,\mathbb{D})$? In 1971, Miles derived the exact formula for $p(5,\mathbb{D})$:

$$p(5,\mathbb{D}) = 1 - \frac{305}{48\pi^2} = 0.356\ldots.$$

Finally, Marckert in 2017 derived exact (but somewhat complicated) formulas for $p(m,\mathbb{D})$ for an arbitrary $m$. For example, for $m = 6$,

$$p(6,\mathbb{D}) = 1 - \frac{305}{24\pi^2} - \frac{473473}{11520\pi^4} = 0.134\ldots.$$

The following table lists numerical values for $p(m,\mathbb{T})$, $p(m,\mathbb{L})$ and $p(m,\mathbb{D})$ for $4 \le m \le 7$.

| $m$ | 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| $p(m,\mathbb{T})$ | 0.666 ... | 0.305 ... | 0.101 ... | 0.0251 ... |
| $p(m,\mathbb{L})$ | 0.694 ... | 0.340 ... | 0.122 ... | 0.0336 ... |
| $p(m,\mathbb{D})$ | 0.704 ... | 0.356 ... | 0.134 ... | 0.039 ... |

As expected, we see that the probabilities decrease fast even for small values of $m$. This is bad news for our machine learning application, because it shows that new points will most likely be in the convex hull of other points. However, *all these results are in the plane*, which corresponds to a (toy) machine learning system with just two attributes. Any real ML system has significantly more attributes, hence we should study Problem 1 in higher-dimensional spaces. In the next section we show that separability properties of random points in higher dimensions are dramatically different from those computed for our low-dimensional example.

## 3. The Effect of Higher Dimension

We start our analysis of Problem 1 in higher dimensions with a simple special case. Let $\mathbb{B}_n$ be the closed unit ball in $\mathbb{R}^n$. We first consider the case when points $X_1,\ldots,X_m \in \mathbb{B}_n$ are fixed, and $Y \in \mathbb{B}_n$ is selected uniformly at random in $\mathbb{B}_n$. In 1986, Elekes [Ele86] proved that for any $m$ points $X_1,\ldots,X_m \in \mathbb{B}_n$, we have

$$\frac{\mathrm{Vol}(\mathrm{conv}(X_1,\ldots,X_m))}{\mathrm{Vol}(\mathbb{B}_n)} \le \frac{m}{2^n}, \tag{1}$$

where conv is the convex hull, and Vol denotes the $n$-dimensional volume. This implies that $Y$ can be separated from $X_1,\ldots,X_m$ by a hyperplane with probability at least $1 - m/2^n$. This probability is greater than $1-\delta$ provided that $m/2^n < \delta$, or

$$m < \delta 2^n. \tag{2}$$

Now assume that we select $m$ points independently and uniformly at random in $\mathbb{B}_n$. Let $E_i$ be the event that the point $X_i$ is inside the convex hull of the remaining points. Then Elekes's theorem implies that the probability of $E_i$ is at most $(m-1)2^{-n}$, and the probability of the event $E = \bigcup_{i=1}^m E_i$ is at most $m(m-1)2^{-n} < m^2 2^{-n}$. Hence with the probability greater than $1 - m^2 2^{-n}$ every point $X_i$ is separable by a hyperplane from the remaining points. This probability is greater than $1-\delta$ if $m^2 2^{-n} < \delta$, or

$$m < \sqrt{\delta}(\sqrt{2})^n. \tag{3}$$

The upper bound (3) was originally proved by Bárány and Füredi in 1988. Complementing this result, Bárány and Füredi also proved that, for all $n \ge 100$, the probability that

$$m = 20n^{3/4}(\sqrt{2})^n$$

independent uniformly distributed points in $\mathbb{B}_n$ are all vertices of their convex hull is less than $2e^{-10}$. Hence, the bound (3) is quite tight. In particular, the result is no longer true if $(\sqrt{2})^n$ in (3) is replaced by $(\sqrt{2}+\epsilon)^n$ for any $\epsilon > 0$.

The following table shows, in various dimensions $n$, the upper bounds for $m$ in (2) and (3) with $\delta = 0.01$, ensuring the separability with 99% probability.

| $n$ | Upper bound in (2) | Upper bound in (3) |
|---|---|---|
| 10 | 10.24 | 3.2 |
| 30 | $1.07 \cdot 10^7$ | 3276 |
| 50 | $1.12 \cdot 10^{13}$ | $3.35 \cdot 10^6$ |
| 100 | $1.26 \cdot 10^{28}$ | $1.12 \cdot 10^{14}$ |

We see that in dimension $n = 30$, a random point is separable from millions of other points with probability over 99%, and thousands of random points are all separable. In dimension $n = 50$, millions of points all become separable. In other words, if we select 3 million uniformly random points in ball $B_{50} \subset \mathbb{R}^{50}$, then with probability over 99% they are all vertices of their convex hull. This observation is in sharp contrast with our low-dimensional intuition.

This effect is not limited to the uniform distribution in the unit ball $\mathbb{B}_n$. In fact, when we say "uniform distribution in the unit ball," we actually mean a *family* of distributions, one for each dimension: the uniform distribution on the interval $[-1, 1]$ in $\mathbb{R}^1$, the uniform distribution in the the disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ in $\mathbb{R}^2$, and so on. In the theorems below, the dimension will not be fixed but will be a variable, and in this case we need to consider a family

$$\mathbb{P} = \{\mathbb{P}_1, \dots, \mathbb{P}_n, \dots\}$$

of probability measures, where $\mathbb{P}_n$ denotes the probability measure on $\mathbb{R}^n$.

**Definition 1.** [GGG+18] The family of joint distributions of points $X_1, \dots, X_m$ in $\mathbb{R}^n$ has **SmAC property** if there exist constants $\epsilon > 0$, $A > 0$, and $B \in (0, 1)$, such that for every positive integer $n$, any convex set $S \in \mathbb{R}^n$ such that

$$\frac{\mathrm{Vol}(S)}{\mathrm{Vol}(\mathbb{B}_n)} \le \epsilon^n,$$

any index $i \in \{1, 2, \dots, m\}$, and any points $Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_m$ in $\mathbb{R}^n$, we have

$$\mathbb{P}(X_i \in \mathbb{B}_n \setminus S \mid X_j = Y_j, \forall j \ne i) \ge 1 - AB^n. \qquad (4)$$

Condition (4) says that, with probability exponentially close to 1, a random point lies inside the unit ball, but outside of any convex set of exponentially small volume. In other words, SmAC property holds for the distributions without (i) heavy tails and (ii) sharp peaks in sets with exponentially small volume. Indeed, any bounded or light-tailed distribution can, after appropriate shift and rescaling, be located essentially inside $\mathbb{B}_n$, while for heavy-tailed distributions there is a significant probability that $X_i \notin \mathbb{B}_n$, hence (4) fails. The name SmAC is an abbreviation of "SMeared Absolute Continuity" and comes from analogy with absolute continuity: the absolute continuity means that the sets of zero measure have zero probability, and the SmAC condition requires that convex sets with exponentially small volume should not have high probability.

The theorem below states that if a family of distributions has the SmAC property, then exponentially many points are in convex position with high probability.

**Theorem 1.** [GGG+18] *Let $\{X_1, \dots, X_m\}$ be a set of random points in $\mathbb{R}^n$ from a distribution satisfying the SmAC property. Let $\delta \in (0, 1)$ be fixed. Then there exists constants $a > 0$ and $c > 1$ such that if $m < ac^n$ then points $\{X_1, \dots, X_m\}$ are in convex position with probability greater than $1 - \delta$.*

The SmAC condition is very general and holds for a large variety of distributions. As an illustration, consider a special case of i.i.d. data. If probability measures $\mathbb{P}_n$ in family $\mathbb{P}$ have support in the unit ball $\mathbb{B}_n$ and density $\rho_n$, then the SmAC condition holds provided that

$$\frac{\rho_n(x)}{\rho_{\mathrm{uni}}(x)} \le CR^n, \qquad \forall x \in \mathbb{B}_n \qquad (5)$$

where $C > 0$ and $R > 0$ are some constants independent of the dimension, and $\rho_{\mathrm{uni}}(x)$ is the density of the uniform distribution in $\mathbb{B}_n$. In other words, the density $\rho_n(x)$ is allowed to differ from the uniform density by an exponentially large factor, and the exponent $R$ must be a constant independent of $n$ but can be arbitrarily large.

For example, let $A_n$ be a bounded measurable set in $\mathbb{R}^n$. Then it is not difficult to see that (5) is true for the uniform distribution in (a possibly scaled and shifted) $A_n$ provided that

$$\frac{\mathrm{diam}(A_n)}{\sqrt[n]{\mathrm{Vol}(A_n)}} \le R\sqrt{n} \qquad (6)$$

for some constant $R < \infty$. In particular, if $A_n$ is the unit cube in $\mathbb{R}^n$, then $\mathrm{Vol}(A_n) = 1$, $\mathrm{diam}(A_n) = \sqrt{n}$, and (6) holds with $R = 1$. Hence Theorem 1 implies that exponentially many points selected uniformly at random from the unit cube are in convex position with high probability.

## 4. Computing Separating Hyperplanes

Under SmAC condition, exponentially many random points $X_1, \dots, X_m$ in $\mathbb{R}^n$ are linearly separable with high probability: for each $i \in 1, \dots, m$, there exists a hyperplane $H$ passing through $X_i$ such that all other points are on the same side from $H$. If $(x_{j1}, \dots, x_{jn})$ are the coordinates of point $X_j$, $j = 1, \dots, m$, then we can explicitly find $H$ by solving the quadratic program

$$\min_{c_1, \dots, c_n, v} ||c||^2, \quad \text{subject to} \qquad (7)$$

$$\sum_{k=1}^{n} c_k x_{jk} + v \le -1, \quad j \ne i; \quad \sum_{k=1}^{n} c_k x_{ik} + v = 1.$$

If $c^* = (c_1^*, \dots, c_n^*, v^*)$ is the solution to (7), then

$$H = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : 1 - v^* = \sum_{k=1}^{n} c_k^* x_k \right\}.$$

The above program is a version of the well-known maximal-margin classifier or a support vector machine.

This quadratic program has $m$ constraints and $n + 1$ variables. Worst-case computational complexity of solving this problem scales as $O(\max(n + 1, m) \min(n + 1, m)^2)$ [Cha07]. When $m$ is potentially exponentially large in $n$, the worst-case complexity grows exponentially with $n$.

The other issue with finding separating hyperplanes through solving (7) is that this approach requires full knowledge of all points $X_j$, $j = 1, \dots, m$. Whilst such knowledge might be available in some tasks, it is hardly practical in the task of *correcting AI errors*. In this context, $X_i$ represents an AI "error" that has already been detected and is to be removed, and $X_j$, $j \neq i$ stand for "correct or expected" past and possibly future AI behavior. The fact that some or all $X_j$ are unknown makes solving (7) hardly possible. The question, therefore, is:

**Problem 2.** How to construct $H$ separating $X_i$ from the remaining points without knowing their positions?

In the next sections we show that, for appropriately high dimension $n$ and under some mild assumptions, there are simple closed-form expressions defining hyperplanes separating $X_i$ from $X_j$, $i \neq j$ with probability close to 1.

### 4.1. One-shot separability: Fisher separability.

In order to develop the intuition for Problem 2, let us return to the simplest example, when the points are selected uniformly at random from the $n$-dimensional unit ball $\mathbb{B}_n$. Any hyperplane $H$ through $X_i$ divides $\mathbb{B}_n$ into pieces with volumes $V_1 \leq V_2$. To maximize the chance that hyperplane $H$ separates $X_i$ from all other points, we aim to select $H$ such that volume $V_1$ is the minimal possible. The optimal choice of $H$ is the hyperplane orthogonal to $OX_i$, where $O$ is the centre of $\mathbb{B}_n$; see Figure 5. If $A$ is the event that point $X_j$ belongs to the piece with volume $V_1$, then a straighforward calculation shows that

$$\mathbb{P}(A) = \mathbb{E}[I_A] = \mathbb{E}[\mathbb{E}[I_A|X_j]] = \mathbb{E}[R^n] = \frac{1}{2^{n+1}}, \quad (8)$$

where $I_A$ is the indicator function of the event $A$, the second equality is the law of total expectation, the third equality follows from the fact that $I_A|X_j$ is equal to 1 if and only if $X_i$ belongs to a ball with radius $R = |OX_j|/2$, and the last equality follows from the fact that $R$ is a random variable with cdf $\mathbb{P}[R \leq r] = \mathbb{P}[|OX_j| \leq 2r] = (2r)^n, \quad 0 \leq r \leq 1/2$.

Now, if we have $m$ i.i.d. points from $\mathbb{B}_n$, there are $m(m-1)$ ordered pairs of points. Hence, the probability that we can find some pair $X_i, X_j$ such that the corresponding event $A$ happens is at most $m(m-1)2^{-(n+1)} < m^2 2^{-(n+1)}$. This probability is less than $\delta$ provided that

$$m < \sqrt{2\delta}(\sqrt{2})^n.$$

Remarkably, this bound is even less restrictive than (3), while the conclusion is stronger: not only are this many points in convex position with probability greater than $1 - \delta$, but in fact each point $X_i$ can be separated from the

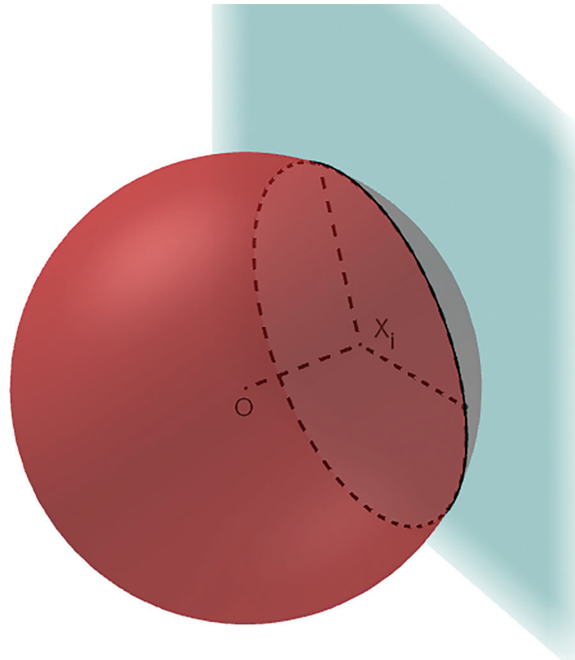

**Figure 5.** One-shot separability in a sphere.

other ones by the *specific* hyperplane tangent to $OX_i$, which is independent from other points, and can be constructed exponentially faster than solving the program (7).

It turns out that this simple idea to choose hyperplane $H$ tangent to $OX_i$ solves Problem 2 for surprisingly many families of distributions, and is known as Fisher separability [GGG+18].

**Definition 2.** A point $X \in \mathbb{R}^n$ is **Fisher-separable** from $Y \in \mathbb{R}^n$ with threshold $\alpha \in (0, 1]$ if

$$\alpha\|X\|^2 \geq \sum_{i=1}^{n} x_i y_i. \quad (9)$$

We say that $X$ is Fisher-separable from a finite set $F \in \mathbb{R}^n$ with threshold $\alpha$ if (9) holds for all $Y \in F$.

The question is: how do we know that $X_i$ is Fisher-separable from $X_j$, $i \neq j$? An answer to this question follows from the next statement.

**Proposition 1** ([GGG+18]). *Let $\alpha \in (1/2, 1]$, $1 > \delta > 0$, let $X$ be drawn from a distribution supported on $\mathbb{B}_n$ whose probability density satisfies* (5) *with some $C > 0$ and $R \in (1, 2\alpha)$, and let $Y$ be a finite set in $\mathbb{B}_n$ with*

$$|Y| \leq \delta \left(\frac{2\alpha}{R}\right)^n \frac{1}{C}.$$

*Then the point $X$ is Fisher-separable from the set $Y$ with probability at least $1 - \delta$.*

Several interesting observations stem immediately from Proposition 1. It appears that construction of separating hyperplanes does not always require complete knowledge

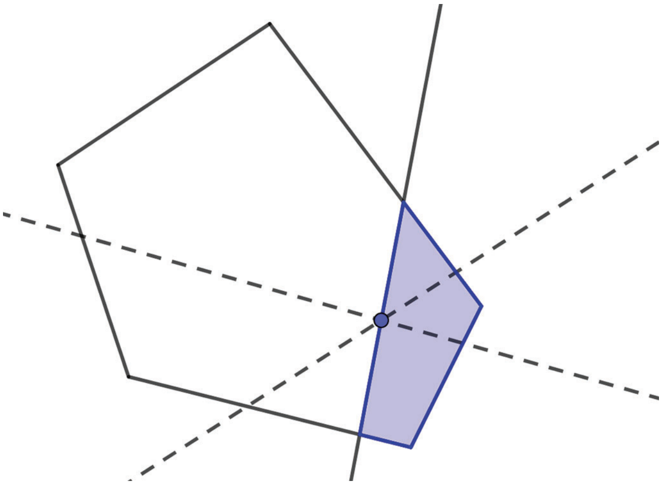

**Figure 6.** One-shot separability: we select a hyperplane that divides the probability measure into two maximally unequal parts.

of sets that are being separated. Some rough information such as the value of the point $O$, the fact that all $X_j$, $j \neq i$ are in a unit ball centered at $O$, and that $X_i$ is drawn from a SmAC distribution suffices. The resulting hyperplane $H$ separates $X_i$ from $X_j$, $j \neq i$ with probability at least $1-\delta$, and with some guaranteed margin $(1-\alpha)\|X_i\|$. Note, however, that this margin is not necessarily maximal as requested by program (7).

It turns out that Fisher separability for exponentially many points holds for many important families of distributions, including rotation invariant log-concave distributions and product distributions whose components have bounded support or very fast-decaying tails [GGT21]. At the same time, there are examples of product distributions with identical log-concave components for which this is no longer true [GGT21]. It is hence natural to ask if and how similar simple solutions could be derived for such distributions with "heavier" tails.

4.2. **One-shot separability: general case.** Now we formulate the same idea in general. Let $\mathbb{P}_n$ be an arbitrary probability measure in $\mathbb{R}^n$, and let $X \in \mathbb{R}^n$ be an arbitrary point. Problem 2 asks to construct a hyperplane separating $X$ from other $m$ points selected at random from $\mathbb{P}_n$ without knowing their positions. Every hyperplane divides $\mathbb{R}^n$ into two half-spaces, say $H_1$ and $H_2$, whose probability measures are $p_1 = \mathbb{P}_n(H_1)$, and $p_2 = \mathbb{P}_n(H_2)$, respectively. We would like $X$ and the remaining $m$ points to belong to different subspaces, say $X \in H_1$, and other $m$ points to belong $H_2$. The probability of the latter event is $p_2^m = (1-p_1)^m$. This probability is maximized if $p_1$ is minimized. Hence, the idea is to select the halfspace containing $X$ whose probability measure is minimal; see Figure 6. Formally, let $\mathbb{H}_X$ be the set of halfspaces of $\mathbb{R}^n$ containing $X$, let

$$\phi(\mathbb{P}_n, X) = \inf_{H \in \mathbb{H}_X} \mathbb{P}_n(H) \tag{10}$$

be the minimal measure of a halfspace containing $X$, and let $H^*(X)$ be the minimizer[2] in (10). Function $\phi(\mathbb{P}_n, X)$ is known as Tukey's halfspace depth.

The probability that $H^*(X)$ separates $x$ from $m$ points is $(1-\phi(\mathbb{P}_n, X))^m$. We would like this probability to be greater than a given constant $1-\delta$ even if $m$ grows exponentially fast with $n$. To ensure this, $\phi(\mathbb{P}_n, X)$ should decrease exponentially fast with $n$. This may not be the case for all $X$: for example, if $\mathbb{P}_n$ is the uniform distribution in the ball, and $X$ is the center of the ball, then $\phi(\mathbb{P}_n, X) = 1/2$. However, there is a hope that $\phi(\mathbb{P}_n, X)$ decreases fast on average, for random point $X$. In other words, we need expected value

$$c(\mathbb{P}_n) = \mathbb{E}[\phi(\mathbb{P}_n, X)]$$

to decrease exponentially fast with $n$.

**Definition 3.** Let $\mathbb{P} = \{\mathbb{P}_1, \dots, \mathbb{P}_n, \dots\}$ be a family of probability measures, where $\mathbb{P}_n$ is the probability measure on $\mathbb{R}^n$. We say that $\mathbb{P}$ has **exponential one-shot separability** if

$$c(\mathbb{P}_n) \leq a_{\mathbb{P}}(c_{\mathbb{P}})^n$$

for some constants $a_{\mathbb{P}} < \infty$, $c_{\mathbb{P}} \in (0,1)$.

In this section, we overview our recent results that establish exponential one-shot separability for a large class of product distributions, and discuss a conjecture that this property holds for all log-concave distributions.

Let us now be a bit more formal. We say that density $\rho_n : \mathbb{R}^n \to [0,\infty)$ of random vector $X = (x_1, \dots, x_n)$ (and the corresponding probability measure $\mathbb{P}_n$) is *log-concave*, if set

$$D = \{z \in \mathbb{R}^n \mid \rho_n(z) > 0\}$$

is convex and $g(z) = -\log(\rho_n(z))$ is a convex function on $D$. For example, the uniform distribution in an arbitrary convex body is log-concave. Let $C$ be the variance-covariance matrix of $X$, that is, matrix with components $c_{ij} = \mathrm{Cov}(x_i, x_j)$. Because the log-concavity of $\mathbb{P}_n$ and the definition of $c(\mathbb{P}_n)$ are invariant under invertible linear transformations, we may assume that $\mathbb{E}[X] = 0$ and $C = I_n$ is the $n \times n$ identity matrix. Such distributions are called isotropic. Quantity

$$L_{\mathbb{P}_n} = \left(\sup_{z \in \mathbb{R}^n} \rho_n(z)\right)^{1/n}$$

is called the isotropic constant of $\mathbb{P}_n$. Very recently, Brazitikos, Giannopoulos, and Pafis [BGP22] proved that

$$c(\mathbb{P}_n) \leq \exp\left(-\frac{an}{L_{\mathbb{P}_n}}\right) \tag{11}$$

[2] *Each halfspace can be identified with its normal unit vector, the set of all such vectors is a compact set, hence there must be a halfspace that achieves the minimum.*

for some absolute constant $a > 0$. A famous conjecture in convex geometry predicts that

$$L_{\mathbb{P}_n} \geq \epsilon \tag{12}$$

for some constant $\epsilon > 0$ independent from the dimension. This conjecture has been made in 1986 by Jean Bourgain [Bou86] in the form that "There exists a universal constant $\epsilon > 0$ (independent from $n$) such that for any convex set $K$ of unit volume in $\mathbb{R}^n$, there exists a hyperplane $H$ such that the $(n-1)$-dimensional volume of the section $K \cap H$ is bounded below by $\epsilon$," and since then is known as the Hyperplane conjecture. It turns out that this conjecture is equivalent to (12), and in fact has many other equivalent formulations. Recently, Chen [Che21] made a breakthrough and proved that

$$L_{\mathbb{P}_n} \geq n^{-f(n)}$$

for some function $f$ tending to 0 as $n \to \infty$. Even more recently, Klartag and Lehec [KL22] improved this to $L_{\mathbb{P}_n} \geq b(\log n)^{-5}$ for some absolute constant $b > 0$. In combination with (11), a full proof of conjecture (12) would imply that any family of log-concave probability measures has exponential one-shot separability.

Our next example is a family of product distributions. Specifically, for each $n$, let $\mathbb{P}_n$ be the the product measure of one-dimensional probability measures $\mu_{1,n}, \dots, \mu_{n,n}$. For any distribution $\mu$ on $\mathbb{R}$, define

$$\psi_\mu(x) = \inf_{c \in \mathbb{R}} \mathbb{E}[\exp(c(Z-x))], \; c_\mu = \mathbb{E}[\psi_\mu(X)],$$

where $Z$ and $X$ are random variables with distribution $\mu$. Then we have proved [GGT] that $\mathbb{P}_n$ has exponential one-shot separability provided that $c_\mu < 1$ for each component distribution $\mu$. This property holds for a large variety of distributions. For example, we have the following sufficient condition [GGT].

**Proposition 2.** *Let $Z$ be a random variable with distribution $\mu$. Assume that $Z$ is non-constant and $M_Z(t) := \mathbb{E}[e^{tZ}] < \infty$ for some $t \neq 0$. Then $c_\mu < 1$.*

When our data are non-negative, Proposition 2 implies the following corollary.

**Corollary 1.** *Let $Z$ be a non-constant non-negative random variable with distribution $\mu$. Then $c_\mu < 1$.*

For log-concave distributions, we have the following explicit and uniform upper bound [GGT].

**Proposition 3.** *For any log-concave probability distribution $\mu$ on $\mathbb{R}$, we have*

$$c_\mu < 1 - 2 \cdot 10^{-5}.$$

Proposition 3 implies the following result.

**Theorem 2.** *Let $\mathbb{P} = \{\mathbb{P}_1, \dots, \mathbb{P}_n, \dots\}$ be a family of product distributions such that all component distributions are log-concave. Then $\mathbb{P}$ has exponential one-shot separability (see Definition 3) with parameters $a_{\mathbb{P}} = 1$ and $c_{\mathbb{P}} < 1 - 2 \cdot 10^{-5}$.*

We did not try to optimize the upper bound for $c_\mu$ in Proposition 3. Instead, we pose the problem of finding the optimal upper bound as an open question. Specifically, if $\mathcal{F}$ is the class of all log-concave distributions on $\mathbb{R}$, then what is the value of

$$c_{\mathcal{F}} = \sup_{\mu \in \mathcal{F}} c_\mu?$$

Proposition 3 provides the upper bound $c_{\mathcal{F}} \leq 1 - 2 \cdot 10^{-5} < 1$. On the other hand, example of Laplace distribution shows that

$$c_{\mathcal{F}} \geq \frac{3}{4} + \frac{e}{16} \int_1^\infty \frac{e^{-t}}{t} dt = 0.7872\dots.$$

While the upper bound is clearly non-optimal, it may be that $c_{\mathcal{F}}$ is equal to the lower bound.

## 5. Conclusions

A phenomenon known as curse of dimensionality states that many methods and techniques that are efficient in low dimension become infeasible is high dimension. Stochastic separation theorems are examples of the opposite phenomenon, blessing of dimensionality, which states that some aspects become easier in higher dimensions. The theorems state that if we have $m$ random points in $\mathbb{R}^n$, then, with high probability, every point can be separated from all others by a hyperplane. This is true even if the number of points grows exponentially fast with dimension.

While being interesting from purely mathematical perspective, stochastic separation theorems could be a stepping stone for the development of much-needed error correcting mechanisms [GGG+18], algorithms capable of learning from just few examples [GGM+21], approaching the challenge of continuous learning without catastrohpic forgetting in machine learning and AI, and to produce new notions of data dimension [GMT19]. The theorems imply that if the number of attributes is moderately high, AI errors may be corrected by adding simple linear correctors, that are fast, easy to compute and implement, and do not destroy existing functionality of the system. The simplest corrector is based on Fisher separability discussed in Section 4.1. Deeper one-shot separation theorems discussed in Section 4.2 make the method applicable even for distributions for which Fisher separability fails.

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# Tropical Combinatorics

*Felipe Rincón, Ngoc Mai Tran, and Josephine Yu*

Tropical mathematics arises from the max-plus semifield. The max-plus algebra, especially max-plus linear algebra and applications to computer science, combinatorics, and optimization, have been studied since 1970s by Cuninghame-Green, Kleene, Zimmermann, and others. However, much of the development in tropical geometry in the last 20 years is due to the *tropicalization* process, which turns algebro-geometric objects into combinatorial ones.

Tropicalization of algebraic sets, also known as Maslov dequantization or logarithmic limit sets, was introduced by Bergman to study the "exponential behavior at infinity" of algebraic varieties, by Viro to construct real plane curves with prescribed degree and topology, by Mikhalkin to count algebraic curves, and by Sturmfels for solving systems of polynomial equations.

Tropicalization has led to numerous recent breakthroughs in diverse areas of mathematics such as topology of moduli spaces of curves [Cha21] and optimization [ABGJ21]. Moreover, tropicalization gives us constructions, intuition, and analogies for studying purely combinatorial objects as well, even if they do not arise as shadows of algebraic geometry. This is the case, for example, in the development of combinatorial Hodge theory, which


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contributed in great part to the recent Fields medal award given to June Huh.

In this article we introduce some of the basic constructions in tropical geometry, focusing on linear spaces and Grassmannians for their combinatorial significance. We give pointers to some recent research frontiers and discuss applications in matroid theory, phylogenetic trees, and auction theory.

In Section 1 we provide background on tropicalization of algebraic sets. In Section 2 we discuss tropicalizations of linear subspaces, their connection to matroids, and tropicalizations of Grassmannians, which are parameter spaces for the set of all linear subspaces of a fixed dimension in an ambient vector space. In Section 3, we look beyond the tropical Grassmannian and study the Dressian as a parameter space of all valuated matroids, not just those arising from linear subspaces. The Dressian provides a unifying language for applications in economics, which we discuss in Section 4.

## 1. Tropical Foundations

In this section we explain how tropicalization uncovers combinatorial structure of algebraic objects, such as *Newton polytopes* of polynomials and their *subdivisions*. More generally, we discuss how tropicalizations of algebraic sets are piecewise linear objects with rich combinatorial structure. We refer to the book [MS15] for proofs and more details.

*Tropical* or *max-plus* algebra is algebra over the real numbers $\mathbb{R}$ with tropical addition

$$a \oplus b = \max(a, b)$$

and tropical multiplication

$$a \odot b = a + b.$$

The operations satisfy associative, commutative, and distributive laws. The multiplicative identity is 0. We may optionally adjoin $-\infty$ if we desire an additive identity, but there are no additive inverses.

Tropical operations arise from limits of logarithms. To build intuition, consider two monic polynomials $F, G \in \mathbb{R}[x]$. If $x$ is very large, then $F, G$ are each dominated by the monomial of highest degree, so $F(x) \sim x^a$, $G(x) \sim x^b$ for some $a, b > 0$. Then, $(F \cdot G)(x) \sim x^{a+b}$, and if $a \neq b$, then $(F + G)(x) \sim x^{\max(a,b)}$. In other words, if we work with large values of $x$, then multiplication and addition of polynomials corresponds to addition and taking maxima of exponents. Let us now consider a richer variant of the above. A *non-Archimedean valuation* on a field $\mathbb{K}$ is a map $\nu : \mathbb{K} \setminus \{0\} \to \mathbb{R}$ satisfying

1. $\nu(ab) = \nu(a) \odot \nu(b)$
2. $-\nu(a + b) \leq -\nu(a) \oplus -\nu(b)$.

For example, $\mathbb{K}$ can be the field $\mathbb{C}((t))$ of Laurent series with complex coefficients, which are formal power series where exponents can start at a negative integer, or its algebraic closure, the field of Puiseux series $\bigcup_{n\geq 1} \mathbb{C}((t^{\frac{1}{n}}))$. Then a valuation could be the lowest exponent of a term with nonzero coefficient.

We now have *two* different ways to tropicalize. We can *tropicalize* a polynomial over $\mathbb{K}$ by replacing the algebraic operations with tropical operations, and replacing the coefficients with (negative of) their valuations. On the other hand, we can *tropicalize* a subset of $(\mathbb{K}\setminus\{0\})^n$ by taking (negative of) valuations coordinate-wise. For example, consider the univariate polynomial

$$F(x) = x^3 - (t^{-4} + t^{-3} + t^{-2})x + (t^{-5} + t^{-4})$$

with coefficients in the field of Laurent series $\mathbb{C}((t))$. The three roots of $F$ are the Laurent series $t^{-2}$, $-t^{-2} - t^{-1}$, and $t^{-1}$. We can tropicalize $F(x)$ to obtain the polynomial

$$f(x) = \operatorname{trop}(F(x)) = 0 \odot x^{\odot 3} \oplus 4 \odot x \oplus 5. \qquad (1)$$

We can also take (negative of) valuations of the three roots, obtaining the real numbers 2, 2, and 1. These two ways of tropicalizing are compatible, if we define roots of tropical polynomials appropriately. This is the content of the Fundamental Theorem of Tropical Algebraic Geometry, which we will now work towards.

Defining the roots of a tropical polynomial $f$ by "solving" for $f = 0$ or $f = -\infty$ is not a very useful notion, due to the fact that tropical algebra has no additive inverses. However, there are still good ways to define tropical roots, and more generally, tropical hypersurfaces, varieties, and their algebraic companions, ideals.

To motivate the definitions, consider the tropical polynomial $f$ from Equation (1) above. We could try to factor $f$ into linear factors to define its roots. This is not possible, though, as the degree-2 terms cannot be canceled out in a

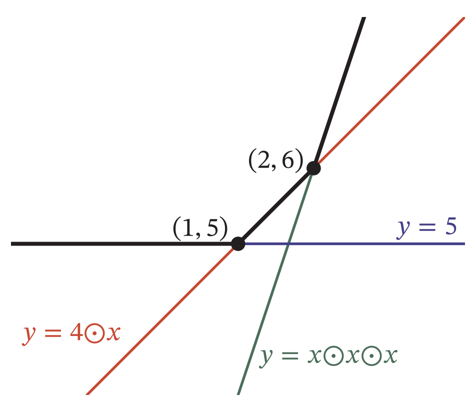


**Figure 1.** The graph of the tropical polynomial $f$ defined in (1). Each tropical monomial is a usual affine function, and the tropical polynomial is the maximum of affine functions. The tropical roots of $f$ are 1 and 2, which are the values of $x$ where the graph bends.

tropical product of lower-degree polynomials. Nonetheless, as real functions, we have the equality

$$f(x) \equiv (x \oplus 2) \odot (x \oplus 2) \odot (x \oplus 1). \tag{2}$$

Tropically multiplying a function by the linear polynomial $(x \oplus a)$ translates its graph vertically by $a$ units and then bends the graph up by slope one for $x \geq a$. This means that the factorization is determined by where the slopes change in the graph of $f$. Thus (2) is the *unique* factorization of $f$ into monic linear factors, which motivates us to say that the *tropical roots* of $f$ are 2 and 1, with 2 being a root of multiplicity two. These are exactly the values of $x$ where the maximum (tropical sum) is attained by at least two of the three tropical monomial terms that make up the tropical polynomial $f$ in the expression (1). The three tropical monomials are usual linear functions, shown by the lines in Figure 1. Note that the three tropical roots are also the valuations of the three roots of the original polynomial $F(x)$. More generally, we have the following.

**Definition 1.1.** The *tropical hypersurface* $\mathcal{V}^{\text{trop}}(f)$ of a tropical polynomial $f$ in $n$ variables is defined as the locus of points $\mathbf{x} \in \mathbb{R}^n$ for which the maximum is attained at least twice among the tropical monomial terms of $f(\mathbf{x})$. Equivalently, it is the corner locus of the piecewise linear function $f$.

Let $\mathbb{K}$ be a field with a nontrivial non-Archimedean valuation $\nu$. The tropicalization of a polynomial over $\mathbb{K}$ is obtained by replacing addition and multiplication with their tropical counterparts and replacing the coefficients with minus their valuations. The tropicalization of a point $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{K}^n$ is

$$\text{trop}(\mathbf{x}) = (-\nu(x_1), \dots, -\nu(x_n)).$$

For a subset $S \subset \mathbb{K}^n$, we can define its tropicalization as

$$\text{trop}(S) = \overline{\{\text{trop}(\mathbf{x}) \mid \mathbf{x} \in S\}},$$

where the bar on the right-hand side denotes the closure in the Euclidean topology.

Thus, from a polynomial $F \in \mathbb{K}[\mathbf{x}]$ we get two tropical objects: the tropicalization of $F$, and the tropicalization of its zero locus in $\mathbb{K}^n$. The following theorem of Kapranov says that this latter set is precisely the set of tropical roots of $\text{trop}(F)$.

**Theorem 1.2** (Kapranov, 1990s). *For any polynomial $F$ we have*

$$\mathcal{V}^{\text{trop}}(\text{trop}(F)) = \overline{\{\text{trop}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{K}^n, F(\mathbf{x}) = 0\}}$$

*where $\mathbb{K}$ is an algebraically closed extension of the field of definition of $F$ with a nontrivial non-Archimedean valuation, and the closure is taken in the Euclidean topology of $\mathbb{R}^n$.*

What information about the tropical polynomial $f$ does the tropical hypersurface $\mathcal{V}^{\text{trop}}(f)$ retain? Written using regular arithmetic, a tropical polynomial $f$ in $n$ variables $x_1, \dots, x_n$ is a function of the form

$$f = \max_{\mathbf{a} \in A}(c_{\mathbf{a}} + \mathbf{x} \cdot \mathbf{a}) = \max_{\mathbf{a} \in A}((\mathbf{x}, 1) \cdot (\mathbf{a}, c_{\mathbf{a}}))$$

where $\mathbf{x} = (x_1, \dots, x_n)$. The set $A \subset \mathbb{Z}^n$ consists of the exponents of monomials appearing in $f$ and is called the *support* of $f$. The coefficients $c_{\mathbf{a}}$ are real numbers.

We can think of the point $(\mathbf{a}, c_{\mathbf{a}}) \in \mathbb{R}^{n+1}$ as the point $\mathbf{a} \in \mathbb{R}^n$ lifted to height $c_{\mathbf{a}}$ in a new dimension. Then $f$ is the function that sends $\mathbf{x} \in \mathbb{R}^n$ to the maximum dot product of $(\mathbf{x}, 1)$ with the lifted points $(\mathbf{a}, c_{\mathbf{a}})$ for $\mathbf{a} \in A$. The tropical hypersurface $\mathcal{V}^{\text{trop}}(f)$ consists of all the points $\mathbf{x}$ where this maximum is attained at least twice.

Convex geometry tells us that, when maximizing a linear function on a set, the possible locations of the points achieving the maxima are the *faces* of the convex hull of the set. When maximizing the dot product with vectors of the form $(\mathbf{x}, 1)$ on the lifted points $(\mathbf{a}, c_{\mathbf{a}})$, the maxima can only occur on the faces on the upper part of the convex hull, since these faces have an upward-pointing normal vector, i.e., an outer normal vector with positive last coordinate. See Figure 2 for an example of lifted points and their upper convex hulls.

This means that the tropical hypersurface $\mathcal{V}^{\text{trop}}(f)$ is determined by the faces in the upper convex hull of the lifted points $(\mathbf{a}, c_{\mathbf{a}})$ with $\mathbf{a} \in A$. More concretely, if we take the upper convex hull of the lifted points and project its faces back down to $\mathbb{R}^n$, we obtain a decomposition of the *Newton polytope* of $f$, which is the convex hull of the support of $f$, as a union of smaller polytopes. This decomposition is called the *regular subdivision* of the Newton polytope of $f$ induced by the lift to heights given by the coefficients of $f$. The tropical hypersurface $\mathcal{V}^{\text{trop}}(f)$ is a polyhedral complex whose faces or cells are in (inclusion-reversing) bijection with non-singleton faces of this regular subdivision. This statement is often referred to as the *duality* between tropical hypersurfaces and regular subdivisions. Compare Figures 2 and 3.

**Example 1.3.** Consider a tropical polynomial in two variables $x$ and $y$ of the form

$$f(x, y) = (u_{00} \odot 0) \oplus (u_{10} \odot x) \oplus (u_{01} \odot y) \oplus (u_{11} \odot x \odot y).$$

Its support is $\{(0,0), (1,0), (0,1), (1,1)\} \subset \mathbb{Z}^2$, and thus its Newton polytope is the unit square $[0,1]^2$. Its regular subdivision is obtained by lifting up the four corners of the square to the heights $u_{ij}$, taking the upper convex hull, and projecting back down to the square. If $u_{00} + u_{11} < u_{01} + u_{10}$, then we obtain the subdivision of the square into two triangles as shown on the left in Figure 2. If the inequality goes the other way, then we obtain the subdivision shown

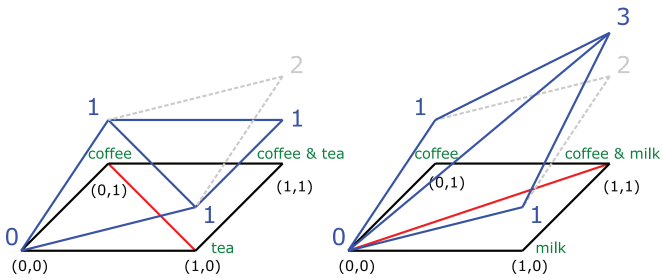


**Figure 2.** Regular subdivisions of the unit square corresponding to the tropical polynomials $0 \oplus (1 \odot x) \oplus (1 \odot y) \oplus (1 \odot x \odot y)$ and $0 \oplus (1 \odot x) \oplus (1 \odot y) \oplus (3 \odot x \odot y)$. The gray dashed line shows the linear function $u(x, y) = x + y$, for reference.

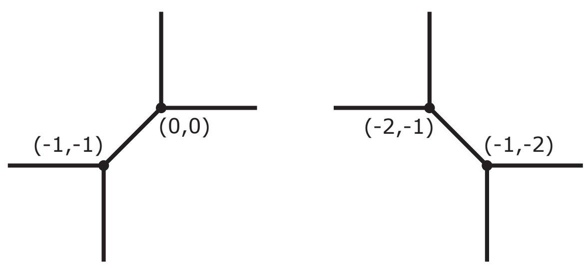


**Figure 3.** Tropical hypersurfaces of the tropical polynomials in Example 1.3. Compare the vertices in these figures with the normal vectors to the faces in the upper convex hulls in Figure 2.

on the right. If there is equality, we obtain the full square unsubdivided.

These subdivisions turn up in auction theory of indivisible distinct goods. Here, the coefficient map $u : \{0,1\}^2 \to \mathbb{R}$ is called a bid function, with $u_{ij}$ representing how much a bidder is willing to pay for the goods bundle consisting of $i$ copies of goods 1 and $j$ copies of goods 2. The regular subdivision $\Delta_u$ represents the relationship between the two items: on the right of Figure 2, the two items are *complements* (such as milk and coffee), since together they are worth more to the bidder than the sum of their individual values. In other words, the bidder would strongly prefer having both milk and coffee, over having milk alone or coffee alone. In the other case, on the left, the two items are called *substitutes* (such as coffee and tea). Some open questions in auction theory were resolved using tropical geometry, by studying the combinatorial types of the bidders' functions. We discuss this in Section 4.

We now consider the case of multiple polynomials. For a single polynomial, its Newton polytope is a natural polyhedral object carrying some discrete invariants such as the degree and the asymptotic behavior of the corresponding hypersurface. When we have a polynomial ideal instead of a single polynomial, what could an analogous polyhedral object be? Tropical geometry provides an answer.

If $I \subset \mathbb{K}[x_1, \dots, x_n]$ is an ideal with variety $V(I) = \{\mathbf{x} \in \mathbb{K}^n \mid F(\mathbf{x}) = 0 \text{ for all } F \in I\}$, we define its tropical variety as the tropicalization of $V(I)$:

$$\mathrm{trop}(V(I)) = \overline{\{\mathrm{trop}(\mathbf{x}) \mid \mathbf{x} \in V(I)\}}.$$

The Fundamental Theorem of Tropical Algebraic Geometry generalizes Kapranov's theorem to any ideal.

**Theorem 1.4** (Fundamental Theorem of Tropical Algebraic Geometry)**.** *Suppose $\mathbb{K}$ is an algebraically closed field with a nontrivial non-Archimedean valuation, and $I \subset \mathbb{K}[x_1, \dots, x_n]$ is an ideal. Then*

$$\mathrm{trop}(V(I)) = \bigcap_{F \in I} \mathcal{V}^{\mathrm{trop}}(\mathrm{trop}(F)).$$

That is, the two ways of tropicalizing are compatible—taking (minus) the valuations of points in the variety $V(I)$, and intersecting the tropical hypersurfaces of the tropicalizations of all polynomials in the ideal $I$. This intersection can in fact be taken to be finite, and a finite subset of $I$ that suffices is called a *tropical basis*—every ideal in a polynomial ring has a tropical basis. There are other ways of describing the tropicalization of an algebraic variety, using logarithmic limits, Gröbner theory, or Berkovich analytifications.

What kind of objects are tropical varieties? Although not immediate from the definition, tropicalizations of algebraic varieties are polyhedral or piecewise linear objects. This was known since Bergman's work and is equivalent to the existence of tropical bases. Bieri and Groves showed in 1984 that the tropicalization has the same dimension as the original variety. Moreover, the *Structure Theorem of Tropical Geometry* says that the tropicalization of an irreducible variety is connected through codimension-one faces and that it satisfies a *balancing condition*. In Figure 3, the balancing condition means that, locally around every vertex, the outgoing direction vectors sum to zero, if weighted appropriately. The balancing condition is a generalization of this statement to higher dimensions. Connectedness through codimension-one faces means that the polyhedral object remains connected even after removing all codimension-two faces.

The connectedness part of the Structure Theorem was recently strengthened—the tropicalization of a $d$-dimensional irreducible variety is connected through codimension-one faces even after removing any $d-1$ pointed maximal faces from it [GHM+21]. This provides a new tool for the *realizability problem* of determining whether a given polyhedral object arises as the tropicalization of an irreducible algebraic variety.

## 2. Tropicalized Linear Spaces and Grassmannians

Linear subspaces are some of the simplest algebraic varieties. It turns out that their tropicalizations are quite rich, with an interesting connection to phylogenetic trees. We now take a quick tour into this world.

A *tropical hyperplane* in $\mathbb{R}^d$ is the tropical variety of a tropical linear function $f = \bigoplus_{i=1}^d a_i \odot x_i$. However, it is not so obvious what the notion of a more general tropical linear space should be. Classically, there are many equivalent characterizations of linear subspaces: as the linear span of a set of vectors, as an intersection of hyperplanes, and as a nonempty subset that is closed under linear combinations, to name a few. However, the absence of additive inverses makes these notions quite different in tropical geometry. As it turns out, the right notion of tropical linear space arises when considering the Plücker embedding of Grassmann variety.

In the 19th century, Julius Plücker realized that the set of planes in 4-dimensional affine space $\mathbb{K}^4$ can be nicely parametrized by a quadratic subvariety of $\mathbb{P}^5$. His work was later generalized by Hermann Grassmann, who found a way of parametrizing all subspaces of $\mathbb{K}^n$ of a fixed dimension $d$ by a projective variety that we now know as the Grassmannian.

Concretely, any $d$-dimensional subspace $L$ of $\mathbb{K}^n$ can be written as the row space of a $d \times n$ matrix $A$. The *Plücker coordinates* of $L$ are the $\binom{n}{d}$ maximal minors of $A$, and they form a point in $\mathbb{P}^{\binom{n}{d}-1}$. These Plücker coordinates depend only on the subspace $L$ and not on the chosen matrix $A$, since row operations on the matrix $A$ only change its maximal minors by a global scalar multiple. The Grassmannian $\mathrm{Gr}(d, n)$ is the subvariety of $\mathbb{P}^{\binom{n}{d}-1}$ consisting of the Plücker coordinates of all $d$-dimensional subspaces of $\mathbb{K}^n$. Any linear subspace $L$ is completely determined by its vector of Plücker coordinates, and thus the Grassmann variety $\mathrm{Gr}(d, n)$ serves as the parameter space for all $d$-dimensional subspaces of $\mathbb{K}^n$.

The Grassmannian $\mathrm{Gr}(d, n)$ is a variety of dimension $d(n-d)$—much lower than that of its ambient projective space, $\mathbb{P}^{\binom{n}{d}-1}$. This variety is defined by polynomials known as the *Plücker relations*. For example, $\mathrm{Gr}(2, 4)$ is defined by the unique quadratic relation satisfied among the six maximal minors of a $2 \times 4$ matrix:

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0, \tag{3}$$

where $p_{ij}$ denotes the maximal minor corresponding to the submatrix consisting of columns $i$ and $j$.

This correspondence between linear subspaces and points in the Grassmannian turns out to carry on into the tropical world. For simplicity, let us fix a valued field with a surjective non-Archimedean valuation onto $\mathbb{R}$. Under this setup, tropicalizing the Grassmann variety $\mathrm{Gr}(d, n)$ under its Plücker embedding in $\mathbb{P}^{\binom{n}{d}-1}$ produces a tropical variety whose points parametrize the set of tropicalizations of all $d$-dimensional subspaces of $\mathbb{K}^n$. In other words, the tropicalization of a linear subspace of $\mathbb{K}^n$ is determined by—and also determines—the valuations of all its Plücker coordinates.

**Theorem 2.1** (Speyer and Sturmfels, 2004)**.** *The following diagram commutes, with the horizontal maps being one-to-one correspondences:*

$$\begin{tikzcd}
\mathrm{Gr}(d,n) \arrow[r, leftrightarrow] \arrow[d, "\mathrm{trop}"'] & \{d\text{-dim subspaces of } \mathbb{K}^n\} \arrow[d, "\mathrm{trop}"] \\
\mathrm{trop}(\mathrm{Gr}(d,n)) \arrow[r, leftrightarrow] & \left\{\begin{array}{c} d\text{-dim tropicalized} \\ \text{linear spaces in } \mathbb{R}^n \end{array}\right\}.
\end{tikzcd}$$

The tropicalization of any $d$-dimensional linear subspace is a pure $d$-dimensional polyhedral complex that is balanced when assigned multiplicities equal to 1 to all its maximal cones. Furthermore, these polyhedral complexes are tropical varieties of degree 1: the number of points (counted with multiplicity) in their intersection with a generic tropical linear subspace of the complementary dimension is always equal to 1. However, as we will discuss in Section 3, the class of tropical varieties of degree 1 consists of more than just tropicalizations of linear subspaces, and it is tightly connected to the study of (valuated) matroids.

**Tropical Grassmannians and phylogenetics trees.** A phylogenetic tree $T$ is a tree on $n$ labelled leaves $\{1, 2, \dots, n\}$ where the internal (non-leaf) edges are weighted by positive numbers and the leaf edges are weighted by real numbers. Such a tree $T$ produces a *pairwise dissimilarity* vector $d(2, T) = \{d_{ij}(T) : 1 \le i < j \le n\} \in \mathbb{R}^{\binom{n}{2}}$ where $d_{ij}(T)$ is the sum of edge weights along the unique path from leaf $i$ to leaf $j$ in $T$. An arbitrary vector $d \in \mathbb{R}^{\binom{n}{2}}$ equals $d(2, T)$ for some tree $T$ if and only if it satisfies the four-point condition: for each set of four distinct points $\{i, j, k, \ell\} \subseteq [n]$, the tuple $(d_{ij}, d_{ik}, d_{i\ell}, d_{jk}, d_{j\ell}, d_{k\ell}) \in \mathbb{R}^6$ lies on the tropical variety cut out by the polynomial

$$(d_{ij} \odot d_{k\ell}) \oplus (d_{ik} \odot d_{j\ell}) \oplus (d_{i\ell} \odot d_{jk}). \tag{4}$$

That is, the maximum among the above three terms is achieved at least twice. Note that (4) is the tropicalization of the quadratic Plücker relation (3)! In general, we have the following theorem.

**Theorem 2.2** (Pachter and Sturmfels, 2005; Speyer and Sturmfels, 2004)**.** *The space of pairwise dissimilarity vectors of phylogenetic trees with $n$ leaves equals the tropical Grassmannian* $\mathrm{trop}(\mathrm{Gr}(2, n))$.

Theorem 2.2 is at the heart of the tropical approach to phylogenetics. An important problem in phylogenetics is to infer the tree $T$ given noisy measurements of the dissimilarity vector $d(2,T)$. For example, suppose that from different data sets one can obtain dissimilarity vectors $d_1,\dots,d_k$, each corresponding to a different phylogenetic tree $T_1,\dots,T_k$ on the same $[n]$ leaves. One would like to aggregate the information across these trees, and output a single "best estimator" $\hat{T}$. Unfortunately, the mean of the dissimilarity vectors $\bar{d} = \frac{1}{k}\sum_{j=1}^{k} d_j$ may not be a dissimilarity vector itself. Instead, one could formulate an optimization problem over the space of trees to find a tree $\hat{T}$ that minimizes the "average distance" to the observed trees $d_1,\dots,d_k$. Solving this optimization problem is an active research area, and the choice of metric on the tree space plays an important role. Here, the geometry of $\operatorname{trop}(\operatorname{Gr}(2,n))$ suggests that the tropical Hilbert metric is a natural choice [MLYK18].

One major quest in theoretical applications of tropical geometry to phylogenetics was to generalize the Pachter–Sturmfels theorem to higher-order dissimilarity vectors, which assign a number to each $r$-subset of leaves of a phylogenetic tree $T$. Recently, it was shown in [CGMS21] that for $2 \le r \le n-2$, the set of *weighted* $r$-order dissimilarity vectors of phylogenetic trees on $n$ leaves is the tropicalization of a natural subvariety of $\operatorname{trop}(\operatorname{Gr}(r,n))$, whose tropical basis generalizes the four-point condition (4).

## 3. Tropical Linear Spaces and Matroids

The tropicalization of the Grassmannian $\operatorname{Gr}(d,n)$ depends in general on the ground field. However, we still get a combinatorially meaningful space if we consider the set of points in $\mathbb{R}^{\binom{n}{d}}$ satisfying just the tropical quadratic Plücker relations, and not necessarily higher-degree relations among Plücker coordinates.

**Definition 3.1.** The Dressian $\operatorname{Dr}(d,n)$ is the space of real-valued functions $p$ on $d$-element subsets of $\{1,2,\dots,n\}$ satisfying the following tropical quadratic Plücker relations: for any $A,B \subset \{1,2,\dots,n\}$ with $|A| = d-1$ and $|B| = d+1$, the maximum

$$\max_{i\in B\setminus A} p(A\cup i) + p(B\setminus i) \text{ is achieved twice.} \qquad (5)$$

For example, if $d=2$ and $n=4$, then (5) is exactly the four-point condition (4), and the Dressian and the tropical Grassmannian coincide. However, in general the tropical Grassmannian is a proper lower-dimensional subset of the Dressian, as it is cut out by the tropicalizations of all relations among the maximal minors of a $d\times n$ matrix, not just the quadratic ones.

The tropical quadratic Plücker relations (5) encode the basis exchange axiom that defines valuated matroids, and thus the Dressian $\operatorname{Dr}(d,n)$ turns out to be exactly the space of valuated matroids of rank $d$ on the ground set $\{1,\dots,n\}$. This section elaborates on this fundamental connection.

Matroids are combinatorial objects that abstract and generalize several notions of independence in mathematics such as linear independence among vectors in a vector space or algebraic independence among elements of field extension. If a $d$-dimensional linear space $L$ is the row span of a $d\times n$ matrix $A$, then a collection of $d$ columns of $A$ are linearly independent if and only if the corresponding Plücker coordinate of $L$ is nonzero. In other words, the matroid recording the linear dependencies among the coordinate functions on $L$ encodes the zero-pattern of the vector of Plücker coordinates of $L$.

Matroids have been studied extensively since their introduction by Whitney and Nakasawa in the 1930s, and have found tight connections to several areas such as graph theory, optimization, and coding theory. Valuated matroids are an elegant generalization of matroids introduced by Dress and Wenzel in 1992, in which every maximal independent set $B$ is assigned a valuation $p(B) \in \mathbb{R}$. For example, a $d\times n$ matrix of rank $d$ over a valued field $\mathbb{K}$ gives rise to a valuated matroid where for any linearly independent $d$-subset of columns $B \subset \{1,2,\dots,n\}$, the value of $p(B)$ is the valuation of the corresponding $d\times d$ maximal minor.

Importantly for tropical geometry, the recipe that recovers the tropicalization of a linear subspace from the valuation of its Plücker coordinates directly generalizes to all valuated matroids, allowing us to associate a *tropical linear space* to every valuated matroid, not just those represented by a matrix over a field. In fact, the combinatorics of valuated matroids is perfectly compatible with that of tropical geometry, in such a way that the set of tropical linear spaces turns out to be exactly the set of tropical varieties of degree 1, as shown by Fink in 2013. In this sense, (valuated) matroids are the mathematical object that provides the answer to what a tropical linear space should be.

A perspective on matroids that has recently gained prominence in great part due to tropical geometry is that of their associated polytopes. Given a matroid $M$ on the ground set $\{1,\dots,n\}$, its associated *matroid polytope* is the polytope in $\mathbb{R}^n$ whose vertices are the 0/1 indicator vectors of the bases (i.e., maximal independent sets) of $M$. For example, the matroid polytope of the uniform matroid $U_{d,n}$ in which any $d$-subset of $\{1,\dots,n\}$ is a basis is the hypersimplex $\Delta_{d,n}$, whose vertices are the $\binom{n}{d}$ vectors with $d$ coordinates equal to 1 and all other coordinates equal to 0. From this polyhedral point of view, matroids can be elegantly axiomatized as follows.

**Theorem 3.2** (Gelfand, Goresky, MacPherson, Serganova, 1987). *A polytope in $\mathbb{R}^n$ with vertices in $\{0,1\}^n$ is a matroid polytope if and only if all its edges have the form $e_i - e_j$ for $i,j \in \{1,\dots,n\}$.*

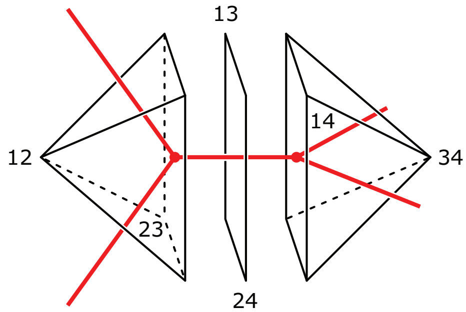


**Figure 4.** A tropical line (in red) dual to a matroid subdivision of the regular octahedron $\Delta_{2,4}$. The Dressian, which is a parameter space for tropical linear spaces or valuated matroids, consists of regular subdivisions that do not introduce a new edge. It also occurs naturally in auction theory.

Vectors in the Dressian $\mathrm{Dr}(d,n) \subset \mathbb{R}^{\binom{n}{d}}$ can be characterized in polyhedral terms as well. They are exactly the height vectors that induce a regular subdivision of the hypersimplex $\Delta_{d,n}$ into matroid polytopes; in other words, a subdivision of $\Delta_{d,n}$ that does not introduce any new edges. Furthermore, the tropical linear space associated to a vector $p \in \mathrm{Dr}(d,n)$ turns out to be a polyhedral complex that is dual to a particular subcomplex of this regular subdivision; see Figure 4. In this way, the combinatorial properties of tropical linear spaces are tightly linked to those of matroid polytope subdivisions.

It is sometimes said that tropical geometry provides a tool for degenerating algebraic varieties into *simpler* polyhedral objects. However, already in the case of linear subspaces, we see that, while all subspaces of a vector space are quite "simple" geometrically, their tropicalizations carry somewhat intricate information about their intersections with the coordinate subspaces, in the form of valuated matroids. In fact, very natural questions about the combinatorics of tropical linear spaces—or dually, matroid polytope subdivisions—remain unanswered, such as the maximal number of faces that a tropical linear space can have. This particular question is known as the $f$-vector conjecture, stated below in terms of matroid polytope subdivisions.

**Conjecture 3.3** (Speyer, 2008)**.** *Any regular subdivision of the hypersimplex $\Delta_{d,n}$ into matroid polytopes has at most $\frac{(i-1)!}{(n-i-1)!(d+i-n)!(i-d)!}$ interior faces of dimension $i$.*

The $f$-vector conjecture has been proven to hold in particular cases, such as regular subdivisions corresponding to valuated matroids that lie in the tropical Grassmannian (over $\mathbb{C}$), or in the case of maximal faces, when $i = n-1$. However, the general statement remains open.

**Combinatorial Hodge theory.** This tropical point of view on matroids that we have discussed has been extremely fruitful in the last few years. The local building blocks of tropical linear spaces, i.e., those subcomplexes consisting of cells containing a single fixed cell, are called Bergman fans of matroids. Combinatorially, a Bergman fan has the structure of a geometric lattice, which is a partially ordered set with special properties. Topologically, it is a cone over a bouquet of spheres. However, the particular embedding in $\mathbb{R}^n$ arising from tropical geometry makes Bergman fans very potent tools.

Inspired by toric geometry, Feichtner and Yuzvinsky in 2004 used this embedding to associate a certain commutative Artinian ring, called the Chow ring, to every matroid. More recently, Adiprasito, Huh, and Katz studied this ring more in depth [AHK18], and showed that in fact it has a very rigid "Hodge structure," in the sense that it resembles the cohomology ring of a smooth projective variety.

Using this powerful algebraic theory, they were able to prove long-standing conjectures about the log-concavity of certain integer sequences associated to a matroid, like the coefficients of the characteristic polynomial and the number of independent sets of a given size. Similar approaches that use other tropical spaces associated to matroids have succeeded more recently in settling other long-standing log-concavity questions in matroid theory [Ard18].

**Algebraic foundations of tropical geometry.** Over the last few years there has been an effort to develop the algebraic foundations of tropical geometry analogous to scheme theory in algebraic geometry. Contrary to classical algebraic geometry, where information about algebraic varieties is thought of in both geometric and algebraic terms, tropical varieties have traditionally been considered as purely geometric objects. In their foundational paper [GG16], Giansiracusa and Giansiracusa introduced a novel algebraic structure attached to the tropicalization of an algebraic variety, which plays the role of a "coordinate semiring" for tropical varieties. It was later understood that this algebraic information is equivalently encoded by the tropicalization of the polynomials in the ideal defining the algebraic variety, as defined in Section 1.

Given these developments, it is natural to aim to develop algebraic foundations for tropical geometry purely on the tropical side, without having to rely on the process of tropicalization of classical varieties or ideals. One possibility would be to study the class of all ideals in the semiring $\mathbb{T}[x_1, \dots, x_n]$ of tropical polynomials, where the coefficients are taken from the set of tropical numbers $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. However, it turns out this semiring is not Noetherian, general ideals in it are not finitely generated, and their associated varieties are not necessarily polyhedral objects. This class thus extends beyond the realm of tropical geometry.

The problem with general ideals of $\mathbb{T}[x_1, \dots, x_n]$ stems from the fact that arbitrary modules over $\mathbb{T}$ do not necessarily behave like tropical linear spaces. Maclagan and Rincón have thus proposed in [MR18] the following notion as a sensible class of ideals for the study of tropical geometry.

**Definition 3.4.** An ideal $I \subset \mathbb{T}[x_1, \dots, x_n]$ is a *tropical ideal* if for every degree $d \geq 0$, the $\mathbb{T}$-module $I_{\leq d} := \{f \in I : \deg(f) \leq d\}$ is a tropical linear space in the space $\mathbb{T}[x_1, \dots, x_n]_{\leq d}$ of tropical polynomials of degree at most $d$.

The class of tropical ideals still contains the tropicalizations of all classical ideals, but it is in general much larger. This phenomenon is analogous to the fact that the class of all matroids is in general much larger than just the matroids arising from classical linear subspaces.

As shown in [MR18], tropical ideals have indeed more desirable properties than general ideals of $\mathbb{T}[x_1, \dots, x_n]$. The fact that tropical linear spaces, which make up each graded piece of a tropical ideal, have a well-behaved notion of dimension means that tropical ideals have a natural notion of Hilbert function. Just as for classical ideals, this Hilbert function eventually agrees with a polynomial, and thus, for instance, tropical ideals have a natural notion of dimension, given by the degree of this Hilbert polynomial. In addition, the varieties they define are always finite polyhedral complexes. In fact, it was proved recently in [MR20] that the variety of a tropical ideal is always a polyhedral complex of dimension equal to the dimension of the tropical ideal, and furthermore, these varieties are always balanced polyhedral complexes, which generalizes part of the Structure Theorem on tropicalizations of algebraic varieties.

The algebraic foundation for tropical geometry in this direction is inherently combinatorial, as it is closely tied to tropical linear spaces and thus to matroids. Even though the theory is only in its beginnings, tropical ideals provide a strong bridge between combinatorics, algebra, and geometry, and they equip tropical varieties with richer structures beyond purely polyhedral ones.

## 4. Tropical Geometry, Matroids and Auctions

Auctions with indivisible goods have a strong connection to tropical geometry. Fundamental economics concepts such as utility, demand set, and competitive equilibrium can be translated into questions about tropical hypersurfaces and their corresponding regular subdivisions. With this bridge, some authors have used tropical geometry, matroid theory, and convex geometry to answer open problems in economics [JKS21, Tra21, HLSV22]. This section gives a flavor of these connections.

The simplest auction is a sealed bid auction for one good, such as an art work or a house. By the deadline, each potential buyer (agent) submits their bid. The seller announces a price $p$ and assigns the good to a bidder, who would then pay this price. The seller could offer a discount, so $p$ could be less than the highest bid, but it is expected that the highest bidder will be assigned the goods; otherwise the highest bidder will perceive the game as unfair for them.

Multi-unit combinatorial auction or product-mix auction are versions of this where there are multiple types of indivisible goods on sales. A bundle of $n$ types of goods is represented as a point in $\mathbb{Z}^n$. Each agent's bid is no longer a single number, but a so-called valuation function $u : A \to \mathbb{R}$, where $A \subset \mathbb{Z}^n$ is the set of available bundles and $u(a)$ is how much this agent is willing to pay for bundle $a \in A$ or how much the bundle $a$ is worth to the agent. The goal for the seller is to partition the goods bundle for sale $a^* \in \mathbb{Z}^n$ into a sum of goods bundles $a^* = a^1 + \cdots + a^J$, where bundle $a^j$ is assigned to agent $j$, *and* to find the price $p_i$ per unit of good $i$ to charge the agents, so that the game is fair for all. That is, at the announced prices $p \in \mathbb{R}^n$, there is no agent who would prefer a different bundle from what they were assigned.

Economists have long known that already for two good types and two agents, there are combinations of valuations $\{u^1, u^2\}$ where *no* fair pricing exists for some goods bundle $a^*$ (cf. Example 4.1). A central problem is thus to find reasonable rules on the auctions that restrict the set of valuations $u$ that the agents can submit, so that a fair pricing is always guaranteed, and that the allowed set of valuations is still large enough to model different types of preferences.

Recently, economists Baldwin and Klemperer [BK19] made three important observations. First, the utility or profit function of an agent, which is the maximum over all available bundles of the difference between the valuation and the price, is a tropical polynomial in the (negative of) prices $p$. Second, the regular subdivision that this tropical polynomial induces on its Newton polytope tells us which goods bundles an agent would ever want to consider buying. That is, this tropical polynomial and its combinatorics store important information on demanded bundles and fair pricing. Third, when there are multiple agents, their aggregated or total utility is the tropical product of the agents' individual tropical polynomials. The regular subdivision corresponding to the product of the tropical polynomials is called a regular *mixed* subdivision. It stores the combinatorial information we need to know about existence of fair pricing for all goods bundles we might want to consider in $\mathbb{Z}^n$. These observations allow us to translate economics questions into combinatorial questions about regular mixed subdivisions.

**Example 4.1.** Let us construct an example of a no-fair-pricing auction based on a simplified version of Figure 2. For simplicity, suppose we have only two types of goods, tea and coffee, and two agents, Left and Right. Left wants to buy either tea or coffee, will not settle for nothing, and does not want to buy both. Right wants to buy either nothing or both coffee and tea. Left bids \$3 for a 1kg pack of coffee only and \$2 for a 1kg pack of tea only. Right bids \$3 for both and \$0 for nothing. Their utility functions are tropical polynomials in the negatives of prices:

$$f_L = (3 \odot (-p_c)) \oplus (2 \odot (-p_t))$$
$$f_R = 0 \oplus (3 \odot (-p_c) \odot (-p_t))$$

where $p_c$ and $p_t$ are the prices for a 1kg package of coffee and tea, respectively. The Newton polytopes of these tropical polynomials are the two line segments in Figure 5, and the Newton polytope of the aggregate utility function $f_L \odot f_R$ is the square on the right. The point $(1,1)$ is not in the support of the aggregate utility function. This implies that if we want to sell exactly 1kg of coffee and 1kg of tea, then there is no way we can assign something to both Left and Right such that each of them gets the product bundle that maximizes their utility, at any price.

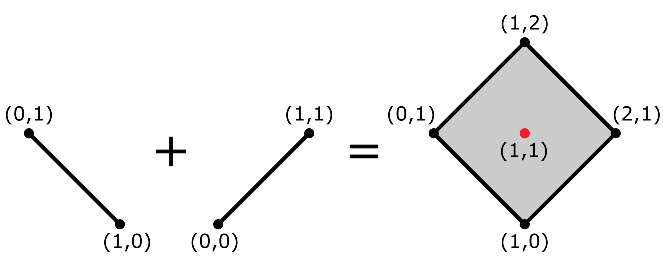


**Figure 5.** Newton polytopes of utility functions for an auction of two agents with two items. See Example 4.1.

On the other hand, the point $(1,1)$ belongs to the Newton polytope of the aggregate utility function. This implies that, if the amount of coffee and tea we sell to an agent is not a discrete but a continuous quantity, and if our agents are willing to buy a convex combination of what they demanded, then there *is* a solution: we can set the price of coffee at $p_c = \$2$ per kg and of tea at $p_t = \$1$ per kg, and assign 500g of tea and 500g of coffee to each buyer. Then each of them gets a convex combination of the product bundles that maximize their utility: Left makes a profit of \$1 and Right makes a profit of \$0, which is the most each of them can make under such pricing. The price vector $(2,1)$ is precisely the intersection of the tropical hypersurfaces of the two tropical polynomials $f_L$ and $f_R$.

**Connections to matroids.** Economists are interested in conditions on the valuations $u^j$ and the pricing function $p$ (for example, beyond linear pricing) that ensure fair pricing (technically known as *competitive equilibrium*) is guaranteed to exist for *all* admissible goods bundles $a^*$. The ideal theorem has the format: if the valuations $u^j$ belong to some function class $\mathcal{V}$ and if the pricing functions $p$ belong to some function class $\mathcal{P}$, then competitive equilibrium always exists at any admissible goods bundle $a^*$. An early and influential result is due to Walras, which says that if $\mathcal{V}$ has the *gross substitutes* property and $\mathcal{P}$ is linear, then competitive equilibrium always exists. Figure 2, left, is an example of a function with the gross substitutes property. Interestingly, gross substitutes are certain generalizations of rank functions associated to matroids. They are called $M$-concave functions in Murota's work on discrete convex analysis. They have several equivalent characterizations, but for our purposes, the following is the most relevant. Compare this to Theorem 3.2.

**Definition 4.2.** A function $u : \mathbb{Z}^n \to \mathbb{R}$ has the *gross substitutes* property if each edge of the corresponding regular subdivision is parallel to one of the vectors in the set $\{e_i - e_j : 1 \leq i, j \leq n \text{ and } i \neq j\} \cup \{e_i : 1 \leq i \leq n\}$.

In particular, functions with the gross substitutes property on certain subsets of the 0/1 cubes are dehomogenized versions of points on the Dressian. This rich connection between Dressians and gross substitutes was instrumental in the solution of the matroid-based valuation conjecture in auction theory [Tra21, HLSV22].

Recent work concerning competitive equilibria extends the tropical framework to go beyond linear pricing, by considering embeddings of the lifted Newton polytope in higher dimensions. With this technique, [BHT21] showed that for combinatorial auctions, competitive equilibrium always exists when both the pricing function and the agents' valuations are *quadratic* instead of linear; that is, $p(a) = \sum_{i \in a} p_i + \sum_{i,j \in a} p_{ij}$ where $p_i$ is the price for one unit of item $i$, and $p_{ij}$ is the "discount" for buying the pair $(i,j)$ together. In a different direction, [JKS21] significantly extended the results on the straight jacket auction by translating revenue optimizations in combinatorial auctions into questions about generalized permutohedra and finding roots of a system of polynomials. The connections between convex geometry and economics also go two ways: the Oda Conjecture in toric geometry can be restated in terms of product-mix auctions [TY19]. These results attest to the rich connections between these areas.

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# Convolutional Neural Networks and their Applications in Medical Imaging: A Primer for Mathematicians

*Kyle Hasenstab*

## Introduction

Deep neural networks, specifically convolutional neural networks (CNNs), have become extremely popular in the field of medical imaging for their ability to automate radiological tasks across a variety of imaging modalities (e.g., X-ray, CT, MRI, ultrasound) with state-of-the-art accuracy. According to a recent study, the number of articles on PubMed containing the keyword phrases "medical imaging" and "deep learning" has exponentially increased from 45 to 1,006 between 2016 and 2020 [WWH$^+$22]. The increased popularity of CNNs is attributed, in part, to the diversity of tasks they are able to perform, which include pathology detection and classification [JMY$^+$22], anatomical segmentation [JMY$^+$22], image-to-image translation [CFK$^+$18], and accelerated image registration [HTY$^+$22]. The purpose of this article is to provide a fundamental understanding of CNNs and a summary of their applications in medical imaging. We also provide a brief description of how CNNs are trained and optimized.

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## Neural Network Basics

**What are artificial neural networks (ANNs)?** ANNs are models that approximate (i.e., learn) nonlinear functional

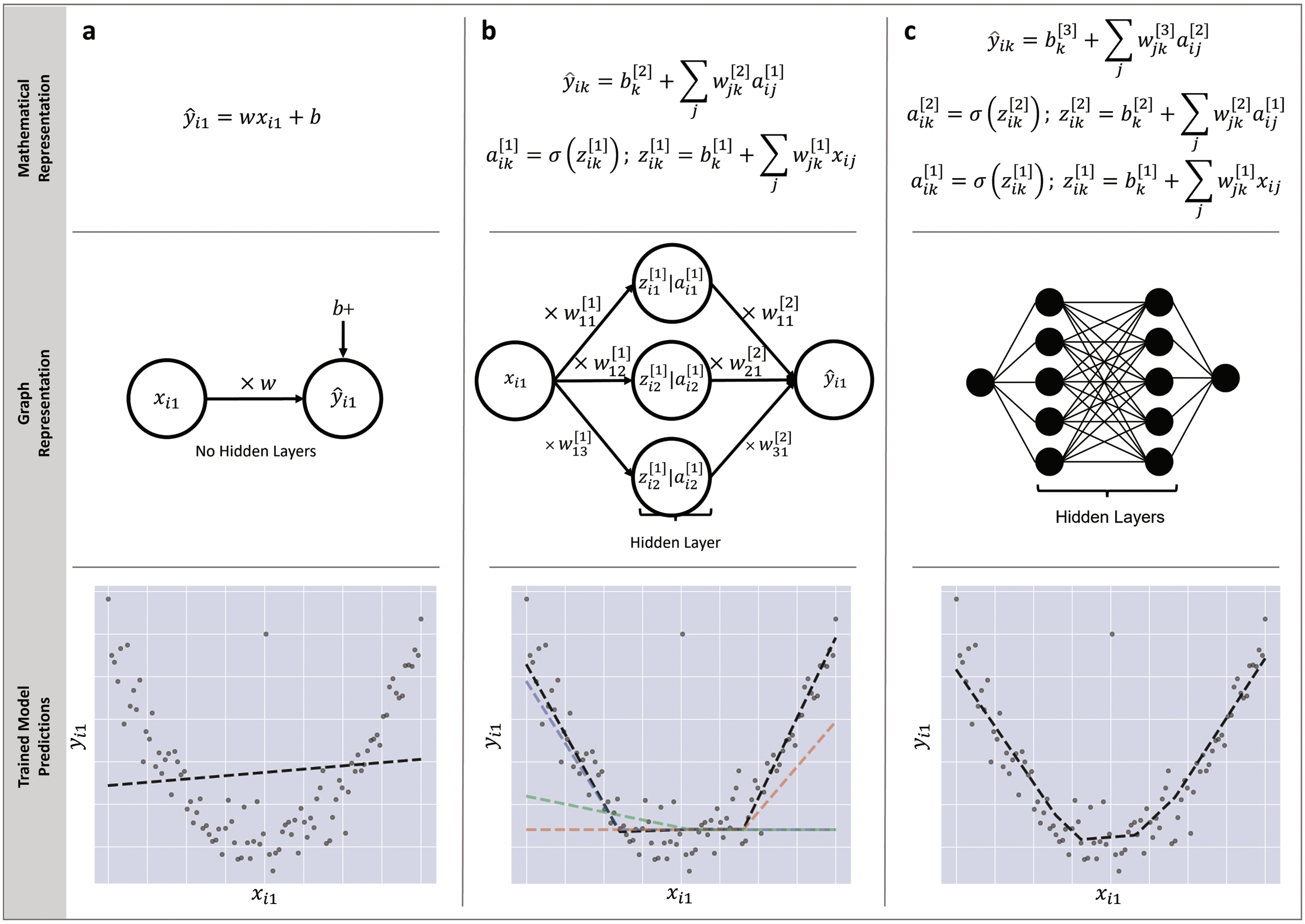


**Figure 1.** *Mathematical and graphical representations of fully-connected neural networks and their corresponding predictions (black dashed lines) in relation to network capacity.* (a) A simple network architecture (linear equation) without hidden layers; (b) A fully-connected network with a single hidden layer containing three hidden neurons; (c) A fully-connected network with two hidden layers, each containing five hidden neurons. Artificial neural networks approximate the nonlinear relationship between a set of inputs and outputs using hidden layers and nonlinear activation functions. The blue, green, and orange dashed lines in (b) illustrate the basic nonlinear transformations using the ReLU activation function, which are subsequently combined in the final layer to form the improved prediction. The inclusion of more hidden layers and neurons increases the network's capability of modeling more complex nonlinear relationships.

relationships between a set of inputs (i.e., features) and outputs. Inputs and outputs can be categorical or continuous and can assume a variety of data types, including tabular data, images, text, and audio. ANNs automatically learn nonlinear structure through two fundamental mechanisms: 1) *hidden layers* and 2) nonlinear transformations or *activation functions*. To illustrate this point, we begin with a simple 1D example aiming to predict a quadratic outcome from a sample of 100 observations $(x_{i1}, y_{i1}), i = 1, \ldots, 100$ (Figure 1a). That is, we would like to approximate some nonlinear function $f$ that maps inputs $x_{i1}$ to outputs $y_{i1}$ [i.e., $y_{i1} = f(x_{i1})$] as accurately as possible. Here, ordered pairs are sampled from the true "unknown" function $y_{i1} = x_{i1}^2 + 0.5 + \varepsilon_i, x_i \in [-1, 1]$ (Figure 1a), where $\varepsilon_i$ is a random error term.

**A basic neural network architecture.** Prior to approximating the functional relationship (i.e., training an ANN), we must specify an ANN architecture. The simplest architecture is the equation of a line,

$$f(x_{i1}) = b + wx_{i1}, \tag{1}$$

where $b$ and $w$ are the intercept (i.e., bias) and slope of the line, respectively, commonly referred to as *weights* or *parameters*. Although ANN architectures can be represented mathematically, as in Equation 1, they are often represented as feed-forward directed acyclic graphs, as shown in Figure 1a, to facilitate communication. Circles in the graph represent *neurons* and arrows (i.e., edges) represent network weights. Weights are multiplied by the value of their corresponding input neuron, with exception to the bias, $b$, which is added. The weights within a network architecture are learned through model training, where they are iteratively adjusted to minimize a loss function that measures performance using gradient descent optimization. For additional details on the training process, we refer the reader to the Training and Optimization section of this exposition.

Figure 1a shows predictions from the network specified in Equation 1 after training. As expected, we observe poor predictive performance due to the linear restriction of our initial architecture. To improve performance, we may consider including $x_{i1}^2$ as an additional input neuron to model the nonlinear relationship. However, this approach relies on our understanding of the nature of the functional relationship (quadratic), which is typically not the case for higher-dimensional data, such as images, where exploratory visual analysis between inputs and outputs is less tractable.

**Expanding the architecture.** To accommodate the unknown nonlinear relationship, we include intermediate neuronal connections between the inputs and outputs, known as hidden layers. In addition, we apply basic nonlinear (activation) functions to each hidden neuron to encourage nonlinear behavior in the predictions. In effect, hidden layers allow us to synergistically combine basic nonlinearities to form more complex nonlinear functions. A common activation function used for this purpose is the Rectified Linear Unit (ReLU) defined as $\sigma(x) = \max\{0, x\}$. Figure 1b shows an example architecture illustrating this concept. This network contains three hidden neurons in a single hidden layer and applies the ReLU activation function to the output of each hidden neuron. The following is the corresponding mathematical representation of this network,

$$\hat{y}_{ik} = b_k^{[2]} + \sum_j w_{jk}^{[2]} a_{ij}^{[1]} \tag{2}$$

$$a_{ik}^{[1]} = \sigma\left(z_{ik}^{[1]}\right); z_{ik}^{[1]} = b_k^{[1]} + \sum_j w_{jk}^{[1]} x_{ij} \tag{3}$$

where $z_{ik}^{[\ell]}$ is the linear value of the $k^{th}$ hidden neuron in the $\ell^{th}$ hidden layer for observation $i$, $a_{ik}^{[\ell]}$ is the result of applying the activation function to $z_{ik}^{[\ell]}$, $b_k^{[\ell]}$ is the bias for the $k^{th}$ neuron in the $\ell^{th}$ hidden layer, and $w_{jk}^{[\ell]}$ is the weight coming from the $j^{th}$ neuron in the $(\ell-1)^{th}$ layer and contributing to the $k^{th}$ neuron in the $\ell^{th}$ layer. With exception to the input neurons, each neuron also receives a bias term, but these are omitted from the graph for readability. Note that we did not apply an activation function to the output neuron in this case since output activations differ depending on the predictive task.

Following network training using gradient descent optimization, we observe predictions better resembling a quadratic function (Figure 1b). Output of the hidden neurons ($a_{ik}^{[1]}, k = 1, 2, 3$) illustrates the basic nonlinear transformations using the ReLU activation function, which are subsequently combined in the final layer to form the improved prediction. Increasing the number of hidden layers and neurons expands the capability of the network to model more complex nonlinear relationships, such as the network shown in Figure 1c, which includes two hidden layers each containing 5 neurons.

Architectures can be readily extended to multiple inputs and outputs by incorporating additional neurons in the input and output layers along with their corresponding weights. If inputs to these networks are 2D or 3D images, the images are simply "flattened" into a 1D vector before being propagated through the network (Figure 2a). Generally, these ANNs are called fully-connected networks since each neuron in a given layer receives input from every neuron in the previous layer.

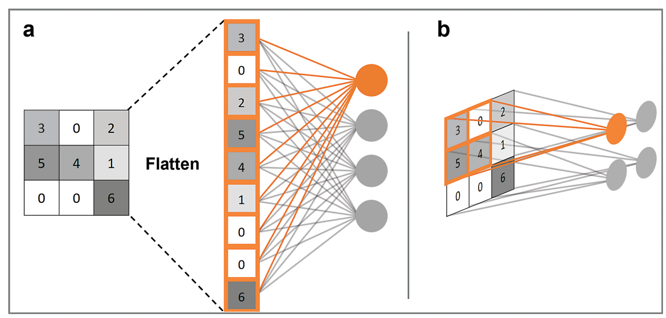


**Figure 2.** *Difference between (a) fully-connected networks and (b) CNNs when applied to images.* Fully-connected networks require images to be flattened prior to being propagated through the network. Each neuron is connected via a weight to every neuron in the prior layer. In a CNN, each neuron is connected to a small area on the input image; output neurons are arranged in a manner that preserves spatial information.

**Fully-connected networks are not ideal for imaging tasks.** Fully-connected networks are well-suited for modeling nonlinear relationships between a smaller number of inputs and outputs (1000s) and when a systematic spatial or temporal relationship within the set of input (or output) neurons is unknown or does not exist, as is often the case for tabular data. However, when applied to images, where spatial structure is present, fully-connected networks become inefficient for several reasons. First, they do not preserve or take advantage of the spatial structure within images since the input image is indiscriminately flattened into a 1D vector. This makes a predictive task much more challenging for the network as it must attempt to learn spatial structure from the data. Second, images can be quite large, causing fully-connected networks to quickly exceed memory limitations during training due to a large number of model weights. For example, a 2048x2048 chest X-ray contains 4,194,304 pixels (i.e., input neurons). A simple network with a single hidden layer containing 100 neurons has over 400 million weights, which is more than five times the number of weights in state-of-the-art networks containing >100 hidden layers. Finally, assuming memory limitations have not been exceeded, the large number of weights can make these architectures very slow to train and prone to overfitting, negatively impacting network performance.

## Convolutional Neural Networks

CNNs are a subclass of ANNs designed to overcome the drawbacks of fully-connected networks when applied to imaging tasks. In contrast to the dense connections of fully-connected networks, each neuron in a CNN is connected only to a small area of the input image, thus preserving spatial information and requiring far fewer network weights, as illustrated in Figure 2b. CNNs consist of a sequence of convolutional layers, but also commonly include pooling, fully-connected, upsampling, and other layers, depending on the architecture and application.

**Convolutional layers.** Convolutional layers comprise a set of *cross-correlation* operations applied to the layer's input, followed by a nonlinear activation function. Note the use of the word "convolution" is a common misnomer. Convolutional operations in CNNs are actually "cross-correlations," which are similar in calculation. However, we will refer to these operations as convolutions for the remainder of the article given the term's prevalence in the CNN literature.

In image processing, a convolution is a mathematical operation involving the inner product of an array, referred to as a *filter* or *kernel*, and localized patches of an image. The purpose of convolutions in this context is to extract imaging features (e.g., edges, texture, shape, etc.) useful for performing a predictive task. Mathematically, a convolution is expressed as

$$K \circ I(x, y) = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} K(i, j) I(x + i, y + j), \tag{4}$$

where $K$ is a $kxk$ filter and $I(x, y)$ is the pixel value at location $(x, y)$ of the original image $I$ being convolved. If the image contains multiple channels, such as the red-green-blue channels of color images, then a filter must match the corresponding channel dimension of the image, such that

$$K \circ I(x, y) = \sum_{\ell=0}^{c-1} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} K(i, j, \ell) I(x + i, y + j, \ell), \tag{5}$$

where $K$ is a $kxkxc$ filter and $I(x, y, c)$ is the pixel value at location $(x, y)$ in channel $c$ of image $I$. The channel axis is collapsed to a single channel as a result of the convolution.

Figure 3a–b illustrates the process of convolving a 8x8x1 image depicting a "happy face" with a 3x3x1 horizontal edge filter. First, the filter is overlaid on the image, starting on the top-left, and the inner product is calculated and stored in the output array. The filter is then shifted by a single pixel and the process repeated until the inner products across all local image patches have been obtained. The amount over which a filter is shifted is called the *stride*, which is often set to one. The convolution results in another image known as a *feature map* (Figure 3c). An activation function is then applied to each pixel in the feature map (Figure 3d) to introduce the nonlinearity. In this specific case, the feature map is smaller in dimension than the original image. To enforce consistent dimensions, inputs to a convolutional layer are often first padded with zeros. Using this particular filter, the feature map after activation highlights horizontal edges (i.e., the mouth) in the image.

Traditionally, investigators explicitly designed filters to extract imaging features thought to be useful for a predictive task. An image would be convolved with a set of handcrafted filters, each producing feature maps highlighting distinctive characteristics of the input image. Feature maps would then be reduced in dimension and used as inputs to a machine learning classifier.

In contrast to these traditional approaches, CNNs consider filter values as unknown network weights. That is, *CNNs learn the image filters useful for a predictive task from data through model training, resulting in large improvements in predictive performance.*

**Pooling layers.** A convolutional layer is often followed by a pooling layer. Pooling layers downsample input feature maps by propagating maximally activated neurons within localized patches of feature maps to subsequent layers. For example, Figure 4a illustrates the pooling operation on a 6x6x1 image using a pooling size of 2x2 and a stride of 2. A 2x2 patch is overlaid on the feature map, starting on the top-left. The maximum within the patch is then calculated and stored in the output array. The patch is then shifted by 2 pixels (i.e., stride) and the calculation is repeated until the entire image has been processed. This procedure is referred to as *max pooling* since the maximum is calculated within each patch. If input to the pooling layer has multiple channels, the max pooling operation is applied separately for each channel. Alternatively, *average pooling* involves calculating the average within a patch, however max pooling is typically used as it is more robust.

*Global pooling* (Figure 4b) is an operation where the pooling size is equal to the dimension of the input, hence reducing the input to a single neuron per channel. Global pooling is typically used near the final layer of a network. Unlike convolutional layers, pooling layers do not contain any trainable weights.

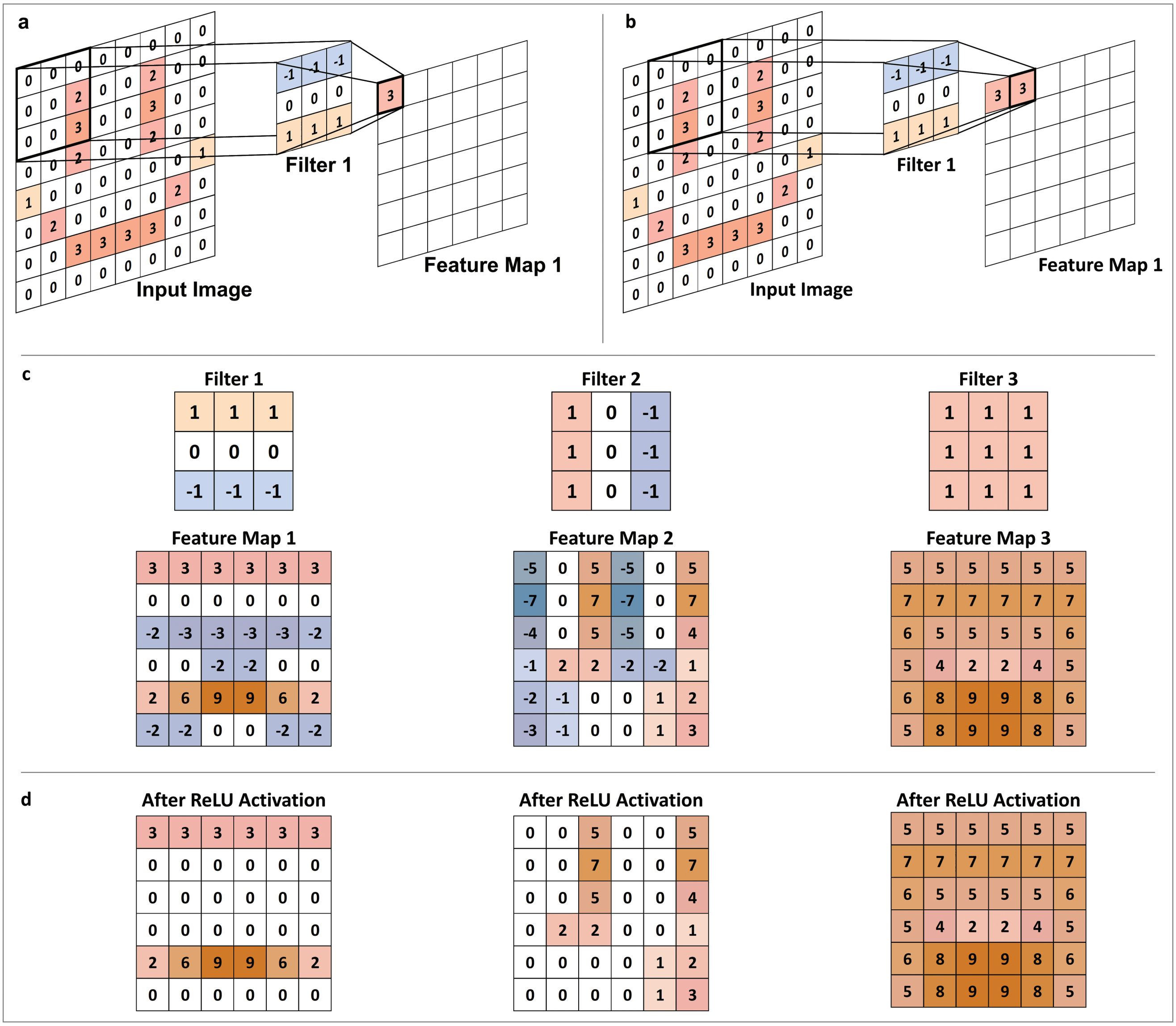


**Figure 3.** *Diagram illustrating the operations within a convolutional layer.* (a) A 3x3x1 filter comprising network weights is overlaid on the 8x8x1 image depicting a "happy face," starting on the top-left. The inner product is then calculated and stored in the output array. (b) The filter is shifted by a predetermined stride (e.g., 1) and the process is repeated. (c) Convolving the input image with different filters produces feature maps highlighting different characteristics of the input image. Filter 1 highlights the horizontal edges (mouth), filter 2 highlights the vertical edges (eyes), and filter 3 smooths the image. (d) The ReLU activation function is then applied to the feature maps to introduce a nonlinearity. Note the weights in the filters are learned from data during CNN training.

Why is pooling important? Although convolutional layers are effective spatial feature extractors, they are still sensitive to spatial variability. Translations or rotations in an input image can produce different feature maps, and therefore different predictions, despite representing the same class or object. For example, a face rotated 15 degrees is still a face, but this simple rotation may adversely affect a network's ability to identify the image as a face. Pooling layers mitigate the effects of this type of localized spatial variability by propagating larger activations across a region to subsequent layers. Pooling layers also have the added benefit of reducing memory consumption due to their downsampling property.

An alternative to max pooling is strided convolutions, which convolve images using a stride greater than one. For example, convolving the image in Figure 3a with a stride of two would result in a 4x4 feature map if zero padding is used. In this case, the strided convolution downsamples the image while simultaneously detecting horizontal edges. A benefit of using strided convolutions is the pooling operation is learned via the filter weights.

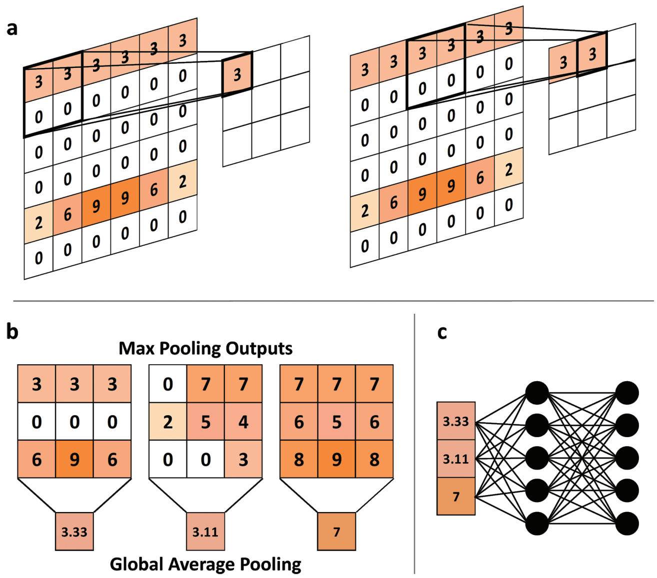


**Figure 4.** *Pooling and fully-connected layers of a CNN.* (a) A 2x2x1 patch is overlaid on a 6x6x1 input feature map and the maximum within the patch is stored in another array. The patch is then shifted by a stride of two and the process is repeated. (b) Applying max pooling to the three feature maps from Figure 3 results in three 3x3x1 max pooling outputs. Feature maps can either be flattened to a 27-dimensional vector or globally averaged to a three dimensional vector. (c) Globally averaged outputs can then be propagated through a series of fully-connected layers prior to the network prediction.

**Fully-connected layers.** Sequences of convolutional and pooling layers (i.e., downsampling layers) produce a set of low-resolution feature maps at the end of the network, which are either flattened or globally pooled into a 1D array. This 1D array can be considered a latent representation of the high-level features of the input image. For example, a neuron in the 1D array may represent the curvature of a smile on the face, and another neuron may represent the symmetry of a face. These latent features are often propagated through one or more fully-connected layers (Figure 4c) to approximate the nonlinear functional relationship between these latent features and the network output, similar to fully-connected networks.

**Upsampling layers.** Upsampling layers are useful for networks designed to predict outputs with spatial structure (e.g., images). Upsampling layers can be viewed as the converse of pooling operations, separately increasing the resolution of feature maps in each channel, often by a factor of two, using image interpolation methods. Figure 5b and 5c show examples of upsampling a 3x3x1 feature map using bed of nails and nearest neighbors interpolation. Similar to pooling layers, upsampling layers do not have trainable weights.

An alternative to standard upsampling layers is transposed convolutions, which convolve an image with a filter to broadcast pixels to a larger array. Figure 5d shows a transposed convolution using a 3x3x1 filter with a stride of two. The top-left scalar value within the input feature map is multiplied by the filter and stored in an array. The filter is then shifted by a predetermined stride and the process is repeated but stored in a separate array. After processing each pixel in the input feature map, the separate arrays are added, producing the output of the transposed convolution. As with strided convolutions, the nature of the upsampling is learned via filter weights.

**Skip connections.** Although not considered an isolated layer in a network, skip connections have become an important and standard component in many CNN architectures. Skip connections forward propagate feature maps from earlier layers in the network to other noncontiguous layers in the network through addition, multiplication, or concatenation operations. As a result, image features learned in earlier network layers are "reused" in later layers of the network, allowing low- and high-level features to interact, which can improve network performance. More importantly, skip connections, for reasons beyond the scope of this paper, mitigate the *vanishing gradient* problem, a situation that arises during optimization that prevents the training of very deep networks.

## Prototypical CNN Architectures

CNNs developed for imaging tasks often follow two prototypical architectures (Figure 6) comprising the network components described in the prior section.

**Encoder architectures.** Encoder architectures map an input image to a 1D output vector and are therefore ideal for classification and regression tasks. A prototypical encoder architecture is shown in Figure 6. An input image is first convolved with a large set of filters, producing a set of feature maps (one per filter) containing localized feature information (edges, brightness, texture, etc.). The features are then propagated through a pooling layer, which reduces the feature maps' dimensions and preserves larger feature map activations. This process is continued for a predetermined number of convolutional-max pooling blocks, with each block producing feature maps with increasingly complex feature patterns useful for the predictive task. The final set of feature maps are then flattened or globally pooled and propagated through a set of fully-connected layers to produce the model prediction. This architecture is sometimes referred to as an encoder network since the image is encoded onto a lower-dimensional space spanned by latent features useful for the predictive task.

**Encoder-decoder architectures.** Encoder-decoder architectures map an input image to another image by first encoding the input image onto a lower-dimensional feature space and subsequently decoding the image back to image space. A prototypical encoder-decoder

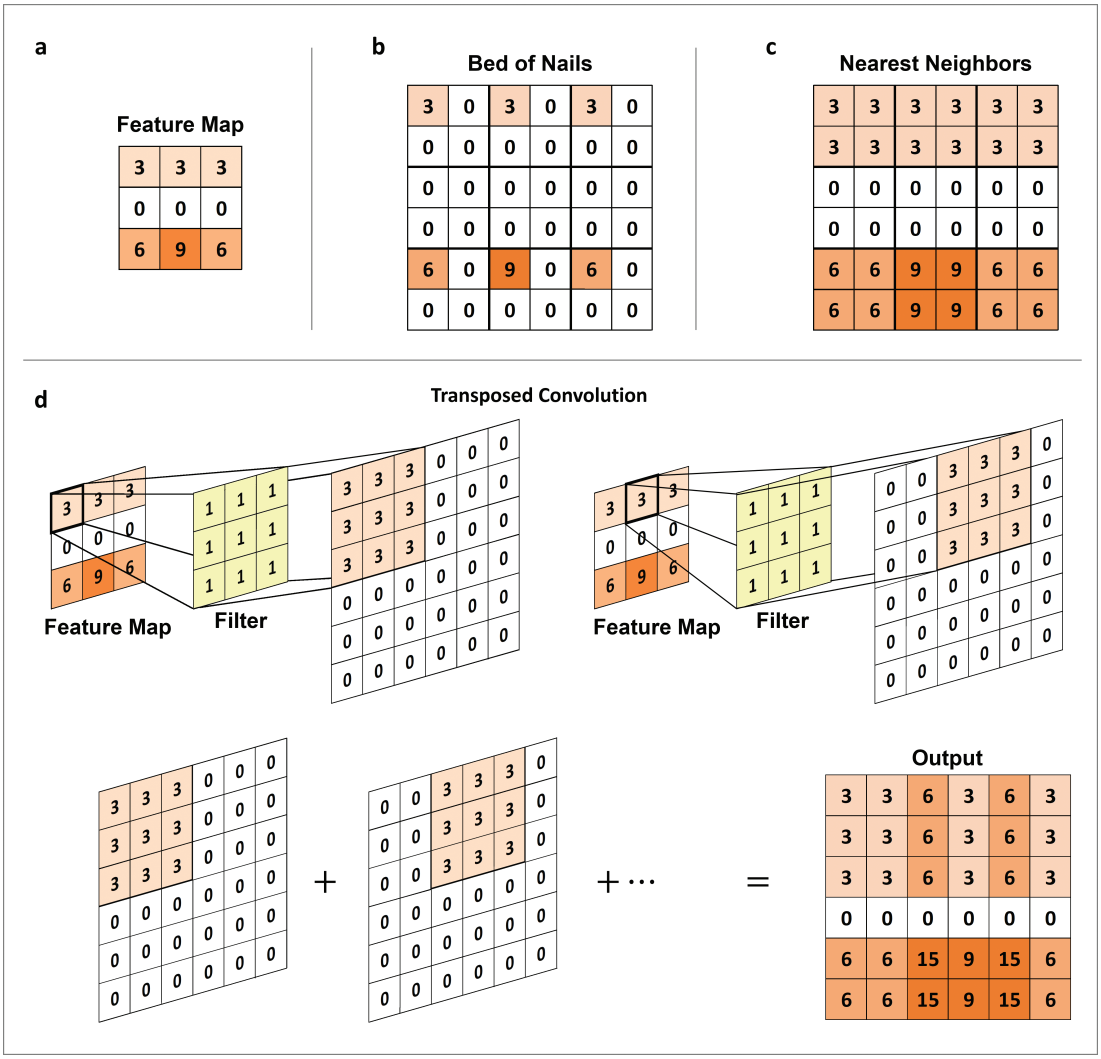


**Figure 5.** *Examples of upsampling a single (a) feature map using (b) bed of nails, (c) nearest neighbors, and (d) transposed convolutions.* The transposed convolution is performed using a 3x3x1 filter and a stride of two. Note the weights within the transposed convolutional filters are learned from the data during CNN training.

architecture is shown in Figure 6. An input image is first propagated through a sequence of convolutional-max pooling blocks in the encoder branch, producing a set of low-resolution feature maps representing complex feature patterns referred to as the *bottleneck layer*. The latent feature representations in the bottleneck layer are then propagated through the decoder branch, which comprises a sequence of convolutional-upsampling blocks, to map the feature maps back to image space.

Skip connections are often included in encoder-decoder architectures via concatenations to link various feature maps across the two branches. These connections have the dual purpose of encouraging the reuse of lower-level features in the terminal layers while mitigating the vanishing gradient problem during training. Since encoder-decoder networks typically do not contain fully-connected layers, these are also referred to as *fully-convolutional* networks.

## CNN Applications

The prototypical architectures in Figure 6 can be modified, extended, or combined to perform a variety of imaging tasks.

**Classification.** The prototypical encoder architecture in Figure 6 (left) is commonly used for image classification tasks. Classification CNNs aim to categorize input images into a set of classes (Figure 7), such as to classify an image of a liver lesion as benign or malignant. Prior to training a classification CNN, output class labels must be *one-hot encoded* into 0-1 values, where the presence of a class is given a value of one and the absence of a class is given a value of zero. The corresponding network must then contain one output neuron for each class.

Since class labels are in the set $\{0, 1\}$, an additional activation function is placed on the output neurons to restrict predictions to the interval $(0, 1)$. The *softmax* activation

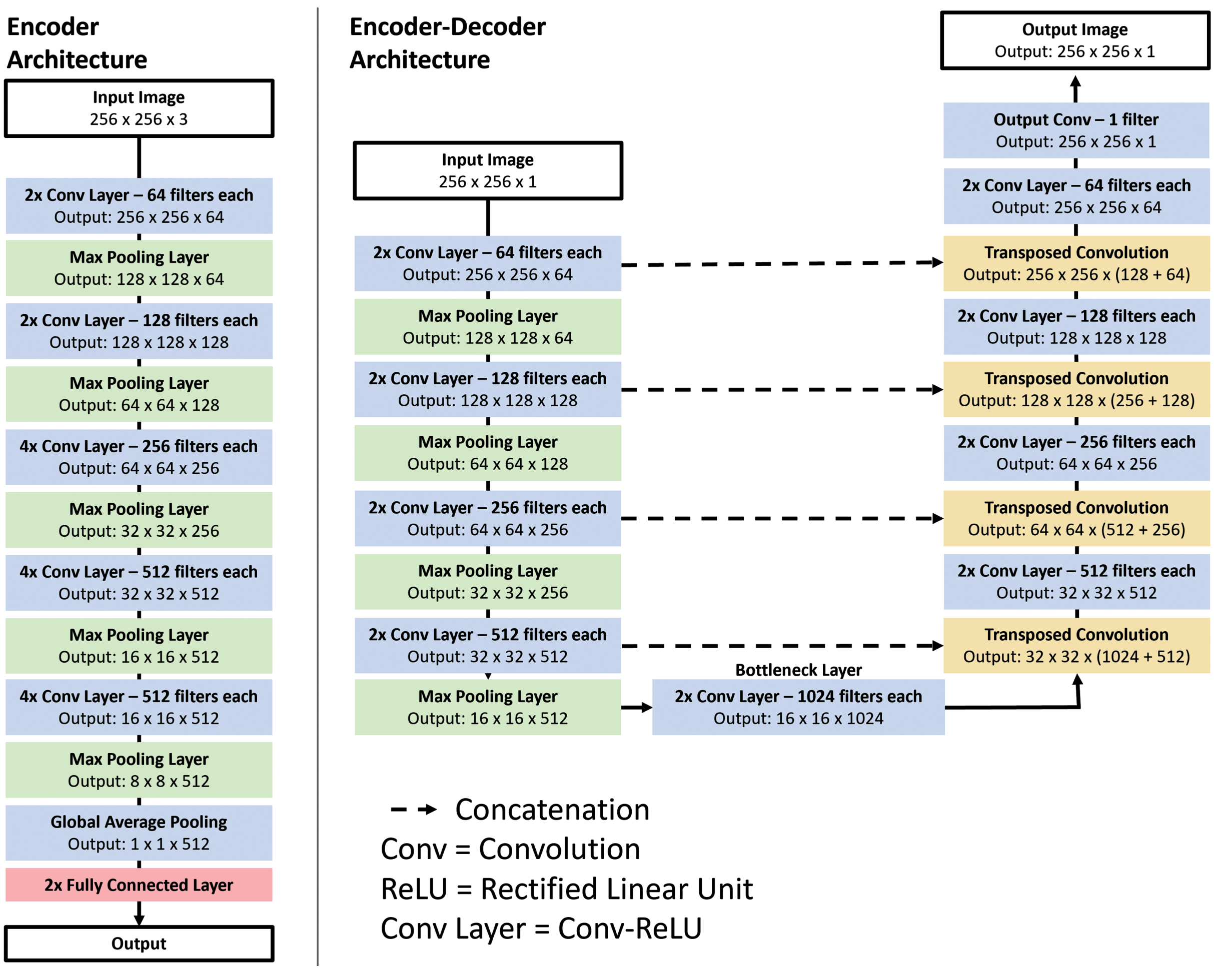


**Figure 6.** *Prototypical encoder and encoder-decoder architectures.* Encoder networks (left) map an input image to a 1D space, which is ideal for classification and regression tasks. Encoder-decoder networks (right) map an input image to another image by first encoding the input image onto a lower-dimensional space (bottleneck layer) and then upsampling the bottleneck feature maps back to image space.

function is used when the goal of the network is to categorize a set of inputs into nonoverlapping classes. Softmax normalizes the output of the two neurons to a probability distribution, such that the sum of their predictions is equal to one. Let $z_j \in (-\infty, \infty)$, $j = 1, \dots, K$, be the predictions from each output neuron across the $K$ classes. The softmax activation for the $i^{th}$ output neuron is

$$\sigma(z_1, \dots, z_K)_i = \frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}. \tag{6}$$

The predicted class of an input corresponds to the output neuron with the largest softmax score.

The *sigmoid* activation function is often used to categorize a set of inputs into overlapping classes (i.e., multilabel classification), such as to classify if a patient has obstructive lung disease or pneumonia, conditions that can co-occur. Using the sigmoid function,

$$\sigma(z_i) = \frac{e^{z_i}}{e^{z_i} + 1}, \tag{7}$$

predictions for each class are mapped to the interval (0,1). A threshold (e.g., 0.5) is then placed on the value of each output neuron to determine class membership. Note that the predictions do not sum to one, and inputs may be classified as members of multiple categories.

Many state-of-the-art architectures have been proposed for classification tasks. Although these networks were developed using the ImageNet visual database [DDS+09], they are often repurposed for classification tasks in medical imaging. One such architecture is the VGG network, which made groundbreaking improvements in classification performance [SZ15]. The network contains 19 layers with trainable weights, making VGG one of the first "deep" networks. However, due to its fully-connected layers, each containing 4096 neurons, VGG is computationally time-consuming, memory-consuming, and over-parameterized. To eliminate the need for fully-connected layers, many architectures now apply a global pooling layer to the final set of feature maps and make predictions directly from

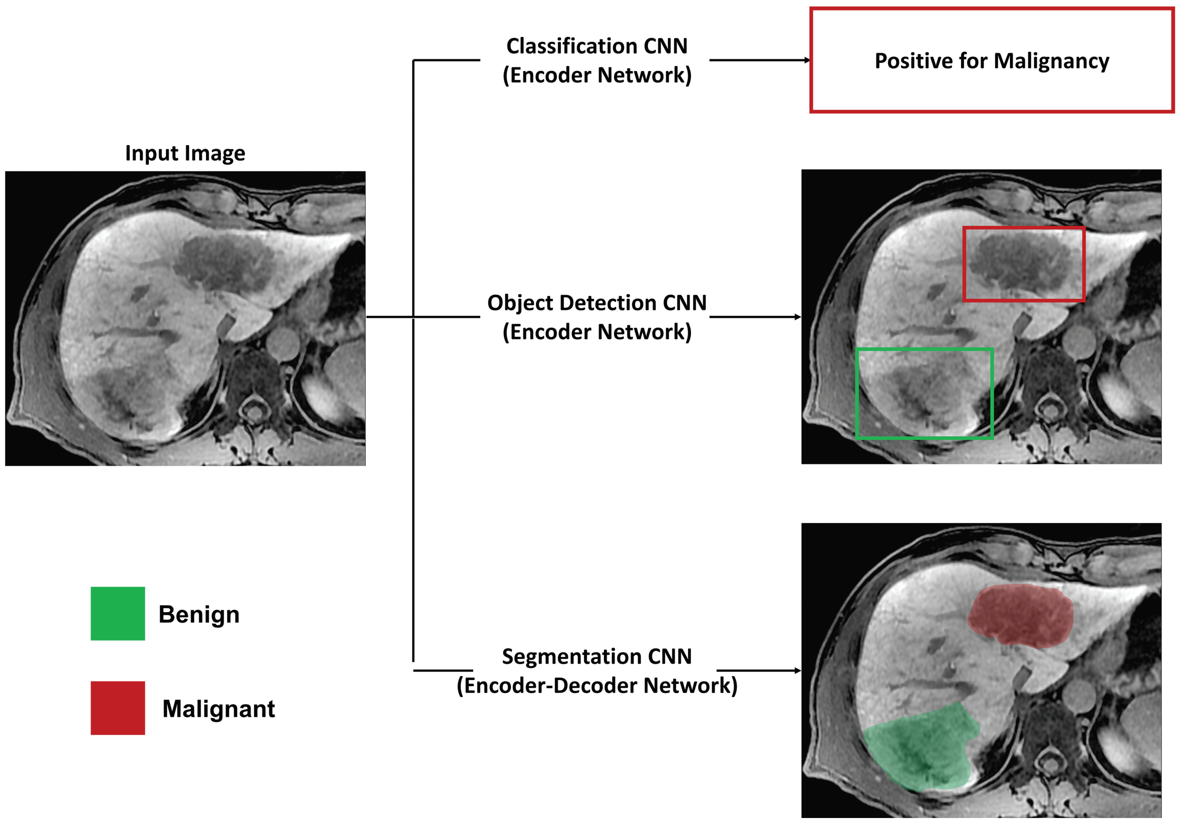


**Figure 7.** *Diagram showing the differences between commonly performed medical imaging tasks.* Classification categorizes the entire input image into a set of finite classes. Object detection identifies objects within an image, localizes the objects using a bounding box, and determines the class of the object within the same architecture. Segmentation partitions an image into meaningful classes at the pixel level, which can be used to calculate anatomical measurements, such as lesion size.

these features. In addition, newer architectures incorporate skip connections, via additions or concatenations, enabling the training of very deep networks (>100 layers with weights). The reduced number of weights and large number of nonlinearities, introduced through the increased depth of networks, have resulted in massive gains in performance. Notable architectures include GoogleNet [SLJ$^+$15] and ResNet [HZRS16], among others.

**Regression.** Regression CNNs aim to map an input image to a 1D vector comprising continuous values, such as predicting the percentage of fat in the liver using a liver CT image. Regression CNNs use the same architectures as classification CNNs, with exception to the output layer, which does not contain an activation function (i.e., linear activation).

**Object detection.** Object detection (Figure 7) extends the task of classification to identifying both the class and location of objects in an image. Early CNN-based object detection algorithms used sliding window methods, which classify patches extracted from many locations across an image into various categories. For example, to detect liver lesions, an encoder CNN would be pre-trained to classify an image patch as a benign lesion, malignant lesion, or no lesion. During test time, patches via a sliding window would each be propagated through the pre-trained CNN to determine the class and object bounds. Since objects may be different in size, patches of various sizes and aspect ratios may be used. However, this approach is computationally intensive, requiring the forward propagation of thousands of patches through a network. Moreover, if the selected patch size and aspect ratio does not match that of an object, the bounding box for that object becomes inaccurate.

The R-CNN family of object detection algorithms improves upon this approach by processing the feature map representation of an input image as opposed to the image itself, making the algorithm much faster than sliding window methods [RHGS15, HGDG17]. An image is propagated through a pre-trained network (e.g., VGG19 trained on ImageNet) and the low-resolution feature maps from the final convolutional layer are extracted. Sliding patches of the feature maps are then mapped to a set of fully-connected layers that predict 1) if the patch contains an object and 2) the bounding box coordinates of the object location. Patches from the feature maps can be used to detect objects since the spatial locations on these feature maps correspond to larger spatial locations on the original image.

These methods are further improved by the You Only Look Once (YOLO) algorithm [RDGF16], which avoids sliding patch-based processing and requires only a single pass through the network. An image is first partitioned into nonoverlapping patches. Using an encoder network, YOLO predicts whether or not each patch contains an object. If an object is contained within a patch, YOLO also predicts the object's bounding box coordinates and specific class. Although the input image is partitioned into patches, all patches are processed simultaneously by the network. For example, if an image is partitioned into a 4x4 grid, the model predicts 16 separate object classifications and bounding boxes with one forward pass, making object detection much faster than prior algorithms.

**Segmentation.** Segmentation aims to partition an image into meaningful classes at the pixel or voxel level (i.e., each pixel or voxel receives one or more classifications). For example, the segmentation CNN in Figure 7 is designed to segment benign lesions and malignant lesions simultaneously. Classes can be constrained to be nonoverlapping, such as the benign and malignant classes (i.e., a malignant lesion cannot be benign), or classes can be allowed to overlap, such as for the liver and lesions (i.e., a lesion is also part of the liver). Since the segmentation task is essentially a classification problem at the pixel level, output class labels must be one-hot encoded into 0-1 values, similar to the classification CNN. In addition, a softmax or sigmoid activation function is applied to the output image at the pixel level to bound the predictions to the interval (0,1). By far, the most popular architecture used for this purpose is the U-net [RFB15], similar to the encoder-decoder network in Figure 6 (right). Segmentation is a very important part of medical imaging and treatment as measurements (e.g., organ volume, average lung attenuation, etc.) using segmentations are often used for clinical diagnosis, prognosis, and research.

**Image-to-image translation.** The encoder-decoder architecture, specifically the U-net, is an extremely versatile network architecture capable of performing a variety of image-to-image translation tasks. In addition to segmentation, U-nets have been used for super-resolution to enhance noisy acquisitions [CFK+18], translation of modalities (e.g., predicting a CT from an MRI) to maximize the utility of information contained within standard examinations [MSP-NdLGAL21], and image reconstruction to make image acquisition more efficient [YJA+2206]. Essentially, any tasks involving the prediction of output images from input images, assuming a viable functional relationship exists between the inputs and outputs, can be performed by these architectures.

**Generative modeling.** CNNs are also commonly used for generative modeling. In particular, most CNN-based generative models are generative adversarial networks (GANs), systems of networks designed to synthesize realistic images for various purposes. In medical imaging, the pix2pix GAN is commonly used for image synthesis or prediction [IZZE17].

Pix2pix is a conditional GAN framework comprising a system of two networks, a generator and discriminator (Figure 8a). The generator is a U-net network trained to synthesize "fake" images that are indistinguishable from "real" images when evaluated by the discriminator. In contrast, the discriminator is an encoder network trained to determine if the synthetic image produced by the generator is "real" or "fake." Training is conducted adversarially, where each training iteration is alternated between the generator and discriminator. As the discriminator's ability to detect "fake" images improves, the generator's ability to synthetize realistic images also improves, resulting in more realistic and accurate results than when training a U-net without the discriminator. Several additional details regarding the training process are excluded for brevity but can be found in [IZZE17].

Since CNNs require large amounts of data to train, GAN-synthesized images can be used to supplement the existing set of training data to boost model performance. GANs have also been shown to improve performance over U-net-only frameworks for image-to-image translation tasks. However, GANs in general are known for being challenging to train. Note that pix2pix requires paired images (i.e., each input image must have a corresponding output image). Image synthesis for unpaired data is commonly performed using the CycleGAN [ZPIE17].

**Image registration.** Image registration is the process of aligning two images based on their corresponding imaging features (e.g., anatomical landmarks, pathologies, etc.). Specifically, registration aims to estimate a mapping, represented as a 2D or 3D vector field, that transforms the coordinate system of one image, referred to as a "moving" image, to the coordinate system of a reference image, referred to as a "static" or "fixed" image. Estimation of the mapping is fundamentally an optimization problem, with the goal of minimizing (or maximizing) some loss or energy function that quantifies the difference (or similarity) between features of the static image and mapped (i.e., registered) moving image. Traditional algorithms using these optimization techniques are quite accurate. However, these algorithms must perform the optimization each time an image pair is to be registered, requiring minutes to even hours to perform a single registration.

To expedite the registration task, CNN-based algorithms have recently been proposed as an alternative to traditional registration. These algorithms use a CNN to

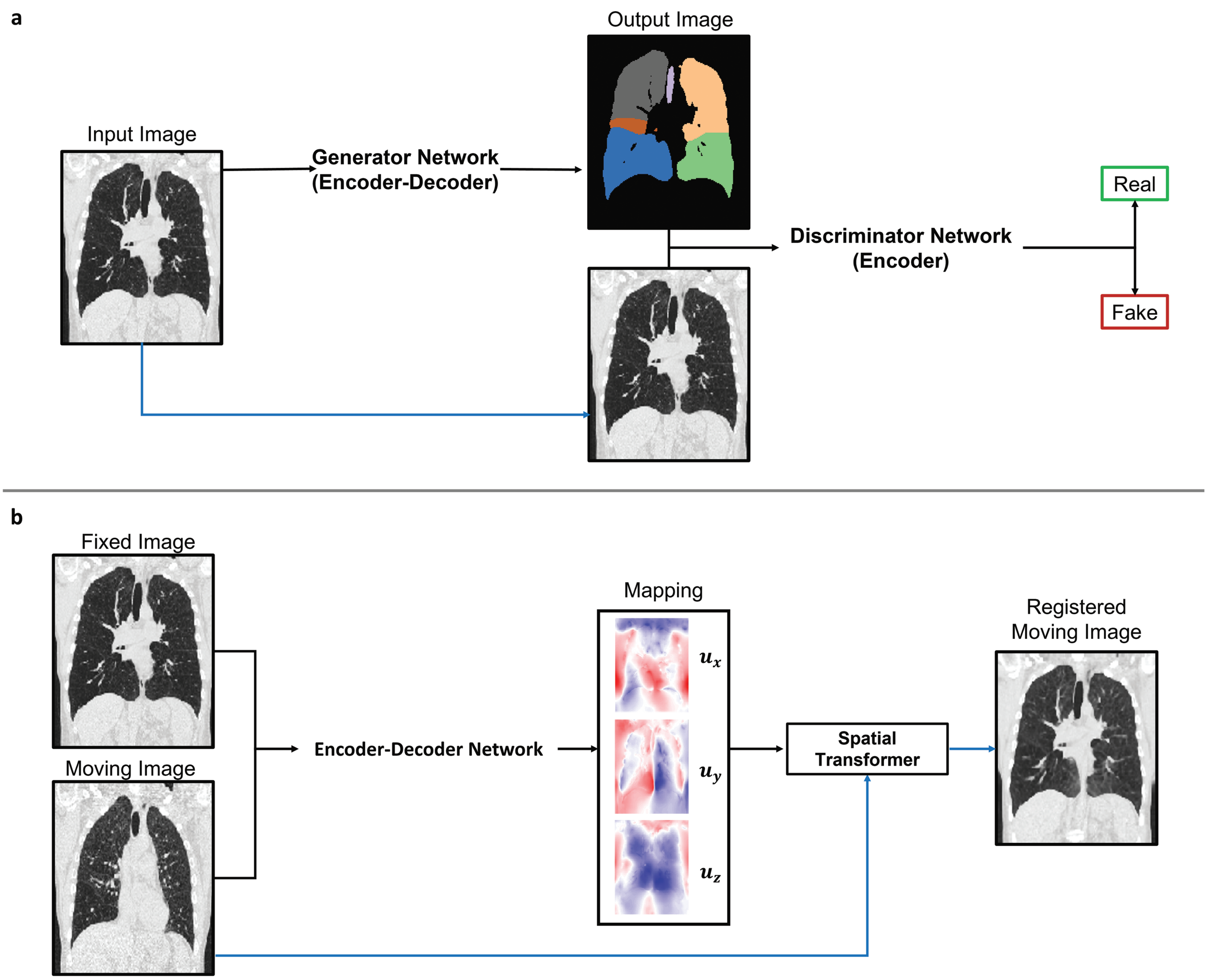


**Figure 8.** *(a) Image-to-image translation GAN and (b) CNN-based registration framework.* The pix2pix GAN is often used to synthetize images to supplement training. In the registration framework, an encoder-decoder architecture is trained to predict a spatial transformation represented by a vector field (i.e., mapping) that is used to map the moving image to fixed image space.

predict the optimal mapping using the image pairs as input, performing the optimization only a single time during CNN training. During test time, when the CNN-based algorithm is applied to new image pairs, the fixed and moving images are simply propagated through the network to predict the mapping needed to register the images, reducing the task to seconds on a CPU and less than a second on a GPU.

Figure 8b shows an example of a CNN-based algorithm designed to register inspiratory (fixed) and expiratory (moving) lung images. The fixed and moving images are propagated through an encoder-decoder to predict a mapping. The mapping and moving image are then propagated through a spatial transformation layer, which uses the vector field to map the moving image to the static image. During training, constraints can be placed on the geometric properties and smoothness of the vector field to enforce realistic anatomical transformations. Image registration is an important part of medical imaging as it facilitates the characterization, monitoring, and surveillance of pathologies and diseases within and across exams.

## Training and Optimization

**Loss functions.** A loss function is a differentiable, objective metric used to quantify CNN performance. The loss guides the iterative adjustment of weights during training to improve predictive accuracy. Table 1 shows a list of commonly used loss functions, among which, mean squared error (MSE) for regression and categorical cross entropy for classification are most common.

MSE loss measures the disparity between predicted and observed outputs through the average of their squared differences, where a larger MSE indicates poorer model performance. Cross entropy loss is a popular choice for classification tasks and is used in conjunction with the softmax or sigmoid activation functions in the final layer. Under these conditions, $y_i \in \{0, 1\}$ and $\hat{y}_i \in (0, 1)$ imply $-y_i \log(\hat{y}_i) \in (0, \infty)$. When $y_i$ and $\hat{y}_i$ are in agreement, the cross entropy loss is close to zero, indicating better model performance.

Losses are typically summed or averaged across observations to quantify total error of the model. If a network contains multiple output neurons, the loss for a single observation is the sum of losses for each individual neuron. Generally, a smaller loss implies better performance. If this is not the case, losses are often negated to enforce this behavior.

**Gradient descent.** Once an architecture and loss function have been selected, the network is ready for training. CNNs are trained with the *gradient descent* algorithm, which uses the gradient of the loss function to iteratively "step" in the steepest downward direction of the loss by adjusting its weights accordingly. The goal of this process is to find a set of weights that minimize the loss function.

Figure 9 visualizes this process using a hypothetical loss for a network with a single weight. If the derivative of the loss with respect to the weight is negative, the weight is increased to reduce the loss. Similarly, if the gradient is positive, the weight is decreased. These weight adjustments are performed iteratively until the loss is minimized or stops improving. The magnitude by which the weight is adjusted is determined by a preset *step size* $\gamma$ (i.e., *learning rate*). Mathematically, the weight update is represented as

$$w_{r+1} = w_r - \gamma \nabla L(w_r), \tag{8}$$

where $w_r$ is the value of the weight at iteration $r$ and $\nabla L(w_r)$ is the gradient of the loss with respect to the weight $w_r$.

Current state-of-the-art models contain hundreds of layers, each with nonlinear activations, creating complex functional hierarchies with millions of weights. Amazingly, optimization of these weights is greatly simplified using the chain rule. For the simple network defined in Equations 2 and 3, we can apply the chain rule to define our loss gradient for each of its weights.

$$\nabla L(w_{jk}^{[1]}) = \frac{\partial L(y_{i1}, \hat{y}_{i1})}{\partial \hat{y}_{i1}} \times \frac{\partial \hat{y}_{i1}}{\partial a_{ik}^{[1]}} \times \frac{\partial a_{ik}^{[1]}}{\partial z_{ik}^{[1]}} \times \frac{\partial z_{ik}^{[1]}}{\partial w_{jk}^{[1]}}, \tag{9}$$

$$\nabla L(b_k^{[1]}) = \frac{\partial L(y_{i1}, \hat{y}_{i1})}{\partial \hat{y}_{i1}} \times \frac{\partial \hat{y}_{i1}}{\partial a_{ik}^{[1]}} \times \frac{\partial a_{ik}^{[1]}}{\partial z_{ik}^{[1]}} \times \frac{\partial z_{ik}^{[1]}}{\partial b_k^{[1]}}, \tag{10}$$

$$\nabla L(w_{jk}^{[2]}) = \frac{\partial L(y_{i1}, \hat{y}_{i1})}{\partial \hat{y}_{i1}} \times \frac{\partial \hat{y}_{i1}}{\partial w_{jk}^{[2]}}, \tag{11}$$

$$\nabla L(b_k^{[2]}) = \frac{\partial L(y_{i1}, \hat{y}_{i1})}{\partial \hat{y}_{i1}} \times \frac{\partial \hat{y}_{i1}}{\partial b_k^{[2]}}. \tag{12}$$

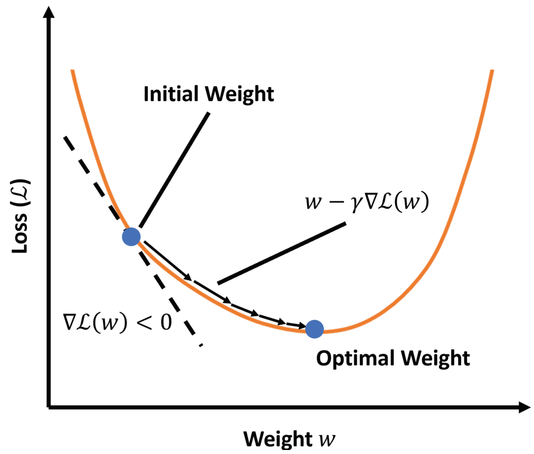


**Figure 9.** *Illustration of the gradient-descent optimization algorithm.* Prior to network training, weights are randomly initialized. The gradient of the predetermined loss function is then calculated with respect to the weights and evaluated for each training observation. Weights are adjusted by the product of the gradient and a preselected step size $\gamma$. The process is iteratively performed until the loss is minimized or stops improving.

These gradients are used to update each of the corresponding weights. Since gradients are propagated backwards to earlier layers to update weights within those layers, this process is referred as *back propagation*. Note that the gradients are a function of the inputs ($x_{i1}$) and outputs ($y_{i1}$), which are known, and network weights ($w_{1k}^{[1]}, b_k^{[1]}, w_{jk}^{[2]}, b_k^{[2]}$), which are initialized. Conventionally, biases are initialized to zero and the remaining weights are initialized using random samples from a probability distribution (e.g., Glorot uniform, Gaussian, etc.). The value of the learning rate $\gamma$ varies by network but is often set to a power of ten between $10^{-6}$ and 1.

## Other Considerations

**Mini-batch gradient descent.** Training a network for imaging tasks often requires a large number of images, which may exceed the memory capacity of computing hardware. Therefore, gradient updates are typically performed on random mini-batches of data. This is known as *mini-batch gradient descent*. A mini-batch of size one is referred to as *stochastic gradient descent*. A set of mini-batch gradient updates that pass through the entire dataset is an *epoch*. In addition to reducing the memory load, mini-batch optimization has favorable convergence properties.

**Optimizers.** In contrast to the depiction of a convex loss function in Figure 9, in practice, CNN loss functions are high-dimensional and highly nonconvex, containing saddle points and local minima. This loss landscape presents additional challenges during optimization, as the gradients near these areas are very close to zero, which can slow training, falsely indicate convergence, or converge to a

| Loss Function | $L(y,\hat{y})$ | Task |
|---|---|---|
| Mean Squared Error | $\frac{1}{N}\sum_{i=1}^{N}(y_i-\hat{y}_i)^2$ | Regression / Image Prediction |
| Mean Absolute Error | $\frac{1}{N}\sum_{i=1}^{N}\lvert y_i-\hat{y}_i\rvert$ | Regression / Image Prediction |
| Binary Cross-Entropy | $-\frac{1}{N}\sum_{i=1}^{N}[y_i\log(\hat{y}_i)+(1-y_i)\log(1-\hat{y}_i)]$ | Classification / Segmentation |
| Dice | $-\frac{1}{N}\sum_{i=1}^{N}\frac{2\sum_{p=1}^{P}y_{ip}\hat{y}_{ip}}{\sum_{p=1}^{P}y_{ip}+\sum_{p=1}^{P}\hat{y}_{ip}}$ | Segmentation |

**Table 1.** *Commonly used loss functions for CNN training and performance evaluation.* Here, $y_i$ is the true output value or label for observation $i$, $\hat{y}_i$ is the corresponding ANN prediction, and $N$ is the sample size (or batch size). For image prediction and segmentation, ANN outputs are entire images, and the outputs and predictions should therefore be modified to $y_{ip}$,$\hat{y}_{ip}$ for pixel $p$. For multi-class tasks, losses are summed across the $K$ categories.

suboptimal minimum. *Optimizers* mitigate this problem by incorporating loss gradients from prior iterations into the weight update equation. Sun et al. provide a thorough survey of several optimizers [SCZZ20]. Among these, Adapative Moment Estimation is often used.

**Overfitting and generalization error.** The primary goal of CNNs for supervised tasks is prediction. That is, a CNN should generalize well to data outside of the data on which it was trained. As the capacity of a network is increased by incorporating more hidden layers (i.e., more weights), the network may be subject to *overfitting*, a situation in which the loss error of external data is much larger than the loss error of the training data.

To determine whether a model is overfitting, data are randomly partitioned into training/validation/testing sets. The training set is used to train several architectures, which are then evaluated using the validation set. The model that minimizes error on the validation set is then selected as the final model. The testing set is used to measure a model's ability to generalize to outside data and should not be used to evaluate generalization error until a final network has been selected. The proportional split into training/validation/testing sets varies by the amount of data available, but a 70%/15%/15% partition is common. When a network is over capacity, as is often the case, the validation set is typically used to stop training early before overfitting occurs. This is known as *early stopping*. Other methods, such as L1 or L2 regularization, dropout [SHK+14], and data augmentation can also be used to mitigate overfitting.

**Data preprocessing.** Prior to training, both inputs and outputs are either normalized (scaled to [0,1] or [-1,1]) or standardized (subtract mean and divide by standard deviation) to ensure a similar data distribution between inputs, outputs, and network weights. This stabilizes gradient calculations, resulting in faster convergence.

**Batch normalization.** In contrast to the standardization of data prior to model training, batch normalization is used to standardize the feature maps produced throughout the network during training [IS1507]. However, standardization using the global mean and standard deviation of feature maps across the entire dataset is impractical. Therefore, standardization is performed at the training batch level using a global set of parameters learned during the training process. Similar to the standardization of input images, batch normalization results in faster convergence and better stability during training. Batch normalization layers are now often included within Convolution-BatchNormalization-ReLU (or Convolution-ReLU- BatchNormalization) blocks in many architectures.

**Transfer learning and pretrained networks.** CNNs typically require large amounts of training data (hundreds or thousands) to achieve strong performance. However, in medical imaging, training data is limited to a patient population of interest, which can often be sparse, leading to network overfitting and poor generalization error. Rather than starting training using randomly initialized weights, one can use a network that has been *pretrained* to perform a different task. The process of training a network using pretrained weights is known as *transfer learning*. Transfer learning speeds training and improves generalizability since training is not initiated from a random starting point.

Earlier layers in CNNs learn low-level features such as edges, textures, and intensities and hence tend to be similar across CNNs despite being trained to perform different tasks. Therefore, weight updates during transfer learning are often constrained to the terminal layers of the pretrained CNN. That is, weights in the earlier layers are "frozen" (i.e., no longer updated during training). If training a pretrained CNN to perform a similar task but using a different dataset, a much smaller learning rate is used to avoid large deviations from the original pretrained weight distribution, a process known as *fine-tuning*.

**Software and hardware.** Currently, Python is the most popular programming language for developing and operationalizing deep neural networks. However, other languages, such as MATLAB and R, have similar capabilities. Within Python, a variety of deep learning libraries are available, including Keras, TensorFlow, PyTorch, Theano, and Caffe, among others. Keras, in particular, is a high-level user-friendly API that interfaces with Tensorflow to enable fast experimentation of deep neural networks. Keras, Tensorflow, and PyTorch are the most popular. As an alternative to programming languages, point-and-click interfaces are beginning to emerge to increase accessibility of these algorithms to individuals with less technical expertise.

Training a deep neural network is a computationally intensive process involving millions of floating point operations and repeated I/O of large training batches. Although some smaller networks can be trained on a standard CPU in a reasonable amount of time, most state-of-the-art networks require a GPU to expedite training, which may still require hours to days to complete. Since networks developed for medical imaging tasks typically involve high-resolution 3D (or even 4D) images, GPUs with larger memory capacities (e.g., >12GB) are often necessary. Training can be performed on either a personal workstation or one of the various cloud-based services available (e.g., Google Cloud, Amazon AWS, Microsoft Azure).

## Explainability and Translation to Clinical Practice

Although CNNs have accomplished state-of-the-art performance on medical imaging tasks, translation of this technology into clinical practice is challenged by the difficulty of explaining their decisions. Understandably, radiologists would like to understand why a CNN made a particular decision, or more importantly why a CNN made an incorrect decision, before adopting the technology in practice. In response to this need, an entire collection of algorithms under the umbrella term "Explainable AI" have been proposed. Among the most popular approaches in medical imaging are attribution methods and feature map visualizations, which highlight the salient regions on an input image used by a CNN to make its prediction. A review of explainable methods in medical imaging can be found here [SSL20].

## Conclusion

Deep neural networks are transforming the field of medical imaging in both research and clinical practice. They are highly versatile algorithms capable of performing a multitude of medical imaging tasks across imaging modalities and subspecialties. Despite the remaining challenges pertaining to their transparency, deep neural networks have the potential to improve patient care and reduce radiologist burnout.

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Kyle Hasenstab

**Credits**

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# Ability and Diversity of Skills

*William Geller and Michał Misiurewicz*

## 1. Introduction

The aim of this paper is to build a simple model of problem solving, both by single agents and by teams. We realize that our model is crude and of course far from universal. Yet the results we get seem to us quite illuminating, and show the importance of both ability and diversity of skills.

The question of how to measure effectiveness of problem solving by individuals and (even more importantly) by teams, and how to choose the best individual/team, has been a subject of a lot of research. We can give as examples papers [GJIB, HP2, KI, KR], and the literature cited there. We do not address explicitly the problem of choosing a team, but our findings may serve as the basis for further research in that direction (in cases where it seems that our model may be applicable).

For a single agent, or a team of agents, we try to measure the probability of success as a function of the difficulty of a problem (or rather the easiness of the problem, measured by a variable $p$; the larger $p$, the easier the problem). In Section 3 we show that the probability of success is concave as a function of $p$.

In Section 4 we show that in our model for a single agent specialization is better than versatility.[1] We also show that comparing agents is difficult. In most cases, for a given agent and chosen values of $p$, there can be another agent, who is better at solving problems with easiness $p$ for those chosen values, but worse at solving problems with all other values of $p$.

In Section 5 we consider teams of agents. We get what can be considered the main result of the paper: whenever our model can be applied, both abilities of the team members and the diversity of skills in the team matter. If any of those increases, so does the probability of success. For simplicity, we consider teams with two members, but it is clear that similar results should hold for larger teams. Interestingly, there is an example where for easier problems ability is more important, but for more difficult problems diversity is more important.

In Section 6 we show how our model can be applied to a situation where the agents are trying to defend an organization against an attack. In this application, diversity is even more important than for general problem solving.

Some of our ideas came from studying the model of L. Hong and S.E. Page [HP1]. Our model is much simpler, and can be easily investigated both by pure mathematical means and by computational means. Moreover, we avoid the main deficiency of the Hong–Page model, where high-ability teams consist basically of clones of the same agent (and as a result, ability excludes diversity).[2]

## 2. Preliminary Model

If an agent will be trying to solve problems that are not known in advance, her expected performance can be measured by an average over various possible problems.

Our first, preliminary model is as follows. An agent has some set of skills. This set is a subset $S$ of $N = \{1, 2, \dots, n\}$. An immediate problem is represented by a subset $P$ of $N$ of cardinality $p$. An agent can make progress if the intersection $S \cap P$ is nonempty. Clearly, the difficulty of the problem is measured by $p$; problems with smaller $p$ are more difficult.

When we want to measure the ability of an agent, we average performance of an agent over all problems of a given difficulty (that is, with a given cardinality $p$). The result clearly does not depend on a concrete set $S$ of skills, but only on its cardinality $s$ (the skillfulness of the agent).

For given $n, s, p$ it is easy to compute the probability of making progress. If $p + s > n$, this probability is 1. If $p + s \le n$, out of all $\binom{n}{p}$ possible sets $P$ only $\binom{n-s}{p}$ result in failure. Therefore the probability of success (that is,

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[1] *Thus a strategic agent might choose to specialize.*

[2] *A reader interested in the discussion about that model may want to look at* [T1, P, T2].

making progress) is

$$1-\frac{\binom{n-s}{p}}{\binom{n}{p}}=1-\frac{(n-s)!\,(n-p)!}{n!\,(n-p-s)!}$$
$$=1-\frac{n-p}{n}\cdot\frac{n-p-1}{n-1}\cdot\ldots\cdot\frac{n-p-s+1}{n-s+1}.$$

In Figure 1, we can see how the probability of success varies with the difficulty of the problem, for agents with various numbers of skills. If the problem is easy, the skillfulness of an agent does not matter much (provided the agent has some minimal number of skills). However, for difficult problems it matters a lot.

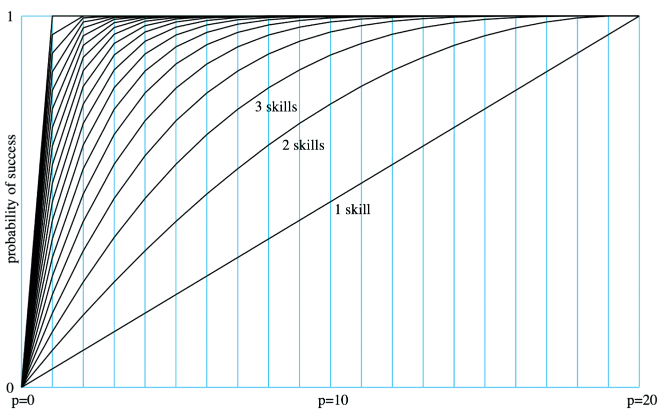


**Figure 1.** For $n = 20$, graphs of the probabilities of success as functions of the difficulty of the problem for various numbers of skills of the agent. As we move to the right, the difficulty of the problem decreases (that is, $p$ increases).

## 3. Main Model

The preliminary model is very crude, because for each skill an agent either has it or does not. However, one should allow an agent to have partial skills. Then $S$ becomes a *strength function* $S : N \to [0,1]$. If $k \in P$, then the probability that the agent can make progress using skill number $k$ is $S(k)$. We assume that those probabilities for different $k$ are independent. This means that it is easier to use in computations the *weakness function* $R = \mathbb{1} - S$, where $\mathbb{1}$ is the constant function 1. Then the probability $F(R,P)$ of failure for a given agent and given problem is equal to the product of the numbers $R(k)$ over $k \in P$.

Often instead of speaking of the strength and weakness functions we will speak of the *strength and weakness vectors* $(S(1), S(2), \ldots, S(n))$ and $(R(1), R(2), \ldots, R(n))$.

For a given $R$, the sum of $F(R,P)$ over all sets $P \subset N$ of cardinality $p$ is equal to

$$\sum_{|P|=p}\prod_{i\in P}R(i),$$

where $|P|$ denotes the cardinality of $P$. Observe that this number is equal to the coefficient for the polynomial

$$Q_R(x)=\prod_{i=1}^{n}(x+R(i))$$

of $x^{n-p}$. Thus, the average probability $F_R(p)$ of failure over all sets $P$ of cardinality $p$ is equal to this coefficient divided by $\binom{n}{p}$. Note that $\binom{n}{p}$ is the coefficient of $x^{n-p}$ for the polynomial $Q_{\mathbb{1}}$. This in particular means that if an agent has no skills (so $R = \mathbb{1}$), her probability of failure is 1 no matter what.

Clearly, if $\sigma$ is a permutation of the set $N$ then $F_R(p) = F_{R\circ\sigma}(p)$. Therefore we may assume that $R(1) \le R(2) \le \ldots \le R(n)$. Sometimes, if we do not want to make this assumption, we will say that $S \circ \sigma$ is a permutation of $S$.

Of course, if $R$ takes only values 0 and 1, we get the previous model.

Let us investigate some basic properties of the function $F_R$.

**Proposition 1.** *We have $F_R(p+1) \le F_R(p)$, and equality holds only if either both numbers are equal to 0 or $R = \mathbb{1}$.*

*Proof.* Replace each subset $P \subset N$ of cardinality $p+1$ by $p+1$ pairs $(P,j)$, where $j \in P$. Then the average of $\prod_{i\in P} R(i)$ over all such pairs will be equal to $F_R(p+1)$. Similarly, when we replace each subset $P \subset N$ of cardinality $p$ by $n-p$ pairs $(P,j)$, where $j \in N\setminus P$, the average of $\prod_{i\in P} R(i)$ over all such pairs will be equal to $F_R(p)$. However, there is a natural one-to-one correspondence between the pairs of the first and of the second type. Namely, if $|P| = p$ and $j \in N\setminus P$, then $|P\cup\{j\}| = p+1$ and $j \in P\cup\{j\}$. Since always $\prod_{i\in P\cup\{j\}} R(i) \le \prod_{i\in P} R(i)$, we get $F_R(p+1) \le F_R(p)$.

Suppose that we have the equality. Then for every $P \subset N$ of cardinality $p$ and every $j \in N\setminus P$ we have either $\prod_{i\in P} R(i) = 0$ or $R(j) = 1$. If for every $P \subset N$ of cardinality $p$ we have $\prod_{i\in P} R(i) = 0$, then $F_R(p+1) = F_R(p) = 0$. Otherwise, there exists $P \subset N$ of cardinality $p$ with $\prod_{i\in P} R(i) > 0$, so we have $R(j) = 1$ for every $j \in N\setminus P$. Unless $R = \mathbb{1}$, there is $k \in P$ for which $R(k) < 1$. Choose one $j \in N\setminus P$ and consider the set $V = P\cup\{j\}\setminus\{k\}$. Then

$$\prod_{i\in V\cup\{k\}}R(i)=R(k)\prod_{i\in V}R(i)<\prod_{i\in V}R(i),$$

a contradiction. This proves the second part of the proposition. □

**Proposition 2.** *We have*

$$\frac{F_R(p+2)+F_R(p)}{2}\ge F_R(p+1), \tag{1}$$

*so the function $F_R$ is convex (and the success function, $\mathbb{1} - F_R$, is concave). Equality holds if and only if either $F_R(p) = 0$ or there is at most one $i \in N$ such that $R(i) \ne 1$.*

*Proof.* In a similar way as in the proof of Proposition 1, we get the following four equalities (in the first one, we have to look at the set $N \setminus \{j\}$ instead of $N$):

$$\underset{j\in N}{\forall} \quad (p+1)\sum_{\substack{|V|=p+1\\ j\notin V}}\prod_{i\in V}R(i) = \sum_{\substack{k=1\\ k\neq j}}^{n}\sum_{\substack{|W|=p\\ j,k\notin W}}R(j)\prod_{i\in W}R(i), \tag{2}$$

$$(n-p-1)\sum_{|V|=p+1}\prod_{i\in V}R(i) = \sum_{j=1}^{n}\sum_{\substack{|V|=p+1\\ j\notin V}}\prod_{i\in V}R(i), \tag{3}$$

$$(n-p)(n-p-1)\sum_{|W|=p}\prod_{i\in W}R(i) = \sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{\substack{|W|=p\\ j,k\notin W}}\prod_{i\in W}R(i), \tag{4}$$

$$\begin{aligned}(p+2)(p+1)&\sum_{|P|=p+2}\prod_{i\in P}R(i)\\ &= \sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{\substack{|W|=p\\ j,k\notin W}}R(j)R(k)\prod_{i\in W}R(i).\end{aligned} \tag{5}$$

Since $(1-R(j))(1-R(k)) \geq 0$, we have

$$1 + R(j)R(k) \geq R(j) + R(k). \tag{6}$$

Now, from (5), (4), and (6), we get

$$\begin{aligned}&(p+2)(p+1)\sum_{|P|=p+2}\prod_{i\in P}R(i)\\ &\quad + (n-p)(n-p-1)\sum_{|W|=p}\prod_{i\in W}R(i)\\ &= \sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{\substack{|W|=p\\ j,k\notin W}}(1+R(j)R(k))\prod_{i\in W}R(i) \qquad (7)\\ &\geq \sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{\substack{|W|=p\\ j,k\notin W}}(R(j)+R(k))\prod_{i\in W}R(i).\end{aligned}$$

From (2) and (3), we get

$$\begin{aligned}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{\substack{|W|=p\\ j,k\notin W}}R(j)\prod_{i\in W}R(i) &= \sum_{k=1}^{n}(p+1)\sum_{\substack{|V|=p+1\\ k\notin V}}\prod_{i\in V}R(i)\\ &= (p+1)(n-p-1)\sum_{|V|=p+1}\prod_{i\in V}R(i).\end{aligned} \tag{8}$$

From (7) and (8) (note that (8) holds also with $j$ and $k$ switched) we get

$$\begin{aligned}(p+2)(p+1)\sum_{|P|=p+2}\prod_{i\in P}R(i)+(n-p)(n-p-1)\sum_{|W|=p}\prod_{i\in W}R(i)\\ \geq 2(p+1)(n-p-1)\sum_{|V|=p+1}\prod_{i\in V}R(i),\end{aligned}$$

that is,

$$\begin{aligned}(p+2)(p+1)\binom{n}{p+2}F_R(p+2)+(n-p)(n-p-1)\binom{n}{p}F_R(p)\\ \geq 2(p+1)(n-p-1)\binom{n}{p+1}F_R(p+1).\end{aligned}$$

Since

$$\begin{aligned}(p+2)(p+1)\binom{n}{p+2} &= (n-p)(n-p-1)\binom{n}{p}\\ &= (p+1)(n-p-1)\binom{n}{p+1},\end{aligned}$$

we get

$$F_R(p+2) + F_R(p) \geq 2F_R(p+1).$$

This proves the first part of the proposition.

To prove the second part, notice that by (7), equality in (1) holds if and only if for every $W \subset N$ of cardinality $p$ and every $j,k \in N \setminus W$ such that $j \neq k$, either $R(j) = 1$, or $R(k) = 1$, or $\prod_{i\in W} R(i) = 0$.

If $F_R(p) = 0$, then for every $W \subset N$ of cardinality $p$ we have $\prod_{i\in W} R(i) = 0$. If there is at most one $i \in N$ such that $R(i) \neq 1$, then $j \neq k$ implies $R(j) = 1$ or $R(k) = 1$. In all those cases we get equality in (1).

Now assume that $F_R(p) \neq 0$ and $R(i) \neq 1$ for at least two indices $i \in N$. Then there are two possible cases. Either there are two or more zeros among $R(i)$, $i \in N$, or there is at most one zero there. In the first case, we can choose $j,k \in N$ such that $j \neq k$ and $R(j) = R(k) = 0$. Then, since $F_R(p) \neq 0$, there is $W \subset N \setminus \{j,k\}$ of cardinality $p$, such that $\prod_{i\in W} R(i) > 0$. In the second case, we choose $j,k \in N$ with $j \neq k$ and $R(j), R(k)$ as small as possible, and again there is $W \subset N \setminus \{j,k\}$ of cardinality $p$, such that $\prod_{i\in W} R(i) > 0$. In both cases, there is no equality in (1). □

## 4. Specialization and Versatility

We would like to be able to measure the skillfulness of an agent in our model. There may be various ways of doing this, and as we will see in Theorem 5, we cannot expect to find a perfect one. We will settle on what seems the most natural way of doing it, by defining it to be $S(1) + S(2) + \cdots + S(n)$, where $S$ is the strength function of the agent.

Within our model, one of the first questions that comes to mind is what is the best distribution of strengths given the skillfulness of an agent. The agent can be more specialized or more versatile. We will show that in our model specialization is better than versatility.

The simplest case is when we have two agents, one with $S_1(1) = S_1(2) = 1/2$ and $S_1(k) = 0$ for $k > 2$, and the other one with $S_2(1) = 1$ and $S_2(k) = 0$ for $k > 1$. The first agent is more versatile and the second one more specialized. We have

$$\begin{aligned} Q_{R_1} &= (x+1/2)^2(x+1)^{n-2} = (x^2+x+1/4)(x+1)^{n-2} \\ &= Q_{R_2} + \frac{1}{4}(x+1)^{n-2} \end{aligned}$$

and

$$Q_{R_2} = x(x+1)^{n-1}.$$

Thus, $F_{R_1}(p) = F_{R_2}(p)$ for $p < 2$, and $F_{R_1}(p) > F_{R_2}(p)$ for $p \geq 2$. Let us make exact computations.

The coefficient of $x^{n-p}$ for the polynomial $x(x+1)^{n-1}$ is the same as the coefficient of $x^{n-p-1}$ for the polynomial $(x+1)^{n-1}$, that is, $\binom{n-1}{n-p-1}$. Thus,

$$F_{R_2}(p) = \frac{(n-1)!}{(n-p-1)!\,p!} \cdot \frac{(n-p)!\,p!}{n!} = \frac{n-p}{n} = 1 - \frac{p}{n}.$$

To get the coefficient of $x^{n-p}$ for the polynomial $(x+1/2)^2(x+1)^{n-2}$, we have additionally to add the coefficient of $x^{n-p}$ for the polynomial $\frac{1}{4}(x+1)^{n-2}$, so

$$\begin{aligned} F_{R_1}(p) &= 1 - \frac{p}{n} + \frac{1}{4} \cdot \frac{(n-2)!}{(n-p)!\,(p-2)!} \cdot \frac{(n-p)!\,p!}{n!} \\ &= 1 - \frac{p}{n} + \frac{1}{4} \cdot \frac{p(p-1)}{n(n-1)}. \end{aligned}$$

This means that while the graph of the probability of success as a function of $p$ lies on the straight line from $(0,0)$ to $(n,1)$ for $R_2$, it lies on a parabola from $(0,0)$ to $(n,3/4)$ for $R_1$ (see Figure 2).

In this example specialization is better than versatility (see Figures 3 and 4 for other examples).

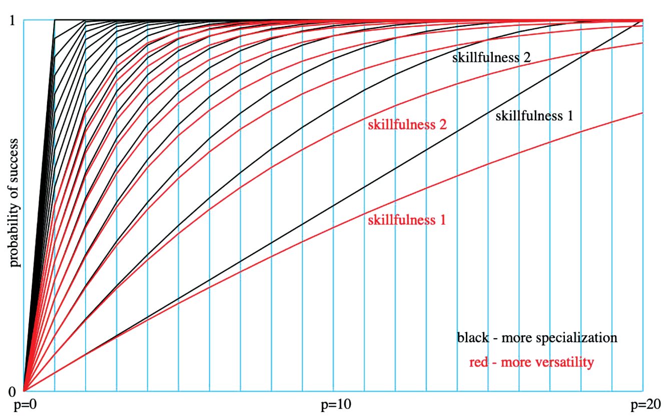


**Figure 2.** The same picture as in Figure 1, with additional red graphs showing probabilities of success with skillfulness an integer, from 1 to 10, but $S$ taking only values 0 and $1/2$. Black graphs represent more specialization and red ones more versatility.

We considered only a simple example, but it turns out that in more complicated situations the result is the same.

**Lemma 3.** *Let $a, b \in (0,1)$, $n \geq 2$. Let a strength function $S_1$ be such that $S_1(1) = a$ and $S_1(2) = b$. If $a+b \leq 1$, set* 

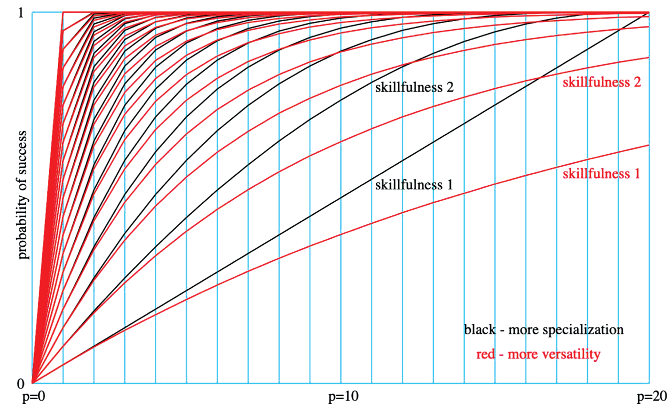


**Figure 3.** The same picture as in Figure 2, but the red graphs showing probabilities of success with skillfulness an integer, from 1 to 20, spread evenly (that is, $S(i)$ is the same for all $i$). This represents even more versatility than in the preceding figure.

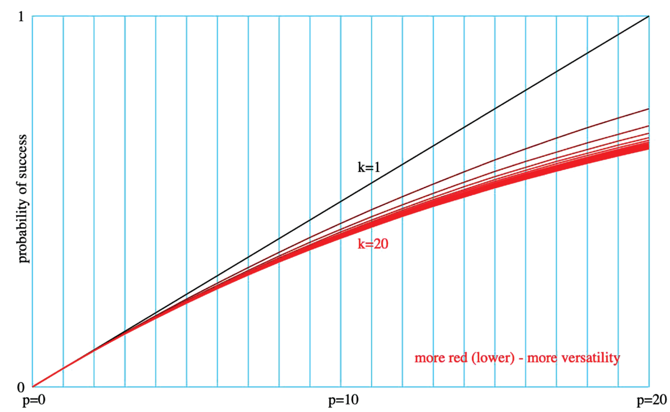


**Figure 4.** Here skillfulness is 1, but it is spread equally into $k$ skills, and $k$ varies from 1 (the highest graph) to 20 (the lowest graph).

*$S_2(1) = a+b$, $S_2(2) = 0$ and $S_2(k) = S_1(k)$ for $k > 2$. If $a+b > 1$, set $S_2(1) = 1$, $S_2(2) = a+b-1$, and $S_2(k) = S_1(k)$ for $k > 2$. Then, in both cases, $F_{R_1}(p) \geq F_{R_2}(p)$ for all $p$ and $F_{R_1}(p) > F_{R_2}(p)$ for at least one $p$.*

*Proof.* Assume first that $a + b \leq 1$. Then for some polynomial $T$ of degree $n-2$ with nonnegative coefficients we have

$$\begin{aligned} Q_{R_1} &= (x+1-a)(x+1-b)T(x) \\ &= \big(x^2 + (2-a-b)x + (1-a-b+ab)\big)T(x) \end{aligned}$$

and

$$\begin{aligned} Q_{R_2} &= (x+1-a-b)(x+1)T(x) \\ &= \big(x^2 + (2-a-b)x + (1-a-b)\big)T(x). \end{aligned}$$

Thus, $F_{R_1}(p) \geq F_{R_2}(p)$ for all $p$, and $F_{R_1}(p) > F_{R_2}(p)$ for at least one $p$.

Assume now that $a+b>1$. Then the formula for $Q_{R_1}$ stays the same, and we have

$$Q_{R_2} = x(x+2-a-b)T(x) = \big(x^2+(2-a-b)x\big)T(x).$$

We have $0<1-a-b+ab$, so the result is the same as in the first case. $\square$

The way we can restate this lemma is that if an agent has at least two strengths other than 0 and 1, then we can change her strength function, keeping the same skillfulness, in such a way that none of the probabilities of failure $F(p)$ increases, and at least one of them strictly decreases. This change of the strength function is in the direction of specialization.

Observe that given a skillfulness $\xi$, there is a unique (up to permutations) strength function with at most one value in $(0,1)$ giving this skillfulness. Let us denote this function by $S_\xi$. This is the strength function of the most specialized possible agent of skillfulness $\xi$.

**Theorem 4.** *Given a strength function $S$ with skillfulness $\xi$ and probabilities of failure $F(p)$, and the function $S_\xi$ with probabilities of failure $F_\xi(p)$, we have $F_\xi(p) \le F(p)$ for every $p$, and unless $S$ is a permutation of $S_\xi$, there is at least one $p$ for which $F_\xi(p) < F(p)$.*

*Proof.* Use Lemma 3 inductively. $\square$

We can interpret this result as saying that for an individual problem solver in our model, specialization is better than versatility.

In Figures 2 and 3 we see pairs of graphs of the probability of success (extended piecewise linearly to functions on $[0,n]$) that have intersections not only at 0 (more such graphs can be seen in Figures 6 and 8). For each such pair there is only one intersection apart from 0. Thus, a question arises whether this is a general phenomenon. We now show that this is very far from being true.

**Theorem 5.** *Let $R : N \to (0,1)$ be an injection, and let $\tau$ be a function from $N$ to $\{-1,+1\}$. Then there exists a function $\widetilde{R} : N \to (0,1)$ such that $F_R(p) > F_{\widetilde{R}}(p)$ if $\tau(p) = -1$ and $F_R(p) < F_{\widetilde{R}}(p)$ if $\tau(p)=+1$.*

*Proof.* We may assume that $R$ is a strictly increasing function. We have $F_R(p) = a_p / \binom{n}{p}$, where

$$x^n + \sum_{p=1}^{n} a_p x^{n-p} = Q_R(x) = \prod_{i=1}^{n}(x+R(i)).$$

If $\varepsilon>0$ is sufficiently small, then the zeros of the polynomial $x^n + \sum_{p=1}^{n}(a_p+\varepsilon\tau(p))x^{n-p}$ are all real and contained in the interval $(-1,0)$. Thus, there is a function $\widetilde{R} : N \to (0,1)$ such that this polynomial is equal to $Q_{\widetilde{R}}(x)$. We have $F_{\widetilde{R}}(p) = (a_p+\varepsilon\tau(p))/\binom{n}{p}$, so $F_R(p) > F_{\widetilde{R}}(p)$ if $\tau(p)=-1$ and $F_R(p) < F_{\widetilde{R}}(p)$ if $\tau(p)=+1$. $\square$

This theorem illustrates the difficulty of measuring the ability of agents (see also, for example, [HP2, KR]). Theorem 5 shows that given a typical agent and a specified set of problem difficulties, there can be another agent who is better at solving problems with those difficulties but worse at solving problems with all other difficulties.

One can ask whether we can remove the assumptions that $R$ maps $N$ to the open interval $(0,1)$ and that it is an injection. The answer is "no," as the following simple examples show.

Let $n=2$ and consider two weakness vectors, $R = (1/2,1)$ and $\widetilde{R} = (a,b)$. We have $Q_R(x) = x^2+(3/2)x+1/2$ and $Q_{\widetilde{R}}(x) = x^2+(a+b)x+ab$. If $a+b \ge 3/2$ then $1/2 \le a \le 1$, so

$$\begin{aligned} ab &\ge -a^2+\frac{3}{2}a \ge \min\left(-\left(\frac{1}{2}\right)^2+\frac{3}{2}\cdot\frac{1}{2}, -1^2+\frac{3}{2}\cdot 1\right) \\ &= \min\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

Thus, we cannot have $F_R(1) < F_{\widetilde{R}}(1)$ and $F_R(2) > F_{\widetilde{R}}(2)$.

Similarly, if $R=(1/2,1/2)$ and $\widetilde{R}=(a,b)$, then $Q_R(x) = x^2+x+1/4$ and $Q_{\widetilde{R}}(x) = x^2+(a+b)x+ab$. If $a+b\le 1$ then by the inequality between the geometric and arithmetic means, $\sqrt{ab} \le (a+b)/2 \le 1/2$, so $ab \le 1/4$. Thus, we cannot have $F_R(1) > F_{\widetilde{R}}(1)$ and $F_R(2) < F_{\widetilde{R}}(2)$.

## 5. Teams

Let us consider now what our model tells us about teams of agents. Suppose we have a team of two agents,[3] and for skill $i$, their weakness is $R_1(i)$ and $R_2(i)$, respectively. We assume independence (in the sense of probability theory), so the probability of not making progress on a problem using skill $i$ is the product $R_1(i)\cdot R_2(i)$. As for a single agent, we will speak of the strength and weakness vectors of a team.

We can take the diversity of a team to be the lack of overlap of their strengths. While this is not a formal definition, we can often say which of two teams has larger diversity. Similarly, we can speak of the ability of the team. Here we can use the skillfulness as the measure, although Theorem 5 suggests that it is not an ideal measure. However, again we can often say which of two teams (or members of the team) has larger ability.

Let us consider the simple example where there are two agents in the team, and each of them has two skills of strength 1/2. There are three possibilities: two, one, or none of the skills coincide. Then we get for the team three possible strength vectors: $(3/4,3/4,0,\dots,0)$, $(3/4,1/2,1/2,0,\dots,0)$, and $(1/2,1/2,1/2,1/2,0,\dots,0)$. Figure 5 illustrates the results. We see that with the same levels of abilities of the members of the team, more diversity in

[3] *The situation should be similar for larger teams.*

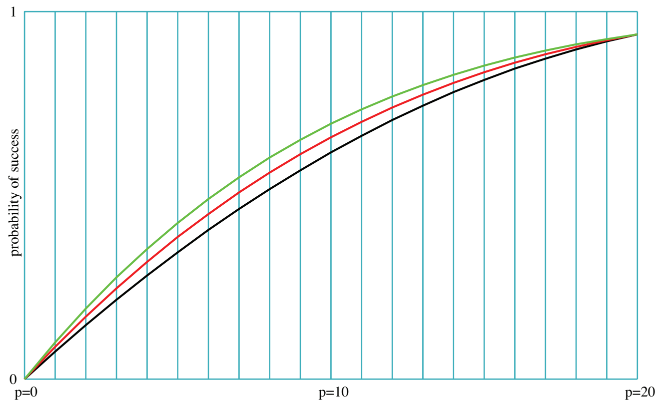


**Figure 5.** Team of two agents, each with two skills of strength 1/2. The black graph corresponds to a team whose skills coincide, the red graph to a team sharing one skill, and the green one to a team with no skills in common.

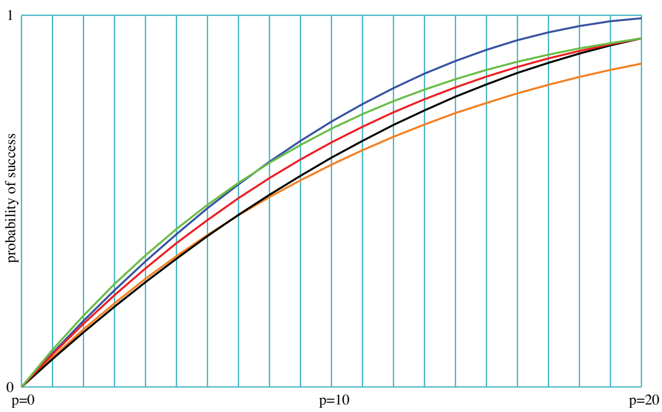


**Figure 6.** Like Figure 5, but with two additional graphs. The dark blue graph corresponds to the team strength vector $(0.91, 0.91, 0, \dots, 0)$ (that is, strength of skills $0.7$, but no diversity), the orange graph to $(0.4, 0.4, 0.4, 0.4, 0, \dots, 0)$ (that is, strength of skills $0.4$ and maximal diversity).

their skills gives better results, except for the easiest problems.

This phenomenon is easy to explain. Consider two vectors $R_1$ and $R_2$, the same except the values at some $i, j$. They come from a team of agents with prescribed skills, slightly differently placed. For $R_1$ we have more diversity, so $R_1(i) = a \in (0,1)$ comes from one agent, and $R_1(j) = b \in (0,1)$ from another agent. For $R_2$, diversity is smaller, so $R_2(i) = ab$ and $R_2(j) = 1$. Then there is a polynomial T, with nonnegative coefficients, such that $Q_{R_1}(x) = T(x)(x+a)(x+b)$ and $Q_{R_2}(x) = T(x)(x+ab)(x+1)$. We have $(x+a)(x+b) = x^2+(a+b)x+ab$ and $(x+ab)(x+1) = x^2+(ab+1)x+ab$ and $(ab+1)-(a+b) = (1-a)(1-b) > 0$, the coefficients of the polynomial $Q_{R_2}$ are strictly larger than the coefficients of the polynomial $Q_{R_1}$ (except the first and last coefficients, for which we have the equality). This

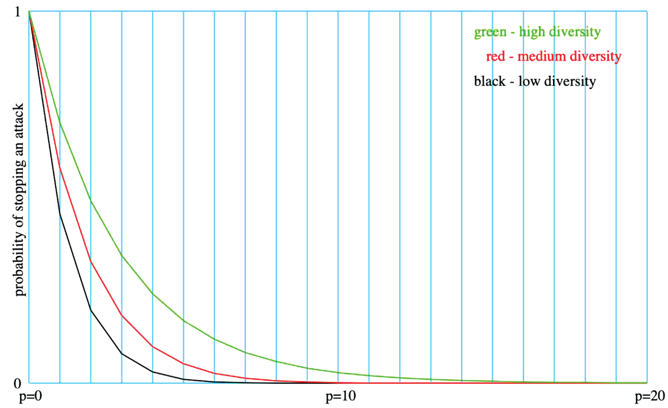


**Figure 7.** Team of two agents for the security problem. The black graph corresponds to the strength vector consisting of ten strengths $0.91$ and ten $0$; the red graph to the vector consisting of five strengths $0.91$, ten $0.7$, and five $0$; and the green one to the vector consisting of twenty strengths $0.7$.

means that if ability is kept constant, by increasing diversity of skills we get better chances for success.

This result differs from what we saw about specialization and versatility. This is because the skillfulness of a team is usually smaller than the sum of each member's skillfulness.

We can ask what happens if we change the abilities of the members of the team. In Figure 6 we added two graphs. One of them corresponds to larger abilities but no diversity; the other one corresponds to smaller abilities but larger diversity. By comparing the two lowest graphs with each other, and two highest graphs with each other, we see that to some degree ability and diversity of skills are exchangeable. However, in this example, for easier problems ability is more important, while for more difficult ones diversity is more important. Of course, we do not know how this applies to real life situations, since our model may or may not fit them (cf. Theorem 5).

However, if the ability of one or more team members increases (and no other changes are made), the coefficients of the polynomial $Q$ for the team decrease, so we get better chances for success for problems of all difficulties.

## 6. Security

Let us look at a possible adaptation of our model to a security problem. Here the agents are trying to defend an organization against an attack (for instance, by hackers).

The attacker has $p$ possible lines of attack (out of $n$ possible), and for each of them the skillfulness of the agent gives us the probability of stopping this line of attack. Thus, the average probability of stopping an attack of strength $p$ is $F_S(p)$. Note that for problem solving large $p$ meant an easy problem; here large $p$ means a strong attack.

In earlier sections we wanted to minimize our probability of failure $F_R(p)$. In contrast, we want now to *maximize*

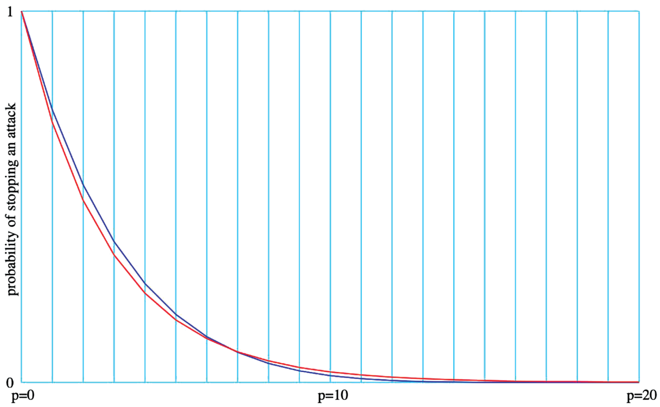


**Figure 8.** Team of two agents for the security problem. The red graph corresponds to the vector consisting of twenty strengths 0.7; and the blue one to the vector consisting of four strengths 0.97, twelve strengths 0.9, and four strengths 0.

the attacker's chance of failure $F_S(p)$. Therefore, by the results for problem solving, for security purposes versatility is better than specialization.

This also means that in this application diversity of skills in a team plays an even larger role than for problem solving. Diversity corresponds to more uniform spread of strengths, which for the security problem is useful even for one agent. An example similar to the one from Figure 5 is illustrated in Figure 7. We consider a team consisting of two agents, each of them having 10 strengths 0.7. Then we compare three possibilities: all, half, or none of the strengths coincide. We get for the team three strength vectors. The first one consists of ten strengths 0.91 and ten 0; the second one of five strengths 0.91, ten 0.7, and five 0; and the third one of twenty strengths 0.7.

Here also diversity and ability are to some degree interchangeable. For example, if we have two agents, one with sixteen strengths 0.7, and the other one with four strengths 0.7, then in the most diverse case we get for the team twenty strengths 0.7. If the first agent's strengths are 0.9 instead of 0.7 and we consider the least diverse case, we get for the team four strengths 0.97, twelve strengths 0.9, and four strengths 0. The first team will be better for strong attacks, but the second one will be better for weak attacks (see Figure 8).

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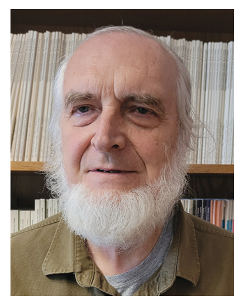
William Geller



Michał Misiurewicz

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**The Early Career Section** offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section with assistance from Early Career Intern Katie Storey. In our next issue, we will feature articles in celebration of Black History Month. All Early Career articles organized by topic are available at `https://www.angelagibney.org/the-ec-by-topic`.


# More Good Ideas

## Don't Give a Terrible Talk

*Elena Giorgi*

We have all been to terrible talks. To avoid being the one who delivers a terrible talk, we should first know what it is that makes a talk terrible, and then do the opposite.

### A terrible talk...

#### ...gets too technical

The most common mistake that a speaker makes is believing that the audience knows as much as he/she does on the subject. You should always remember that the background of the audience can vary greatly: there will probably be people who are very close to your work (and may have invited you to give the talk for example), but there could also be those working in adjacent fields who want to attend the talk because they may be interested in some aspects of your work. A good way to avoid misjudging the background of the listeners is to ask the organizers beforehand about the audience members: how heterogeneous is the audience? are there people from other fields? how many students or postdocs? Having this information will help you prepare the material accordingly.

Another common mistake is to assume that the audience is listening to your talk with full attention for its entire duration, for example having clear in mind at the end of the talk something you said at the beginning. Don't be afraid to repeat concepts, especially the ones you introduced in a different part of the talk which may be useful in a subsequent moment. Even the audience members who listened carefully will benefit from your recap.

In order to make your talk more understandable, whenever possible you should include images, as well as graphs, tables, arrows, and any visual aid you can think of. Explore these tools and use them to make your talk more enjoyable.

#### ...is not accessible to nonexperts

Those who are in the audience but are not expert in your field of research may not be able to appreciate the crucial technicalities that make the core of your work and that you spent time developing. Even though you really want to share the technical aspects of your work, try to elevate the argument as much as possible and start by presenting a simplified version of the main ideas. If anyone is interested in more technical details, you will be able to present them in a private discussion after the talk, which is also a great way to start a conversation with a fellow mathematician.

Always remember: your goal is to make people understand, not to impress them. This is true even in a job talk: be understandable and the audience will be impressed as a consequence.

#### ...does not have a clear narrative

A clear narrative is crucial in every aspect of human learning, and that is true in mathematics as well. A talk is not just a list of new ideas and proofs: it is a story, and as all narrated stories it should be engaging to the listeners. Keep in mind that the narrative may not be the linear history of your attempts at solving a problem. The story of your talk

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could be something you realize about your research project only months or years later, where two distant concepts are connected by an idea, or a method can be used in different settings uncovering something new.

Spend some time identifying the story of your talk: the main problem, its genesis, the relevant attempts, its resolution, and what is missing from its current status. Make the narrative clear during your talk: when giving a definition or presenting a lemma, frame it as part of the story by alluding to its role in it. Presenting something you tried but did not work is also a good example of constructing a narrative, and it is very instructive for your audience who may be wondering about similar directions.

A narrative helps the audience follow your talk and pay attention. Humor is also a great way to keep people engaged. Always remember: to deliver a great talk, you are as important as the words you will be saying. Use your body and tone of voice to be emphatic and passionate; be a pleasure to listen to.

### ...overlooks the motivations and the conclusions of your work

Why are you doing what you are doing? It may seem obvious to you, so obvious that you forget to mention it, but it is one of the most important parts of your talk. Start your talk with a presentation of the motivation for your work, not only your personal motivations but also those that could be inspiring for other people, both internal and external to your field of research.

Give some history of your problem by positioning it in the larger context of your field. Include a long introduction where you can touch on the take-away message of your talk. End your talk with clear conclusions: a brief summary of your talk is particularly helpful for those who may have gotten lost at a certain point.

### ...does not respect the audience

Nobody wants to feel stupid, so don't make them. Saying things like "it is trivial" or "it is obvious" is disrespectful towards your audience, who may not consider as trivial or obvious any of the things you mentioned, as they have not been thinking about them as much as you have.

It is important to show real openness to questions: ask often if anybody has any questions, and ask it repeatedly, with particular attention to the younger audience who may be frightened to ask. Another way to respect the audience (in the room or outside) is to always refer to the relevant work of other people.

### ...goes over time

If you don't want to give a terrible talk, don't go over time. There are various reasons why your talk can end up running over time, but it is important to find a way to avoid it, no matter the reason.

The most common reason is that the talk was not adequately prepared: you thought you were able to say more than actually possible in the given amount of time. There is an easy fix to this: plan in advance with realistic expectations, and if uncertain, it is better to make it shorter to avoid looking stressed and anxious when trying to finish on time.

It is also possible that you are not able to finish the talk on time due to no fault of your own, for example because you received lots of questions from the audience. It is normally a good sign if the audience is engaged with your talk and asks many questions. If this happens, when the time is almost up and you realize you didn't have time to get to the points you wanted, summarize the most important take-home message and try to connect them to some of the previous questions. Be at peace with yourself: answering the questions of your audience is far more effective at explaining your work than finishing at all costs your preplanned talk.

Elena Giorgi

### Credits

Photo of Elena Giorgi is courtesy of Columbia University: April Renae.

# Making Accessible Documents Using LaTeX

*Eric Larson and Isabel Vogt*

In order to disseminate mathematics as widely as possible, it is desirable to produce documents which are accessible to people with visual impairments. Indeed, there is a long history of successful blind mathematicians (including Euler late in life [J]), yet visual impairements can pose an obstacle to the written communication of mathematics. In the modern era, screen readers are a major way that blind individuals interact with written (electronic) documents. Unfortunately, without special effort, screens readers will typically garble equations, often beyond comprehensibility.

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Many guides exist for producing accessible documents in software such as Word—this is, in fact, a standard topic at many university-wide teacher-training workshops—but similar guides for mathematical content and LaTeX are difficult to find.

Since mathematics is often distributed as PDFs, we intended to write an article about producing accessible PDF documents in LaTeX. But this turns out to be incredibly difficult! The basic issue is that PDF documents produced in LaTeX do not, by default, contain extra information that *could* be used by a screen reader to make equations accessible (i.e., the LaTeX code, the pronounciation, or anything else besides the visual appearance). And since very few PDF documents *not* produced by LaTeX contain any equations, very little work on the screen reader side has been done for equations in PDF documents. Thus, even if you produce a PDF with this extra information, it is unlikely to be properly read by a screen reader. (And even if you manage to make it work in one situation, it might fail with a different operating system, different PDF viewer, or a different screen reader).

By contrast, it is relatively straightforward to make accessible HTML pages using LaTeX! The reason for this is two-fold:

1. A standard method for including equations on websites is mathjax, which works by embedding LaTeX code in the HTML file, so screen readers have been developed that take advantage of this.
2. There are web browsers that can be installed on any operating system and have their own functional screen readers specifically for HTML documents, for example the ChromeVox extension for Chrome. This eliminates the uncertainty of how an unknown general-purpose screen reader will react to your document.

We therefore begin, in Section 1, by giving a brief guide to producing accessible HTML pages from LaTeX source. We then discuss, in Section 2, some incremental steps you can take to make your PDFs more accessible for visually impaired students, like setting PDF metadata and choosing colorblind friendly color palettes. Finally, in Section 3, we discuss some of the more serious obstacles to creating truly accessible PDFs.

Note to the reader: We are not experts on this topic—just sympathetic users who are beginning to grapple with these issues ourselves. In addition, this article describes some steps that can be taken as of 2022... we hope that development on accessibility software continues at a rapid pace, making this article out of date in the near future!

## 1. Accessible HTML Documents

Currently, the most functional tool for creating HTML documents from LaTeX source is `tex4ht`, which can be easily used via the `make4ht` build system. On a UNIX-based system, to compile the document `mydocument.tex`, using a configuration file `myconfigfile.cfg`, and using mathjax to display equations, one simply executes the command:

```
--$ make4ht -c myconfigfile.cfg mydocument.tex mathjax
```

**1.1. Communication inside and outside of math mode.** The major issue with this approach—which also affects the visual appearance of the document—is that the processing of math mode is done by `mathjax`, while `tex4ht` itself processes the remainder of the document. Therefore communication failures between `tex4ht` and `mathjax` can arise. For most documents, this manifests in two ways:

- A macro defined in your preamble will be processed by `tex4ht`, and not be available inside of math mode, so the corresponding symbols will be replaced by error messages. (There are no issues with user-defined macros outside of math mode.)
- A `\label` on an equation in math mode will be processed by `mathjax`, and not be available for use by a corresponding `\eqref` outside of math mode, so the reference will render as (??).

A simple way to fix these issues is to first place your macros that will be used in math mode in a separate document `mymacros.tex`. So that your document will compile normally with TeX or pdfTeX, create a file called `mymacros.sty` containing the following code:

```
\ProvidesPackage{mymacros}
\input{mymacros.tex}
\endinput
```

and add the following line to your preamble:

```
\usepackage{mymacros}
```

Then, to make `mathjax` aware of your macros, place the following code in your configuration file (`myconfigfile.cfg` referenced above):

```
\Preamble{xhtml,mathjax}
\Configure{@BODY}{\IgnorePar
\HCode{\detokenize{\(}}
\special{t4ht*<mymacros.tex}
\HCode{\detokenize{\)}}
\par}
\begin{document}
\renewcommand\eqref[1]{\NoFonts\HChar{92}eqref\{\detokenize{#1}\}\EndNoFonts}
\EndPreamble
```

This code (on the second-to-last line) also redefines the `\eqref` command so that it will properly link up with a `\label` defined inside of math mode.

**1.2. Alternative text for figures in HTML.** When using a figure to illustrate a mathematical argument, it is essential that a complete proof is written in words. This is not just for visually impaired mathematicians—in private communication with the second author about a previous *Notices* article on the topic of mathematical writing, Jean-Pierre Serre jokingly said that he is sometimes frustrated that "a picture needs a thousand words" to be understood!

If you have any figures, you can also easily provide alternative text for screen readers to read in place of the image. In the most recent version of TeXLive, this can be done as follows:

```
\includegraphics[alt={my alternative text}]
{myimage}
```

## 2. Incremental Steps to Make PDF Documents More Accessible

**2.1. Specifying document language and title in metadata.** One simple step you can take is to specify the language of the document in the metadata (so a screen reader will be able to infer the pronounciation of words). It is also helpful to specify the title of the document, so that the screen reader can communicate easily to the user which document is being read. These two, as well as other metadata, can easily be set with the `hyperref` package as follows:

```
\usepackage{hyperref}
\hypersetup{
pdflang={en-US},
pdftitle={Making Accessible Documents Using
LaTeX},
pdfauthor={Eric Larson and Isabel Vogt},
pdfsubject={Mathematics},
pdfkeywords={Accessibility, LaTeX}
}
```

**2.2. Colorblind friendly color palettes.** If you choose to include color in your figure, you should try to make sure that color is not the *only* distinguishing attribute and be sure to select a palette that will appear distinct to colorblind mathematicians.

There are several excellent online guides aimed at scientists and mathematicians that contain sample palettes and tools to simulate what your own choice of palette would look like to a colorblind person [OI, N]. These guides are written in terms of RGB values of colors. To define a color `mycolor` in LaTeX from its RGB value, include the following in your preamble:

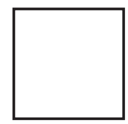
```
\usepackage{xcolor}
\definecolor{mycolor}{RGB}{myrvalue,
mygvalue,mybvalue}
```

For the reader who is already familiar with modern RGB color theory (red–green–blue primaries), we remark that a simple rule of thumb is that complementary colors in RGB color theory are usually easy to distinguish; the same is not true for complementary colors in traditional color theory (red–yellow–blue primaries). Moreover, different levels of the blue component are most important (since the peak spectral sensitivities of the red and green cones are much closer). For example, the complementary color to green in modern color theory is *magenta*, which appears contrasting to nearly everyone; by contrast, green and *red* can be difficult for many colorblind individuals to distiguish.

**2.3. Alternative text for figures in PDFs.** You can specify alternative text to be read in place of a figure in a PDF using the `pdfcomment` package. In the preamble, you add:

```
\usepackage{pdfcomment}
```

You can then add alternative text to a figure using the `pdftooltip` command. For example, to draw this figure:



You would use the LaTeX code:

```
\pdftooltip{
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0, 1) --
(0, 0);
\end{tikzpicture}
}{Here is a square!}
```

## 3. Hard Obstacles to Accessible PDF Documents

**3.1. Navigational data in PDFs.** PDF documents can contain *tags*, a type of metadata that screen readers can use to infer the structure of the document. Several packages are under development to generate tagged PDFs using LaTeX; the most mature such package is `tagpdf` [F], but even that package is still experimental.

**3.2. Where does what data about equations go in the PDF file?** In order for a screen reader to verbalize equations, the PDF must contain some sort of structured data describing the equations or their verbalizations, perhaps via the `/Alt` or `/ActualText` fields. Unfortunately, there is no standard method of doing this; as a consequence, even if this data *is* embedded in a PDF, a screen reader is unlikely to read it properly.

So far, the only package under development to do this is `axessibility` [CCABKMAB], which embeds the LaTeX code for the equations in the `/ActualText` field. For two different setups (operating system + PDF viewer + screen reader), the authors were able to get the screen reader to read the equations... but only after hand-coding two different math dictionaries. In fact, the screen readers in question already had such dictionaries, but didn't recognize that the embedded LaTeX code was LaTeX code, because it is not (yet) standard to do this.

This illustrates the benefit of including such data, in a way as standardized as possible, *even if* screen readers will not be able to use it: Until including this data becomes standard, nobody will write screen readers that take full advantage of it.

At the moment, of course, it is not clear what the best way of including such data is. We urge continued experimentation from TeX developers on this important project.

ACKNOWLEDGMENTS. We would like to thank Juna Gjata for many helpful discussions and for patiently testing various documents produced with LaTeX on screen readers with us.

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Eric Larson


Isabel Vogt

## Credits

Photos of Eric Larson and Isabel Vogt are courtesy of Lori Nascimento, Brown University.

# How to Referee a (Math) Paper

## *Álvaro Lozano-Robledo*

You can run but you can't hide. Eventually, an editor will find you and send you a referee request.

*Now what?*

If you do not have any previous experience on what to do next, this article is for you. I would also recommend Arend Bayer's "Writing, and reading, referee reports," and Brian Katz's "What makes a good PRIMUS review."

**Why do we referee papers?** Refereeing papers is a service that mathematicians provide to the community. Math papers can be long and complicated, and the refereeing process gives you the opportunity to have other research mathematicians proofread your paper carefully for correctness and for suggestions, before it is published. It is a hard job, it can take many, many hours, and it is unpaid. But we publish papers, and others referee our papers so we return the favor by refereeing other mathematicians' papers.

This article is not about the academic publication system, which deserves an entire different piece. Here I will limit myself to the task of refereeing a paper, and we will leave the editorial commentary on journals, predatory journals, "publish or perish," the tenure system and the need to publish, etc., for another piece.

**When do mathematicians start refereeing?** When you receive a request to referee a paper, there are several important factors to consider. But before we go into such factors, let us first address the question "when should you start taking on referee jobs?" Or, more generally, "who should be a referee?"

The most important qualification in order to be a referee is that you need to be an "expert" in the topic of the paper, which usually means that (i) you have enough background to follow and digest the arguments and techniques used in the paper under review, and (ii) you are familiar with the literature on the subject, enough to know how this result fits into the published record. If you are invited to referee, then the editor believes you are suffciently qualified to write a review of the paper, so now it is up to you to decide if you are a good fit for the job.

In light of all this, typically, mathematicians start refereeing after (a) they graduate with a PhD, and (b) they have published at least one paper. And the first paper you are asked to referee is probably related to your thesis, or to the topics of your first papers.

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Note that grad students are sometimes asked to referee papers... I do not think this is, in general, a good idea or fair to the student or the author. Perhaps a better idea would be for the PhD advisor and grad student to collaborate on a review, which would provide a training opportunity in refereeing papers, but I am not sure this is a great idea either, since someone else's paper and career is on the line.

**Should I accept the referee job?** When you receive a referee request, you will be able to see a copy of the paper so you can decide if you can accept the refereeing job at this time. Go ahead, have a look, and then consider the following factors:

- *You are under no obligation whatsoever to referee papers.* As I mentioned above, this is an unpaid job, so you can always politely decline an invitation to referee. However, if you are a research mathematician who publishes papers, then you should consider reviewing papers as part of your service to keep the community going.
- *Is this a journal you know about and mathematicians should be refereeing for?* Please be aware that there are many publications out there that are "fake" or dishonest, so only accept referee jobs from reputable journals.
- *Is there a conflict of interest that disqualifies you for this job?* If you cannot be an impartial referee then you should not accept the job. Simply let the editor know, and bow out. Here is a list of common conflicts: the paper is by your advisor, one of your students, a close collaborator, a family member, a close friend or a person you have a personal conflict with; the paper's results are very much like a paper you are writing yourself; etc. If you think there might be a conflict of interest, there is probably a conflict. You can always consult with the editor of the journal, and let them decide. Note that some fields are really small, so there are a lot of connections that may be unavoidable. In summary: if for whatever reason, you think you will not be an impartial referee, then please decline the job.
- *Do you have time for this job?* As I mentioned above, refereeing is a hard job and to do it well, it takes time (probably many hours). Ask the editor when the referee job is needed by, and if you cannot possibly have it ready by their deadline or soon after, then let them know you are too busy to take on this job at this time. Please keep in mind that some mathematicians' careers, particularly graduate students and postdocs, are in the balance here, and timely refereeing can make a huge difference in their next job search.
- *Are you refereeing too much?* If you accept too many jobs, then it might jeopardize your time for your own research. As a general rule, I referee about twice or three times as many papers as I submit to journals. Why? Many journals require two different referees, so I figure that two people kindly took the time to referee my paper, so I need to give back to the community two refereeing jobs for every paper I publish myself. If I have already accepted a refereeing job(s), and I am too busy with it, I will simply let the editors know that I am not available at this time to write a good and timely report.
- *Are you a good fit for the job?* Have a look at the paper and try to get a sense of the topic, and the techniques used in the proofs. If you are unfamiliar with them, then you may not be the "expert" they are looking for, and it may take you an enormous amount of time to familiarize yourself with the techniques and the literature on the subject, so you should be honest with the editor and simply say that this is far from your area of expertise, and you are not a good fit. Refereeing is not the time to learn a new area, when someone else's career is on the line. Of course, the paper is most likely brand new research, so you will learn a lot reading and reviewing the paper! But you shouldn't referee a paper which is too far afield.

**What kind of a referee job am I being asked to do?** Typically, editors will ask for one of two types of referee jobs: a quick opinion, or a full referee report. In a quick opinion, you are only asked to evaluate if the paper is a good fit for the journal, and the results are interesting enough for the refereeing process to continue ahead. Usually, this opinion is not even shared with the authors, so it is an internal editorial process, and the editors just want a quick note back from you (one paragraph or two) about the paper with your first impressions (see below for more comments about how to evaluate the fit of a paper).

**Should I reject a paper right away?** Assuming you have answered yes to the questions in the bullet points above, then it is time to get started: accept the job, download the paper, and start lightly browsing its contents.

The first decision you need to make is if the paper should be rejected right away because, in your opinion, it is not a good fit for the journal. This is a hard call to make, so you can ask the editor for more information on what kind of papers they are looking to publish. Another good idea is to go through the journal's archives, and look for other papers in the same area that they have recently published. Is the paper under review, in principle, at about the same level or above that of recent papers that have appeared in the same journal? If so, then go ahead with the job. If the paper is clearly not a good fit, if the result is known, if the combination of results and techniques are not strong enough for the journal, if the paper needs a huge amount of work,... then reply to the editors with a rejection. The sooner the better, and if you can, please offer a quick explanation of why the paper is not a good fit, and suggestions of better journal fits.

Please do not (ever!) be mean when you reject a paper, or if you write a quick opinion. Harsh words are completely unnecessary. Just be professional, and imagine you are the one at the receiving end of the rejection letter. Be honest and direct, but always try to offer some constructive suggestions.

If you are in the middle of refereeing and you find a big problem with a proof, then stop right away, and consider for a while if it's a mistake that cannot be salvaged. If so, you may need to reject the paper on those grounds. Or at least ask the authors for clarification.

By the way, never contact the authors directly. All communication should go through the editor and the online editorial system. The anonymous nature of refereeing ensures that referees can be impartial and honest.

**How much time should I spend refereeing this thing?** Refereeing can take many hours, and if the paper is long, it can be months of work. Make sure the editors have given you a deadline that is reasonable so that you don't have to put everything else aside to review the paper. Let the editors know what is a manageable deadline to have a report ready.

That said, once you start refereeing, if the job is taking longer that you imagined, then there might be other factors to consider. If it is taking too long because something came up, you might want to let the editors know so they can reassign the job if needed. If it is taking way too long because the paper is just not well written, or the arguments are confusing, or you are spending too much time fixing small steps of their proofs... then consider rejecting the paper on those grounds.

Note that a rejection is not necessarily a death sentence for the paper. Most journals offer sending back the paper to the authors for "light revisions" or "major revisions." If you don't want to quite reject the paper, but you think that it needs a great amount of work before it is ready for you to review it again, you can send it back with an initial set of general comments indicating what the authors would need to do for you to reconsider it. For example, you can ask the authors to restructure the paper, to add more detail in the proofs, to add more results in a certain direction that seems to be conspicuously missing from the paper, etc.

**What am I actually looking for while refereeing?** You are now in the thick of it, reading the paper, it looks like a good fit, and the paper seems worth looking at in detail. What now? What are an editor and an author actually looking for?

- *The amount of detail and time you put into a report is a personal choice.* The bare minimum amount of work a referee needs to do is to check that all the arguments are mathematically correct. In other words, make sure the proofs are correct, and the theorems are stated correctly. However, most of us go an extra mile, and give feedback to improve the paper in several additional ways.
- *Should I worry about grammar and sentence structure?* This is optional, because it can be a very time-consuming job to go into this level of detail. I do care about this, and I can't let it go, so I will go into all sorts of grammar comments, but that's just me. The key is that I want the paper to be readable, and easily understandable by others, so if bad sentence structure is getting in the way of the math, then I will definitely comment on it and suggest alternative sentences that would make a clearer, easier to digest argument.
- *Should I check every piece of math line by line?* This is tricky. You need to check that the arguments are mathematically sound, so you need to go into enough detail to ascertain as much. If you are not checking certain arguments in the paper (e.g., because they are standard, or not the main point of the paper) then let the editor know, or simply write it in the referee report.
- *Should I provide suggestions?* Yes!! Absolutely. The reason you are doing this job is because you are an expert in the field. You are the target audience! So any suggestions you may have, are very much welcome, and that's the kind of referee report that enriches the refereeing experience and improves papers. You can offer references, alternative proofs, short cuts, examples, or any other kind of suggestion that you think would improve the quality of the paper (particularly if it improves its readability). However, you cannot expect that the authors will overhaul the paper with your suggestions... after all, it is their paper and you are not a coauthor.
- *Should I be tough?* No. Do not, in any way, write comments that can be construed as offensive. You should be an impartial, professional, honest, and direct referee. So if things are missing, or if there are glaring mistakes, simply point them out in a plain way, and let the authors deal with the mistakes. If your comment is going to read like "the authors should know that..." then remove that comment and think of a way to point the problem out in a neutral way.
- *I am in a pissy mood. Should I referee at this time?* No. It will not go well. You will be annoyed by every single little thing, and you might end up rejecting the paper for some minor thing. Step away, relax, watch a movie, go for a walk, sleep on it, and when you are back in a constructive mood, go back to the paper and keep going.
- *Should I be really nice?* You do not have to go out of your way to be complimentary to the authors, but (negative) referee reports can be hard pills to swallow, particularly for early stage mathematicians. So I try to sound encouraging about the good parts, and offer constructive criticism and ideas whenever possible. The key is to strike a balance so that your report is useful.
- *I found a mistake. Should I reject the paper?* Not yet. How big of a mistake is it? Is it a simple error that can be fixed? Offer a solution (though you are not obligated to do so). Is it a complicated issue that you cannot fix yourself in a reasonable amount of time? Write it down in the report, and let them deal with it (this may be a minor or major revision, depending on the size of the gap in the proof). Is it a catastrophic error? Then, yes, contact the editor, let them know there is a serious issue with the paper, and reject it.
- *Should I evaluate the overall quality of the paper?* Yes. This is a very hard thing to do, but yes, absolutely, the editor

will want to know your overall impression after you have looked at the entire paper. First, I gain an impression of the paper, enough to decide whether the paper is a good fit for the journal and I am going ahead with the process. And then I wait until I have read the paper in detail to decide on an overall opinion of the paper.

- *How do I actually referee?* That's your personal choice, but I print a hardcopy of the paper, and write all my comments on the paper itself and in the margins, so that when I am ready to write, I go comment by comment and expand on it in the report.

**The referee report.** It is time to write all your comments and feedback on the actual report. Your name, affiliation, and email address should not appear anywhere in the report. Make sure the report is anonymous and that you are not writing your comments in a way that will easily identify yourself. Consider adding the following components to your report:

- Title and authors of the paper under review.
- Journal where the paper is submitted (this is mostly for your records, because sometimes you get to referee the same paper twice for different journals!).
- Overview: a summary of the results of the paper, so the editor and authors know that you have actually read the paper. It is also a place to state the main results in your opinion, which may differ from the results that the authors think are the main results! This section is a neutral zone, however, so you are just stating results without colorful commentary.
- Recommendation: a narrative of the strengths and weaknesses of the paper, in your expert opinion, which concludes with a recommendation for the editors: reject, accept, needs minor revision, major revision, etc. You can include big items that the authors need to address before the paper is accepted, and general comments about the paper.
- Detailed comments: this is an itemized list of comments. Please include pages and lines and theorem numbers that you are referring to, so that the authors know exactly what you are talking about.
- Conclusion: any other general comments that may improve the paper, or thoughts about the paper itself. Once you are done, send the anonymous referee report to the editors, in their preferred contact method, probably through their online editorial system.

**What happens then?** After the report is sent back to the editors, the editorial team may be waiting for other referees to also send in their reports. Once they have all the reports on the paper, they will make a decision. If they ask the authors for a revision, they might ask you to look at the paper one more time. I usually agree to look at the revised version because it is efficient, since I am already familiar with the paper, but again, it is your call if you are available or busy at the time.

Finally, thanks for taking the time to do a great job refereeing papers! Authors definitely appreciate the hard work of a referee.

ACKNOWLEDGMENTS. I would like to thank Keith Conrad and an anonymous referee for their helpful comments and suggestions.

Álvaro Lozano-Robledo

### Credits

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# Be Inspirable

## *Allison Henrich*

What is it that you'd like to do in your career? Would you like to become a better teacher? Become a powerful advocate for others? Contribute something significant to your research field? Perhaps you have your sights set on someday obtaining a large grant that could be transformative for your career, your institution, or the math community as a whole. Or maybe—and this may be the most exciting scenario—you don't know what your dreams are. You have a set of skills, interests, and values, and you don't know how you can combine them to achieve something great. Whatever you want to accomplish, I am a big believer that you can be successful at achieving your goals—both those you are concretely aware of and those that are a collection of ephemeral ideas—if you do one thing: Be inspirable.

What in the world does it mean to be "inspirable"? (That's not even a real word!) It means to be open to inspiration. Being inspirable means putting yourself into situations where you will meet new people or discuss new ideas with old friends. It means learning new things, brainstorming, considering what is possible. And, crucially, to be inspirable means that you are open to getting *really* excited about good ideas, so much so that you feel compelled to act on that excitement.

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Inspiration can come from a variety of situations. Some of these situations are ones you'd expect to be inspiring: attending a conference or a professional development workshop. Others are situations where inspiration might be more rare, but possible: a mandatory faculty meeting, a FaceTime date with a friend, or even watching an incredibly confusing talk or lecture.

In the math community, one common source for inspiration for new faculty members is Project NExT. This intensive professional development program exposes its fellows to new ideas in teaching, conversations about equity in mathematics, the nuts-and-bolts of grant writing, and so much more. It is probably far more rare for a fellow to go through the NExT program and *not* be inspired to take some positive action than otherwise. As a Project NExT Red08 dot, I took the opportunity to see Joe Gallian give his workshop "Getting Your Research Off to a Good Start" three times over the course of several years. During this workshop, Joe would point out that his more expository articles had far greater readership and vastly more citations than his traditional research papers. This inspired me to think about writing for MAA journals about topics that a broader spectrum of people would find interesting. Since then, I have coauthored papers that have appeared in the *American Mathematical Monthly, the College Math Journal, Mathematics Magazine,* and *Math Horizons*. It may not have occurred to me to try and write in such venues if Joe hadn't planted that seed in my mind early on.

For those who haven't had the opportunity to participate in Project NExT, Section NExT can be a wonderful source of new ideas. Kate Kearney from Gonzaga University, for one, feels inspired after attending Section NExT meetings. She writes, "Possibly one of the most useful things about Section NExT has been the opportunity to meet with, talk with, and learn from people across the section at many different kinds of schools and at many levels of the academic hierarchy. It's always interesting and informative to hear from people at a variety of different types of schools (liberal arts, state schools, community colleges, big schools, small schools, urban schools, very very rural schools). I can't think of anywhere else that I have as rich of a resource of contacts with diverse perspectives on many different teaching situations." One concrete change that Kate was encouraged by Section NExT to make was to try mastery-based grading in Calculus 2.

Of course, getting the most out of conferences has been more difficult in the COVID era, but it is certainly still possible. For instance, in our virtual Pacific Northwest Section NExT meeting in June 2021, we discussed a number of teaching techniques we discovered during the pandemic and shared with each other the ones we thought would be most valuable to keep moving forward. In this conversation, I learned about software I'd like to try out, assessment techniques I'll use, and I got ideas for course policies I'll implement that are more supportive of students.

AMS and MAA sectional meetings (and other local meetings of national organizations) can also be fertile ground for inspiration. When I was at an AMS sectional meeting in Charleston several years back, I heard Harrison Chapman give a talk related to knot theory that wasn't really in my specific research area. Sometimes, in talks that aren't squarely in my mathematical subdiscipline, I don't expect to derive any new ideas. This was such a fantastic session with such engaging speakers, however, that I was paying close attention to each talk. In Harrison's talk, he mentioned tangentially some way of viewing knots where the trefoil knot could be interpreted as being "the same as" the figure-eight knot. This made me wonder which other pairs of knots are "the same." Fast forward a few years, a paper I coauthored devoted to exploring this question, "Knots Related by Knotoids," was the lead article in the *Monthly* and went on to win the Halmos–Ford Award.

Getting involved in organizations you believe in—for instance, NAM, AWM, MAA, CUR, SACNAS, SIAM, PME, AMS, and Math Alliance—can lead you to meet people you might not have met otherwise and think about how to magnify your impact on the math community. Pamela Harris from the University of Wisconsin Milwaukee had the following to say about her involvement in the Society for Advancement of Chicanos/Hispanics and Native Americans in Science (SACNAS): "I have been inspired by the mentoring at the annual SACNAS conference. Mathematics faculty attend the conference and spend the entire weekend mentoring students and early career faculty. Their unending mentoring has inspired me to find moments to always be supportive of others, regardless of how busy I may be. This means that when I travel and meet people I make an effort to get to know them and their aspirations. This helps me share relevant opportunities with them when I encounter them, but it also helps me feel connected to the mathematical community."

Robin Wilson from Loyola Marymount University was similarly inspired by his involvement in the National Association of Mathematicians (NAM): "A lot of my interest in outreach and general interest in supporting undergraduate and graduate students has to do with me wanting to give back to the mentorship I received from the NAM members. I had people that I knew would look out for me, and I want to be able to do the same for others." Since he initially found a home in NAM as an undergraduate, Robin has contributed to his community in countless ways. Recently, he was a co-PI for a \$1 million grant from the NSF to increase the number of underrepresented minorities pursuing Ph.D.s in mathematics. The project, called BAMM! Bolstering the Advancement of Masters in Mathematics, is a joint effort between three CSU campuses, led by Oscar Vega at Fresno State—an incredibly inspirable person himself!

Speaking of grants, perhaps the most inspirable person I know, Michael Dorff, the most recent past president of the MAA, has earned many. For instance, Michael received grants to launch and support both the Center for Undergraduate Research in Mathematics (CURM) and the Preparation for Industrial Careers in Mathematical Sciences (PIC Math) program. Michael recounts, "About 15 years ago, a colleague of mine, Tyler Jarvis, and I were discussing an NSF program that focused on helping students make a transition from one critical phase of their career to the next phase. In writing a proposal to NSF, we knew that it would be better to think of projects we had familiarity and success with. In connection with this, the BYU mathematics department was having a lot of success with its undergraduate research program that paid students to work on research during the academic year. We also knew that math faculty at smaller universities and colleges did not have the experience or capacity to get funding from NSF to do small undergraduate research projects. This led to the creation of the CURM model which was further enhanced by reaching out to a large group of colleagues at other institutions through Project NExT to find out what they thought about the idea and what suggestions they had on how to improve it. Two significant ideas that came from that were to have a training workshop for the participating faculty and to make the program so that it was not something more to add to the already-full plate the faculty had (which is why we incorporated course buyouts into CURM)."

Inspiration can also come from unexpected places, if you are open to it. When I began my job at Seattle University, I was excited to become a part of such a vibrant math department, but unsure of how I could contribute to the Jesuit mission of the university. Initially, I thought, "If I just care for my students as people and do a good job teaching, this should count as my contribution." During the mission-oriented part of the mandatory faculty orientation, I expected the conversation to be more applicable to philosophy, religion, and law faculty—that is, until we started talking about social justice and service-learning. This conversation led me to realize that a math class could incorporate service-learning in a way that benefitted both the community and the students in the class. Since then, I have regularly taught a course I developed called Quantitative Literacy & Social Justice with a service-learning component that has had a measurably positive impact on my students' attitudes about math and has provided hundreds of math tutors (my students) to local schools over the years.

Motivation for change can also come from observing things going wrong and learning from those things. For instance, I am interested in hearing mathematics talks of all varieties. I learn not only from watching speakers who have spent years honing their public speaking skills and give well-crafted talks, but also from those who give talks that could be improved. I had a feeling that I wasn't the only one who has picked up some speaking tips by observing less-than-perfect academic talks. Friends and colleagues reported to me that they learned the following from watching "bad" talks: (1) never go over time, (2) don't put too much information on a slide, (3) practice your talks so that you know what you're supposed to say next and how you want to explain concepts, (4) watch out for using too many filler words, like "like," (5) try not to flip back and forth between your slides, and (6) don't copy/paste a proof from a research paper into your slides and go through every gory detail. For me, watching others' (both "good" and "bad") talks has helped me become a better speaker over the years. But more recently, I have had the opportunity to give some recorded virtual talks that have enabled me to watch and analyze my *own* performance. This has spurred me to make several improvements that will make me a better speaker.

I have dozens of other anecdotes I could share about how being inspirable has benefitted me in my career. Talking with others, attending events, reflecting on past practices, and being open has allowed me to be a research mentor in two REUs and codirect an REU on my own campus, it has led me to coauthor and coedit several papers, books, and blogs. It has transformed the way I teach. Can you think of a time you were inspirable? What impact did it have on you? How could you put yourself in situations and a frame of mind to have more great ideas and develop more connections with people?

Allison Henrich

### Credits

Photo of Allison Henrich is courtesy of Seattle University/ Yosef Chaim Kalinko.

# Jacques Tits (1930–2021)

*Richard M. Weiss*

## Introduction

**Figure 1.** Jacques Tits at age 19.

Jacques Tits was born in Uccle, a municipality of Brussels, on August 12, 1930, and died on December 5, 2021. The son of a mathematician, Tits displayed extraordinary mathematical ability at an early age. He received his doctorate at the University of Brussels in 1950 and spent the following year at the Institute for Advanced Study. In 1964, he moved from the University of Brussels to a professorship in Bonn, and then in 1973 to the Collège de France, where he remained for the rest of his career. For almost thirty years he held courses and seminars at the Collège de France and for nineteen years, Tits was editor-in-chief of the *Publications Mathématiques de l'IHES*.

Tits made many fundamental contributions to our understanding of the structure of semisimple algebraic groups and finite simple groups and did more than anyone to explore and reveal the geometric nature of these subjects.

When Tits was young, Chevalley had shown that semisimple algebraic groups over an algebraically closed field are classified up to isogeny by Dynkin diagrams. Tits made a careful study of various structural features of groups defined over an arbitrary field, much of which appeared in a famous collaboration with Borel. Tits then used this work to show that semisimple algebraic groups over an arbitrary field are uniquely determined by combinatorial data in the form of a *Tits index* (a Dynkin diagram endowed with certain decorations) and an *anisotropic kernel*, the two things knitted together by Galois descent as described in his lecture notes from a meeting that took place in Boulder in 1966. This deep result is in the spirit of the theorem of Wedderburn that says that a simple associative ring which is finite-dimensional over its center is isomorphic to $\mathrm{Mat}_n(D)$ for some division ring $D$, $n$ and $D$ being analogs of the Tits index and the anisotropic kernel. Another important analog is the theorem of Witt that says that a finite-dimensional quadratic form is uniquely determined by its anisotropic part and the dimension of its hyperbolic part.

Tits is best known for the theory of buildings. A building is a geometric structure defined by a few simple axioms involving a notion of dimension called the *rank*. The simplest example of a building of rank $n$ is the projective space associated with a vector space of dimension $n+1$. Further examples arise when the vector space carries a quadratic or Hermitian form. Together these are the buildings associated to the classical groups.

Buildings have distinguished substructures called *apartments*. A building is *spherical* if its apartments are finite. A building is *irreducible* if it is not a direct product. The classical buildings are all spherical and irreducible.

Tits introduced the notion of a BN-pair (also known as a Tits system) and used it together with the structural features revealed in his work with Borel to show that to every absolutely simple algebraic group $G$ of positive $k$-rank $n$ for a given field $k$, there is an irreducible spherical building of rank $n$ on which the group $G(k)$ acts. When $G$ is classical, then so is the building. These buildings are particularly fascinating objects in the case when $G$ is exceptional.

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**Figure 2.** At the induction of Jacques Tits into the Order Pour le Mérite, Bonn, 1996.

In a celebrated volume of Springer Lecture Notes published in 1974, Tits gave the classification of irreducible spherical buildings of rank at least 3. His classification shows that every such building is either classical or exceptional or belongs to a unique family defined over an imperfect field in characteristic 2.

Inspired by the work of Iwahori and Matsumoto, Tits went on to investigate *affine* buildings. Affine buildings are those in which the apartments have a natural representation as an affine space. Every affine building $X$ of rank $n+1$ has a boundary which carries the structure of a spherical building of rank $n$ (and is called the *building at infinity* of $X$). Tits observed that for every absolutely simple algebraic group of positive $k$-rank for a field $k$ that is complete with respect to a discrete valuation, there is an affine building on which the group $G(k)$ acts and that its boundary is precisely the spherical building associated with $G(k)$. His famous lectures on the structure of absolutely simple groups isotropic over a local field and their affine buildings were delivered at a meeting in Corvallis, Oregon in 1977. In two monumental volumes of the *Publications Mathématiques de l'IHES* (and in lecture notes from a conference on Lake Como in 1984), Tits and Bruhat completed the classification of irreducible affine buildings of rank $n+1$ for $n \geq 3$. Central to this classification is the notion of a *valuation of a root datum* of the building at infinity.

Tits and Borel had shown that a semisimple algebraic group of positive $k$-rank has a configuration of subgroups, which Tits called a *root datum*. Tits identified a corresponding property for spherical buildings he called the *Moufang condition* in honor of Ruth Moufang, a pioneer in the study of projective planes. Every irreducible spherical building of rank at least 3 satisfies this condition, and every spherical building satisfying this condition possesses, in a suitable sense, a root datum.

An irreducible spherical building of rank 2 is simply a connected bipartite graph in which every vertex has at least three neighbors and $g = 2m$, where $g$ is the girth, $m$ is the diameter of the graph, and the minimal circuits are the apartments. Tits called such graphs *generalized m-gons*.

The *residues* of a building are certain distinguished subbuildings. Every building is, in a suitable sense, an amalgam of its irreducible rank 2 residues; and if the building is spherical, then it is, in fact, uniquely determined by these subbuildings. The proof of this was a crucial step in Tits' classification result for spherical buildings.

There is now an enormous literature on the subject of generalized polygons, especially finite generalized polygons. Generalized polygons are, however, too numerous to classify (every projective plane can be viewed as a generalized triangle, for example), but Tits observed that the irreducible rank 2 residues of an irreducible spherical building of rank $n \geq 3$ all satisfy the Moufang property as do all the spherical buildings associated to an absolutely simple algebraic group of $k$-rank 2. In 2001, Tits and Weiss classified generalized polygons that satisfy the Moufang property. They are almost all the spherical buildings associated with an exceptional or classical group, but this time the list of exceptions is longer.

Affine buildings are CAT(0)-spaces uniquely determined by their boundary whenever the boundary satisfies the Moufang condition and the field is complete. It is this property that makes affine buildings a subject of great interest in geometric group theory. It also points toward a possible connection to physics through the holographic principle.

Jacques Tits and Mark Ronan introduced and developed the notion of a twin building. This notion was inspired by Tits' work on Kac–Moody groups which points to another possible connection with physics. Tits also extended the Moufang condition to buildings of rank 1 with the notion of a *Moufang set*. Moufang sets have proved to be an essential tool in the study of absolutely simple algebraic groups of $k$-rank 1.

Tits maintained a keen interest in the classification of finite simple groups as it unfolded. As a tool for identifying the finite groups of Lie type, spherical buildings played an essential role in the classification. Later a theory of "diagram geometries" based on older ideas of Tits' was introduced by Francis Buekenhout and others with the goal of including the sporadic groups in this geometric picture. This led, in turn, to Tits' "local approach" to buildings. Tits wrote papers on Griess's construction of the monster and moonshine and on several other sporadic groups as well and he proved the simplicity of ${}^2F_4(2)'$, now called the Tits group.

**Figure 3.** Jacques Tits wearing the medal of the French National Order of the Legion of Honor, Paris, 1995.

Tits introduced the notion of the Coxeter complex associated to a Coxeter group (and the term *Coxeter group* itself) including roots, projection maps, and other essential features of these complexes. He proved fundamental results about the structure of the automorphism group of a tree, the simplest of all affine buildings. Tits analyzed geometric structures associated with the Suzuki and Ree groups, showed that these groups are classified by "Tits endomorphisms" of the corresponding field, and proved their simplicity even when the field is imperfect.

In 1970, Tits proved that in characteristic 0, every finitely generated linear group contains either a solvable subgroup of finite index or a non-abelian free group. Now known as the *Tits alternative*, this result has inspired a host of generalizations.

In 1964, Tits proved a remarkable result about the simplicity of the subgroup of the group of rational points of a $k$-simple algebraic group generated by certain unipotent elements. Some remarks in this paper gave rise to the Kneser–Tits conjecture. His 1968 paper on quadratic forms became the starting point of the *Book of Involutions*. In 1971, Tits determined all the $k$-irreducible linear representations of a reductive group over an arbitrary field. He also took the first steps in the theory of pseudo-reductive groups.

In what is now known as the Tits-Kantor-Koecher construction, Tits obtained Lie algebras from arbitrary Jordan algebras. He devised the Freudenthal-Tits magic square which forges a Lie algebra out of a composition algebra and a degree 3 Jordan algebra, yielding all exceptional Lie algebras if the field is algebraically closed. He also produced the first and second "Tits constructions" which play a central role in the structure theory of Jordan algebras.

This brings to a close our attempt to name the highlights of Tits' mathematical career, but no brief summary can encompass them all, nor can we sufficiently describe the influence Tits' mathematics has had on group theory and all its many neighboring disciplines.


**Figure 4.** Jacques and Marie-Jeanne in Oslo, 2008.

# Tributes

## *Jean-Pierre Bourguignon*

Jacques Tits and I met for the first time in the early 1970s in Bonn. Friedrich Hirzebruch, who had convinced him to take a position there, introduced me to him in the tea room on the ground floor of Beringstrasse 1. This remains a special memory for me because of the many precious opportunities I had to meet him later in my life.

Indeed, after I became director of the IHÉS in 1994, we developed a close and trusting relationship, something I am highly grateful for because of the thoroughness with which he approached questions I posed to him. His sense of humour and his gentle way of talking to people were legendary.

Jacques Tits was one of the very first visitors to IHÉS, shortly after its creation in 1958. He lectured several times at the Institute's first location Rond-Point Bugeaud, near Place de l'Étoile in the heart of Paris's XVI$^{\text{th}}$ arrondissement. Here is what he wrote in a letter to Léon Motchane, the founder of IHÉS and its first director, dated 15 July 1961: *"Pour autant que l'on puisse juger de son propre travail, je compte les deux séjours que j'ai faits à l'Institut des Hautes Études Scientifiques parmi les périodes les plus productives de ma carrière scientifique."*

Later, when the Institute had moved to Bois-Marie in Bures-sur-Yvette, where it still is, Jacques came for several long visits, staying with his wife Marie-Jeanne at the Ormaille Résidence.

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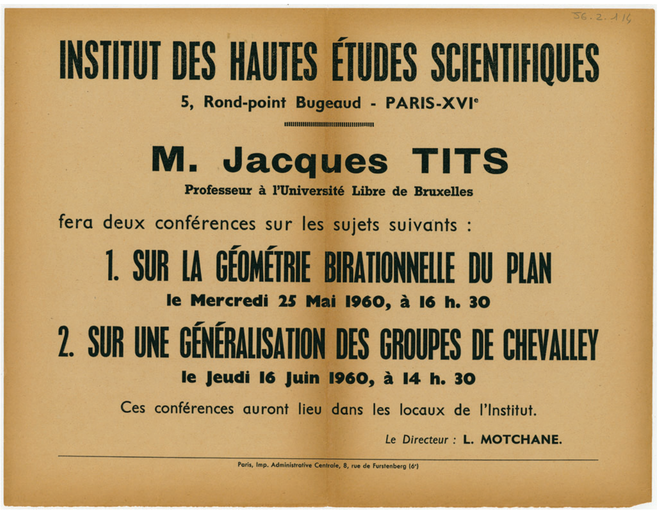


**Figure 5.** Announcement of Tits' first lecture at IHES.

Jacques Tits was asked to become editor-in-chief of the *Publications Mathématiques de l'IHÉS* in 1980, succeeding Jean Dieudonné who had held the position for 20 years. He was an extraordinarily dedicated editor-in-chief. In the interview Pierre Deligne gave on the occasion of the conference held in 2000 to honour Jacques[1] when he retired from the Collège de France and shortly after having left the helm of the *Publications Mathématiques*, Pierre Deligne says: *"C'était une situation idéale. Tits faisait énormément de travail... Il était un despote éclairé. Il jouait son rôle parfaitement et savait prendre des décisions quand il fallait mais il consultait d'abord."* The advice Jacques Tits gave to Étienne Ghys when he took over the editorship from him is revealing: *"Vous savez, c'est très facile, il suffit d'aimer la revue."*

His extremely careful checking of all articles to be published there contributed certainly to the high recognition the journal enjoys in the mathematical community. During his editorship, the journal attracted a number of landmark articles, some of them quite long.

On a number of occasions during my time as Director of IHÉS, I reached out to him for advice on issues related to scientific initiatives IHÉS should take concerning mathematicians to invite or events to organize. His in-depth knowledge of the mathematical community at a high level was very valuable. Earlier, he had been helpful in securing some financial support for the IHÉS from the Belgian government.

His later years were not easy due to a difficult health condition which confined him to his apartment. At the end of his life, Tits needed care the around the clock, but his sense of humor persisted in spite of all the hardships and his mind remained clear and agile.

Several mathematicians paid him regular visits. Jean-Pierre Serre would come to see him every three weeks more or less. I accompanied Misha Gromov on some visits, as Jacques appreciated my providing some "translation" of what Misha said.

His funeral was very simple. Besides the testimony of a family representative, Étienne Ghys and Michel Broué presented accounts of their admiration for him. I had the privilege of reading short testimonies received from five Fields medallists. Jacques Tits made the Foundation Hugot of the Collège de France his sole legatee.

[1]*The movie shot on this occasion by Jean-François Dars and Anne Papillault on behalf of the CNRS is entitled "A Jacques Tits." It can be found at:* `https://images.cnrs.fr/video/1168`.

## *Pierre Deligne*

It has been my good fortune that Tits was a professor at the ULB (Université Libre de Bruxelles) in the early sixties. He gave me two crucial pieces of advice: "Do what you want" and "Go to Paris." The latter was easier said than done. Tits made it possible by introducing me to Grothendieck (at the Fall 1964 Bourbaki seminar), who, together with his colleagues, enabled me to become "pensionnaire étranger" at the ENS (École Normale Supérieure in Paris).

Every Thursday afternoon during one of my last years of high school, I would bicycle to the ULB to attend Tits' course on Lie groups. I vividly remember the day he wanted to define the adjoint group. He began a pedestrian proof that the center is an invariant subgroup, then stopped to say (rough translation): "In fact, this is obvious. As I can define the center, it is stable by any automorphism, hence by inner automorphisms." For me, this interrupted proof was a revelation of the power of "transport of structures." It also shows how symmetry was never far from his mind.

I also fondly remember the day when I bicycled to his house in the pouring rain to show him some mathematics, and how, arriving unannounced, I was warmly received–and dried–by him and Marie-Jeanne.

In 1964, Tits left Brussels for Bonn. It was only in 1973 that I again saw him regularly. Attending his course at the Collège de France was one of the highlights of my week.

Tits was a perfectionist. When he succeeded Dieudonné as editor of the *Publications Mathématiques de l'IHÉS*, he devoted a lot of energy to it, but he enjoyed the result, and the beautiful typography. Tits resigned when the composition was computerized.

He had a great interest in languages. He learned Japanese to better enjoy his visits and Chinese to read classical

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**Figure 6.** The Tits family: Léon (nicknamed Pap), Jacques (Yaak), Jean (Coc), Yvonne (Vonne), Ghislaine (Ghaine), and Louisa (Mam).

poetry. In French, he suffered when I failed to use a needed imperfect subjunctive, and regularly chided me for my Belgicisms, correcting my "rouler en vélo" to "rouler à vélo."

He and Marie-Jeanne were inseparable. When walking became difficult, he leaned on her. Her death was a shock from which he did not recover.

His death, during the covid epidemic, came unexpectedly. I could only find some solace by reading from his *Collected Works*, where his spirit remains.

## Jean-Pierre Tignol

As a thesis advisor, Jacques Tits was always supportive and benevolent to me, and I benefited immensely from his approachable demeanor and generous personality. In our infrequent work sessions, I had the privilege to witness the workings of his mind and to appreciate his unfailing, often self-deprecating, sense of humor. Even though the problem he suggested to me was purely algebraic, his line of thought was infused with geometric insights.

This unique opportunity bestowed on a student in Belgium by an illustrious mathematician from Bonn University who was about to move to the Collège de France was a result of Tits' attachment to his country of birth. While he lived abroad, he regularly returned to visit not only his family, but also his colleagues in the mathematics department of the Université Libre de Bruxelles, from which he had graduated and which had offered him his first position. Tits had to become a French citizen in order to take his chair at the Collège de France, but he kept an enduring connection with Belgium. He once recounted that on an official visit at the Collège de France the French president Valéry Giscard d'Estaing asked him where he came from (*"Et vous, d'où sortez-vous ?,"*) expecting as a reply the name of any of the prestigious French *grandes écoles*. Tits replied: "Er... from Belgium."

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## Pierre-Antoine Absil

Jacques Tits was the brother of my maternal grandfather. I met him on rare occasions, but my mother Janine Tits was his closest relative during the latter period of his life. Together with her brother André Tits, she was of great help in gathering the family memories that are shared in this contribution.

Born on August 12, 1930 Jacques was a lively, joyful child, curious about everything. He lived in the family home at 21 Avenue Victor-Emmanuel III in Uccle, Belgium, with his parents, his older brother Jean, and his older sisters Ghislaine and Yvonne. As a child, he dreamt of becoming a tramway driver: he loved watching the driver doing his thing. Jacques' father, Léon Tits (born in February 1880), was employed as an assistant in the mathematics department at Université Catholique de Louvain (UCL). At the time, he was a Catholic priest, like most professors and many assistants back then at UCL. By 1914 though, he was in disagreement with the clergy. He left the priesthood and was forced to resign from the university. The Catholic Church made it difficult for him to find employment elsewhere. His parents and many of his relatives rejected him as well. In 1917, he married Louisa André, a remote cousin, a warm, honest person, who worked as a piano teacher. The family lived happily, though with limited means. Léon died of Parkinson's disease in 1943, in the midst of World War II. After Léon's passing, Jacques' older brother Jean became the family's breadwinner and took over his father's private tutoring.

In 1941, Jean was starting as an engineering student at the Université Libre de Bruxelles (ULB). As Jacques would hear his dad and brother discuss integration, he wanted to understand. It was shortly before Léon's death, when Jean told his bed-ridden father "now I know" that Jacques is truly exceptional. Jacques started to teach university-level mathematics to his brother's classmates who were encountering difficulties. He soon decided, on the encouragement of his mathematics teacher, Charles Nootens, to attempt the entrance examination to the ULB's engineering school.

In preparation for the entrance examination, Jacques had to learn trigonometry, so Jean lent him his 60-page textbook. The next day, Jacques *knew* it all. His secret: start

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**Figure 7.** Marie-Jeanne and Jacques, Rome, 1953.

from the end; if you understand the end, you can reconstruct the rest. He passed the entrance examination with flying colors. At age 14, he entered the ULB as a student in mathematics.

In his third year at the ULB, under the direction of Paul Libois, he obtained startling results in algebra, in particular on 3-transitive groups. He obtained his bachelor's degree at age 18, then his PhD before age 20. I recall seeing a framed newspaper article at my grandparents place praising his accomplishment as the youngest Belgian Doctor of Science.

In 1949, Jacques met then-Princeton mathematician Emil Artin at the *Colloque d'algèbre et théorie des nombres* in Paris. Artin invited him to visit Princeton. This would be Jacques' first trip outside of Europe.

Also in 1949, Jacques' first niece, my mother Janine, was born. "Yaak" became "Oncle Yaak." He loved taking care of Janine. In his late years, he confided to Janine that he would have loved to have children of his own and that he considered her as his daughter.

In 1953, in Rome, Jacques met Marie-Jeanne Dieuaide, herself an FNRS Fellow from Belgium. Her field was history. Jacques and Marie-Jeanne were housed in the same dormitory building. Marie-Jeanne later confided to us that, before Jacques' arrival, she had joked with other FNRS fellows: "A mathematician is joining us? I hate math! Too serious and boring for me!" She soon changed her mind and proceeded to make him see other horizons. Marie-Jeanne and Jacques got married in Brussels on September 8, 1956.

From 1956 to 1962, Jacques taught extensively at the ULB. In 1964, Jacques and Marie-Jeanne left for Bonn, where they would remain for ten years. Jacques kept close contacts there for the remainder of his life.

A close friendship had been formed between Jacques and Jean-Pierre Serre. Serre wished to have Jacques with him at the Collège de France, and succeeded after Jacques changed his citizenship to French, at that time a requirement for obtaining a professorial position at the Collège de France. In 1975, Jacques gave his inaugural lecture at the Collège de France. This lecture was addressed to a general audience, and Jacques succeeded in making it seemingly understandable, even exciting, to the "person in the street," bringing to life the central role played by symmetry in mathematics.

In 2008, several family members had the privilege of attending the Abel Prize award ceremony in Oslo. Jacques, in a wheelchair, peppered his speech with the humorous touch that characterized him.

In the latter portion of his life, Jacques had health problems. Multiple times, Marie-Jeanne contacted Janine, head-pharmacist at Verviers Hospital, asking her to consult with Jacques' doctors concerning his ailments. Several times Janine, in close consultation with her cousin Claude, had both Jacques and Marie-Jeanne urgently hospitalized in Paris. In spite of all these travails, Jacques never complained. Always accepting his fate, smiling, full of great charm and humor, he had an amusing anecdote for everyone.

My wife Tatiana Sirbu recently accompanied my mother to Paris. Originally from Moldova, she speaks fluent Russian. Jacques wanted to hear about her home country, her youth in the USSR, her current research work on deportations and transfers of populations during the Soviet era; they even had long conversations in Russian together. That day Jacques was especially witty. Beside being fluent in English and German, he could converse in Russian and Italian, and was in the process of learning Spanish (he wanted to read *Don Quixote* in the original) from his then chief homecare person, Madame Rodriguez, plus an Assimil book. He also studied Chinese and Japanese. He once confided to my mother and André that he still had some to-be-written mathematics papers in his head.

Jacques and Marie-Jeanne never had children. Their child was their research, their life was the Collège de France. An idea emerged: Would Jacques bequeath his entire estate to the Collège de France, specifically to its Fondation Hugot? Jacques was delighted at such a thought. Jean-Pierre Serre contacted the Fondation Hugot and soon Jacques wrote a will, before two witnesses: his dear friends Jean-Pierre Serre and Jean-Pierre Bourguignon (then President of the European Research Council). This being settled, Jacques was serene. He received the promise that he would never have to leave his apartment, and Florence Terrasse-Riou, director of Fondation Hugot, told him that, down the road, his apartment would remain the "Apartment Jacques Tits" and would be made available as housing for visitors.

Jacques never showed interest in using new technologies. He lived without a TV or even a radio; newspapers,

**Figure 8.** Jean-Pierre Bourguignon, Jean-Pierre Serre, Jacques Tits, Florence Terrasse-Riou, Claude Piret, and André Tits in the Tits' apartment in Paris, 2017.

**Figure 9.** Franz Bingen and Jacques Tits at the wedding of Tits' sister Yvonne and Bingen's brother Roald in Brussels, June 19, 1954.

magazines, and books were sufficient for him. For his 91st birthday (on August 12, 2021), André offered him a laptop so that, with the help of Stéphanie, his chief homecare person at the time, he could read emails we sent him and interact with us on Skype.

On December 1, 2021, Janine had a pleasant Skype conversation with Jacques and Stéphanie. Jacques sent her a virtual kiss. He would leave us four days later at dawn.

Always generous, charming, smiling, and joking, Jacques expressed interest for all things. We keep from him enchanted memories, a life model for future generations.

## *Franz Bingen*

Jacques Tits was born in 1930 as the youngest in a family of four surviving children. As a child, he played a lot with his sister Yvonne, who preceded him by eighteen months. The two felt like twins. They kept this special complicity throughout life. There was a mathematics gene in the family. Jacques' father was a high school mathematics teacher. He taught Jacques how to calculate at the age of four. Jacques made rapid progress and skipped grades in elementary school. His father quickly realized his uncommon mathematical gift and did his best to develop it. Unfortunately, he died as Jacques was approaching thirteen. Jacques found his own way to help his mother to make ends meet. He gave lessons in mathematics to students preparing for the entrance exam to the Faculty of Applied Sciences at the Université Libre de Bruxelles (or ULB). He took that opportunity to take the exam himself. Jacques came out first in the exam, and this allowed him to start early working on a bachelor's degree in mathematics at the Faculty of Sciences of the ULB. He got his BA at eighteen. Two years later, he defended a PhD thesis prepared under the direction of Paul Libois, who had been his geometry professor throughout his studies. After that, he obtained postdoctoral support from the science foundation in Belgium (FNRS). This gave him the opportunity to present his habilitation and to start an academic career at the ULB. In particular, he assisted Paul Libois by contributing to the teaching of the projective geometry course for second-year students in mathematics.

I became a student at the ULB in 1950. Projective geometry was one of the courses I had to take and Jacques Tits was the professor. I enjoyed his very personal style of teaching. In the tradition of Enriques as later developed in a two-volume book by Veblen and Young, one usually started from the axiomatics of the projective plane and deduced its main properties. After this one climbed a dimension higher and if time allowed one reached general projective space. Jacques Tits began with the projective line and the characterization of the group of projectivities on the line among triply transitive groups. In dimension 2, he introduced the nearly fourfold transitive groups and again established the relationship with the group of projectivities of the projective plane. Then he could deduce the traditional properties more easily. In spite of the difficulty of the material, he succeeded in keeping his lectures understandable to the students. We owed this to his very communicative style of teaching, which was very different from the distant manner adopted by most professors at that time in universities. His eyes sparkled with intelligence and above all he radiated kindness.

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Three years later, I met Jacques again in very different circumstances. My brother Roald went to a summer camp in Austria. Holidays did not go according to plan for him, as an angina forced him to take to his bed. One young lady paid special attention to him during his convalescence. She was, by a nearly miraculous coincidence, Jacques' sister Yvonne. She and Roald soon became engaged and they married on June 19, 1954. At the wedding, Jacques (24 years old) and myself (two years younger) were of course present. We raised our glass to the health of the newlyweds and also, to a certain extent, to a new friendship that would last for the rest of our lives. Obviously, Jacques felt very comfortable in our family. We were soon separated, however, by our various scientific stays abroad.

The early sixties was an exciting time at the ULB. Jacques became a full professor, while I was appointed lecturer in the Flemish section of the university. We found it important, at a time when the number of students, and therefore of professors, researchers, and assistants, was growing significantly, to activate research in mathematics at the university. Our contribution took the form of a seminar devoted to a current scientific subject. The first year was devoted to Banach algebras. Jacques wanted to better understand the link between a commutative Banach algebra with unit and its compact spectrum. Lucien Waelbrouck, who had studied continuous inverse algebras, was playing the third wheel at the seminar, which was rapidly named the BTW seminar (BTW is the acronym for value added tax in Dutch). Through Georges Papy, we got to know another young mathematical prodigy, Pierre Deligne, still on the benches of secondary school. To interest him, we oriented the second and third seminars towards algebraic geometry and Lie algebras, this time with the help of Guy Valette and Firmin Bratslavski, two geometers.

In 1964, Jacques Tits obtained a chair of mathematics at the University of Bonn, better tailored to his mathematical interests than his assignment in Brussels. Jacques came to Belgium regularly to visit family, in particular his dear sister Yvonne and her children. Here is how Christine, a daughter of Yvonne, describes her relationship with her uncle: "…for us, Uncle Jacques was above all this super-funny uncle, extremely simple and kind, who came to visit us once a year according to the availability of his conference life, staying with us for the weekend. He told us extraordinary stories, experienced during his travels around the world. Magical moments for the children that we were, where he had this mischievous side, disarming with candor alongside an immense sweetness. This is the image that, I am sure, my sister and brother will keep with me of this uncle we loved a lot and who made us laugh and dream so much."

**Figure 10.** Hendrik Van Maldeghem, Gopal Prasad, Pierre-Emmanuel Caprace, Jef Thas, Bertrand Rémy, Jean-Pierre Serre, Ernie Shult, Bernhard Mühlherr, Jacques Tits, Francis Buekenhout, Marie-Jeanne Tits, Richard Weiss, and Mark Ronan at a colloquium in honor of Jacques Tits' 75th birthday at Ghent University, October, 2005.

Much, much later, around 2008, I started meeting him again, this time in his apartment in Paris. My wife and myself went several times a year to the ballet at the Paris Opera and always took the opportunity to visit Jacques and his wife Marie-Jeanne Dieuaide. Jacques had developed Parkinson's disease. He had his *Complete Works* on his bedside table. He leafed them through with us and asked for the latest news in his family. Jacques passed away peacefully in December 2021. His friends retain the image of a brilliant mathematician with a charming personality.

## *Michel Broué*

I would just like to tell here how sad many of us are, who have known Jacques Tits professionally and personally. A peculiar intuition, a source of exceptional ideas, an original and quite productive point of view, and even a kind of library, have disappeared. This is quite a loss.

But the first feeling which comes to my heart when I think of him is: kindness. To chat with him was always pleasant, reassuring, quiet. He also expressed — I am not sure I find the right word — a kind of unusual modesty. A kind of modesty always spiced up with a soft and constant sense of humor. Once he was giving a lecture at the Bourbaki Seminar, and at one point he had to mention a theorem known all around the world as "Tits' Theorem;" he talked about *"le théorème de moi."*

He had been very precocious, defending the equivalent of a Habilitation at the age of 20. Years later, he explained to me that the main hardship for a mathematician is

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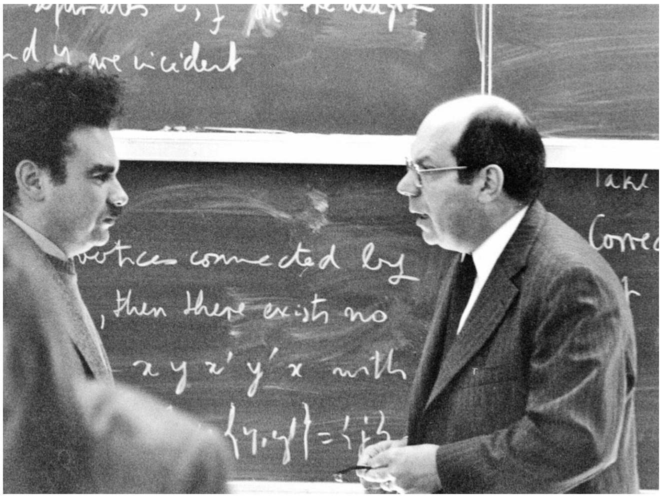


**Figure 11.** Michel Dehon and Jacques Tits at the conclusion of a talk by Tits in Ghent, October 25, 1979.

always to understand (and to smoothly accept) that one day you will find someone quicker, brighter, "better." Once this is accepted, a mathematician's life is marvellous, he added.

He was both attached to traditions and opened to rational reasons (he was a mathematician…). A student of mine submitted a "Note aux Comptes-Rendus de l'Académie des Sciences" which was good. But the author insisted on writing *"je définis," "je démontre,"* etc. and at first Tits would not accept this. He wanted the usual *"nous définissons," "nous démontrons,"* etc. The student insisted that no one else but he had defined and proved, and he added that only the late Kings of France would speak of themselves with "nous." Tits accepted "je."

The Collège de France was profoundly renovated at the end of the last century. One day the room where he was supposed to deliver the first lecture of his annual course was unavailable, and there were signs on the main door which directed the audience to another room. Tits arrived from the rear and did not see the signs. *"Voilà, personne pour mon cours, je savais que cela arriverait un jour"* was what he immediately thought, and when he eventually found the right room he was still pale. Needless to say, though, that Jean–Pierre Serre, among others, never missed one of his lectures.

## *Alain Valette*

The late seventies were an exciting time to study mathematics in Brussels. The two universities (the French-speaking one, ULB, and the Dutch-speaking one, VUB) were sharing the same campus, and there was intensive collaboration between the two mathematics departments, with a number of professors enjoying dual affiliations. People like Jean Bourgain and Ingrid Daubechies were active at VUB, and at ULB we enjoyed regular visits of extraordinary alumni like Pierre Deligne and most frequently Jacques Tits. We were lucky, as undergrads, to have two young geometry teachers, Francis Buekenhout and Jean Doyen, who strongly encouraged us to attend research seminars. So, from my 4th and final undergraduate year (1979–1980), I enjoyed following Tits' seminar talks, in Brussels and in Ghent. Even if I did not always have the prerequisites, I was always impressed by his clarity, and there was always something deep to extract from his beautiful lectures. That same year, in spring 1980, my mathematical inclinations were leading me towards operator algebras; I applied for a PhD thesis scholarship from the Belgian Fund for Scientific Research (FNRS), and I was lucky to get it. Simultaneously, my official thesis supervisor Lucien Waelbroeck had a severe accident that kept him away from academia for a full year. So I found myself in the embarrassing situation of having a scholarship but no supervisor. To help me out of this unpleasant situation, Buekenhout arranged an appointment for me with Tits. I was extremely intimidated, and trying to make me more comfortable Tits said, waving his hands about 60 cm from each other: "Oh but I know you, you were like that first time I saw you." He was alluding to the fact that he met me as a baby boy, back in 1959, when my father Guy Valette was doing his PhD thesis with him. (My father, born in 1934, was Tits' first PhD student.) Even more intimidated, I nevertheless succeeded in explaining my thesis project. Tits exclaimed: "Young man, if you want to do operator algebras today, there is one saving grace: go to Paris and work with Alain Connes!" With the recklessness of youth, I went to find Connes in Paris and indeed became his unofficial PhD student. Two years later Connes got the Fields medal. In retrospect, Tits gave me the best advice in my career.

Since my thesis was on $C^*$-algebras associated with real or $p$-adic simple Lie groups, I frequented group theory conferences where I would occasionally meet Tits. Sometimes I had the good fortune to be invited to his table for lunch or dinner and got to experience how sweet and gentle he was, but also how funny and witty he could be, with a typically Belgian sense of self-mockery. Tits' style of writing was akin to his style of lecturing: a model of clarity and exposition. Un grand monsieur.

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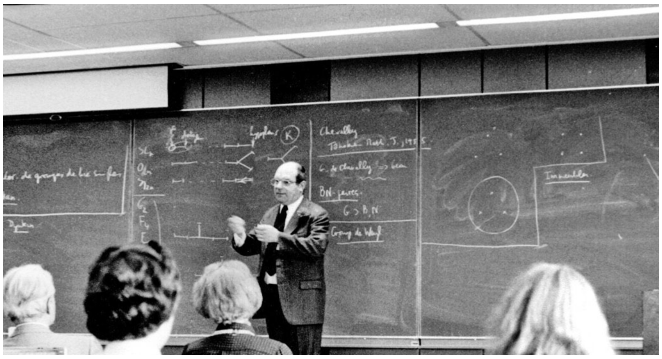

**Figure 12.** Tits lecturing at the 80th birthday conference for Tits' advisor Paul Libois at the Université Libre de Bruxelles, April 1, 1981.

## *Michel Racine*

Jacques Tits was a kind and witty friend. I met Jacques and Marie-Jeanne in the fall of 1966. He was a visiting professor at Yale and I, a beginning graduate student. I did not have the prerequisites to attend his course on algebraic groups but was looking for the opportunity to speak French. They were both extremely kind with a good understanding of what belonging to a French-speaking minority meant. This developed into a life-long friendship. A few years later while they were visiting Ottawa, I introduced my future wife to them. I had tried to explain to her Jacques' mathematical importance without much success until I told her he received his PhD at 19. She asked Jacques if this was true. "Yes." "But you are a genius!" His answer was yes but with a connotation of there are things in life that can't be helped. Marie-Jeanne looked aghast and said "Jacques!" All three of us broke into laughter and Lise looked nonplussed. What made it so funny was that it was really out of character.

At the 1974 ICM in Vancouver, he began his talk with: Pick a group. Any group. Your favorite group. Let's say $E_8$. Looking around at those who were laughing or smiling, you could tell who would enjoy the talk. In 1988–89, we spent a sabbatical in Paris. Marie-Jeanne was helpful in finding us a place to stay. Early on, Jacques made the rounds of the mathematical libraries to introduce me to the librarians. I thought, what a waste of time, we could have discussed math instead. But, of course, he knew what he was doing. Without his personal intervention, I would not have been allowed to use these institutions.

Tits was proud to be a foreign member of the German order Pour le Mérite founded by Frederick the Great. There are no more than 40 German members and 40 foreign ones.

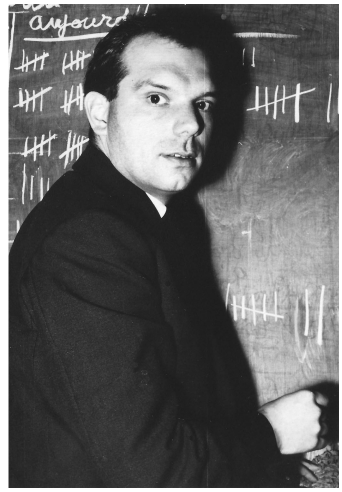

**Figure 13.** Tits teaching at the Université Libre de Bruxelles his first year as an Assistant Professor, 1957.

In the preface of his book [2], Nathan Jacobson wrote: "I am greatly indebted to Jacques Tits who took time off from his own important researches on algebraic groups to derive, via the theory of algebraic groups, the elegant constructions of exceptional Jordan algebras which we have given in Chapter IX."

Jacques and Marie-Jeanne were inseparable. Her illness and death were the great tragedy of the end of his life. When they spent a few weeks in Ottawa, I rented a two-bedroom suite in a nearby hotel. The staff cleared one of the bedrooms and installed two banquet tables side by side so they could work together. In our conversations, "What are you reading?" was a frequent question. Once Jacques answered "We are rereading Proust." "Are you reading the same thing?" "Of course. We read in bed; one of us reads aloud until we feel sleepy."

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## *Roger Howe*

When I think of Jacques Tits, I think of a kind and generous person. Among the leaders of French mathematics of his generation, he stands out as the one who saw value in what I was doing, and took steps to further my career.

I met Jacques somewhat by accident, for me a very happy accident. We both spent 1971–72 at IAS and we both had apartments in the IAS visitors apartment cluster. My walk to Fuld Hall took me past the Tits' apartment. Apparently, my whistling while walking by attracted the attention of Jacques and his wife Marie-Jeanne. (Perhaps because it was off key. I am not at all musical, but they were too polite ever to say that.)

Jacques arranged to have me invited to visit the Sonderforschungsbereich run by Friedrich Hirzebruch at the University of Bonn for 1973–74. In Bonn, I had two significant mathematical interactions with Jacques. Günter Harder was also party to these. The first concerned the orbit structure of pairs of classical groups acting on the tensor product of their standard modules. This led me to the idea of dual pairs in the symplectic group, which has been

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my main research focus in the half century since. I first reported on the basic ideas at the Arbeitstagung conference that marked the end of the academic year for the Sonderforschungsbereich.

The second was about a question of Harish-Chandra, who was trying to establish properties of character distributions, which had been so central to his work on real reductive groups, in the $p$-adic situation. Harish-Chandra wanted to know if certain constructions could be guaranteed to be well-behaved relative to nice maximal compact groups. I knew how to show this was so when the size of the residual field was large enough. Jacques, with his understanding of algebraic groups as group schemes, was able to show that this meant it could always be done. He communicated the results to Harish-Chandra, who incorporated them into his work on characters.

I saw Jacques next in the summer of 1977, at the AMS Symposium on "Automorphic Forms, Representations and L-Functions" in Corvallis, Oregon. He gave a set of plenary lectures on buildings and their implications for reductive algebraic groups over local fields. I also gave a talk in Corvallis, sketching how the ideas conceived in Bonn had developed since 1974 and some implications for $p$-adic representation theory. The main facts were mostly still quite conjectural, but Jacques again was supportive, and in the following year he invited me to give a talk at the Collège de France.

Over the following decade plus, I had the pleasure of seeing Jacques and Marie-Jeanne in New Haven, when Jacques would visit Yale. He had substantial interests in common with Nathan Jacobson (Jordan algebras) and Walter Feit (finite groups). A most enjoyable feature of these visits was the farewell dinner at the Union League Cafe, generally considered the best restaurant in New Haven.

I regret that I hardly saw Jacques after 1990. What remains strong is gratitude for the substantial help and encouragement he gave, and appreciation for the person he was.

## *Jef Thas*

In 1969, I followed a series of lectures on "Groupes de Chevalley" at the University of Brussels; the main organizers were Francis Buekenhout and Franz Bingen. There I learnt about the work of Jacques Tits on BN-pairs. A few years later, I read the book of Peter Dembowski on finite geometries, and learned about generalized polygons. These objects were defined by Tits in his famous 1959 paper on triality. In Dembowski's book, I found the description of certain Tits' generalized quadrangles $T(O)$ arising from ovoids. I generalized the construction of these $T(O)$ and gave a lecture on it at a summer school in Italy in 1972. I sent the paper to Jacques Tits, and he answered me from Princeton that he would present my work to *Geometriae Dedicata*. So my first contacts with Jacques Tits started in the period 1972–1974. That was the beginning of the many handwritten letters I received from him. It was also the starting point of my research on generalized polygons, one of the topics on which I am still working.

In 1976, Jacques Tits sent a letter to Stanley Payne, Francis Buekenhout, and me. In his letter he included the preprint "Quadrangles de Moufang, I." He also mentioned that due to his moving from Bonn to the Collège de France his collection of reprints and preprints was "in a shamble" and so he was not sure that the results in his text were new. He asked us if the results were known up to the best of our knowledge. He also said that "the interest of the paper is certainly quite limited." In fact this paper is part of his huge achievement, the classification of all Moufang polygons. This letter also shows his kind way of dealing with much younger researchers. (I was 32 and was of course very honored.) In February 1979, Jacques Tits was awarded the title of Doctor Honoris Causa by Ghent University. It was a great honor for me to be his promoter and to introduce him during the ceremony. It was the first time that my wife and I met Jacques Tits and his wife Marie-Jeanne Dieuaide. We talked about a lot of things, not only mathematics. We told him that we liked Italy very much and that we stayed twice at the Academia Belgica in Rome. It appeared that in 1953 Jacques stayed at the Academia and that he met Marie-Jeanne there while she was doing research on Medieval History.

In the winter of 1979, Jacques Tits and his wife visited Ghent University again, and Jacques gave a talk on diagram geometries. Jacques Tits and his wife were very fond of Ghent. While being in Ghent, Marie-Jeanne consulted the archives of the city in the frame of her research about the Flemish cities in the Middle Ages. They always stayed in their favorite hotel, the "Cour St Georges" in an eighteenth-century building.

In 1981, they visited Ghent again, and now Tits talked about groups and Kac–Moody algebras. At the same time my colleague Stanley E. Payne, then professor at Miami University, Ohio, was visiting me. We were busy working on our book *Finite Generalized Quadrangles*. In 1977, Payne published two long papers proving the uniqueness of the generalized quadrangle of order 4. Before including it in the book, we discovered that the proof was incomplete. We mentioned this to Tits who immediately started to write on the blackboard in my office. The next day he

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came up with a proof that certain configurations could not exist. This allowed us to complete the missing part in the proof. His reasoning was very original, and I am not sure that Payne and myself would have found it. Tits was just in time visiting me in Ghent to save us!

For Tits' 60th birthday, the Belgian Mathematical Society organized a conference in 1990 at the Palace of the Academies in Brussels. The rector of our university invited Jacques, Marie-Jeanne, my wife and myself for an informal lunch, just the five of us in a room adjacent to the meeting room of the University Board. The rector had to preside at a meeting of the Board, which was supposed to start right after the lunch, but the rector was so charmed by the Tits family that he let the Board know that they could start without him.

For my 50th birthday, some of my colleagues organized a two-day conference as a surprise. An even bigger surprise was that Jacques and Marie-Jeanne showed up. Jacques gave a beautiful talk on Moufang polygons.

In 1996, we organized a conference in honor of the 65th birthday of Jacques Tits. Then in 2003, I had the opportunity to see him again in Brussels at a conference in honor of my good colleague Francis Buekenhout. Some years later, my colleagues Van Maldeghem and Mühlherr organized a meeting for the 75th birthday of Jacques.

The last time I spoke briefly to Jacques was in 2008 at the Palace of the Academies in Brussels, during a ceremony in honor of Jacques Tits and Pierre Deligne being awarded the Abel prize respectively the Wolf Prize. His health was not good anymore and he needed a wheelchair.

Jacques Tits was a great man, not only as a mathematician but also as a human being. We all will miss him.

## *Hendrik Van Maldeghem*

The first time I heard about Jacques Tits was in a lecture for second-year undergraduates at Ghent University. It was a course in projective geometry, and the professor (Julien Bilo), nearing his retirement, was more concerned about telling stories than presenting mathematical results. One story was about a little boy wearing short pants that he met at Brussels University, who amazed his professors with his knowledge and mathematical insight. Jacques was barely 14 when he entered university.

Some years later—I think I was still a student, or perhaps a first-year PhD—Jacques Tits visited Jef Thas and I saw Jacques in real life. Jacques was an honorary doctor at our university (on the initiative of Jef Thas) and paid regular visits. The first talk I heard him present was in Mons in the same year, I think it was 1983. I remember him writing down the correct order of the monster on the blackboard, excusing himself for knowing it by heart by pretending that this huge number consisted of his telephone number, then his bank account number, then his social security number, etc. The talk was in French, as far as I remember, but it did not matter. In fact, that is one of the many things I always liked about Jacques' talks: he spoke and pronounced very clearly, using simple words; it did not matter in which language he was speaking—and he spoke many languages! His explanations always made the audience feel that they understood everything; that was his special gift.

One especially charming feature about Jacques was that he always made people feel important; for him all mathematicians were equal, he never looked down on lesser gods. I experienced this myself several times. For example, after finishing my PhD, which was about the special class of affine buildings of type $\widetilde{A}_2$, I wrote a letter to Jacques explaining what I did (no email at that time; it was 1984). In the same year, Jacques classified all affine buildings of irreducible type and rank at least 4. His reply to my letter started with the sentence "It seems that we have been working along the same lines this year." As a second example, many years later, in 1994, I invited Jacques Tits to present a talk at the conference celebrating the 50th birthday of Jef Thas, my mentor at Ghent University. We were publishing the proceedings and I persuaded Jacques to submit a paper. He wrote one about the Moufang condition for generalized polygons and the relation with root systems. He wrote this by hand, and I committed myself to put the text into LaTeX. Doing this, I discovered a small oversight in one of the formulations. (He'd overlooked that the root groups of the smallest Suzuki group are abelian.) Jacques was very pleased, and at the conference he started his talk by thanking the organizers, as usual, but added, "if you ever write a paper full of mistakes, just send it to Van Maldeghem to type it out, and he will not only do this, but also correct all your errors."

I remember Jacques as someone who was very generous and thankful. In the 90s, I followed several courses of his at the Collège de France. Every Tuesday in winter, I drove 630 kilometers from Ghent to the center of Paris and back to follow his lectures. Sometimes PhD students joined me, either for the full course, or on a sporadic basis; one time my two sisters even joined me (they were math teachers). And at the end of every course, Jacques invited me and everyone else who joined me regularly, to an extended lunch in Paris (more like a dinner at noon). He was so thankful that we came from so far just for him—but of course the pleasure and the added value were entirely ours.

Jacques' lectures were very pleasant to follow.

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**Figure 14.** Francis Buekenhout, Jean-Pierre Serre, Jacques Tits, and Hendrik Van Maldeghem in the first row, at the conference in honor of Jacques' 75th birthday in Ghent, 2005.

His style was informal but in a way very efficient. For instance he talked about "a path with hair on" to mean a path together with all neighbors of its inner vertices, which was immediately clear to everyone and fine for us. But Serre did not agree with his informality and often interrupted him asking for more precise and mathematically sound definitions and expressions. Jacques explained everything in a rather geometric way, which I liked. He only struggled when there was a choice between two alternatives, like plus or minus, inside or outside, left or right. I remember I corrected him once (no big deal in fact), and then for the rest of the lecture he called on my help to decide every dichotomy he encountered....

There were three conferences in honor of Jacques held in Belgium, one in Brussels and two in Ghent. The first was on the occasion of his 60th birthday, the second for his 65th birthday, and the third for his 75th birthday. I was not involved in the organization of the first one (that was Jef Thas), but I was the main and local organizer of the other two. We celebrated his 65th birthday in fact one year late, in 1996. At that conference he gave a double talk, one on Friday and one on Saturday. In the first talk, my sister (a teacher in mathematics in high school) came into the room with two dozen school girls, on a school trip to see one of the greatest mathematical minds in action. Jacques was absolutely not disturbed by that and made his young audience feel welcome with a few jokes. The next day, he continued his lecture, but he was so into it, that he lost track of time. When he looked at his watch after one hour and a half (he was supposed to speak for 50 minutes, but nobody minded), he exclaimed "My God, look at the time, when did I start?" To which Francis Buekenhout dryly replied "Yesterday."

Soon after that conference, we celebrated Francis's 60th birthday with a special session in the one-week conference "Finite Geometry and Combinatorics." Jacques came over to give a talk on the new class of Moufang quadrangles that Richard Weiss had just discovered. This triggered the following example of Jacques' humor (a humor that, in contrast with some professional jokers, complimented people instead of insulting them or making fun of them). With this new class, Jacques confessed with a little bit of drama "my friend Richard disproved my old conjecture, and so he proved me wrong," and then he continued along the line "but luckily I have two other friends, Bernhard and Hendrik, because they saved my conjecture by showing that the new quadrangles fit into the broad picture of generalized Galois descent, so all Moufang quadrangles are of algebraic origin after all."

At all conferences that I organized and invited Jacques, I had the pleasure of accompanying him to lunch and dinner, and even of inviting him to my home. These were always very joyful experiences for me.

The conference celebrating his 75th birthday was the last one in which I saw him participate. When I took him to the train station, he immediately asked for a wheelchair, and he apologized to me saying "It must be awful to see a friend be discharged in a wheelchair like that, but do not worry, I am getting used to it." His Parkinson's had become worse (during one of the conferences that he organized on Algebraic Groups in Oberwolfach, he confided in me that this illness prevented him from riding a bicycle, which he would have loved to do). It was also the last time I saw Jacques in Ghent.

One of the greatest honours in my scientific career was to be a co-editor of Jacques' *Collected Works*. One of the highlights for me was the day that the four editors spent in Paris together with Jacques asking him all sorts of questions. Jean-Pierre Tignol produced a transcript of these interviews. We didn't use it for the *Collected Works*, but it is now an invaluable treasure to me. Jacques talked nineteen to the dozen about all kinds of aspects of his life and career. Near the end of the production process of the *Collected Works*, I was the one making contact with Jacques through Jean-Pierre Serre. I delivered two copies of the four books of his *Collected Works* to his apartment in Paris, on Thursday January 30, 2014. That day I had lunch with Serre at 13:30 and coffee with Tits and his wife Marie-Jeanne at 16:00. Marie-Jeanne told an interesting story. She said that when historians meet (she was an historian) and discuss scientific matters, at the end of the day when they separate they each have their own original ideas and beliefs. When mathematicians meet and started discussing various matters, at the end of the day they all agree, no matter what their original belief was.

Marie-Jeanne and Jacques were together at many conferences. She accompanied Jacques as frequently as possible. When I had an appointment with Jacques after a lecture at the Collège, I noticed that he always first called

**Figure 15.** Jef Thas, Arjeh Cohen, Dan Hughes, Francis Buekenhout, Jacques Tits, Ernie Shult, and Antonio Pasini at a conference in honor of Francis Buekenhout in Brussels, November, 2003.

Marie-Jeanne just to say his lecture went well and ask how she was doing. She also took great care of Jacques when his illness became worse. Sadly, Jacques' life companion passed away too soon, on Tuesday February 2, 2016.

On Tuesday April 23, 2019, Bernhard Mühlherr and I presented Jacques with a hard copy of the *Complement to the Collected Works of Jacques Tits* [6]. This was the last time that I saw him.

A few months before Jacques died, there was an initiative among my department to compose a booklet containing trivia about the math professors. One of the items was what is considered their greatest scientific achievement. You could read great theorems there, proofs, prizes, and other concrete accomplishments in that rubric. On my page, it just mentioned my friendship with Jacques Tits.

Thank you, Jacques, for your beautiful mathematics, for your beautiful personality, and for your beautiful friendship. An architect died, but what he built will live on.

## *Bernhard Mühlherr*

My first encounter with Jacques Tits was in January, 1989. Tits was giving a course on twin buildings at the Collège de France. I was in Brussels working on buildings for my Diplom thesis and my advisor, Francis Buekenhout, recommended that I attend Tits' course. I was surprised that I was able to understand so much of his lectures despite my rudimentary knowledge about buildings. Only much later, did it become clear to me that Tits possessed an extraordinary talent for describing the central ideas of his mathematics on a very concrete level. Since Buekenhout had let Tits know that I would be attending his course, Tits offered that we could meet for an hour after one of his lectures so

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**Figure 16.** Jacques Tits, Bertrand Rémy, Gopal Prasad, Bernhard Mühlherr, and Jean-Pierre Serre at the meeting in honor of Jacques' 75th birthday in Ghent, 2005.

that I could ask him questions about buildings. I looked forward to this hour, but I was anxious that I didn't understand enough and the meeting would just waste his time. It turned out that my fears were completely unfounded. Tits listened carefully to the ideas I was working on, made valuable suggestions and encouraged me to continue with my project.

Looking back, it is clear that those lectures influenced me more than anything else in my mathematical training. Around this time, Tits formulated several open questions about twin buildings and in the following years I made a number of contributions to their solution. Throughout this time, we stayed in regular contact. Whenever I had some progress to report, it sufficed for him that I would give the general idea; we never talked about the details. For me, these discussions were principally a kind of mathematical compass. At one point they were decisive in suggesting that I should pursue a vague idea I'd described. This idea brought a breakthrough in the classification of twin buildings. Tits invited me to give a lecture about my results at the Collège de France. This invitation and the fact that on another occasion Tits referred to me as his student, are among the greatest honors of my research career.

I remember well a number of meetings with Tits. Once, when I was in Paris with Hendrik Van Maldeghem for one of Tits' lectures, the three of us met in Tits' office, where there was a table piled high with manuscripts and preprints. He told us that these were all the things that he still needed to work his way through and we'd better not steal anything! This was just one of the many small moments when I got to enjoy Tits' very special sense of humor. Tits was always interested in languages. We generally spoke in German to each other and he would say something if I used a construction that wasn't familiar to him. Because of my South German origins, this occurred fairly often and he liked to joke in these situations that my French was better than my German.

Jacques Tits' mathematics opened exciting perspectives to mathematicians of the next generation. He enjoyed seeing others work with his ideas and he was generous with his support and encouragement. I remember well a conversation he had with my first PhD student Pierre-Emmanuel Caprace at the conference in Ghent in honor of Tits' 75th birthday. Caprace had given a talk about his thesis in which he combined and modified many of Tits' ideas to solve the isomorphism problem for Kac–Moody groups. Tits asked questions about the details, but as usual only a few general remarks were all he needed to appreciate what Caprace had accomplished. Tits' pleasure at seeing his mathematical ideas woven together in new ways and bearing valuable fruit was particularly clear that day.

**Figure 17.** Arjeh Cohen, Marie-Jean Tits, and Jacques Tits on a boat during the conference on buildings and diagram geometries by Lake Como, 1984.

## *Richard Weiss*

I first "met" Jacques Tits in the late 70s in the math library of the Free University of Berlin, where I was a post-doc. Killing time looking through recent journals on display, I came across a paper entitled "Non-existence de certains polygones généralisés, Part I" in the latest issue of *Inventiones*, where Tits was editor. In this paper, Tits began the proof that Moufang $n$-gons exist only for $n = 3$, 4, 6, and 8. I didn't know what a Moufang polygon was, but I had been working on generalizations of a theorem of William Tutte that says that finite trivalent graphs whose automorphism group acts transitively on paths of length $s$ but not on paths of greater length exist only for $s = 4$, 5, and 7. I knew that special attention was needed to rule out the case $s = 9$. The coincidence in these numbers was striking and within hours I understood how to prove a more general version of Tits' result by combining a lemma in his paper with results that I had in my drawer. Tits reacted to news of my result with charm and generosity. Not to leave things hanging, he wrote a much shorter version of his Part II using ideas from my paper and the two papers appeared quickly back to back.

In 1992, I was spending a couple of months of a sabbatical visiting Hendrik Van Maldeghem in Ghent. One afternoon Hendrik stuck his head in my office and said he was driving to Paris the next morning to hear a lecture of Jacques Tits about Moufang polygons and would I like to come. In fact, the subject was the theorem that $n = 3$, 4, 6, or 8. This was before Thalys, and Paris was far away. We were on the road at 5:00 AM, merged into the daily traffic jam on the Périférique just as the sun was rising, and then drove through the city, arriving at the lecture room just on time for the 9:00 AM lecture. Tits came in and started to write on the board, but when he turned around and noticed me, he made a startled expression and said "Oh, this is like lecturing on the Riemann hypothesis and discovering that Riemann is in the audience." Tits always knew how to be witty and generous at the same time. Hendrik and I went down every Tuesday for the remaining lectures and on the last day, Tits invited the two of us to a merry and lavish lunch in a nearby restaurant.

In 1993 Dina Ghinelli invited me to hold a series of lectures on Tits' work on Moufang polygons in Rome. For the last part of his course, I worked through his unpublished notes on the Moufang quadrangles that he called "indifferent." At that time, Tits had classified Moufang triangles and octagons and announced the classification of Moufang hexagons, but this unpublished manuscript was all that he'd done with Moufang quadrangles apart from describing examples coming from groups of type $E_6$, $E_7$, and $E_8$ in lectures at the Collège de France. Once I thought I'd really understood Tits' proof in the indifferent case, I grew ambitious and wanted to go farther. After much hesitation, I wrote a letter to Tits proposing that we collaborate to finish the classification and write the whole thing up as a book. I was proposing coming in on a project in which he'd invested years of effort and was quite certain that my offer would be rebuffed. In fact, months went by with no reply. Tits was at Yale for the semester visiting his old friend Nathan Jacobson. Still no reply. I'd mentioned my letter to Diego Benardete who was at Trinity College at that time. Later, I learned that at a tea after a colloquium talk at Yale, Diego marched up to Tits with the words "Professor Tits, you're keeping Weiss waiting!" This did the trick. Days later, shortly before his return to Paris in December, Tits called me at home in Boston to say he agreed to work together. "But I'm very busy with many other projects," he warned, "and it might take us five years!" He was wrong. In the end it took seven.

These were a thrilling seven years. Our collaboration consisted mostly of written exchanges. Laptops were not yet common and Tits never used anything but a fax machine for his communications. Our first goal was to complete the classification of Moufang quadrangles. Pushing the ideas in Tits' indifferent paper, we arrived at the situation where the exceptional Moufang quadrangles should turn up. In this case, we had to invent and classify structures that we later called "quadrangular algebras." When the classification was essentially complete, I noticed a mistake in a lemma asserting the existence of an element of order 4 in one of the root groups when the characteristic is 2. Each repair to the proof fell apart. It turned out that there was, in fact, a new family of Moufang quadrangles whose root groups were all abelian. Tits was thrilled. Within a week of hearing about them, Bernhard Mühlherr and Hendrik Van Maldeghem showed that these new quadrangles filled in a gap in Tits' picture. They arise by descent from a group of type $F_4$, but not one associated with an absolutely simple algebraic group, rather from a split pseudo-reductive group of type $F_4$ defined over a purely inseparable field extension.

I think that Tits was particularly pleased with these developments because they confirmed his well-known attitude about the importance of characteristic 2. Characteristic 2 was, as a rule, historically excluded in the study of quadratic forms, Jordan algebras, and composition algebras. This offended Tits' understanding of the geometric nature of these things. Here is what the authors of the *The Book of Involutions* wrote in their introduction: "Not only was Jacques Tits a constant source of inspiration through his work, but he also had a direct personal influence, notably through his threat—early in the inception of our project—to speak evil of our work if it did not include the characteristic 2 case."

Once or twice a year I was able to spend a month in Paris and often had the use of a small windowless storage room in the Collège de France Annexe as my office. Tits' office was down the hall, but as he'd warned me at the start, he was a very busy man and our meetings were always by appointment. Tits knew my limitations as a mathematician. I think, though, that he had respect for my persistence and trusted me to get around the technical problems that arose on my own. But his guidance about what *ought* to be true was the real driving force behind the project. Tits often joked about seeing things in his crystal ball, but what he was really referring to was his uncanny ability to see a whole world hidden in a Dynkin diagram.

* * *

In the bibliography we have included all the works of Jacques Tits as well as a few other books alluded to in this article. Volume I of [8] also includes a Curriculum Vitae and surveys of Tits' work written during his lifetime, including one by Tits himself.

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Richard M. Weiss

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# Guido L. Weiss (1928–2021)

*Eugenio Hernández and Edward N. Wilson with contributions by Ronald Coifman, Mauro Maggioni, Yves Meyer, Fulvio Ricci, Hrvoje Šikić, Fernando Soria, Anita Tabacco, and Rodolfo H. Torres*

Guido Weiss was born in Trieste, Italy, on December 29, 1928, and died in St. Louis, Missouri, on December 24, 2021. His family emigrated to the USA in 1939 and settled initially in Topeka, Kansas, moving two years later to Chicago. He obtained two degrees in mathematics from the University of Chicago, a bachelor's degree and a PhD under the guidance of Antoni Zygmund.

In 1961, Guido moved from DePaul University to Washington University in St. Louis where he later became the Elinor Anheuser Professor of Mathematics. He married Barbara Gibgot, then a doctoral student in molecular biology, and they had two sons, Paul and Michael.

Guido played a huge role in the development of harmonic analysis at Washington University. In particular, he entertained a host of students and collaborators from around the world. He met R. Raphael (Raphy) Coifman in Geneva in 1964 and started a long-term friendship and collaboration. Coifman moved to Washington University, where he and Guido collaborated on many results, including the beautiful theory of atomic decompositions of Hardy spaces. He met Yves Meyer at Oberwolfach in 1965, and beginning in the 1970s Guido, Meyer, and Coifman developed a broader understanding of Calderón–Zygmund operators. Simultaneously, Guido and Eli Stein were writing the book ([SW71]), which served as an invaluable reference for generations of harmonic analysis students. He and Stein also developed a program studying $H^p$ spaces of several variables.

Interpolation of operators was the subject of Guido's PhD dissertation. He returned to this subject in the 1980s with the theory of interpolation of families of Banach spaces ([CCR+82]). After reading M. Frazier and B. Jawerth's work on phi-transforms and wavelets, Guido became interested in reproducing formulae. From the 1990s until his retirement in 2018, his papers with postdocs and other collaborators were strongly influenced by these developments.

The above paragraphs give only a quick sketch of some of the highlights of Guido's mathematical career. The reader is referred to the article by Susan Kelly and Rodolfo Torres ([KT21]) for additional details. The tributes that follow offer a sample of the teaching style and mathematical achievements of Guido from the viewpoint of some of his former students and collaborators. This is done with the intention of keeping the memory of Guido Weiss alive.

## Ronald Coifman

Guido Weiss was an extraordinary human being and mathematician; he has been my mentor and friend for over 50 years.

We met in 1964 in Geneva, Switzerland. Guido and Barbara came to spend a sabbatical at the university of Geneva.

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I was finishing my thesis under J. Karamata, and as his assistant, I was expected to welcome them and facilitate their arrival. After various misadventures they settled in, and in his usual way Guido initiated both a friendship and mentorship with me. I was going to teach him skiing, and he would teach me Harmonic Analysis.

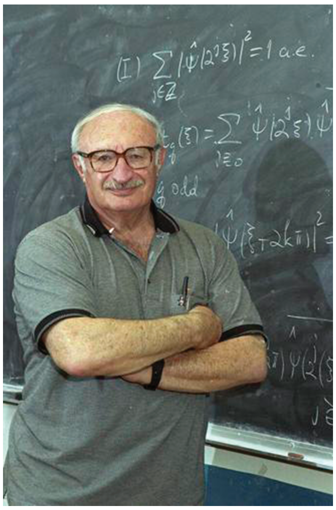

**Figure 1.** Guido Weiss in his office in 2005.

During that year, he was writing his book with Stein on Harmonic Analysis in Euclidean spaces, which I helped to proofread. This exercise allowed Guido to communicate his long-term program in analysis to me, expanding on the vision of his own mentor Antoni Zygmund. This interaction opened a whole new world of mathematics to me, as well as mathematical friendships. We traveled to Paris to meet with Antoni Zygmund and Jean Pierre Kahane, and we went skiing in Zermatt. Guido and Barbara welcomed Lucienne and me to their home. The informality and friendliness of these meetings was quite astounding to us "kids" (coming from a senior professor). Guido would lounge on the floor with a glass of Dole wine in his hand discussing everything from Italian Pizza that he would bake to music, and to mathematics (when we were alone).

While in Geneva, Guido convinced me to go to Chicago for a special year in Harmonic Analysis, while he and Barbara returned to Washington University. We had already started on several papers, on basic analysis ranging from group representations to Blaschke products in complex analysis. These were part of his program to extend real and complex analysis beyond their classical context. Three years later Guido organized a special year on analysis on symmetric spaces with a vision of building interactions between different groups of analysts and geometers.

Lucienne and I move to Washington University in St Louis continuing our friendship with Guido and Barbara and our collaboration; it was our best decision. For the next fifteen years we continued mathematics and "skiing" together.

In 1970, our two families spent the year in Orsay. Guido and I were teaching a Harmonic Analysis course on spaces of homogeneous type. Our goal was to develop a general setting on which many classical tools and methods could be extended and applied, thereby providing a bridge between geometry and multiscale analysis, as it applies to the structural understanding of operators which are not convolutions.

As usual Guido befriended many in our audience: Jacques Peyriere and Yves Meyer, as well as Aline Bonami and Jean Louis Clerc, who participated in writing up our lecture notes. I should note that much current activity in computer science and data geometry on networks is directly related to the vision of blending algorithms (real variable methods) with the geometries of nature as developed in our lecture notes.

At the time, we were challenged by proving $L^p$ estimates; this was always understood as a test of understanding, and our goal as formulated by Zygmund and his school was to develop "methods" of analysis. Guido always stressed this point; it has been the challenge all along, in particular, understanding the role of complex methods to prove inequalities, and the role of geometry to understand complex methods and the power of generalized analytic functions to control singular integrals.

I should add that the combination of an extended view of analysis in mathematics, together with Guido's extraordinary ability to assemble communities of mathematical friends in France, Italy, Spain, China, and more, and collaborating and exploring the world around his vision, led to a remarkable broadening of the Calderon–Zygmund school, and our mathematical horizons.

Guido liked to quote A. Zygmund, who said that you assess the mathematical contributions of a person by "integrating his positive part." This generous philosophy helped generate many friendly, collaborative teams.

## *Eugenio Hernández*

Two people have profoundly influenced my career in mathematics: Miguel de Guzmán (1936–2004) while I was an undergraduate student and Guido Weiss when I was a graduate student and later as a collaborator and friend.

I arrived at Saint Louis in August 1977, with a backpack and a TWA[1] bag of toiletries, because my two suitcases were lost on the trip from Spain. After Guido learned of my situation, he wrote a letter to TWA officials. Two weeks later I had a check for $400 to compensate for the loss. This little story sheds light on the kind of care Guido took of his students.

While I was a second-year graduate student, Guido suggested I prove an interpolation theorem for operators acting on $H^p$ spaces using the atomic decomposition developed by Raphy Coifman. I am sure that as an expert on complex interpolation and $H^p$ spaces, Guido could have proved the result in a couple of hours. It took me several

[1] *Trans World Airlines, a well-established airline at the time.*

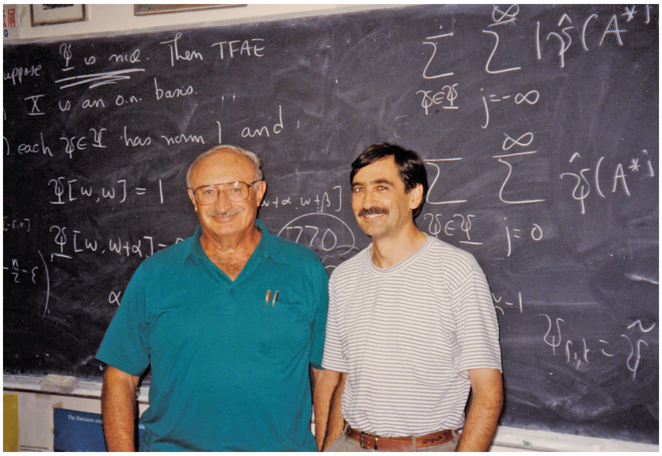

**Figure 2.** Guido Weiss and Eugenio Hernández in 1996.

months, during which he patiently guided me towards the correct proof, which resulted in my first math paper.

Guido always tried to share his math projects with others. After my four-year term as Vice Rector of the Universidad Autónoma de Madrid, I sought his advice on how I should spend a 1994–95 sabbatical year retooling in mathematical research. He soon sorted out the finances and offered me an invitation to visit Washington University and to collaborate with him in "organizing" the mathematical theory of wavelets. It gave me the opportunity to learn this theory and the trials and tribulations of writing a book with the mind always set on finding the best and simplest ways to explain mathematics. Our collaboration produced not only [HW96] but also many papers over the following years together with several of his collaborators, of which I would like to mention D. Labate, H. Šikić, and E. Wilson.

I am thankful to Guido for all the mathematics he taught me, his lessons of life, the tennis matches we played, as partners and as opponents, and for his friendship.

## *Mauro Maggioni*

The contributions of Guido L. Weiss to mathematical analysis, and harmonic analysis in particular, have spanned multiple decades and multiple research directions—from Hardy spaces to wavelets—often tying together different areas of analysis. Guido's work has influenced the work of many mathematicians, and often it was the fruit of his collaborations with researchers from all over the world.

I was very fortunate to be welcomed by Guido at WashU, and become one of his students—at the time some conjectured I would be the last one, but luckily that turned out to be far from being the case—not only because of his scholarship in mathematics and the endless amount of advice I would receive from him, but also because of the many lessons I received, in mathematics and in life. Guido made me feel welcome from the very beginning. He was always generous with his time, with meetings that never felt hurried and in which working on a problem on the blackboard felt natural. He was always available to answer questions and provide interesting research directions together with references to beautiful papers and books to enrich my very limited knowledge of mathematics.

He fostered interactions with a wide variety of colleagues, short- and long-term visitors to the department, and among students. This led to invaluable opportunities to learn mathematics in other areas and to forge new collaborations. These interactions often moved from the department to his home, where he and his wife Barbara offered the kindest and warmest hospitality, to guests from all over the world, speaking many different language— a surprisingly large subset of which Guido could speak proficiently. He consistently refused to speak anything but my mother language with me, which was truly helpful to me especially at my arrival in the United States.

Guido's lectures, always at the chalkboard, had utmost clarity and perfect pacing, only matched by that of his handwriting. His papers and books were superbly written, with a terse prose and a natural organization of the materials. They have influenced generations of mathematicians, researchers, and students alike, and will continue to do so. In fact, I remember that I had to quickly acquaint myself with one of Guido's books: at our first meeting in his office, after about a week at WashU, and after explaining the story behind a small fraction of the memorabilia there, Guido asked me if I knew measure theory and some functional analysis, to gauge the possibility of waiving the first-year analysis course. I answered in the affirmative, quite possibly with an excess of self-assurance. He suggested that I study his book with Eli Stein, *Introduction to Fourier Analysis on Euclidean Spaces*, reconvene in a week to discuss it, and then see if I could indeed waive the Analysis course. As soon as I started reading, I realized I knew close to none of the material in the book, so I stocked up on food for the week, and started studying. While I was far from having understood the book in that one week, Guido generously considered what I had learned enough for a waiver,[2] and we went on discussing the book in greater depth in our subsequent meetings. This book started opening my mind to what the next level of Analysis looked like, and I loved it. Under Guido's direction, I then learned about

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[2] *I only learned much later that the first-year graduate course in Analysis was at a much more basic level than the material in the book; there were multiple lessons that I learned in that first week.*

**Figure 3.** Guido and PhD students from Washington University. Picture taken at the Conference in his honor, May 10–14, 1993.

spaces of homogeneous type, Calderón–Zygmund theory, representation theory, and wavelets; research in these areas inspired me and propelled me to a career as a researcher.

I always remember Guido as setting an example by showing the importance of hard work and humility in tackling difficult mathematical problems, and of kindness and openness in welcoming others in mathematics and life. He will be missed by many in the mathematical community, who were influenced by him personally or through his work.

## Guido Weiss at Oberwolfach 1965

### *Yves Meyer*

I met Guido Weiss in August 1965, at an Oberwolfach conference organized by Paul L. Butzer. I was still a graduate student. Guido was ten years older than me and already a famous mathematician. There were only nineteen participants at this meeting. The old castle still existed, and we were accommodated there. It was deliciously obsolete. I became fascinated by Guido, by his mathematics and by his extraordinary personality. Discussions with Guido were great. He was at the same time a deep mathematician and an accomplished humanist. For Guido, mathematics was a part of human culture. He thought that mathematics should be shared by everyone and should contribute to happiness. Doing mathematics should be as pleasant as playing tennis. This idea was completely new to me. To me doing mathematics was a difficult experience. It was a frustrating confrontation with a few great mathematicians. Guido changed my mind. He was my mentor and often corrected my naive beliefs in politics. He had deep and original views on many issues.

Barbara Weiss and my wife Anne were allowed to participate in the conference without being mathematicians. They did not listen to the mathematical talks. Instead they violated the established rules by arriving ten minutes before lunches and dinners and changing the seating arrangement. As a result, the four of us ate together almost every day. This was forbidden by the organizers since we were supposed to interact every day with new participants. At our table, we spoke French. Guido was fluent in many languages. He was fond of playing with words, which he did in a mixture of English, French, and Italian. His jokes were absurd and hilarious.

I spent 1970 with Guido at Orsay. Then in 1974, he invited me to work with him at Washington University. But while I was there, he was busy with another program. Then Raphy Coifman convinced me to attack Calderón's conjectures. That is another story.

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### *Fulvio Ricci*

Many aspects of Guido Weiss's personality are fixed in my mind, which I often recall with nostalgia, and whose impact I recognize at various moments of my professional life.

Among these aspects, I just want to mention his rigor in research and teaching, his sense of responsibility toward persons around him and institutions, and, going more closely to scientific issues, his approach to problems exclusively following his own mathematical taste. Scientifically, I owe him for having introduced me to Euclidean harmonic analysis.

I was fortunate to meet him in the early days of atomic theory of Hardy spaces. Classical Hardy spaces $H^p$ with $0 < p < 1$ (in the sense of holomorphic functions on the unit disc) had been the subject of my bachelor's thesis. Attending the intensive course Guido gave in Perugia in 1976 about his AMS Bulletin paper with Raphy Coifman, I was shocked to learn that a revolution on this topic had taken place and to understand the tremendous generalizing power coming from Guido's contribution.

This event, followed by a semester at Washington University that Guido had arranged for me, determined the direction that my research has taken ever since. The

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Stein–Weiss book became a basic reference for me and the source of new ideas. The chapter on spherical harmonics combined nicely with what I was learning at the time about analysis on Lie groups and homogeneous spaces. This included the idea of transplanting Calderón–Zygmund theory from $\mathbb{R}^n$ to a Lie group, in particular $SU(2)$, which was the topic of the Springer Lecture Notes volume by Raphy and Guido. Another important source of inspiration for me was the CBMS Lecture Notes on the transference principle by Raphy and Guido.

For various years since then, Guido and I have remained in close contact, though we have not worked together on a specific research project. Following his own taste, Guido became more and more interest in other topics, first in the general theory of complex interpolation, then in wavelet theory. As our interests were gradually diverging, the occasions for meeting unfortunately became less and less frequent over time. When I heard the sad news at the end of last year, my first reaction was regret for not having been closer to him over the last several years.

Another, more important, reason for being grateful to Guido is the great effort and dedication he put into supporting harmonic analysis in Italy. He had a large number of Italian graduate students and promoted, or even just facilitated, many joint initiatives. Altogether, he contributed greatly to the formation of a coherent group of mathematicians with common scientific interests.

## *Hrvoje Šikić*

Guido Weiss was one of the most exceptional people I ever met. How does one describe such a unique and multifaceted person? Perhaps one could do so through some personal recollections.

It was in February of 1996 that I entered Guido's office for the first time. It was located centrally on the second floor of the Cupples I Hall at Washington University in St Louis. His office was packed with memorabilia and in front of the blackboard was a wide olive-green sofa. A cute small Shih Tzu, with the impressive name of Thor, was comfortably settled on the sofa. Guido explained that any mathematics discussed was to be approved by Thor and then pointed toward the upper right-hand corner of the blackboard where the word *stupid* was written in several languages. He emphasized that he had already written the word in Croatian, so that I probably could not add much to it. My response was that the word written in Polish indeed meant stupid, but the Croatian one would be better translated as a fool, so hopefully I could be of some help.

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And it continued like that for the next twenty-five years; mainly mathematics, interrupted by word puns, thumb wrestling, discussions of current events, history, arts, and sports. Guido's language skills were remarkable. I am not sure how many languages he spoke, but I witnessed conversations of his in most major ones.

Throughout his life, Guido encountered a few historical figures and many people who were in positions of power and influence, but it did not prevent him from treating everybody with respect and open arms. After getting to know him better on a personal level and becoming aware that he had lived through some difficult times in his long life, I asked him how he maintained his optimism. He explained in somewhat mathematical terms that he always tried to focus on the *f-plus* of the person's character and ignore the *f-minus* for as long as possible.

My wife and I became friends with Barbara and Guido, and we saw both provide help to many people on numerous occasions. Just to give you an illustration of the level of their generosity, when we arrived for one of our longer stays in St Louis, and our rented apartment was not yet ready, Barbara and Guido took us along with our two-year old daughter into their home for three weeks.

As a mathematician, I like to think of Guido as a theory-builder and a storyteller. Think of mathematics as an infinite onion, where you peel off layer after layer. When working with Guido, we did not rush, our pace was moderate and steady; every day we would try to bring some insight, perhaps very small, but an improvement nevertheless. When we managed to peel off a layer, then we stopped to rethink it. Furthermore, we would remember it and very often revisit it in the future. Guido liked to build a solid house, but the level of understanding was never final; a new point of view was not to be neglected. At one point early in our collaboration, Guido, Manos Papadakis, and I worked on a statement for a few months. Eventually, Manos and I brought a seventeen-page proof to Guido. We all checked it and it was correct. But Guido sent us back to work on it further until we had a reasonable proof; the instruction being that "a reasonable proof" would be clear to us once we reached it. And indeed, it was so; the end result was an elegant one-page proof where every step had a clear and simple meaning.

For Guido, exposition in mathematics was not just a matter of style or format. It was the essence of the content and understanding, a guide through a new theory. As a consequence, his writing and his lectures were highly appreciated everywhere in the world and on every level, from undergraduate lectures to research seminars.

An opportunity to work with Guido also meant joining Guido's academic family, a large and very diverse group of excellent people from every corner of the world. Guido was a natural leader, and the St Louis School of Analysis left its mark in many countries in the world, my own included. Guido visited Croatia on several occasions. One of the visits some twenty years ago was particularly emotional and reminded us again of the horror of the Holocaust. Guido's mother's family was from Pakrac, a small town some fifty miles east of the Croatian capital of Zagreb. Just before WWII and their move to the US, Guido and his family visited their relatives in what was then Yugoslavia; as it happens, never to see most of them again. Almost sixty-five years later we brought Guido to Pakrac, where we found his grandfather's house and managed to trace one of his cousins near Zagreb and reunite them after all these decades. Guido had a long and complex life, and he will remain a symbol of the human spirit and an inspiration for all of us.

**Figure 4.** Guido at his farm with students, colleagues, and his son Michael, in 1983.

## *Fernando Soria*

When I arrived at Washington University, I was not planning to be a student of Guido. But the frequent talks he used to have with us, the Spaniards who came around that time (José Luis Fernández, Juan José Manfredi) and in previous years (Eugenio Hernández, José Dorronsoro, Julián Aguirre, Patricio Cifuentes, just to mention a three-year period), convinced me that indeed it was with him that I wanted to do my doctoral thesis. Thus, after finishing my qualification period, he agreed to be my supervisor jointly with another unforgettable and brilliant mathematician, Mitch Taibleson. Working with them has been the best thing that has happened to me in my entire academic life.

From Guido and his book, written with Eli Stein a few years earlier ([SW71]), I learned the essence of harmonic analysis. There was something about him that has always captivated me. I am talking about the enthusiasm he showed for mathematics, the cleverness of his arguments, and the care with which he used to write his manuscripts. He would not stop until there was not a single doubt about the validity of the arguments, until they were understood by the most profane of mortals. This was true for the courses he gave, as well as for his research work. I remember the care with which he showed us his own work or the work of others on any subject, whether it was interpolation, Hardy spaces, maximal operators, factorization of weights, atomic decompositions, and the many other fields with which he dealt. I keep some of his handwritten notes which, despite the time that has passed, I still use.

It is also worth mentioning the amount of time he spent with his colleagues and students talking about mathematics. It was incredible to see how a person who was immensely busy with issues not always related to pure academics, could squeeze his time in order to have informal work sessions on almost a daily basis. Every guest who came to WashU to work with him or just for a friendly visit (and Guido had lots of friends!) was immediately invited to one of these sessions in his office in Cupples I Hall. One of his students would begin by describing a problem he or she was working on and then everyone present would give their opinion, relating it to similar problems and suggesting techniques to solve it. Guido, as a connoisseur of the ins and outs of the proposed problem, would take the lead. But he never disdained a suggestion, however banal. A phrase he often repeated after an unexpected suggestion was: *well, what you just said it's either true or false. We'll have to see it.*

From a personal point of view, Guido was more than a teacher, a friend, or a colleague. He was the mirror in which I wanted to see myself reflected and from which I learned so many things, professionally and in everyday life. Every time I sit down next to a student to review their master's or doctoral thesis, his image comes to mind. In particular, I think of the immense patience he always had, sitting next to me, while explaining the reason for this or that correction to my work. It was really an unforgettable, and even priceless experience.

As a final thought, I must say that Guido was instrumental in helping mathematical analysis to flourish in Spain, making it easy for almost a generation of Spaniards to go to Washington University, either as doctoral or postdoctoral students and generating collaborations with Spanish institutions. In recognition of his scientific career, the Autonomous University of Madrid organized a conference in his honor in 1993. Undoubtedly, some of the best harmonic analysts of the time came to the congress, many of

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them personal friends of Guido. I think it was a memorable moment that made him immensely happy.

## Anita Tabacco

I met Guido Weiss in October 1982 when I arrived at Washington University to pursue my PhD. I was 22 years old and frightened by having made the decision to leave Italy, where I could have followed a clear and safe road. I was even more frightened by the idea of being evaluated by Professor Weiss, who would immediately notice my knowledge (tiny!) of mathematical analysis.

I still remember my feeling during our first meeting in his office. It seemed that the space was mainly full of books and papers; though it was nothing compared to what I saw the last time I visited Guido in May 2014! Guido was my scientific father and, even more, a very generous mentor. He helped me in all the difficult moments that I had to face as a graduate student. I grew up thanks to him and his example; he had an enormous impact on all the stages of my career.

Guido's lectures were carefully prepared—always looking for a simple way to explain fundamental points. This was not only the case when he was teaching a course, but also when he wanted to introduce a new research topic. He was attentive to details, but with the general picture very clear in his mind. Guido was a Maestro, and his influence on my way of teaching is undeniable. I keep in full view the notes I took in his courses, his lectures, and in front of a blackboard; they are still valuable to me today.

Studying his book written in collaboration with Elias Stein, *Introduction to Fourier analysis on Euclidean spaces* [SW71], and reading the first article he gave me (*Extensions of Hardy spaces and their use in analysis*, coauthored with R. R. Coifman [CW77]) were fundamental for me. In particular, the paper contained all the ingredients that are still contained in my research today, such as spaces of homogeneous type, Hardy and BMO spaces, interpolation theory, doubling measures, Calderón–Zygmund decomposition.

Guido's intelligence, pleasure in life and personal relationships, knowledge, and above all generosity, contributed in a fundamental way to the growth of harmonic analysis in Italy. He played a key role in forming a group of friends and colleagues who met and talked while at Washington University.

I want to remember him as he has always been with all of us: available, full of energy, ideas, and great humanity.

I am happy to have met him on my journey.

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## Rodolfo H. Torres

Guido Weiss had a tremendous influence on my life starting when I was a graduate student. Estela Gavosto (now my wife) and I arrived at Washington University in St. Louis from Argentina with the help of Cora Sadosky. I remember Cora encouraging us to go to WashU because it was "a very special place to be a graduate student" and that "Guido will take very good care of you." Little did we know at the time that in addition to receiving great training and education, we would form many fantastic friendships and relationships with colleagues which continue to this day, due in large part to Guido. This experience has had a huge impact on our professional careers and lives.

Guido and several of our other teachers, in particular Mitch Taibleson and Al Baernstein, were extremely caring to a large group of students coming from China, Italy, Poland, and Spain whose time at WashU overlapped. Guido had established close connections with all these countries and was a magnet for attracting students. In particular, he was one of the first US mathematicians to participate in exchanges with post-Cultural Revolution China, after the reopening of relations with the US.

When we arrived in Saint Louis, Raphy Coifman had moved to Yale from WashU and so the Math Department was eagerly trying to hire other outstanding mathematicians. They succeeded in recruiting Björn Jawerth, Steven Krantz, and later Björn Dahlberg. The atmosphere in the Math Department during this time was incredible. There was a constant stream of visitors from other universities in the US and from all over the world. Practically every week some leading figure in harmonic analysis, complex analysis, several complex variables, or partial differential equations was visiting WashU and giving a lecture. We were encouraged to attend the talks, even if we were not at a point in our education to fully understand everything. The idea was to expose the students to a lot of great math and also to give them a chance to meet people. The accompanying receptions at the homes of faculty, to which graduate students were invited, served the same purpose. The day-long picnics hosted by Guido and his wife Barbara at their farm near St. Louis, were wonderful occasions to build synergy and camaraderie in the department. As were the memorable costume parties that the Italian students used to organize, which several professors would also attend. Guido and his colleagues managed to create a truly unique welcoming environment where we could be both nurtured professionally and also have a good time.

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**Figure 5.** Mitch Taibleson and Guido Weiss in 1994.

Guido was always there to help us with anything he could. In fact, it was typical of Guido to grab the phone and call someone and arrange things for any of us, or just "save the world," as he used to joke while pointing to a big Superman belt buckle he often wore.

When it came time to pick an advisor after the grueling boot camp of first-year qualifying exams, Guido was going to be away for some time, so it was agreed that I would work with Björn Jawerth. Although I was not an official PhD student of Guido, when Björn moved to the University of South Carolina and I decided to stay at WashU, Guido became my closest mentor in the last steps of my thesis work.

In the mid 1980s, harmonic analysis saw the important developments of the T1 and Tb theorems as well as the birth of wavelets. These topics were very central to the focus of much of the research at WashU on the study of singular integrals and functions spaces. Michael Frazier, who was a postdoc at WashU at the time, and Björn developed their $\varphi$-transform theory, producing powerful and unifying decomposition techniques, which include smooth atomic and molecular decompositions for the whole scale of Besov and Triebel–Lizorkin spaces. Many of the applications of these decompositions were then elegantly presented in Guido's collaborative monograph with Michael and Björn [FJW91], which is one of the most cited references on the subject. Björn had suggested to me that I explore a version of the T1 theorem for spaces of smooth functions, in particular those with a lot of regularity, by showing, in the "spirit of St. Louis," that appropriate Calderón–Zygmund operators map atoms into molecules. This would require improving on and developing arguments to extend previous work by M. Frazier, Y-S. Han, B. Jawerth, and G. Weiss along the same lines, but for small regularity. I had obtained some results when Björn learned that Guido and Michael, who were in a special semester on harmonic analysis at the Mathematical Sciences Research Institute (MSRI), were also working on the problem. I got very worried, but Guido arranged for me to visit MSRI and join forces with them. It was great to write my first math paper with him and Michael. Though I never formally collaborated with Guido again, we discussed a lot of mathematics over the years, and his ideas and mathematical contributions inspired much of my other work.

I continued to be in touch with Guido after leaving St. Louis, visiting him many times, and calling him on the phone. It was always a pleasure to meet him in his legendary office, which was often filled with other colleagues, visitors, and students. Guido liked to include everyone in the discussions and ideas he was investigating. I learned a lot from him and not only in mathematics. He was always ready to provide a letter of recommendation, advice about a difficult situation, or just chat about sports, world politics, or joke about life. When I last saw Guido, in his late eighties during a visit to WashU, he complained that he was no longer the super-strong athlete he used to be and could no longer play tennis, another subject he had taught many of his students, but he was still active mathematically and very engaging. He told me about a wavelets project he had been working on with his lifelong friend Ed Wilson, yet another great professor who was a dean when we were students. Guido took me for dinner at one of his favorite Italian restaurants on The Hill. We had a great evening as usual and, as was his wont, he told the server how to correct the name of the Italian dishes on the menu. The multiple languages that Guido spoke were always a source of conversation, amusement, and play on words.

I only had some brief communications with him after my last visit, but I know through Barbara that Guido enjoyed the biographical article my friend and colleague Susan Kelly and I wrote about him shortly before he passed away. I regard that article as a way to express our admiration for Guido. I miss him and think of him in many of my current roles and activities. When facing a difficult problem or trying to help others, I often ask myself what Guido would have done in a similar case. I am extremely thankful for his mentorship and friendship and feel fortunate he has touched my life. He will be forever remembered.

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Figure 4 is courtesy of Fernando Soria.

# Restoring Confidence in the Value of Mathematics by Teaching Undergraduates Math They Will Use

*Tyler J. Jarvis*

## 1. Lost Confidence

Mathematics is deeply beautiful and extremely useful. But even when we mathematicians succeed at helping people see the beauty in mathematics, many don't believe it has value beyond its beauty. Everyone knows that somehow, deep inside some computer somewhere, math is doing something that somebody thinks is useful. But most people don't really believe that math is useful for them. They lack confidence in the value of mathematics.

Students don't hesitate to tell teachers whenever we cover a difficult section or fail to engage them sufficiently in the math we teach: "When am I ever going to **use** this?!?"

**1.1. Even math majors.** But at least college math majors know math is useful, right? Not really. About ten years ago we surveyed the math majors at my university to get a better sense of how to recruit more students into mathematics. We asked them: *Why did you major in math?* and *Why aren't others majoring in math?* Their answer was "I love it, but I know I can't get a job if I don't want to teach. I thought I'd just do something I enjoy now and worry about a job later."

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Their answers should not have surprised me, but they did. I thought math majors knew there were lots of applications of math relevant to them and to their future employment. We had alumni with good jobs that used math, but the students didn't know that—they were just in it for the beauty. Even math majors had no confidence in the value of mathematics.

**1.2. The cost of no confidence.** That lack of confidence in the value of mathematics is a serious problem for at least four reasons:

1. **It diverts away resources.** Funding, faculty lines, and other resources go where administrators and other decision makers think they will create the most value. When we aren't seen as adding value, we lose resources. This is bad for mathematics, bad for us, and bad for students. Conversely, my university has provided our department with additional faculty and other resources as we have built confidence in the value of mathematics and attracted more and happier students.
2. **It damages learning.** Students are more willing to put in the necessary effort to learn if they believe they will get something out of it. They don't all find math as beautiful as we teachers do, and if they don't see it as useful, how long can we keep saying "trust me, you're going to like this" before they quit?
3. **It drives away "world changers."** Many students come to college wanting to change the world. They will not study math if they see it as only beautiful

but not useful. Mathematicians have powerful tools to help overcome poverty, cure disease, and make the world more fair and just. Driving away these students means not only that we have fewer good students, but that these goals will be harder to achieve because the people working on them won't have the tools they need.

4. **It excludes many.** This lack of confidence excludes many students from our major because only a privileged few can afford to spend their college years doing what they like, ignoring their future careers.

**1.3. Wrong solution: just advertise.** Of course math is useful, even for undergraduates needing careers. At the time of our survey our students were getting jobs—some using math. And we knew lots of mathematicians in rewarding math jobs at interesting companies. Maybe, we thought, we just needed to tell students about these jobs. We told them about traditional math jobs in engineering, actuarial sciences, national security, and finance. We also told them about newer math jobs at places like Google, Amazon, LinkedIn, and Pixar.

This sort of worked. The number of math majors started to grow. But it was the wrong thing to do because many were still skeptical of the value of mathematics for them, and, more importantly, it wasn't entirely honest. They were right to be skeptical.

Yes, our alumni got some of those jobs. But not really because of the math we taught them. Maybe they got them partly because of the problem-solving and critical thinking skills we taught. But much more often they got them because of some computer science classes they took on the side. They never got them because they could perform complicated matrix operations by hand or because of their knowledge of unique factorization domains and quadratic reciprocity.

We polled our alumni around this time and they were using math in their jobs—just not the math they learned in their degree. They told us: "I wish my major had prepared me better for my job."

**1.4. Better solution: deliver.** This suggests a better solution: Actually deliver on the promise of mathematical value to our students. By this I mean that we should teach math that students can actually use, and enable them to use it by giving them the necessary skills and tools to work on problems that they truly care about. When we deliver on the promise that math has value, then we can advertise the careers, and then they believe us because then it's true.

Let me be clear here. I am not talking about some sort of job training program. A rich, transformative education in rigorous mathematics and critical thinking (often called a *liberal education*) is a wonderful thing, and we should not give that up. I am talking about integrating the mathematics that is actually used in the real world, whether in computing, biology, engineering, physics, economics, or data science, into that traditional rigorous mathematical liberal education.

What I hope to convince you of is that it is possible to deliver on the promise that math has value, to teach math that students can really use, and to enable them to use it without losing what we love and what is most important about a traditional liberal education in mathematics. Moreover, as explained below, this can be done without a lot of effort from or retraining of faculty whose primary background is in pure mathematics.

## 2. Applied Mathematics is Good Mathematics

Some of you, like me, were raised by true believers in the religion of Bourbaki, Hardy, and Halmos. They taught me to be proud of the uselessness and "purity" of my math and to believe, incorrectly, that only pure math was beautiful, interesting, challenging, or rigorous.

But after 20 years of working in pure math, I finally looked a little closer at applied math, and, to my surprise, I found it to be just as beautiful, interesting, challenging and rigorous as pure math.

The first time I saw how the fast Fourier transform could be used to rapidly find a highly accurate low-degree Chebyshev polynomial approximation of an arbitrary smooth function, I was in awe. It's a glorious combination of beautiful ideas, and it's fabulously useful to boot. If you haven't seen it yet, go look it up—it's fantastic. See [Tre20, Chapter 3] or [HJ20, Section 9.5] for details. Some other things I find beautiful in applied mathematics include Noether's theorem on symmetries and conservation laws; Thompson sampling to optimize the tradeoff between exploration and exploitation in multi-armed bandit problems; and the Metropolis–Hastings algorithm for Markov chain Monte Carlo.

In addition to being beautiful and useful, applied mathematics can also be taught in a way that is at least as rigorous and challenging as the traditional curriculum, and it can be taught in a way that develops mathematical ability and critical thinking skills as well as or better than the traditional major. As evidence for this claim, let me tell you about the undergraduate major at Brigham Young University we call the *Applied and Computational Mathematics Emphasis (ACME)* [ACM22].

## 3. ACME

We started ACME as a way to deliver on the promise of mathematical value to our students. ACME is the brainchild of Jeff Humpherys and is the result of a lot of work by a lot of people, including Jeff Humpherys, Emily Evans, Jared Whitehead, and me, along with scores of other collaborators and generous support from the National Science Foundation. Since we started ACME, our university has provided our department with additional faculty and other resources because we have more, happier students.

I am describing ACME here not to say that everyone should do exactly what we are doing—every school is different, and you'll have to make your own way. But I hope this is a useful proof of concept and that it gives you some ideas of things you could try. And I hope that some of the resources that we have developed can be useful to you.

**3.1. 21st-century mathematics.** The first question we faced was *What mathematics should we teach?* Applied math has traditionally been focused on problems of matter and energy—so much so that in 1998 V. I. Arnol'd insisted that math is a part of physics [Arn98]. But most of our alumni and industry contacts told us that the math they use is less about matter and energy and more about information, data, and computation.

I'll let you decide for yourself whether that's part of physics or not, but it's clear that the mathematics of information and data needs to be a big part of any modern curriculum in applied mathematics. That doesn't mean we abandon traditional applied math, but rather expand its scope to focus primarily on data-driven methods, modeling, and algorithms. These are also the three main components of applied mathematics identified by Weinan E in [E21].

**3.2. Mathematics, not data science.** I need to emphasize that the ACME program is not a degree in data science. It is a rigorous education in the theory and practice of applied and computational mathematics.

What we do in ACME is relevant to data science, but it is not just data science. About a quarter of our students go into data science careers or data science graduate programs. But our students also go into many other careers, and to graduate school in many other disciplines, including pure math, applied math, economics, finance, biology, physics, engineering, computer science, and statistics. And they flourish in those programs because they have a deep and rigorous understanding of mathematics. An alumnus now enrolled in a PhD program in biology wrote about how ACME prepared him for that experience: "I work with machine learning every day, and cookie cutter methods don't necessarily work for the problems I'm trying to solve. I need to be able to read scientific and mathematical papers and really understand how all of the parts of modeling with machine learning fit together. Having some feel for the mathematical foundations of it all really gives me confidence to try things and fail and not be afraid that there is some mysterious mathematics that I don't understand or wouldn't be able to understand if I tried." —Karl Ringger '21

Some students initially think they want more training in data science and and less education in mathematics, but over time they come to appreciate the power of a rigorous education in mathematics. I recently received an email from one of our alumni, currently doing a PhD in network science at Northeastern, about this: "I can't emphasize enough how grateful I am that my background is based in math theory, rather than just knowing how to plug and play with *NumPy* and *scikit-learn*. Thanks for requiring us to learn so much at such a high level, being patient while we complained about it, and encouraging us the whole time." —Cory Glover '19

## 4. Key Features of ACME

From my perspective, the key features of the ACME program are

1. A challenging and rigorous curriculum in mathematics.
2. Lockstep cohorts for the junior and senior years.
3. Computer labs for all advanced theory classes.
4. A student-chosen concentration in an area of application.

The first half of the program is the same as our traditional math major, covering basic mathematical and computer programming prerequisites. But in the junior and senior years students enroll in a lockstep cohort through a rigorous and challenging mathematical core consisting of two theory classes and two lab classes, two hours a day, five days a week, every semester for four semesters. Students also choose a *concentration* of four to five courses in an application area. I'll discuss each of these in more depth below.

**4.1. Challenging curriculum.** The ACME curriculum is at least as intense and demanding as our traditional major. It is built on a mathematically rigorous foundational core with daily (five days per week) homework sets or labs. Details on the curriculum for each course can be found at `https://acme.byu.edu/`.

I know some schools have been thinking about adding a data science program or an applied math program that is less challenging than the traditional major. I strongly recommend against this. First, mathematical tools are powerful, and those who wield them need to understand very well how they work and when and why they don't work. Moreover, there are many benefits of a challenging curriculum to both the students and the program.

*Benefits of a challenging curriculum.* The most obvious benefit is that students learn more, but a challenging curriculum also attracts students, motivates collaboration, develops students' ability to learn, develops ability and confidence to solve hard problems, and brings better job opportunities for graduates.

*Students want a challenge.* Students are attracted by the challenging nature of the ACME program—they don't want a weak, watered-down experience. Here are two typical examples from our anonymous student ratings feedback about this. "I chose ACME because it challenges me." "The most engaging and exhausting mental challenge of my life. I Love It."

*It encourages collaboration.* Another big benefit of the challenging curriculum is that it motivates students to learn to collaborate with classmates. We do take many active steps to foster collaboration, and our cohort system is a big part of that effort (see Section 4.2 below), but the difficult curriculum itself also helps encourage collaboration. I'll let one of the students explain: "I never enjoyed working with other students before ACME, but now I prefer it because I realize that I learn material better when I help others to understand it, and faster when they help me. The high expectations served as a catalyst for good habits that I would not have tried in their absence. With more to be done than I could accomplish on my own, I embraced working with others. The rigor of ACME taught me how to learn and gave me the opportunity to respond compassionately towards my peers. Because the load was challenging we learned—together—the value of working through difficult circumstances and the joy of rising to meet lofty expectations." —Kolton Baldwin '21

*Students learn to learn.* Another benefit of the challenging curriculum is that students learn to learn more rapidly and effectively. Students and alumni often talk about how the ACME program has made them better learners, able to quickly learn new ideas, algorithms, and techniques that their coworkers struggle with. The following quote from a recent graduate is typical: "Mathematics was always a weakness of mine, and I'm now a lot stronger with it. But most of all my ability to soak in mass amounts of new information is what has improved most. It all prepped me for being a quick, efficient learner. I've been set up for life and I'm excited to keep learning." —Lee Woodside '22

*Students learn to solve hard problems.* The challenging curriculum also helps build student ability and confidence to solve hard problems. Here is a comment from a student currently in the program: "Because of ACME, I am no longer afraid of math—math is afraid of me. I'm very grateful for the way that the program has built me into a reliable problem solver." —Sam Goldrup '23

And the following is from an ACME alumna currently working on her PhD at Rice, studying applications of deep learning in medical imaging. "Consistently being challenged by the fast pace of ACME gave me the confidence to apply my deep learning research to imaging physics—something I had no prior background in." —McKell Woodland '18

*Employers want ACME students.* The strong skill set of our graduates means that once someone has hired one ACME graduate, they usually want to hire more of them. Here's an excerpt from an email I recently received from an employer trying to recruit more ACME graduates: "There are many programs out there which claim to prepare students for data science careers only to send them into the job market woefully underprepared.... But ACME students, on the other hand, have passed our technical interviews with flying colors and have shown they have the ability to solve hard problems."

And here is the experience of one alumnus: "Technical leads that have known me now search out for ACME students to hire as a first preference." —Wesley Stevens, '18

*Problems of a challenging curriculum.* I don't want to imply that the challenging curriculum is without its problems. One of the difficulties includes the risk of students' becoming intimidated or developing impostor syndrome. But our lockstep cohort (see Section 4.2) helps to mitigate that, as does the use of objective preparedness measures. It's more likely a student will feel unprepared if we say the prerequisite is "good knowledge of analysis" than if we say the prerequisite is "a B or better in Math 341." ACME faculty and TAs also explicitly coach students about impostor syndrome, why it happens, and how to overcome it.

The opposite problem also occurs, with a few students developing a big ego and a destructive attitude, thinking they are better than students not in the program or students not doing as well in the program. Again, explicit coaching is very powerful, and many students find that although they may be good at one thing (e.g., mathematical proofs), they are not necessarily so good at other things (e.g., computer programming). Needing and getting help from their classmates on the the things they struggle with tends to make them more humble and compassionate in those settings where they excel.

Students also sometimes struggle with time management and the trap of local optimization—focusing too much on one assignment and not enough on the big picture of their learning experience. This is partly helped by coaching from faculty and TAs about better learning strategies and by incentivizing good habits, but sometimes they have to learn it by experience.

Our ACME faculty and TAs regularly meet together to discuss how to best coach and otherwise support the students through these challenges. Managing these different challenges takes real work from the faculty and TAs. But the work is rewarding and brings significant benefits to our students.

"It's HARD, but so powerful." —Jesse Casillas '17

**4.2. Lockstep cohort.** The lockstep cohort starts in fall semester of the junior year and is a fundamental part of the ACME experience. Students take courses with the same classmates for two hours every day for two academic years and study together in common study rooms with those same classmates. They also organize social activities together.

*Benefits of the cohorts.* There are many benefits of the cohorts, both for the students and for the program. These include enabling us to to take advantage of interconnections between the parallel courses, building a sense of teamwork and group support, and building loyal and enthusiastic alumni.

*Cohorts enable interconnections.* Cohorts enable us to take advantage of interconnections between parallel "sister" courses. For example, in one course they learn about orthonormal bases and linear projections, and in the sister course they use that knowledge to understand Fourier series. As another example, in one course they learn about the uniform contraction mapping principle and in another they use that knowledge to prove the stable manifold theorem. Students appreciate the things they learn in one class much more when those things are used right away in another class.

*Cohorts encourage teamwork.* Learning to work together is essential but hard for many math majors. The cohorts help with that. One year a cohort entered themselves in the university intramural frisbee competition and won the championship for the entire university. That was partly because of some expert coaching by one member of the cohort, but it also shows how well they learned to work together.

*Cohorts provide emotional support.* The emotional and social support the cohorts give students is powerful. One student who was struggling with some mental health issues suddenly stopped coming to class and ditched his study groups. His peers recognized he needed help, went to his dorm, and banged on his door until he got out of bed. They told him to get dressed and come with them so they could all work on homework together and get him caught up. This wasn't initiated or even noticed by the faculty until much later, but with the help of his peers he finished the semester strong and is now flourishing in a good graduate program working on his PhD in mathematics. Without the cohort, I don't think he would have finished the semester.

Another student just this month came to consult with me about how to help a classmate struggling with some personal issues. This is a stark contrast to my traditional math classes, where the students mostly don't even know each others' names, despite my best efforts to get them to engage with each other.

*Cohorts grow loyal alumni.* Working together with classmates as a team transforms students into loyal and enthusiastic alumni who stay connected after graduation and generously give time and money to support the students currently in the program. Our university's most recent senior survey indicated that more than 40% of all math majors (both ACME and traditional) were mentored by alumni—significantly more than any other department in our college. ACME students often mention how helpful it was for them to talk to alumni, and this is all a result of alumni volunteering and taking initiative to make themselves available to students. In contrast, before we started ACME we saw almost no alumni mentoring, and alumni of our traditional major still do not mentor students very often.

*Challenges of cohorts.* Of course there are difficulties with a lockstep cohort, including reduced flexibility for both faculty and students. Faculty must coordinate what they teach and when they teach it to be able to build on what is taught in sister courses. And students must take the cohort courses at the time and in the semester that they are taught. This sometimes also requires us to coordinate with other departments to avoid scheduling conflicts. And the schedule doesn't always work perfectly for everyone. Some students need to switch to another cohort, or even take one of the off-ramps we provide to switch back to the traditional major from ACME. Conversely, students who realize late that they want what ACME has to offer can still join a junior cohort for just one semester or one year and use ACME courses to count toward their traditional degree.

The cohort system can also be a challenge for introverts who prefer to work alone or find it difficult to form study groups. Learning to work with others is an important skill even for introverts, but we try to help them overcome some of these hurdles by assisting with study group formation and providing dedicated study spaces and online collaboration tools like *Slack*.

Managing these difficulties takes work, but it's worth it, because of the benefit to the students. And for many students the cohort itself is a strong draw. As one alumnus says, "I chose ACME because of the cohort situation and the in-depth learning about the math for many algorithms used in the industry today." —Wesley Stevens '18

We started the cohort system as an efficiency, to reduce the need for faculty and TA resources, but the benefits of the cohort system are so significant we can't imagine doing ACME without cohorts. In fact, after seeing the power of the cohorts in ACME, our department started a cohort experience for freshmen and sophomores for all our majors (both the traditional major and ACME).

**4.3. Computer labs.** Throughout the junior and senior core, students do a lab every week for each of the two core theory courses they are taking each semester. The general approach we take to labs is that the students first code up an implementation in Python of the mathematical ideas we are treating, then they compare their implementation for speed, scalability, and correctness to the industry standard implementation. Sometimes their code is competitive with the polished industrial version, which they find very rewarding. Finally, they use the mathematical tool to solve an interesting problem.

As an example, one lab involves using the FFT to filter the loud, annoying buzz of the popular vuvuzela noisemaker out of a recording of a World Cup soccer game. Students also experiment with convolution, starting with a recording of a Chopin piano piece played in the studio (a low-echo environment) and then convolving that with a recording of a balloon pop in an echoey stairwell. They get a kick out of hearing how the convolved result sounds like the piano is being played in the echoey stairwell. They often take this to the next level by convolving many of their favorite audio clips with the balloon pop.

Other popular labs include Markov chains for text generation, Perron–Frobenius for PageRank and March Madness brackets, finding Bacon (Erdös) numbers, Monte Carlo integration, Multi-armed bandits, SIR epidemic models, Hidden Markov model speech recognition, Random forests, Kalman filter, HIV treatment, and color quantization with K-means. These labs were developed with financial support from an NSF TUES grant (DUE-1323785) and all of our labs are free and open source [HJE22].

*Benefits of computer labs.* The labs help students learn the math better, improve students' attention to detail, improve students' employability, and motivate students to learn more mathematics.

*Labs improve mathematical learning.* The best way to learn is to teach, and the computer is the dumbest possible student—it does only and exactly what you tell it and never gets the idea, sees the pattern, or fills in the details. To teach the computer the programmer must describe every part of every algorithm and formula and be able to debug all the errors that arise. Doing all that improves the programmer's understanding enormously. As J. Betteridge et al. say, "Learning to use computers well is a very effective way to learn mathematics well: by teaching programming, we can teach people to be better mathematicians" [BCC+22].

*Labs improve attention to detail.* The Python interpreter usually won't run students' programs at all unless they have been careful about every aspect of their code, including syntax, order of operations, and carefully defining variables and methods before using them. Getting immediate feedback on these things in computer labs helps them learn to think more carefully and clearly about similar things in their written mathematics, where feedback is much slower. Importantly, students often seem to respond better to an impersonal error message from a computer than they do to a TA or professor telling them that the their proof is wrong. This helps them learn that mistakes are normal and expected, and that identifying mistakes is essential to growth.

*Labs boost employability.* Computer labs directly build students' programming skills and their ability to convert complex ideas into efficient code. Moreover, the labs help students learn industry-standard tools, improving their employability and giving them a chance to build a portfolio of interesting projects to demonstrate their abilities to prospective employers. The labs focus on using computers and mathematics together, which is not something they can gain just by taking computer courses alongside their math courses, but it is something that employers tell us they want.

*Labs motivate mathematics.* Students are motivated by the applications in labs to learn more mathematics. The theory of Markov chains or the singular value decomposition may feel dry to them, but when they can use the theory to build a cool application, they become more motivated to learn and understand the mathematics.

*Challenges with computer labs.* As with anything, there are challenges with the labs, but so far these have been manageable for us.

The first challenge is limited resources for teaching. Not all our faculty program well, not all that do program well know Python, and faculty are busy with other things. For these reasons we designed the labs to be taught by teaching assistants (both graduate and undergraduate), and that works pretty well for us.

Another challenge is that the lab materials need regular updating, requiring a team of faculty and TAs to review and revise the labs regularly. But the benefits of the labs are so powerful for student learning that they absolutely outweigh the cost to us of managing these relatively small issues.

*Why Python?* We use Python almost exclusively for several reasons. First, students need to learn to program well, so they need to have enough experience in a full-blown programming language to learn it in some depth. Other mathematical and statistical computing tools like *Mathematica*, *MatLab*, *Maple*, and *R* are great for what they do, but as programming languages they are not as versatile nor as widely used outside of the academic community as Python.

Python is the primary language of modern data science and is currently the most popular programming language in use, according to the TIOBE index [TIO]. It is easy to learn, and it is free and open source. So it has been our exclusive tool. There are many packages within Python that can be used for specific applications, and we use many of these in our labs [HJE22], but the underlying tool is always Python. It is possible that Julia will eventually take the place of Python, but Julia is not yet as mature as Python and is not yet widely used.

**4.4. Concentration.** Students are required to do a concentration of four to five courses in an application area of their choice, usually from another department. Because the students have a strong mathematical background, these concentration courses are usually more advanced than a typical minor. Some of the most popular choices include computer science, data science and machine learning, economics, business, biology, and physics.

*Benefits of concentrations.* The concentration helps students learn to communicate across disciplines and see how math is used in a subject they care about. And they use it to prepare for their specific chosen career path, whether that's a job in machine learning, going to graduate school in economics, or starting their own business.

Also, many students are attracted to ACME because the concentration allows them to study both math and another subject they love and use them together, rather than choosing between them. The following quote from a recent graduate is typical of what students tell us about why they chose ACME: "ACME offered me the opportunity to explore biology and mathematics simultaneously.... The idea of having a concentration that was unique to me and my interests really appealed to me." —Karl Ringger '21

*Challenges with the concentrations.* In some concentrations students must fill many prerequisites before they can get to the interesting courses that really use math. Some departments are good about working with us to find alternative paths into those courses, and others aren't. And, of course, the students often need guidance as they choose and navigate their concentration, and that takes faculty time. But it's worth it, because it really helps them.

## 5. Additional Challenges

One big challenge we faced when starting ACME was a lack of suitable curriculum materials. Being naïve, we decided to write our own. The National Science Foundation helped with a grant to support our work; but it really was a lot of work, and I don't recommend it if you can avoid it. I hope that some of what we have done will be useful to you. I've already shared a little about the labs above, but we also wrote some textbooks, which are published by SIAM [HJE17, HJ20]. We were pleased that SIAM produced beautiful hardbound books in full color for less than what it would cost an individual to photocopy the book.

One of the greatest challenges we faced when starting ACME was limited resources, which motivated the lockstep cohort model. The cohort model is efficient, using only two faculty lines to run eight required courses, with labs run by graduate students.

We also had few faculty who knew all the material. To address this we developed our textbooks with faculty in mind as well as students, allowing faculty to learn the material ahead of the students. Many of our faculty also were not proficient at computer programming. To address this we use student TAs to teach the labs, so that the faculty need not code.

Finally, some faculty were suspicious of applied math and were reluctant to support any program that might move resources from pure math to applied math. In fact, when the first draft of the program went to the department curriculum committee for approval, it was unanimously opposed. But eventually the committee and the rest of the department agreed to let us try it, especially when the university academic vice president gave us one faculty line on condition that within five years we meet a target of 40 students enrolled per cohort and 25 graduating per year.

## 6. Results

**Enrollments.** Our first cohort had only 15 students in it, but by the fifth year, we had 70 enrolled in each new cohort and over 60 graduating each year—far exceeding the vice president's requirement for keeping the faculty line. The total number of math majors (traditional and ACME combined, but not math education, statistics, or computer science) has grown from the low 200s to well over 400 (approximately 1.3% of the BYU student body). Since 2014, a year after ACME started, the percentage of minority students in ACME, while still lower than we'd like, has grown by 73%, compared to an increase of only 6% for minorities in the university as a whole.

**Jobs and internships.** We are not a jobs training program, but ACME students generally get much better jobs than our traditional math majors. Data is incomplete because not everyone reports job and salary information back to us, but for those who do report, the highest starting salary of our traditional math majors is roughly the median starting salary for our ACME majors.

We have also seen a large increase in the number of employers coming to recruit our students. And many faculty in other departments now try to recruit our students as research assistants.

**Graduate programs.** Our students have been very successful in top graduate programs in both pure and applied mathematics, but they have also been successful in top graduate programs in other disciplines, including biostatistics, computational biology, computer science, economics, electrical engineering, geology, machine learning, marketing, math education, petroleum engineering, and statistics.

"In my graduate degree [Biostatistics at Berkeley] I have classmates who graduated from Ivy League schools who are constantly baffled by the breadth of topics in computer science, mathematics, and statistics I've been exposed to and the understanding I've retained.... I really am *super* grateful for ACME. It prepared me better than I could have imagined for grad school."—Tyler Mansfield '20

**Faculty.** Although some of our faculty didn't know all the material the first time they taught an ACME course, which meant extra work for them, most of them have loved teaching it. Even most of those who haven't yet taught ACME classes recognize that ACME attracts more good students to math, with a positive spillover into the traditional major and graduate programs. And the university has provided our department with additional faculty and other resources because we have more, happier students.

## 7. First Steps

If you want to start something like this for your students, what should be your first steps? And what is the lowest hanging fruit or most bang for the buck?

*Math + programming.* I feel strongly that the most important thing math majors need is more computer programming in a widely used programming language, ideally merged with their mathematics in a way that enables them to use computers to solve mathematics problems and to implement deep mathematical ideas in efficient code.

One way to start this is to integrate programming labs into linear algebra for math majors. This helps them develop their programming skills, it frees them from the drudgery of solving large linear systems and finding eigenvalues by hand, and it helps them see how tedious computations can be assigned to the computer to let them do interesting things with their mathematics. Students can easily access a powerful computing environment through free tools like Google Colab without any special help or expertise. For an example of how this can be done, see [HSWS22].

Another thing to consider for every math major is a course in algorithms and optimization, where they really think about the mathematics of computation and learn about optimization—the fundamental tool of data science, machine learning, and statistics. That course should also have lots of programming labs. We teach this course using [HJ20] and the labs in [HJE22], but there are many other ways that you could do such a course.

Finally, whether you adopt these math-plus-programing courses or not, consider requiring at least one or two standard computer science courses of all your math majors. Although Python and C++ are some of the most useful languages for mathematicians, courses in other popular languages like Javascript and Java are also useful.

*Concentrations: math + X.* Students benefit from seeing how mathematical ideas are used in other disciplines. But not every student likes the same applications, and not everyone will respond to the specific applications you choose to show in your classes. Consider encouraging or even requiring your students to take classes outside of mathematics in a complementary subject where they can apply their skills to something that interests them. It need not be in a STEM field; the social sciences, business, and other disciplines benefit greatly from the skills that mathematicians bring.

*Cohorts.* Cohorts are extremely powerful for improving the learning experience and helping students learn important soft skills. Even if you find it difficult to form formal cohorts among your majors, consider doing things to approximate cohorts. That could be scheduling two classes that majors typically take concurrently to run in the same classroom, back to back. It could also mean helping them form study groups, and, if possible, dedicating space for them to study together—maybe even in that same back-to-back classroom in the hours before and after the two classes. Almost anything that gets students talking to each other and working together is helpful.

## 8. Conclusion

I don't think that our approach is necessarily the perfect fit for every program. But I hope that I've been able to give you some ideas of both why and how to implement a program that teaches students mathematics that they can use, and prepares them to actually use it.

When the students see that what we have to offer is relevant to their goals, their life, and their ambitions, then they are willing to trust us when we then ask them to do hard things. In the words of one student on our anonymous student ratings form, the experience "globally optimized our learning, happiness, and personal growth. That's all you really need to know, since this is an optimization class."

We can restore confidence in the value of mathematics by delivering what students have been promised—useful math and the practical skills to use it. I hope I've been able to convince you that we can do this without losing what we love about the traditional math major. Applied math is beautiful, and it can and should be taught in a way that is rigorous and challenging. I ask you to consider how you can use these ideas to open the doors for more of your students to enjoy mathematics, to succeed in mathematics, and to use mathematics to make the world a better place.

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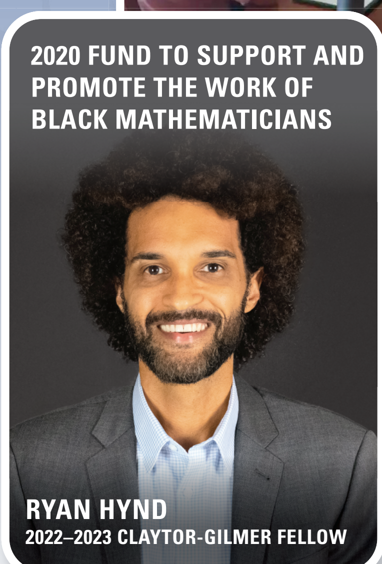


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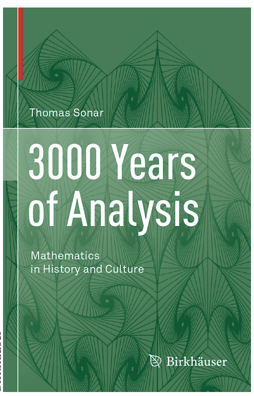
# BOOK REVIEW

# 3000 Years of Analysis

## *Reviewed by Anthony Weston*



Birkhäuser

***3000 Years of Analysis***
Thomas Sonar

*3000 Years of Analysis* by Thomas Sonar provides a mathematical and cultural excursion through the historical development of mathematical analysis from ancient to modern times. The mathematical focus of the text is placed on models of continuous change such as the differential and integral calculus. This focus naturally includes the study of infinitely small quantities and culminates with a discussion of the formal development of nonstandard models of analysis in the twentieth century. Sonar notes that in choosing a duration of 3000 years, he is making a compromise since one cannot say with certitude exactly when notions of analysis began to germinate.

The original German language editions of *3000 Years of Analysis* were published in 2011 and 2016. The focus of this review is on the English translation of the 2016 edition. Sonar states in his preface to the English translation that he was partly motivated by an intention to make "the history of analysis available to interested nonspecialists and a broader audience." The same preface contains an elegantly stated definition of analysis: "In essence analysis is the science of the infinite; namely the infinitely large as well as the infinitely small. Its roots lie already in the fragments of the Pre-Socratic philosophers and their considerations of the 'continuum', as well as in the burning question of whether space and time are made 'continuously' or made of 'atoms'. Thin threads of the roots of analysis reach even back to the realms of the Pharaohs and the Babylonians from which the Greek[s] received some of their knowledge."

Despite the appeal to nonspecialists Sonar makes clear in his preface that the apprehension of analysis does not come for free: "But not later than with Archimedes (about 287–212 BC) analysis reached a maturity which asks for the active involvement of my readership. Not by any stretch of imagination can one grasp the meaning of the Archimedean analysis without studying some examples thoroughly and to comprehend the mathematics behind them with pencil and paper." In practical terms, this means that the purely mathematical passages of *3000 Years of Analysis* require at least a good undergraduate background in mathematics. In other words, the level of mathematical difficulty of *3000 Years of Analysis* is comparable to that of *The Historical Development of the Calculus* by C. H. Edwards, Jr [Edw79]. Indeed, as Sonar makes plain, there are a number of instances where he explicitly follows Edwards.

A unique feature of *3000 Years of Analysis* is that it provides an exquisitely detailed treatment of the history of analysis on three distinct levels: (1) the historical and cultural sweep of the times in which key advances took place is given, (2) ample biographies of the personages behind the advances are given, and (3) the mathematical foundations of the advances are presented. Such an undertaking is both a formidable task and a delicate balancing act. The outcome is a fascinating and valuable addition to existing literature on the history of mathematics. We now pick up some of the main threads of Sonar's monograph beginning with the fourth century BC.

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*Communicated by* Notices *Book Review Editor Katelynn Kochalski.*



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At this time the ancient Greek mathematicians were well aware of the existence of incommensurable geometric magnitudes and the implied limitations of the Pythagorean theory of proportionality. These deficiencies were addressed by Eudoxus of Cnidus (408–355 BC). Eudoxus was a student at Plato's Academy in Athens and he is universally regarded as the greatest mathematician of the fourth century BC, not least because of his theory of proportionality. Eudoxus defines two geometric ratios $a : b$ and $c : d$ to be *proportional*, denoted $a : b = c : d$, if and only if for any two given positive integers $m$ and $n$, it follows that either (1) $na > mb$ and $nc > md$, or (2) $na = mb$ and $nc = md$, or (3) $na < mb$ and $nc < md$.

Eudoxus's extensive theory of proportionality appears in Book V of Euclid's *Elements*. In the case of two incommensurable magnitudes $a$ and $b$, Eudoxus's definition of proportionality partitions the rational numbers into two disjoint subsets $U$ and $O$, where $U$ consists of all rational numbers $m/n$ such that $m : n < a : b$ and $O$ consists of all rational numbers $m/n$ such that $m : n > a : b$. Of this partition Sonar notes: "At the 'interface' between $U$ and $O$ a new number may be defined which obviously has to be an irrational one. We had to wait well until the second half of the 19th century before Eudoxus's theory of proportions could be utilized for the construction of the real numbers. This fundamental step was finally carried out by the mathematician Richard Dedekind (1831–1916) from Brunswick, Germany."

Eudoxus was also the first to state what has since become known as the *Archimedean axiom*. Namely, given two geometric magnitudes $a$ and $b$, there exists a positive integer $n$ such that $na > b$. The germ of this axiom appears in Book V of Euclid's *Elements* (Definition 4) where it is stated rather differently: Two geometric magnitudes "are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another." On the basis of this axiom, Eudoxus was able to prove that if two geometric ratios $a : c$ and $b : c$ satisfy $a : c = b : c$, then $a = b$. Eudoxus's proof is a classical exercise in reductio ad absurdum. Sonar also notes: "With the Archimedean axiom Eudoxus also brought a method for the computation of areas to life: the method of exhaustion."

Lurking in the work of Eudoxus and other ancient Greek mathematicians we see implicit considerations of infinite processes. By adroitly using the *Archimedean axiom* and the *method of exhaustion*, Eudoxus and other ancient Greek mathematicians were able to explicitly avoid taking limits when calculating areas and volumes. Nevertheless, notions of the infinite were hotly debated by ancient Greek scholars. Two schools of thought erupted: atomism (the existence of fundamental indivisible components in nature) and the theory of the continuum (that which is always divisible no matter how often it is divided). Leucippus (5th century BC) and his student Democritus (460–370 BC) are credited with founding atomism. They contended that matter is not infinitely divisible and that it is, indeed, composed of individual discrete particles or atoms that cannot be divided. Aristotle (384–322 BC), who drew a crucial distinction between potential and actual infinities, was a major proponent of the theory of the continuum. For Aristotle, the continuum is a potential infinity: No matter how often a continuum is divided, a continuum will remain.

Foremost in the canon of ancient Greek mathematics are the brilliant works of Archimedes of Syracuse (287–212 BC). In some ways, Archimedes was a proto-engineer whose inventions involving levers, pulleys, and screws drew great acclaim during his lifetime. However, as his extant writings make patently clear, Archimedes's primary focus was undoubtedly on mathematics. These writings focus on area, length, and volume calculations and they hone the *method of exhaustion* into a tool of phenomenal precision. Scholars of Archimedes identify three codices of his writings that are generally referred to as A, B, and C. As Sonar explains: "Already with the writings contained in codices A and B Archimedes could be identified as a great mathematician and physicist, but it is codex C that catapulted Archimedes into the heaven of immortals and gave him a place of honour at the side of Newton and Leibniz."

Codex C was lost for the best part of a millennium only to come to light as partially erased text in a medieval prayer book that resurfaced in 1899. During the summer of 1906, using nothing more than a magnifying glass, the renowned Danish philologist and historian Johan Ludvig Heiberg (1854–1928) determined that the text of the prayer book was written over several treatises of Archimedes, including two previously unknown works: *The method* and *Stomachion*. The greatest revelation in Codex C is undoubtedly *The method* because it presents the mechanical heuristic that Archimedes used to discover some of his most outstanding quadrature and center of gravity results. The rationale behind Archimedes's heuristic was based on levers in equilibrium and he went on to apply the method with extraordinary success by "weighing" indivisibles.

We see in the sixth century AD a tremendous phase transition in the history of mathematics. As the Roman empires crumble a great confluence of Greek, Persian, and Indian mathematical ideas takes place in the emerging Arabic realms. The prophet Mohammed was born in Mecca in 570 AD and went on to found Islam after receiving divine inspiration in the Cave of Hira, Jabal an-Nour (mountain), in 610 AD. Within decades large swathes of the Greek-Hellenistic world, the Iberian Peninsula, and North Africa became subject to rapidly emanating influences of Arabic and Islamic culture. Manuscripts of the ancient Greek, Persian, and Indian mathematicians were translated into Arabic and disseminated throughout the Arabic realms.

The mathematician and astronomer al-Khwārizmī is thought to have lived from 780–850 AD. Not much is known about al-Khwārizmī's life but he worked at the Grand Library of Baghdad under the patronage of the Abbasid dynasty Caliph Ma'mun. During this period Baghdad was a renowned center for the study of the works of ancient Greek, Persian, and Indian scholars. In terms of mathematics, al-Khwārizmī wrote influential textbooks on arithmetic and algebra. His textbook on arithmetic, which only survived to more modern times in Latin translation (*Algoritmi de numero Indorum*), dealt extensively with the Hindu art of reckoning. Manipulation of decimal numerals, positional notation, and a symbol for zero all figure prominently in *Algoritmi de numero Indorum*.

A central figure in the firmament of Islamic Golden Age scholarship is the physicist and mathematician al-Haytham (965–1039 AD). Considered to be a parent of modern day optics, al-Haytham underpinned his investigations into the nature of light and vision with direct experimental evidence. His multi-volume work on optics was translated into Latin as *Opticae thesaurus Alhazeni* in 1270 and subsequently became influential in Western Europe. Al-Haytham was a master practitioner of the *method of exhaustion* for calculating areas and volumes. In particular, al-Haytham obtained nontrivial generalizations of some of Archimedes's celebrated volume results.

The Dark Ages refer to the European period of some four centuries that followed the fall of the Western Roman Empire in 476 AD. During this period of stagnation and decline the scientific posture of Western Europe was seriously corroded and, in fact, barely limped along. The masterworks of the ancient Greeks were reduced to a dim memory during the Dark Ages. Of this period Edwards [Edw79] writes: "Only the Latin encyclopedists preserved any connection, however tenuous, with the intellectual treasures of the past." This statement is beautifully unpacked by Sonar in his treatment of "the great time of the translators." One of the earliest translators of Arabic texts was Adelard of Bath (1080–1152). Sonar notes: "One of the first Latin translations (from the Arabic) of Euclid's *Elements* flew from his quill as did astronomical tables compiled by al-Khwārizmī." And so it happens, punctuated by crusades and other calamities, that a rich mathematical tradition, nurtured and enriched by Arab scholars, slowly finds its way back into late medieval Europe.

Between 1328 and 1350 a group of logicians and natural philosophers at Merton College in Oxford developed a theory, that became known as *latitudes of forms*, to quantify "qualities" such as heat and speed. The Merton scholars introduced rigorous definitions of uniform motion and acceleration, and derived the *mean speed theorem* for uniformly accelerated bodies. Sonar describes the *mean speed theorem* as being the "First Law of Motion." The theorem represents a radical departure from millennia of primarily static mathematical thought and it inaugurates kinematics as a fundamental field of scientific inquiry. Sonar deftly explains how this profound transformation arose from within Scholasticism and he provides vivid insights into some of the key historical figures, including richly detailed passages on Robert Grosseteste, Roger Bacon, Albertus Magnus, Thomas Bradwardine, and Nicole Oresme.

The ideas of the Merton College scholars spread rapidly to France and Italy in the middle of the fourteenth century. The theory of *latitudes of forms* was keenly studied and extended by the Parisian polymath Nicole Oresme (1320/25–1382). Oresme introduced graphical representations of intensities of qualities and provided a geometric verification of the *mean speed theorem*. In Oresme's work we see glimmers of the graphical representation of functional relationships and steps being taken towards the introduction of coordinate systems.

At the beginning of the fifteenth century a clear shift away from the Middle Ages became palpable in Europe. The ensuing Renaissance lasted for roughly two centuries and provided the pathway to modernity. The period of the High Renaissance in the Italian states, which started in about 1495 and lasted for around thirty years, is of central importance to art historians, not least because of the epic artworks of Leonardo, Michelangelo, and Raphael. The vivid cultural flowering of the Renaissance encompassed a period of rapid scientific progress. As Sonar points out, a key driver of this scientific progress was the recently invented Gutenberg printing press.

One may view the Renaissance as embodying a shift away from the Church-centered Scholasticism of the Middle Ages and toward the establishment of a new era of Humanism–a return to the centrality of the individual as emphasized in classical antiquity. Renaissance Humanism included the notion that natural phenomena may be completely explained by science and mathematics. Nowhere was this notion more clearly displayed than in the radical astronomical models of Copernicus, Kepler, and Galileo. Sonar pays particular attention to Johannes Kepler (1571–1630) and this is hardly surprising because Kepler, apart from being one of the most extraordinary figures in the history of science, was a proponent of using infinitesimals as a means to simplify the calculations of areas and volumes. The tumultuous life and times of Kepler are vividly sketched by Sonar over the course of some twenty pages. These biographical musings are followed by a treatment of Kepler's tactics for using geometric infinitesimals to calculate areas (such as Kepler's "barrel rule" for determining the area under a parabola) and volumes of solids of revolution (such as the torus).

The first half of the seventeenth century also witnessed the majestic contributions of René Descartes (1596–1650) and Pierre de Fermat (1607–1665) to the development of analytic geometry and analysis. As Sonar implies, Descartes and Fermat were polar opposites in terms of temperament. For instance, about Descartes, Sonar writes, "After an utterly eventful life as a superb philosopher, mathematician, physicist, bon vivant, mercenary, and wrangler he died on 11th February 1650, shortly before his 54th birthday, in Stockholm." In stark relief we have Fermat, who Sonar describes as being "the most unobtrusive of the great French mathematicians of the 17th century – no known scandals, no life as a mercenary, and no sharp turning points in his life; at least as far as we know. He was, however, one of the most profound thinkers of his age." The historical significance of the new analytic geometry of Descartes and Fermat is succinctly summarized by Edwards [Edw79]: "Whereas the Greek geometers had suffered from a paucity of known curves, a new curve could now be introduced by the simple act of writing down a new equation. In this way, analytic geometry provided both a much broadened field of play for the infinitesimal techniques of the seventeenth century, and the technical machinery needed for their elucidation."

Until the early seventeenth century the construction of tangent lines to curves were, for the most part, a rarity. In the 1630s, aided by the new analytic geometry, Fermat and Descartes introduced novel methods for constructing tangent lines to hitherto unknown classes of curves. Fermat used a poorly explained but presumably infinitesimal based "pseudo-equality" technique to construct tangent lines. In contrast, the "circle method" of Descartes is of a purely algebraic nature. Descartes was pleased to avoid the use of infinitesimal arguments in his construction of tangent lines but it came at a high cost in terms of the monotonous algebraic calculations that had to be carried out. This shortcoming of the circle method of Descartes was largely alleviated in the 1850s by algorithmic advances of Johann Hudde (1628–1704) and René de Sluse (1622–1685) that streamlined the construction of tangent lines.

Descartes's circle method only applies to explicitly defined curves of the form $y = f(x)$, $f^2$ a polynomial. In the mid 1650s de Sluse took things even further and developed an algorithmic procedure for constructing tangent lines to implicitly defined curves of the form $f(x, y) = 0$, $f$ a bivariate polynomial. *Sluse's rule* was published in the 1672 *Philosophical Transactions of the Royal Society* but without any indication of how the rule was obtained. As Edwards [Edw79] points out: "Whatever may have been the means by which Sluse's rule was first discovered, the principal significance of the rules of Sluse and Hudde lay in the fact that they provided general algorithms by which tangents to algebraic curves could be constructed in a routine manner."

In Italy, during the time of the great works of Descartes and Fermat, Galileo's disciple Bonaventura Cavalieri (1598–1647) put forth radical new ideas on how to apply indivisible techniques to solve previously inaccessible quadrature and cubature problems. The most well-known theorem of Cavalieri is the following simplified principle: "If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio." (Quoted from [Edw79, p. 104].)

Cavalieri was also skilled in the manipulation of the cross sectional indivisibles of lone geometric figures. For example, by calculating "sums of powers of lines", Cavalieri was able to determine the area under the curve $y = x^n$ ($n$ a positive integer) on the interval $[0, 1]$, albeit not very rigorously. As Sonar remarks: "Looking at Cavalieri's 'summations' today is breathtaking and hair-raising. There were lines of thickness 0 airily 'summed' and put into ratios; hence it may be not surprising that opposition formed quickly against Cavalieri's method of indivisibles."

During the sixteenth century mounting pressure to simplify tedious arithmetic calculations led to the invention of logarithms by John Napier (1550–1617). In essence, Napier isolated his definition of a logarithm from a series of proto-logarithmic tables and a kinematic model involving points moving on a pair of line segments. In order to construct these tables Napier combined judicious numerical choices together with some subtle nonlinear interpolation schemes. Napier's logarithmic tables were published as a slender volume in 1614, adroitly entitled *Mirifici logarithmorum canonis descriptio*. The impact of Napier's wonderful logarithms was both dramatic and immediate. For example, Kepler used Napier's tables to simplify the computations that led to his discovery of the third law of planetary motion.

At the outset of the seventh chapter of *3000 Years of Analysis* Sonar presents twenty pages of thoughtfully written biographical material on Isaac Newton (1643–1727). Sonar's treatment is concise, deft, and finely balanced. The capacity of Newton to be cantankerous is clearly stated but not overly dwelt upon and, for this, the reader can be most grateful. Sonar charts Newton's difficult childhood days (when great ingenuity was already abundantly evident in the young man), the sublimely productive Cambridge years 1663–1687, and eventual decline as a well paid government official. Of the beginning of the sublime period, Sonar writes: "In a notebook we find Newton's true occupation in 1663: Theorems concerning conic sections following Pappus, remarks concerning geometrical theorems by Viète, van Schooten, and Oughtred, theorems concerning arithmetic by John Wallis, methods of grinding lenses, questions of natural philosophy, theology, and alchemy."

During the period 1663–1687, Newton became an expert manipulator of infinite series. The starting point for Newton was studying John Wallis's (1616–1703)

*Arithmetica infinitorum*. Newton's reading of Wallis led him to formulate the binomial series expansion of $(1+x)^{\alpha}$, $\alpha$ a constant. The case $\alpha = 1/2$ was previously known to Henry Briggs (1561–1630) through his work on logarithms and finite versions of the binomial theorem (when $\alpha$ is a positive integer) had been known since antiquity. Critically, Newton's formulation of the binomial theorem allowed for the free use of negative and fractional exponents.

Newton's interest in the binomial theorem was not idle. In developing the calculus of fluxions, Newton relied on the binomial theorem to unlock implicit differentiation for curves of the form $f(x, y) = 0$, $f$ a bivariate polynomial. As Sonar notes: "Newton thought in terms of motion and velocities when he attempted to compute tangents of curves of the form $f(x, y) = 0$. In the eyes of Newton the curve $f(x, y) = 0$ itself 'results' from the points of intersection of two moving lines which we can interpret as being the velocity components in $x$- and $y$-direction."

Newton wrote up his work on fluxions in a manuscript that is dated October 1666. This so-called *October tract* was circulated to some English mathematicians but it was not formally published. Included in the October tract is Newton's "inverse method of fluxions." This was the first statement of the fundamental theorem of calculus in the history of mathematics. In the words of Sonar: "Modern analysis could only begin with the thorough knowledge that differentiation and integration are inverse operations. Barrow had this result implicitly but it was left to Newton and Leibniz to clearly acknowledge the central place of the fundamental theorem." Newton went on to use his method of fluxions to show that problems of quadrature, constructing tangent lines, rectification of curves, extreme values, and so on, all fall under one umbrella. Newton had thus unified millennia of mathematical analysis into a single coherent whole. Gottfried Wilhelm Leibniz (1646–1716), within the space of a few short years, would independently obtain the fundamental theorem of calculus, but from a different point of view to that of Newton.

Leibniz completed a baccalaureate in philosophy and mathematics at the University of Leipzig in 1663 and was awarded a doctorate of law at the University of Altdorf in 1667. In 1672 Leibniz traveled to Paris in a diplomatic capacity for the Elector of Mainz. This placed Lebniz within the orbit of a coterie of superb European scholars, including the Dutch mathematician Christiaan Huygens (1629–1695). Huygens is credited with bringing Leibniz up to pace on the mathematical literature of the times. In this way, Leibniz was exposed to significant treatises such as *Arithmetica infinitorum* by John Wallis and *Opus geometricum* by Gregorius Saint-Vincent (1584–1667). During his four years in Paris, Leibniz began to assemble and finesse his own invention of the calculus.

In discussing Leibniz's development of the calculus Sonar makes the following preliminary remark: "We are used to present Newton's results in Leibniz's notation simply because it turned out to be more feasible." Indeed, the primacy of Leibniz's notation in calculus is not simply a quirk or historical accident. During his life, Leibniz was intensely interested in finding a universal language or *characteristica universalis* that would allow complicated notions of reasoning to be distilled into simpler components. The program of characteristica universalis is a recurring theme throughout the works of Leibniz and it necessarily involves a preoccupation with symbols and notation. The outcome is summed up by Edwards ([Edw79, p. 232]): "His infinitesimal calculus is the supreme example, in all of science and mathematics, of a system of notation and terminology so perfectly mated with its subject as to faithfully mirror the basic logical operations and processes of that subject. It is hardly an exaggeration to say that the calculus of Leibniz brings within the range of an ordinary student problems that once required the ingenuity of an Archimedes or a Newton."

Section 7.2.4 in *3000 Years of Analysis* deals with the infamous priority dispute that erupted between Newton and Leibniz over the invention of the calculus. Newton had developed the outline and framework for his fluxion-based calculus during the years 1664–1666 but he did not formally disseminate the work until the publication of *Philosophiae Naturalis Principia Mathematica* in 1687. Prior to 1676, Newton's calculus of fluxions and fluents was for the most part unknown outside of England. An exception to this was a letter that Newton had sent to de Sluse in 1672 on the matter of constructing tangents. In contrast, Leibniz developed his own differential-based version of the calculus during the years 1672–1676, almost a decade later than Newton. However, the priority of publication for works on both differential and integral calculus belongs to Leibniz. In 1684 and 1686 Leibniz published articles on differential calculus and integral calculus (respectively) in the Leipzig periodical *Acta Eruditorum*.

Using Henry Oldenburg as an intermediary, Newton and Leibniz corresponded directly about the origins of the calculus during the second half of 1676. Newton addressed two letters to Leibniz that have since become known as *Epistola prior* (13 June 1676) and *Epistola posterior* (24 October 1676). The first letter was open and friendly but the tone of the second letter was more circumspect and it used an insoluble anagram at a critical juncture to secrete the true scope of fluxional calculus from Leibniz. By the end of 1676 some lines had been drawn but there was, as yet, no rancorous priority dispute between Newton and Leibniz. Matters took a turn for the worse in 1684 with Leibniz's publication of *Nova methodus pro maximis et minimis* in *Acta Eruditorum* and were overheated by 1699. As Blank [Bla09] puts it: "Throw in a priority dispute, charges of plagiarism, and two men of genius, one vain, boastful, and unyielding, the other prickly, neurotic, and unyielding, one a master of intrigue, the other a human pit bull, each clamoring for bragging rights to so vital an advance as calculus, and the result is a perfect storm."

Subsequent to giving a measured treatment of the priority dispute, Sonar turns to the actual mechanics of Leibniz's development of the calculus. The extent to which Leibniz used infinitesimals is discussed at length, with particular attention given to the so-called characteristic triangle and its relation to Leibniz's *transmutation theorem*:

$$\int_a^b y\,\mathrm{d}x = xy\Big|_a^b - \int_a^b xy'\,\mathrm{d}x.$$

Leibniz was keenly aware of both the significance and versatility of the transmutation theorem. It put Leibniz in a position to re-derive virtually all previously known plane quadrature results and to provide some brilliant new applications. One such new result was Leibniz's "arithmetical quadrature of the circle" which leads to the heavenly series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

The brightest mathematical star of the Age of Enlightenment was Leonhard Euler (1707–1783). Born and educated in Basel, Switzerland, Euler entered the University of Basel at the age of 13 or 14 and was mentored in mathematics by John Bernoulli (1667–1748). Euler's first mathematical work, *Constructio linearum isochronarum in medio quocunque resistente*, was published in *Acta Eruditorum* in 1726. This paper marks the beginning of a prodigious and wide ranging mathematical output by Euler, the collected works of whom exceeds 70 hefty volumes.

After the immense flowering of infinitesimal analysis in the eighteenth century an unease about rigor started to become pervasive at the beginning of the nineteenth century. Or as Sonar puts it: "After the death of Euler many mathematicians believed that there would not be much left in mathematics worth[y] of study. On the other hand one felt a certain discomfort concerning the foundations of analysis which was triumphant in applications but operated still upon infinitely small quantities or even, as with Euler, upon 'zeros'." It is not surprising then that mathematical research during the nineteenth century focused on consolidation and a search for rigor, at least in terms of analysis. Against this backdrop a new wave of mathematicians began to apply themselves to the determination of a more rigorous basis for (infinitesimal) analysis and related notions such as continuity.

One of the earliest and most important figures in this new wave of rigor-oriented mathematical analysis was Bernhard Bolzano (1781–1848). In 1817 Bolzano published a note in Prague entitled *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*. It is reasonable to assert that this note heralds the advent of nineteenth century mathematical analysis for in it we find (arguably) the first precise formulation of continuity. Bolzano defined continuity as follows: $f$ is *continuous* at $x$ if the "difference $f(x+\omega) - f(x)$ can be made smaller than any given quantity, if one makes $\omega$ as small as one wishes." Bolzano's note goes on to give a rigorous proof of the *intermediate value theorem*.

As Sonar explains, Augustin Cauchy (1798–1857) provided a proof of the *intermediate value theorem* at around the same time as Bolzano. However, whether or not Cauchy formulated continuity in similar modern terms to Bolzano remains moot. Across a number of picturesque passages Sonar evokes the rich tapestry of the life, times and mathematics of Cauchy, writing at one point: "As a teacher Cauchy turned out to be a revolutionary. Since he considered analysis being indispensable to engineers he gave lectures on that topic. He developed a rigorous concept formation of the limit and attached much importance to utmost accuracy which discouraged his students."

Prior to the nineteenth century there was no formal definition of the definite integral as we know it today. In the eighteenth century integration typically entailed finding anti-derivatives in the spirit of either Newton or Leibniz. Of this century, Edwards [Edw79] writes: "Neither limits of sums nor areas of plane sets were sufficiently well understood to provide a solid basis for a logical treatment of the integral." Cauchy was the first to develop a notion of the definite integral that was predicated in terms of limits of sums, rather than anti-derivatives. The "Cauchy integral" of a continuous function on a compact interval was first introduced by Cauchy in his textbook *Résumé des leçons sur le calcul infinitésimal*.

Despite the success of putting integration on a more secure footing, the Cauchy integral was too limited in its scope to deal with basic questions that had already arisen from the works of Joseph Fourier (1768–1830) and Lejeune Dirchlet (1805–1859) on the representation of functions by trigonometric series. This limitation led Bernhard Riemann (1826–1886), who was also profoundly interested in Fourier series, to develop a more general definite integral. Riemann gave necessary and sufficient conditions for a bounded function to be (Riemann) integrable and he pointed out that it is *possible* for a function with a dense set of discontinuities to be integrable.

Edwards [Edw79] mentions that "Cauchy occasionally stumbled conspicuously, as in failing to distinguish between continuity and uniform continuity or between convergence and uniform convergence." One of the first persons to grasp the importance of uniform convergence was Karl Weierstrass (1815–1897). In 1872 Weierstrass stunned the mathematical world by exhibiting a nondifferentiable continuous function. Prior to the publication of Weierstrass's example there had been a common misapprehension that a continuous function may only have isolated points of nondifferentiability. (Bolzano was under no such illusion as he had given an example of a nondifferentiable continuous function in the 1830s. However, in a quirk of fate, Bolzano's example was not published until the 1920s.) Part of the fallout from Weierstrass's example was a realization that the foundations of analysis needed further attention, especially in respect of the construction of the real numbers. This led to a flurry of constructions of the real line in the early 1870s, the most enduring of which are those of Richard Dedekind (1831–1916) and Georg Cantor (1845–1918).

Chapter eleven of *3000 Years of Analysis* deals with the twentieth century renaissance of infinitesimal analysis. Sonar recounts that inklings of this revival may be found in an unpublished "black book" *Vom Unendlichen und der Null – Versuch einer Neubegründung der Analysis* that was written by Curt Schmieden (1905–1991). In the mid 1950s Detlef Laugwitz (1932–2000) became acquainted with the "black book" and this eventually led to a joint paper with Schmieden [SL58]. This paper presents a largely constructive version of nonstandard analysis but it is one in which the (putative) hyperreal numbers $\mathbb{R}^*$ form only a partially ordered ring and for which the "transfer principle" from the reals to hyperreals is quite limited. A considerably more satisfactory version of nonstandard analysis was developed by Abraham Robinson (1918–1974) during the 1960s. Robinson's nonstandard analysis is based on model theory and it is one for which the hyperreal numbers $\mathbb{R}^*$ form a non-Archimedian field equipped with a substantial transfer principle.

Sonar's discussion of nonstandard models of analysis completes an epic walk through several millennia of mathematical discovery. It is apt that Sonar's treatment of the historical development of analysis effectively begins and ends with the continuum. We remark that the sheer depth of historical detail included in *3000 Years of Analysis* sets it apart from other sublime works such as that of Edwards [Edw79]. We further remark that Sonar's monograph is richly illustrated with a plethora of geometric figures, line drawings, engravings, reproductions of historical artworks and photographs.

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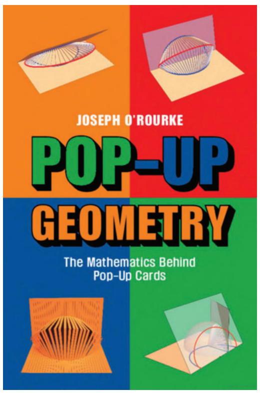




***Pop-Up Geometry***
*The Mathematics Behind Pop-Up Cards*
By Joseph O'Rourke

Most people can identify with the joy that pop-up books or pop-up cards brought as a child (and may still bring to our adult selves). How many of us have opened a pop-up and wondered about the mathematics required for the image inside to jump off the page? *Pop-Up Geometry* lifts the veil to reveal the magic of the mathematics at play.

*Pop-Up Geometry* requires only a high school math background with knowledge of algebra, geometry, and basic trigonometry. Using this as a launching point, it tackles mathematical concepts such as vectors, parametric equations, solving systems of equations, and platonic solids, grounding each topic in its relationship to pop-up cards. The mathematical concepts, theorems, and exercises are all color coded in easy to digest boxes.

Complete with vibrantly colored graphics, companion animations that depict the motion described in the book, and templates for the reader to make the pop-up creations the book is analyzing, this book makes it easy for the reader to engage with the material they are learning. It also includes exercises broken into different categories based on level of difficulty, and the solutions to the exercises can be found in the last chapter of the book. This would be an excellent book for a high school student or undergraduate interested in mathematics and geometry, or to provide inspiration for activities in a math for liberal arts class.

***What are Tensors Exactly?***
By Hongyu Guo

Most complex definitions evolve over time as the theory surrounding them is developed. Tensors are no different in that regard. They were first introduced by J.W. Gibbs in 1884. Woldemar Voigt developed the idea further in the late 1800s and Hassler Whitney defined a tensor product in the late 1930s. As a relatively young topic, the definition of a tensor has taken on many forms in the recent past. To highlight this issue and the dynamic form of the definition, in the first chapter of the book, Guo gives seven definitions, not all of which are equivalent. This book aims to help clarify what tensors are, in a way that is intuitive, yet maintains a high level of mathematical rigor.

Not all tensors are complicated and unfamiliar; linear transformations, for instance, are tensors. The author frequently creates analogies with structures from linear algebra to help the reader better understand. Several chapters of the book are dedicated to exploring how tensors are used in fields such as physics and machine learning. Some of the topics explored include Gibbs dyadics, the inertia tensor, and Riemannian geometry.

This book is written for students at the advanced undergraduate level or higher. The introduction contains a flow chart which indicates which chapters rely on others, allowing readers to skip chapters without missing required background material. The author also includes interesting historical and philosophical notes addressing questions such as "Is math invented or discovered?" These notes coupled with discussions about common misconceptions make it an informative and multi-faceted book.

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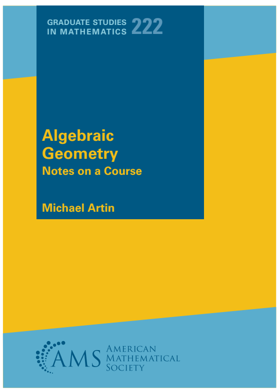



GSM/222, 2022

***Algebraic Geometry***
*Notes on a Course*
By Michael Artin

In his article in "Princeton Companion to Mathematics," Janos Kollar defines algebraic geometry as follows: "Succinctly put, algebraic geometry is the study of geometry using polynomials and the investigation of polynomials using geometry." This attempt to marry algebra and geometry, two quite distant and seemingly unrelated areas of mathematics, led to algebraic geometry, which is a notoriously difficult subject to learn and, therefore, to teach.

One explanation for this difficulty may be that geometric constructions appeal to pictures and images (handled, according to neurologists, by the right brain) whereas algebra involves complicated formal constrictions (handled by the left brain). The struggle between the two approaches is well illustrated by the history of the subject: compare the Italian school (Cremona, Corrado and Beniamino Segre, Castelnuovo, Enriques, Albanese, Bertini, Severi, del Pezzo, among others), which relied heavily on geometric approach and the Grothendieck school (Grothendieck himself, Serre, Deligne, Artin, Tate, Mumford, and many others), which used complicated algebraic constructions. Oscar Zariski is particularly important for his role as a "bridge" between these two schools.

In the last 50 years, there were many attempts to present algebraic geometry to the mathematical community, ranging from books for undergraduates (T. Garrity et al, AMS, 2013) to books for professionals (A. Grothendieck's EGA volumes, IHES, 1960–1967), from books emphasizing geometric aspects (P. Griffiths–J. Harris, Wiley 1978; D. Mumford, Springer, 1976) to books which build algebraic geometry starting with strong algebraic foundations (J. Harris, Springer, 1992; D. Cutkosky, AMS, 2018). Among these books, the most influential ones were those where the authors tried to balance algebra and geometry (I. Shafarevich, Springer, 1974; R. Hartshorne, Springer, 1977). All of these books, and others, have led to spectacular applications of algebraic geometry to many areas of mathematics, including number theory (the proof of Weil's conjectures and Fermat's Last Theorem) and mathematical physics (work by Witten and his school).

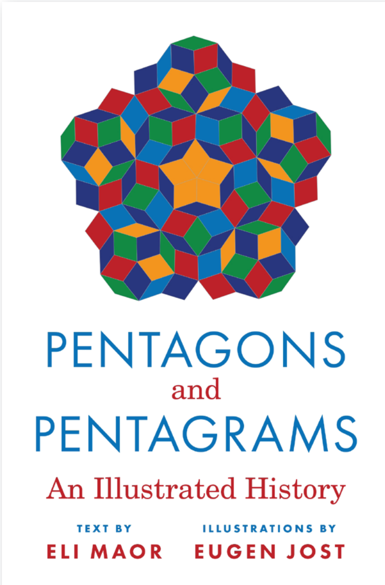
The book *Algebraic Geometry: Notes on a Course*, written by Michael Artin, one of the most active and notable modern algebraic geometers, is the latest attempt to reach the right balance between algebra and geometry. It is based on a course for advanced undergraduates and beginning graduate students that the author taught for many years at MIT, and on the feedback he received from his students and colleagues.

The book starts with a chapter about curves on the plane, which provides instructive examples for the material in later chapters. Chapters 2 and 3 introduce algebraic geometry of affine and projective varieties, respectively. Morphisms of algebraic varieties and their geometric properties are presented in Chapter 4. The structure of affine and projective varieties in the (Zariski) topology are described in Chapter 5. In Chapters 6 and 7, sheaves and their cohomology are studied. Finally, in Chapter 8 the author returns to algebraic curves proving the Riemann–Roch Theorem and using it to define the group law on points of an elliptic curve.

Packing a course on algebraic geometry into less than 350 pages requires making several important choices. Artin's choice is to restrict the exposition to the maximal spectrum of a ring and to varieties of complex numbers saying that, in his opinion, "... absorbing schemes and general ground fields won't be too difficult for someone who is familiar with complex varieties." On the other hand, he discusses in detail such crucial notions as integral morphisms, sheaves and their direct and inverse images under a morphism, and the cohomology of sheaves.

The author says in the preface that his goal in teaching algebraic geometry is "to make the development so natural as to seem obvious." The overall impression when reading this book is that he succeeds in reaching his goal.

*The AMS Bookshelf was contributed by AMS Book Publisher Sergei Gelfand. For questions or comments, please send email to* `sxg@ams.org`.

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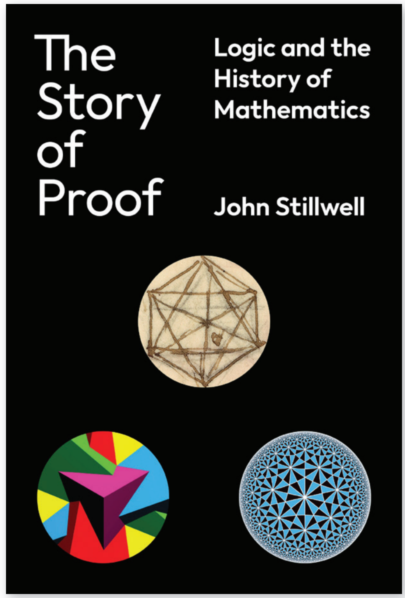


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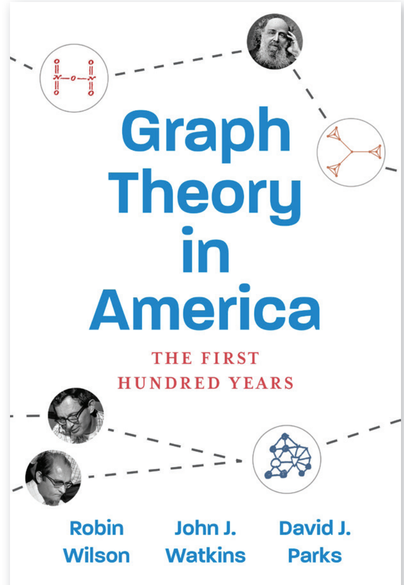


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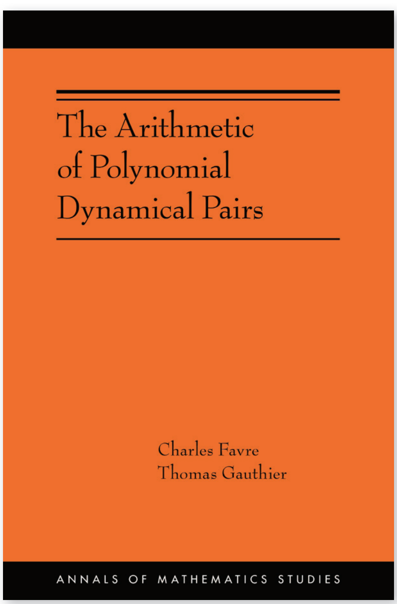


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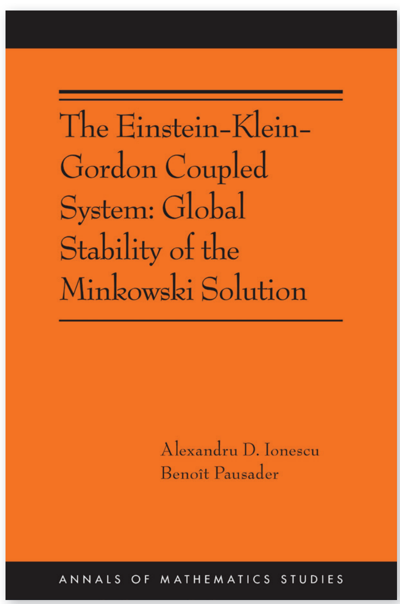


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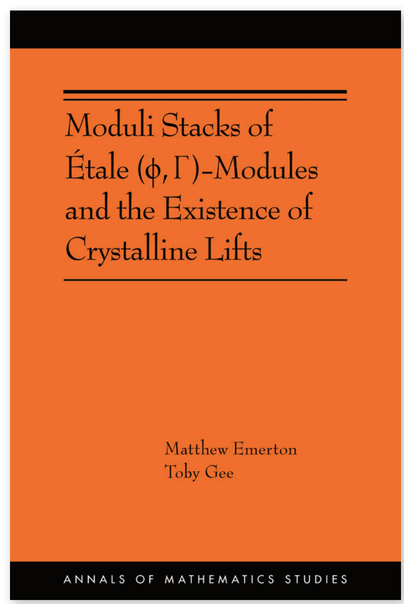


A foundational account of a new construction in the *p*-adic Langlands correspondence

PRINCETON UNIVERSITY PRESS

# A Conversation with Alan Weinstein

*Henrique Bursztyn and Rui Loja Fernandes*

Alan Weinstein has been one of the most prominent and influential differential geometers of the last five decades. He has made fundamental contributions to such diverse areas as Riemannian geometry, symplectic geometry and Hamiltonian dynamics, geometric mechanics, microlocal analysis and quantization, Poisson geometry and Lie groupoids, as well as their various interconnections and applications. Alan will turn 80 years old in June 2023. This conversation took place over Zoom in May and June 2022, while Alan was at his home in Palo Alto, recovering from COVID.

## Growing Up in New York

**R.** You grew up in New York. How was it? Do you remember well the transformations that were occurring there at the time, some of which were portrayed in West Side Story?

**A.** I grew up in the city only until the age of eight and then my family moved to Long Island to a quite well-off neighborhood, which was almost all white. There were a few African-Americans, maybe one or two Hispanics and one Asian family. So the cultural changes portrayed in West Side Story didn't really impact me very much. But it was a time when people were starting to get politically active. In the late 50s, there were starting to be many protest marches for integration, and I got involved in some of that stuff.

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**H.** And how was high school?

**A.** I attended Roslyn High school, where my interest in mathematics was particularly encouraged by a teacher named Anthony diLuna. He had graduated from a master's program at the University of Chicago, which was one of the first places encouraging a more creative approach to teaching math. There is now a scholarship named after him in my high school. I think I had him for advanced algebra, and maybe trigonometry. Apart from that, as a senior, I did an Advanced Placement Calculus class which, in some ways, was kind of a waste, as the teacher wasn't very good, so I was learning calculus on my own. I published my first paper in a journal for high school students [Wei60]. It was on the symmetry of the graph of a cubic equation around its inflection point. I believe that I did it by translating the inflection point to the origin and then showing that the resulting function was odd.

**H.** In high school, did you take part in any math competitions?

**A.** I was a "mathlete." There were high school math teams who met on a regular basis, and we'd go to a different high school for competitions. One would normally spend an hour doing problems, and I did pretty well. At the time there were not yet Math Olympiads [they started in 1959 in Eastern Europe]. Later, at MIT, as a junior and senior, I took part in the Putnam Math Competition and got an honorable mention once.

**H.** How did you get motivation to study math? At home from your parents?

**A.** Not particularly. They were happy enough to have me study math. They might have wished that I had gone into medicine or law, which was typical for parents at that time, but I never had much interest in that.

**Figure 1.** Alan Weinstein.

## Attending College at MIT

**H.** How and why did you end up at MIT?

**A.** I went to MIT partly because I had applied for a Grumman Scholarship, which would have required me to major in engineering. At the time, I thought I might be interested in engineering. I was a runner up for this scholarship, but I went anyway to MIT thinking I'd still major in engineering. Since I didn't win the scholarship, I didn't really have to. I think there were two things that changed my mind. One was two really good teachers I had for calculus in the freshman year. The other was that I didn't much like the laboratories in chemistry and physics. At that time in MIT, every freshman took the same courses: math, physics, chemistry, and something called humanities, which was mostly literature. One of the teachers I had for that, A. R. Gurney, became a quite famous playwright. I also studied Russian for no particular reason, except it was something different.

**R.** I guess that allowed you years later to translate Arnold's famous book on Classical Mechanics.

**A.** Yes, that allowed me to translate Arnold's book, along with Karen Vogtmann, who was a graduate student in Berkeley at the time (and actually did the majority of the work). I haven't read anything in Russian for a long time, although I can read in Cyrillic when something appears in the news these days related to the Ukrainian war.

**H.** How was your experience as a math undergrad at MIT? Who were your main mathematical influences there?

**A.** I had two wonderful semesters of honors calculus taught by James Munkres and Gian-Carlo Rota. This was almost like a real analysis class, where we started by defining the real numbers and proving everything. We used Courant's calculus book, and we started with sequences and limits of sequences, because that was easier to handle than limits of functions where you have to worry about both deltas and epsilons, while here you have only an $N$ and an epsilon. And then it went on. I think there were even some infinite series before we started with functions of a real variable.

Rota was not so well-known at the time, and he was more of a real analyst, before he became a famous combinatorist. But that was after my time at MIT. I had one more course from him, namely probability, an upper-division class.

Another professor whom I remember having a significant influence on me was Irving Segal, who was writing a book with Ray Kunze on *Integrals and Operators*, for a first graduate course on real analysis. I served as an informal "copy editor" for that manuscript.

**H.** Do you remember who taught you differential geometry at MIT?

**A.** The undergraduate class in differential geometry was taught by someone who wasn't at all in differential geometry. But by my senior year I was taking, like many students do, some graduate courses. And I took differential geometry with Sigurdur Helgason, who used his differential geometry book. That was a very good class, and he was a very good teacher, too.

**R.** By Helgason's book, you mean *Differential Geometry, Lie Groups and Symmetric Spaces*?

**A.** Yes. That's the one. It is a large book, and I think we only did about the first five chapters. But it became a reference for me later on. Certainly, an important book, which I kept in Berkeley.

**R.** At MIT did you get to know Victor Guillemin, or did you meet Shlomo Sternberg at Harvard, who later became major figures related to your work?

**A.** No. I don't think Victor Guillemin had arrived at MIT yet. Or if he did, I had no idea of him. I did interact with Henry McKean with whom I took a class in complex variables. Much later, when I visited NYU for a semester, I actually ended up writing a joint paper with him on solutions of the sine-Gordon equation [BMW94].

**R.** Is that paper related to your paper with Andreas Floer which, to our surprise, is your most cited paper on MathSciNet?

**A.** No, the paper with Floer was on the the nonlinear Schrödinger equation [FW86]. The other collaborator on that paper with McKean was Bjorn Birnir. Recently, I saw a reference to our paper so I looked him up, of course, to see what he was doing. And he's been involved in quite practical fluid dynamics, including the transmission of COVID in the air. So I should reach out to him and tell him I just caught it!

## Graduate School at Berkeley

**H.** When did you decide to pursue a PhD in mathematics? Was it a planned decision? Why did you pick Berkeley for graduate school?

**A.** Not really planned. I guess I didn't really think about it. I just did it. You know, by the time I was a junior I realized I would probably go on to graduate school and I started talking to people about it. In addition to Berkeley, I think I applied to either Princeton or Harvard, I don't even remember. I might have gotten in because I did very well at MIT, but I chose Berkeley partly just to get far away from where I had lived until then and see another part of the country.

**H**. Tell us about your experience as a graduate student in Berkeley in the 60s. Math environment? Flower power movement? Anything unmentionable (laughs)?

**A**. Not much flower power, but the Free Speech Movement made a great impression. Also amusing was the subsequent Filthy Speech Movement.

The math culture was great. In those days before the internet, people actually talked with their colleagues. There was a daily geometry lunch at the Student Union attended by many of the faculty, to which graduate students were invited as well. There was a wonderful seminar (run by Smale, I think) going through the proof of the Atiyah–Singer index theorem, which had just come out. Things were much less competitive than they became later, partly I suppose because there was no shortage of jobs in the 60's.

**H**. When you went to Berkeley, did you already know that you were going to study with Chern?

**A**. No, I did not. Actually, I didn't even really know Chern. But I pretty much knew that I wanted to do differential geometry, since I liked the subject so much after

**Figure 2.** Alan Weinstein in Berkeley, *circa* 1972.

the class from Helgason at MIT. Once I got to Berkeley, I took a beginning graduate class from Frank Warner, who again is the author of an excellent text. (It seems that I had classes from the authors of lots of good textbooks. Helgason and Segal at MIT, and Frank Warner at Berkeley.) He helped cement my interest in differential geometry. Then, in my second year, I took a topics course with Chern on integral geometry, which was one of his interests. Anyway, that cemented my interest in differential geometry, and I decided to work with Chern.

**H**. How was it to have Shiing-Shen Chern as an advisor? How did you find your thesis problem?

**A**. Chern was a great advisor. He mostly listened and encouraged. Chern and his wife Shih-Ning were great hosts, and their home was a center for social life in the geometry community. Regarding my thesis, at some point I met with Frank Warner and Hung-Hsi Wu; we used to go to the geometry lunch I mentioned before. Chern occasionally would also go to this lunch and it may have been there that I heard about a problem that Warner and Wu were working on, which was on a Rauch conjecture about conjugate points and cut points on Riemannian manifolds. So I got interested in that and I started thinking about it. Then at some point, they said, "Well, Alan, you're thinking about this, we'll leave it to you." So that became my thesis, and Warner and Wu became members of my thesis committee.

Wu and I became colleagues when I returned to Berkeley as an Assistant Professor. Wu has been at Berkeley the whole time and now he is my office mate!

**H**. How long were you in graduate school?

**A**. I was there for only three years. I was lucky that I found this problem. And I was able to write a 27-page thesis and get out.

**H**. That was a short thesis!

**A**. I guess that was all the length I needed. And maybe I was lazy, too. In fact, in February of my third year, I had basically finished writing the thesis. I don't remember when I turned it in, but I went on to spend some time in Paris. I had a car then, so I drove back to the East Coast, via Los Angeles, where I attended one of the first of the so-called *Geometry Fiestas*. I think they are now called the *Geometry Festivals* and they are centered at U Penn. It's an annual, mostly differential geometry, meeting. This one was an early one at UCLA. So I went there first, and I actually gave a talk on my thesis, the first meeting that I'd ever been to.

**H**. What was your connection with Paris?

**A**. I got to go to Paris because Chern had some connections with IHES, since he had been there many times. The idea of spending time in Paris, came up in the previous summer (that is, the summer of 1966, two years after starting graduate school). I had pretty much worked out the solution of my thesis problem by then, and I knew I just had to write it up. My girlfriend Margo was about to go off to Paris, where she was doing a Middlebury College master's program in French. So I wanted to go to Paris as soon as possible, which turned out to be February (of 1967). We were married that May at the Mairie d'Orsay because I had an apartment in Orsay while visiting IHES. It's now been 55 wonderful years together!

**R**. Was it your first time in France?

**A**. It was my first time in France. In fact, except once or twice on family driving trips to Montreal and Quebec, it was my first time outside the United States.

**H**. And how was your French?

**A**. Well, once I decided that I was going to go to Paris, I started studying French at Berkeley. I audited a French class in the fall semester (of 1966), including the oral language labs.

**R**. So coming back to your visit to Paris, who were your main mathematical contacts there?

**A**. My main contact was Marcel Berger, who had been at Berkeley, and I knew him through Chern. He was, in some sense, my main mathematical contact in Paris, although I did get to see a fair amount of René Thom, who was there too. One of the nice things about IHES, and I think it still happens, was that every day there's lunch in a building at the bottom of the hill. And almost everybody would come to lunch. So sometimes I would be at the table with Grothendieck. Zariski was there also that spring and he talked to Grothendieck a lot. This was before the time of Deligne.

**R**. At that time, you began to change your research a little bit. How did that happen?

**A**. Jeff Cheeger's work was starting to be well known. His work on manifolds of nonnegative curvature and his finiteness theorem for Riemannian manifolds with a bounded curvature, were considered very important. So I remember reading his thesis carefully. At that time, I proved an estimate for the number of homotopy types of positively pinched manifolds [Wei67]. (It was vastly improved by Cheeger.)

I was also working on topics related to Palais's work about actions of compact groups on manifolds. I never actually wrote a paper on that because Palais's paper on proper actions came out, but I attempted to give a talk about that in French; I think I was feeling overconfident. One of the people in the audience was Bernard Morin, a French mathematician who made the first models of turning the sphere inside out. Even though he was blind, he developed an algorithm for doing that. So I was giving my talk and I drew something on the board and he asked me to describe the picture. That was very challenging for my French, so a French person in the audience had to explain to him what was going on.

**H**. And this visit, still as a graduate student, was the beginning of a life-long connection with Paris...

**A**. Yes, Margo and I have been visiting Paris regularly since then. Early on, I came back for a short visit, maybe in 1969, and then for a month or two in 1970. Then in the summer of 72, after our daughter Asha was born, we lived in an apartment in Paris. We went back for a year in 1975–76, and that was back at IHES. It was a really good time. Dennis Sullivan was there and very active. There were several kids who were all about the same age. One was our daughter Asha. One was Michael Sullivan, Dennis's son who is now a mathematician, and one was Christian Gromoll, also now a mathematician, the son of Detlef Gromoll. I think we have a picture of the three of them together. It was kind of fun. By then I was really more interested in curvature-related things and also getting more into symplectic geometry.

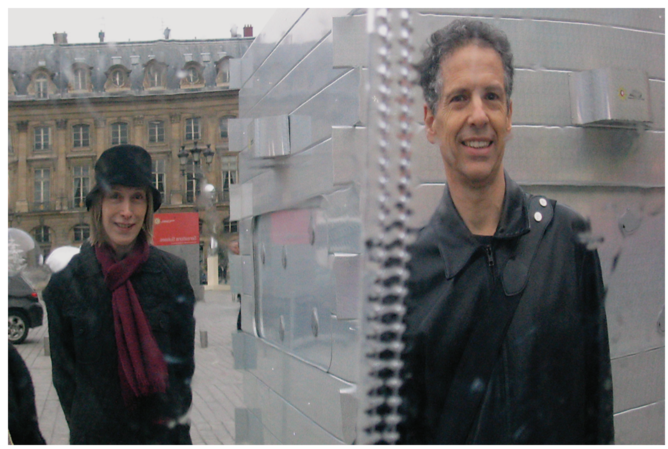

**Figure 3.** Margo and Alan Weinstein in Paris in December 2004.

**H**. As a student, who were the mathematicians that you looked up to, that were particularly inspiring to you?

**A**. Berger, Wilhelm Klingenberg, and Chern of course. They were kind of my mathematical heroes at the time. I was a student, and Riemannian geometry was the thing. I was really interested in curvature, although my thesis wasn't about that. I did write a paper about curvature when I was a graduate student [Wei68]. Klingenberg was also involved in the study of closed geodesics, and so I got very interested in closed geodesics and periodic orbits. Once I got into symplectic geometry, closed geodesics morphed into periodic orbit interest. And that's what led to the stuff I did on periodic orbits, equilibria, and so on.

I also really admired Smale who, as I mentioned before, ran a seminar on the Atiyah–Singer index theorem. Atiyah himself taught a course at Berkeley in the summer of 1968, during an AMS summer conference on global analysis. That was a really great meeting, and Atiyah gave a course on the index theorem. Obviously, I also admired him very much and I had a little bit of contact with him over the years, nothing too close. Later I did a couple of things on the index of Fourier integral operators, though I never got as far as I wanted.

## Postdoctoral Years

**R**. As a postdoc, you went back to MIT and then Bonn, and that was about the time you started to get interested in symplectic geometry. Do you remember when you first heard about symplectic manifolds?

**A**. It was when I was a Moore instructor at MIT, because I was doing this work on conjugate locus and cut locus. Frank Warner had written a paper about the singularities of the conjugate locus, and I got interested in the subject. I realized that the exponential map was a projection of a Lagrangian submanifold of the cotangent bundle. This hadn't played a part in Warner's work and so I got very interested in that. Arnol'd's paper on the Maslov index had introduced me to Lagrangian submanifolds, though I did not meet Arnol'd until many years later.

**R**. But according to MathSciNet you had an earlier paper on symplectic structures on Banach manifolds.

**A**. Yes, from around the same time. This paper used Moser's method. I was getting interested in symplectic geometry and I knew about Moser's paper "On the volume elements on a manifold," where he proved that two volume elements on an oriented compact manifold with the same volume are diffeomorphic. So I figured out how to extend that to symplectic manifolds. This paper on symplectic structures on Banach manifolds, as well as other work I did on normal forms, was all based on Moser's method. For example, I applied Moser's method around Lagrangian submanifolds [Wei71]. By then I was also starting to think about the WKB method, and how it related Lagrangian submanifolds in the cotangent bundle to quantum states. I was also interested in the interface between classical and quantum mechanics, for instance because of relations between the Laplace spectrum and the geometry of Riemannian manifolds, and symplectic geometry turned out to be the right tool for studying that.

**H**. How long were you at MIT as a postdoc?

**A**. I did just a year as a Moore instructor at MIT and then I took a NATO postdoc, also for a year, in Bonn. There my sponsor was Wilhelm Klingenberg, who was very involved in pinching theorems. For example, a complete, simply-connected Riemannian manifold with curvature strictly between 1/4 and 1 must be a sphere. That was one of the first topics I was interested in even as a graduate student, because Berger and Klingenberg, who both did pinching theorems, were in Berkeley as visitors brought by Chern. In Bonn, in addition to Klingenberg, there were two of his postdocs, Detlef Gromoll and Wolfgang Meyer, who were working together on some Riemannian geometry problems. After the year in Bonn, I came back to Berkeley as an Assistant Professor.

## Back to Berkeley as Faculty

**R**. During your first years at Berkeley, now as a faculty member, you wrote one of your most cited works on what is now known as "symplectic reduction" or "Marsden-Weinstein-Meyer reduction" with Jerry Marsden. How did you meet Marsden, and how did you start collaborating with him?

**A.** Jerry and I were attending a class of Smale's on classical mechanics, where of course symplectic geometry is a big part of the story. Smale had proven a version of symplectic reduction for cotangent bundles and lifted actions from group actions on the base. Jerry and I figured out how to do this for general symplectic manifolds and Hamiltonian group actions, and so we wrote that paper on reduction [MW74]. Only later did we learn that Ken Meyer had discovered reduction on his own.

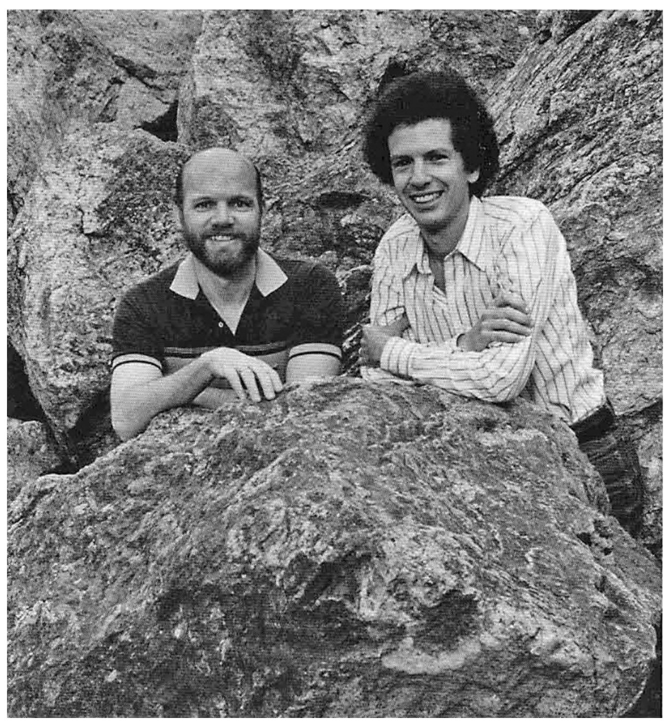

**Figure 4.** Jerry Marsden and Alan Weinstein.

Soon after, Jerry and I wrote another paper on Hamiltonian dynamics [MW81]. It originated from another seminar we both attended, run by a plasma physicist named Alan Kaufman and his student at the time, Robert

Littlejohn. There they talked about some work of Phil Morrison, who had discovered a noncanonical Poisson bracket and Hamiltonian structure for the Maxwell–Vlasov equations. These are equations for a collisionless plasma, i.e., charged particles interacting with each other via the electromagnetic fields that they produce. Morrison's bracket turned out not to satisfy the Jacobi identity, so we looked at it and realized that we could get a Poisson structure by symplectic reduction from a cotangent bundle crossed with another factor to account for the electromagnetic field. So we had a great simplification and a structure that actually satisfied the Jacobi identity. After six months of work, we figured out how to do this in two minutes! That got us into lots of stuff with infinite dimensional systems and the Hamiltonian structure of many other systems, for fluids and so on.

**R.** Something you are interested in to these days, right?

**A.** Something I'm still interested in these days. Moreover, somehow it became clear that Poisson structures were very important, and so I started thinking about Poisson structures in their own right. More or less at the same time appeared the paper of André Lichnerowicz on the subject, which was preceded by an earlier work of Alexandre Kirillov, which also included Jacobi structures, by the way. But I found that one could go a lot further than they had, in various directions [Wei83], and Poisson geometry eventually became a field in its own right. Lichnerowicz was also a visitor at Berkeley, and he gave lectures which had a big influence on me.

**H.** After returning to Berkeley, one can say you turned into a symplectic geometer…

**A.** That's right. Although I still wrote some papers in Riemannian geometry, including on curvature pinching, since Chern was of course there and was still very active. He was the head of differential geometry. Frank Warner had moved to Penn by then, but Hung-Hsi Wu was still there.

**R.** Was it easy to know what was going on in the Russian school?

**A.** Well, Eliashberg came later, of course. But because I knew some Russian, I could read the Russian journals to follow what was going on. And, of course, there were translated versions of the major journals. I also started having some correspondence (in English) with Maslov and Arnol'd from early on.

**H.** What about contact geometry? How did you come up with the "Weinstein conjecture"?

**A.** That came about because, as I mentioned before, I was interested in periodic orbits of Hamiltonian systems, inspired by Klingenberg's and his students' work on closed geodesics. For a Hamiltonian system on a symplectic manifold of dimension $2n$, using a version of Moser's variational method, I was able to prove that in a neighborhood of a nondegenerate minimum of the Hamiltonian, there are $n$ families of periodic orbits [Wei73]. At that time, I also had a student named O. Raul Ruiz, who did a thesis on the existence of brake-orbits in Finsler mechanical systems, which also used variational methods. Then I started looking at convexity, and I was able to use variational methods to prove the existence of periodic orbits for convex Hamiltonian systems [Wei78]. About that time, I was asked to referee a paper by Paul Rabinowitz, where he proved the existence of periodic orbits on star-shaped energy surfaces [Rab79], and somehow just looking at that, I conjectured a wide extension of what he had done.

**H.** Your conjecture is in an appendix [Wei79] of that paper!

**A.** Yes, it is there because I was the referee. I thought maybe what lets you apply variational methods to get periodic orbits is the contact nature of the energy hypersurface. In the published version of the conjecture that now carries my name, I included the hypothesis that the manifold be simply connected. The reason was that I (mistakenly) thought I had a counterexample related to the cotangent bundle of a torus. But it turns out from what people did later that the hypothesis was not needed.

**Figure 5.** Alan Weinstein with his wife Margo, his daughter Asha, and their cats Lucy and Toby in Berkeley around 1985.

**R.** Coming back to the Poisson brackets, at some point groupoids appeared in the picture too…

**A.** They appeared because of geometric quantization and deformation quantization. In deformation quantizing a Poisson manifold, the objects you're deforming are the functions on the Poisson manifold. A WKB approach involves looking at functions on a Poisson manifold as Lagrangian submanifolds in its cotangent bundle. If you had a product on the functions on a Poisson manifold, this gave you a kind of binary operation on Lagrangian submanifolds which might imply that the bilinear "quantized" operation should be associative. By then, I had heard about groupoids at a meeting, the Séminaire Sud-Rhodanien de Géométrie in southern France, where Kirill Mackenzie talked about groupoids and algebroids in their own right, and so I made this connection [Wei87]. It turned out that Mikhail Karasev and Viktor Maslov had done something very similar; also, Stanisław Zakrzewski, independently, had thought of similar things. But I pursued it further. (Zakrzewski passed away very, very young. There's a paper of his which I finished after he died.)

**R.** Although Lie algebroids and Lie groupoids had been around for quite some time, the discovery of their connection with Poisson geometry kind of transformed the subject...

**A.** That is right, they had been around for quite some time but not in symplectic geometry. Kirill Mackenzie had written several papers on the subject, and Rui Almeida and Pierre Molino, whom I first met in the same Séminaire Sud-Rhodanien, had found the first example of a nonintegrable Lie algebroid.

**R.** Can we talk a little bit about your creative process? How do you come up with new ideas, and how do you identify interesting problems?

**A.** I wish I knew!

**H.** That's a key point of the interview, I'm sure everyone wants to learn that!

**A.** OK, I'll try. One thing I remember is that for a long time I was interested in lots of different things. Now I'm much less good at multitasking. But I was very good at multitasking back in the day. So I used to think about various things, and sometimes one of these areas gave me an idea that I could apply to some distant problem. That was partially responsible for the variety of problems that I worked on. If I look back, I started in Riemannian geometry, and I kept working on that for some time after I started getting interested in symplectic geometry, and then in microlocal analysis...

Another method which I have frequently used is to approach a problem by considering simple, even trivial, examples, such as the zero Poisson structure.

**H**. Indeed your research has covered an impressively wide array of topics, including Riemannian geometry, symplectic geometry and Hamiltonian dynamics, semiclassical analysis and PDEs, quantization and noncommutative geometry, Poisson geometry and Lie groupoids, etc. Is there anything that unifies, or a common motivation that explains the breath of your work?

**A**. One thing came from another. I suppose that the classical–quantum transition was responsible for a lot of it. Since very early on, in fact since I took an upper-division physics class at MIT, I was very interested in the relation between classical and quantum mechanics. You can see reflections of that in a lot of the things I've done that involve quantization. For example, the thesis problem of my former student Steven Zelditch, which was centered on Schrödinger's equation, was an attempt to extend to the noncompact case some previous work on closed geodesics in Riemannian geometry and its relation with the spectrum.

**H.** It is remarkable that, many times, you had one of the key ideas in a subject, but you don't really pursue it that much and you let other people work on it. For example, for the Weinstein conjecture in contact geometry, which we talked about before, you posed the conjecture but you did not actually work on it afterwards, although it became a huge thing...

**A.** There is a certain amount of laziness on my part. On the other hand, it often happened that I got interested in something and, fortunately, I had some student who got interested in pursuing it. So I could leave it to him or her.

**R.** Besides conjectures, there are also these philosophical principles that you suggest and then often everyone adheres to, like "everything is a Lagrangian submanifold," for example, which you called "the symplectic creed"!

**A.** I am very proud of that. That principle, it was kind of half a joke. I put it at the beginning of a survey article I wrote [Wei82], and it seemed appropriate for a survey. It turned out to be, obviously, an exaggeration (laughs). But if you think about the Fukaya category, for example, the objects are Lagrangian submanifolds. There are many other examples.

**R.** But the Fukaya category appeared much later than that survey...

**A.** In fact, I think the idea that everything is a Lagrangian submanifold came mostly from from WKB and geometric quantization. Hörmander's paper on Fourier integral operators had a big influence on me. That paper I probably studied more carefully than any other paper. By then, I was talking with Victor Guillemin at MIT and Shlomo Sternberg at Harvard. There was a nice back and forth exchange of ideas, and that's partly what got me more seriously interested in microlocal analysis. Other people who influenced me were Hans Duistermaat, who wrote notes on Fourier integral operators when he was at NYU, and François Treves, who was a professor at Rutgers and also wrote a two-volume text on pseudodifferential operators and Fourier integral operators.

**H.** If one considers symplectic manifolds in broader contexts, like graded or shifted symplectic spaces, then your symplectic creed becomes really far-reaching. For example, Dirac structures are Lagrangian submanifolds in an appropriate sense. How did Dirac structures come about?

**Figure 6.** Alan Weinstein received a honorary doctorate from the University of Utrecht on March 26, 2003. The promoters were Hans Duistermaat (left in this picture) and Ieke Moerdijk (in the background).

**A**. I was initially motivated by some work of Robert Littlejohn, a physicist at Berkeley that I mentioned before. I was on his thesis committee and his work involved Dirac's theory of constraints. Because submanifolds of Poisson manifolds are, in general, neither Poisson nor presymplectic, there should be something more general. So I gave that problem to one of my PhD students at the time, Ted Courant. Eventually, he came up with the basic theory of Dirac structures and wrote his thesis about them. We also wrote a little joint announcement about it [CW88]. But obviously, Dirac structures caught on much more than we ever thought they would!

**R**. So you couldn't really anticipate they would become so important…

**A**. No, not at all. I mean, it seemed like a very good idea. So I pursued it a little bit, writing a paper with Zhang-Ju Liu and Ping Xu where we introduced Courant algebroids [LWX97], and that, together with Dirac structures themselves, is what made the theory explode. Dmitry Roytenberg and Pavol Ševera gave a supermanifold interpretation of Dirac structures as Lagrangian submanifolds, and soon after they appeared in generalized complex geometry. It was first Nigel Hitchin, and then his student Marco Gualtieri [Gua11] who got that subject to take off as a big thing, which I really appreciated. I had met Marco as a student at a conference, and he explained to me what he was doing. Later he visited Berkeley, and we talked a lot, but I never actually did anything much in generalized complex geometry. There is also a connection of Dirac structures with new notions of symmetries, like group-valued momentum maps. These things really made Dirac structures take off!

**R**. Besides your research, you have also been a very dedicated teacher at various levels. You're the author of several calculus books and have advised nearly 40 PhD students. What can you tell us about your life as an educator?

**A**. The idea for the calculus books came as I was playing tennis with Jerry Marsden. (Once we started, I never had time for tennis again!)

Being an advisor was one of many things I enjoyed about being a professor. I really liked working with students, with each of whom I had a different kind of relationship. I mostly let them go on their own, as much as they wanted to. Occasionally I had problems in mind, or general areas. I usually had more than one student at a time, which was nice, because they could also talk to each other without me. I occasionally collaborated with students while they were students, but more often I wound up doing collaborations after they graduated.

**R**. One final question: What occupies your mind these days?

**A.** Besides wondering about where all my time is going, and enjoying doing things with my family, I'm thinking mostly about problems which arise from trying to understand geometric properties of the constraints for the initial value problem in general relativity. This led on the one hand to the discovery of a groupoid symmetry for which the Lie algebroid bracket matched that of the Poisson brackets of the constraints, and on the other hand a theory of compatibility between Lie algebroids over a manifold and presymplectic or Poisson structures on the manifold. Unfortunately, all of this work, which has turned out to be interesting in its own right, has not led to a resolution of the initial question about the Einstein equations, so I'm still trying.

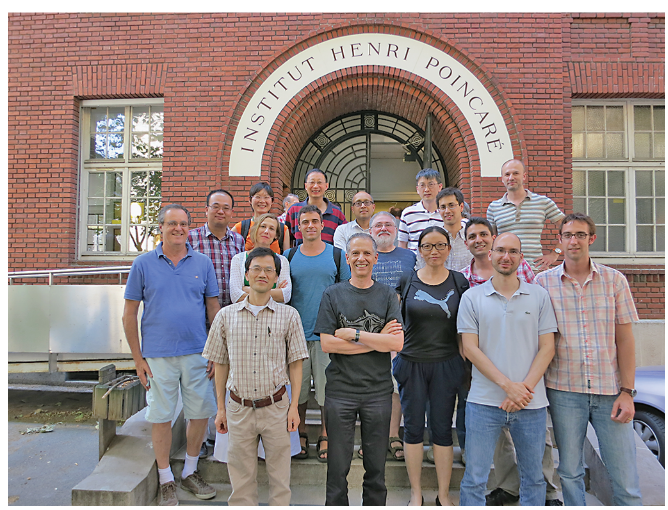


**Figure 7.** Alan Weinstein with some of his former PhD students, during his 70th Birthday Conference at Institut Henri Poincaré, in July 2013.

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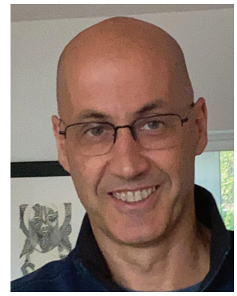
Henrique Bursztyn



Rui Loja Fernandes

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# The Next Generation

*Edward Dunne*

It is time for a big change in MathSciNet. In particular, it is time for a new user interface. The biggest changes in this release are the clean, modern look, the functionality for tablets and phones, and the increased accessibility.

Mathematical Reviews was an early adopter of the internet, recognizing its potential to give greater access to the content that had historically existed only in print. In the 1980s, a segment of the content was available online via the third-party services BRS and Dialog. This database was initially called MATHFILE, then renamed MathSci. It included all bibliographic and subject information on articles and books reviewed in Mathematical Reviews starting from 1973 and reviews starting from 1979. This remained the state of the art for a decade. In 1989, Tim Berners-Lee began working on what became the World Wide Web. The possible interfaces to the web were rudimentary until Mosaic was released in 1993. Standards for the web and HTML were made open starting in 1994, making it easier to set-up web servers and to create web sites. In January 1996, MathSciNet, the web version of Mathematical Reviews was launched. Since then, there have been new features and changes to the layout, but the overall structure of searching has remained the same for 25 years.

Before the pandemic, Mathematical Reviews had begun work on a new user interface (UI) for MathSciNet. The work was interrupted, but resumed in 2021. As I write this, a beta version is available. By the time of publication, the new interface will be available for all users.

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DOI: https://doi.org/10.1090/noti2596

Elements of the new interface have been introduced over the years. We added filters, originally called facets, in late 2016. These are modeled on the filters that show up when searching a retail site or a library catalog. The filters show users how many items in the search results contain a particular element, such as an author's name or a journal. You can then select that element to refine the search results. Filters are powerful because they use the richness of the database itself to point users to productive searches. The profile pages for journals were expanded dramatically in June 2019. The journal profiles became a prototype for the look of the new interface. More significantly, the revised journal profiles advanced the idea of giving users a detailed picture of any journal based on the data from its publication history.

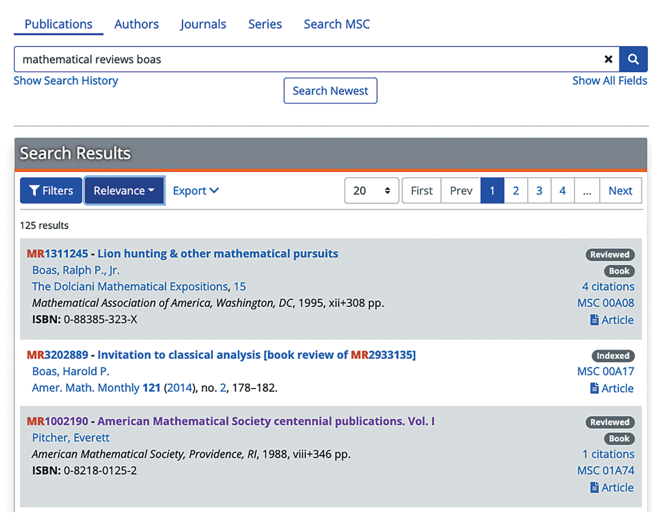
Three years ago, two consecutive columns in *Math Reviews News* ([2] and [3]) described how to use MathSciNet. Those columns gave an overview of the structure of the database, especially the aspects that influence how to search. A lot of that structure remains the same, and is not affected by the change of the user interface.

## Next Generation Searching

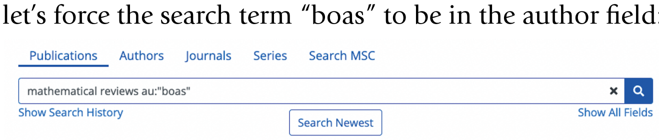
Before diving into details, let me point out that there are four primary searches in new MathSciNet, corresponding to the major parts of the database: authors, journals, series, and publications (papers and books). Previously, MathSciNet only had authors, journals, and publications searches. Each type of search works somewhat differently and returns different results. For instance, with an author search, the results will be a list of people, not their papers. If the author search produces a unique match, you are directed to the author's profile page, rather than to a list containing just that author. Similarly, a journal search will return a list of journals—or take you to a journal profile page if the search has a unique match. With this iteration of MathSciNet, we have added a series search, which looks for book series, such as the *Graduate Studies in Mathematics* from the AMS or *Lecture Notes in Mathematics* from Springer. Series searches include all book series: monograph series, collection/proceedings series, and series that contain both monographs and collections. By far, the most common search is for publications. Indeed, over 80% of the initial searches on MathSciNet have been publication searches. Author searches represent about 15% of all initial searches. Journal searches have made up the rest.

Much has changed since the original release of MathSciNet. Search technology is more powerful. People's expectations for searches and their experiences with databases have evolved dramatically. Google is responsible for some of that change, but so, too, is almost every web site we use, from online shopping to library catalogs to social media. People are as likely to access the internet on a phone or tablet as on a laptop or desktop computer.

The biggest change for MathSciNet is that the publications search is now through a single box. You enter some terms, hit "Enter," and the software searches for those terms in all the primary fields in the database related to that search. Let's search for "mathematical reviews boas." Ralph Boas was an Executive Editor of *Mathematical Reviews*, but also an accomplished research mathematician. This search is trying to focus on what he wrote about *Mathematical Reviews*.

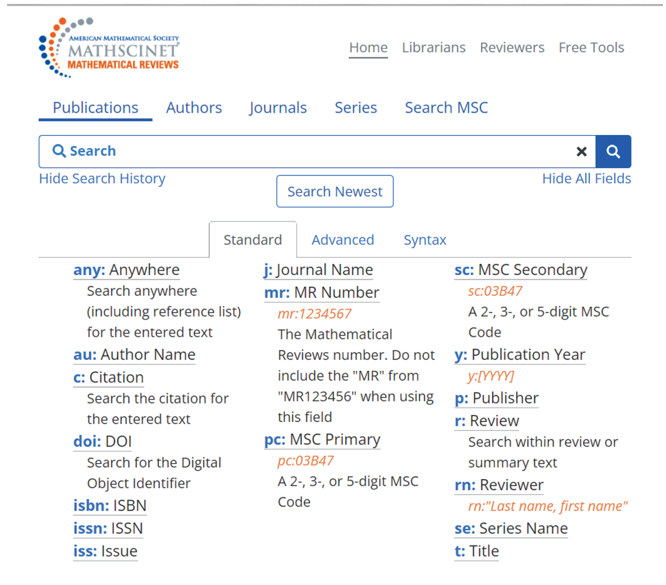



This produces 125 results. The first two have an author named Boas. The third in the list is by Everett Pitcher, and Ralph Boas is mentioned in the text of the review. You can refine the search by introducing field codes. In this case, let's force the search term "boas" to be in the author field:




This results in just 24 matches, all of which have an author named Boas.

The interface makes adding a field fast and easy. Click "Show all fields" over on the right and a list of standard field codes appears, along with a short explanation of each code.

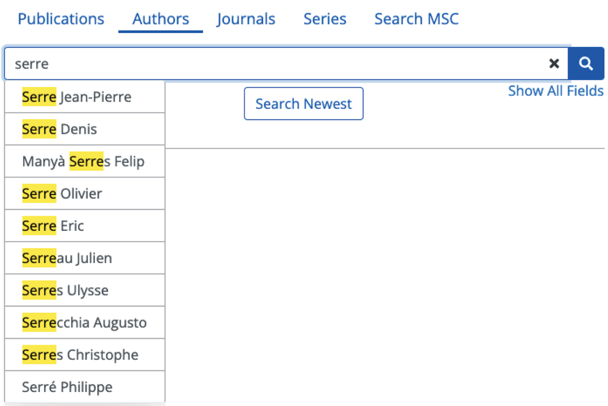



If you are unfamiliar with the field codes, you can leave them displayed, then click on a code. That will insert the code into the search box, add any delimiters needed, and put the cursor in the appropriate point in the search box.

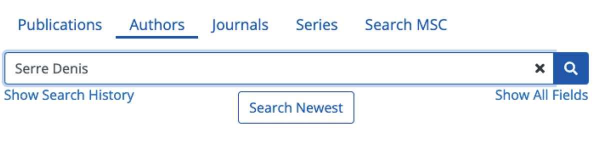

You can also add a field by typing in the code directly. As you do so, we provide autocomplete: if what you are typing matches the beginning of one or more field codes, we offer them up as possible completions:


If you choose one of the completions, the user interface again adds appropriate delimiters and places the cursor so that you can begin typing the search term. These features help you navigate MathSciNet both quickly and precisely.

**Author searches and journal searches in next generation MathSciNet.** MathSciNet already had autocomplete for author searches and journal searches, and this carries over to the new user interface. As soon as you type two or more letters in the search box, the database starts suggesting completions.



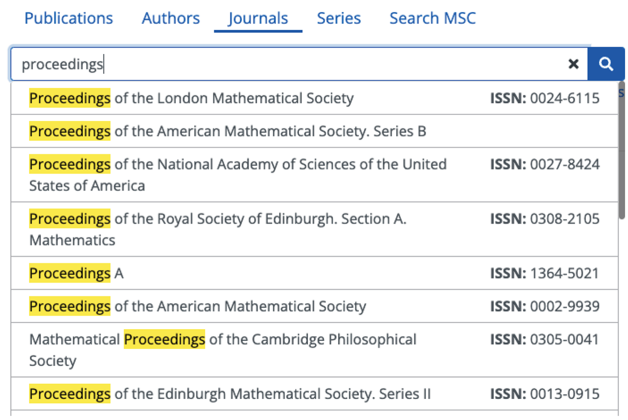

Clicking on a suggestion puts it into the search box.




Then using "Enter" or clicking the search icon executes the search, in this case finding the unique author in the database named Denis Serre.

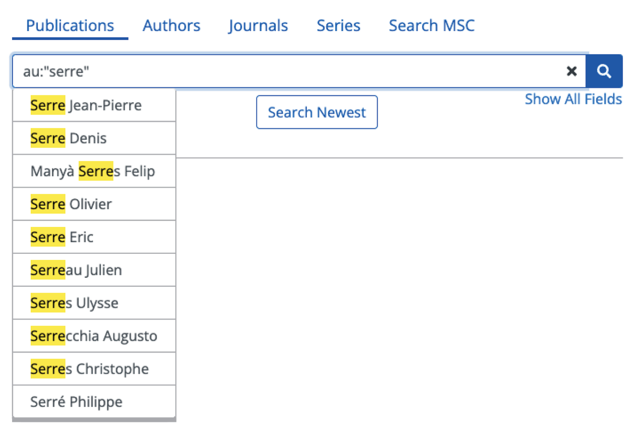

Notice that MathSciNet took us directly to Denis Serre's profile page, rather than to a list of authors named "Denis Serre." That is because there was a unique match.

Autocomplete works similarly for journal searches. Start typing the name of a journal and the database starts offering suggestions:

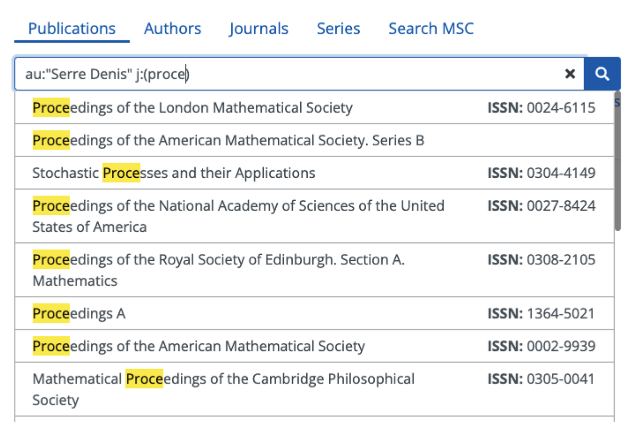



Pick one by clicking on it, then use "Enter" or the search icon to activate the search.

**Autocomplete and publications searches.** With the new user interface, a limited autocomplete is active for publication searches. Specifically, suggestions are offered for terms in the Author Name, Reviewer, Journal Name, and Series Name fields. As you start typing inside **au:" "**, for instance, the database starts suggesting names.




After you pick one, you can add the journal field and start typing. The database will start suggesting completions for the journal name:

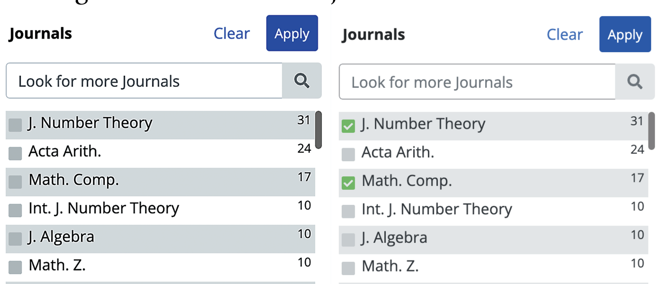


### Search results.

*Sorting.* Results can be sorted by Newest first, Oldest first, Number of Citations, Number of Authors, or Relevance. The first four are carried over from the previous user interface. The fifth, Relevance, is new with this edition of MathSciNet. The relevance ranking is a combination of term frequency (the number of times the term occurs in a particular record), inverse document frequency (a measure of how often the term occurs across the database), and field-length norm (a norm based on the number of terms or words that occur in the field being searched, calculated so that shorter fields have more relevance). The default sorting is newest first. If you select a different sort order, it will follow you as you search and view results.

## Filters, Filters, Filters!

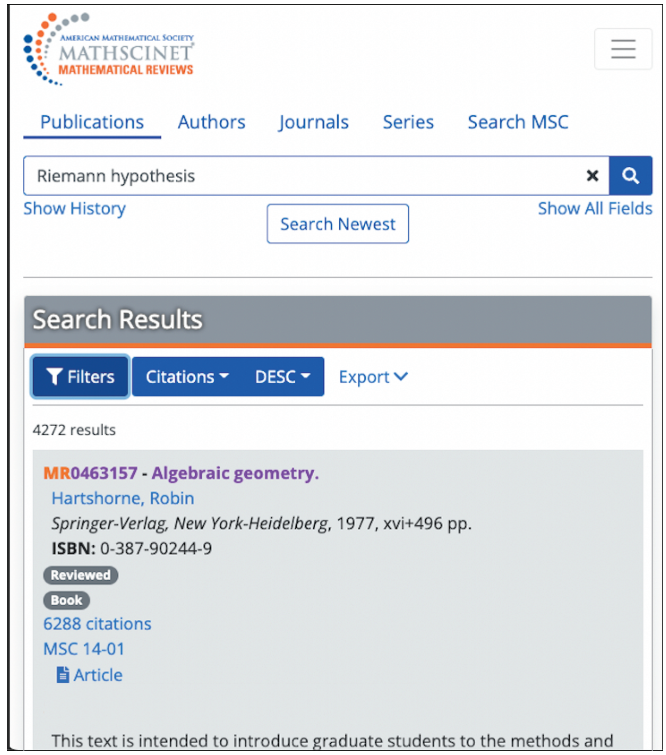
We added filters, or facets, to sidebars in 2016. They provided a particularly helpful way to refine results by using the information from the database. The new implementation adds significant functionality. You can now choose multiple values within a single filter. For example, a publications search for the term "twin primes" turns up 772 results. In the Journals filter, it is possible to pick both *J. Number Theory* and *Math. Comp.*, then filter for results coming from either of those journals.



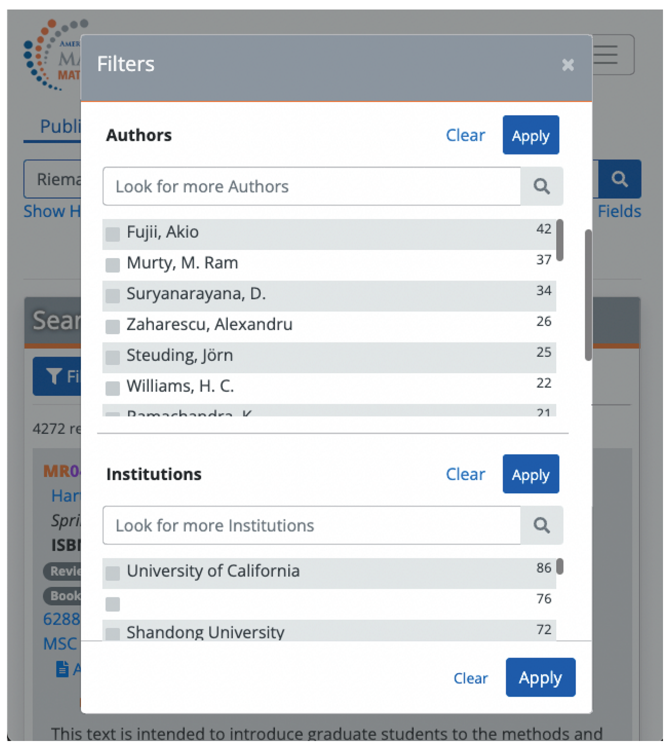

Previously, selecting a filter would instantly apply it. Now, you select one or more values, then click "Apply" to activate the filter, which is the feature that allows you to make multiple selections. But there is more: clicking a value twice turns the check into a minus sign, forcing that value NOT to be found in the filtered results. Finally, clicking a value a third time unchecks its box.

## Works Well on Mobile Devices

From the very first empty screen during development, we were designing with phones, tablets, laptops, and desktops in mind. Layouts, features, and functionality were all designed so that the important features work well in smaller formats. Sometimes this means that a menu collapses to an icon, such as the three stacked lines known to web designers as "the hamburger," or that descriptive labels are abbreviated. For the filters on a small screen, the sidebar is hidden by default, displaying a button above the first result instead.




When you click "Filters," MathSciNet presents them as a scrollable overlay, as shown below:



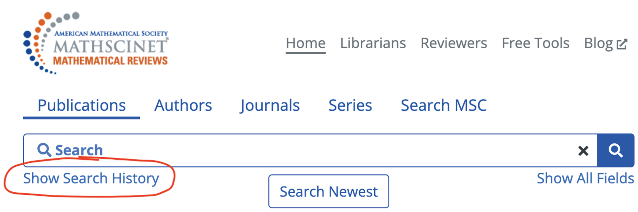

As the devices and their screens become bigger, more of the options are automatically displayed, rather than requiring a tap on the screen to expand them.

## Increased Usability and Accessibility

Choices of colors and contrast levels, fonts, and font sizes were all made for increased clarity and improved accessibility for users with visual impairments. We have also ensured that all navigation is possible by either a mouse or a keyboard, using the Tab or Arrow keys. Headings are implemented for compatibility with screen readers. The new version of filters, with selection followed by activation, helps users working with a screen reader or just with a keyboard. The two-step process also opened the door to allowing multiple selections and a Boolean NOT operator within the filters, which is a powerful new functionality.

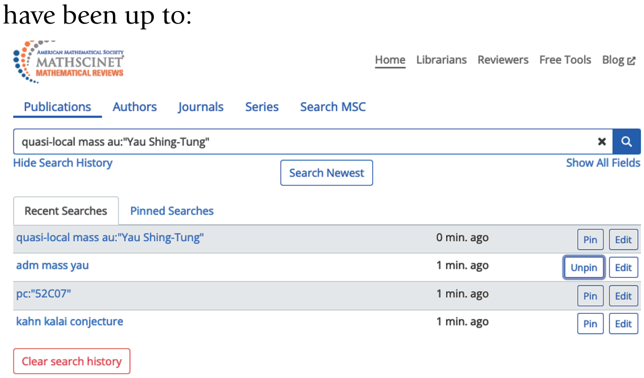
## Some New Features

**Search History and Pinned Searches.** The new version allows users to see their recent Search History, as well as to "pin" a search. By default, the Search History is hidden. Clicking on "Show Search History" opens it.



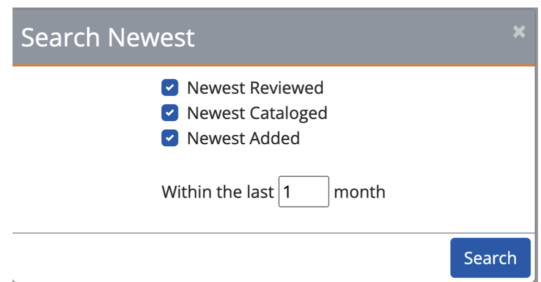

Note that on a small screen, such as a phone, "Show Search History" is abbreviated to "Show History":


After performing some searches, you can see what you have been up to:




If you click on the search itself, MathSciNet will put those terms in the search box and execute the search. If you click on "Edit," MathSciNet will put those terms in the search box, allowing you to modify the search before activating the search. It is quite useful! You can also "pin" a search. This is helpful, for instance, if you are doing many searches, but want to come back to a few of them. The preferred searches might get lost in the Search History, but will be easy to find among the Pinned Searches. You pin a search via the Search History, which is also where you unpin a search.

**Search Newest.** In order to see the latest additions to the database, we have added a "Search Newest" button. By default, this displays the reviews, fully-cataloged items, and preliminary items that have been added in the last month.




This is actually a variation on the Current Publications search that was previously available, but required a few clicks. The new version is easier to find, as well as easier to modify, since it puts the appropriate search into the search box, allowing you to add other searches, such as subject classifications, keywords, or journal names.

## Favorite Features Remain

Some favorite features remain, and some have even been improved. The more than 1.8 million reviews are still present, of course. Reference lists and links to items citing the current item continue to be readily available, with reference lists more easily searched. Links to full text on the publishers' sites are present, including the OpenURL feature that allows you to go through your library to access the material.

The journal profile pages were an early prototype for this next generation of MathSciNet, so remain little changed. Indeed, we were so happy with the layout of the journal profile pages that we mimicked them to create the series profile pages. Author pages still show the date of the author's earliest publication in our database, as well as the counts for publications, related publications, and citations. We have changed the way the coauthors and subject areas are displayed, making them more similar to what was developed for the journal profile pages:


One of the most common tools people use is downloading the BibTeX description for any item in the database. That is now a little more straightforward, and batch downloading multiple items in BibTeX is even easier. For a single item, click on the "Cite" button at the top right. For multiple items, use the "Export" button at the top of the results list. You can then either select every result on the page or check the boxes of the items you want. Then finish with "Get Citations." The new interface also makes it easier to see how to capture a permalink to the listing of an item in MathSciNet. It's an option from the "Cite" button.

**Note**: Counts and screenshots were accurate as of September 2022. As items are added to the Math Reviews Database, new items may be part of sample searches and counts will change.

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Edward Dunne

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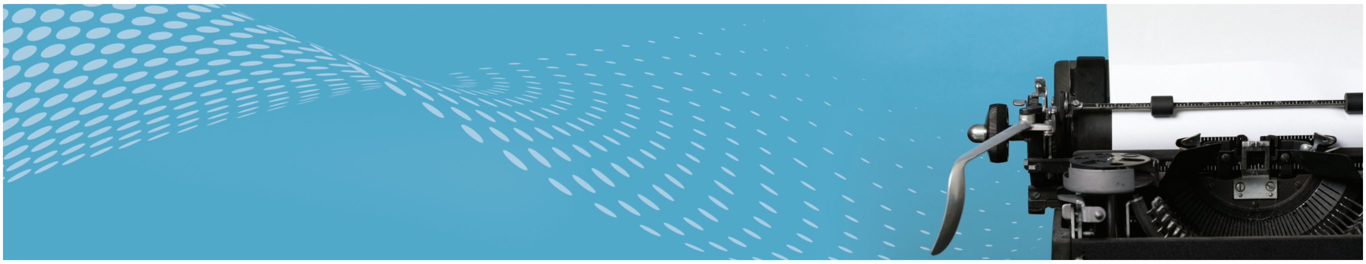
## SHORT STORIES



# Isbell Duality

*John C. Baez*

Mathematicians love dualities. The dual of a vector space $V$ is the vector space $V^*$ of linear maps from $V$ to the ground field. Any linear map $f : V \to W$ between vector spaces gives a linear map going the other way between their duals, $f^* : W^* \to V^*$, given by

$$f^*(\ell)(v) = \ell(f(v)), \qquad \forall v \in V, \ell \in W^*.$$

Composition gets turned around:

$$(fg)^* = g^* f^*.$$

Furthermore, there is always a linear map

$$i : V \to V^{**}$$

given by

$$i(v)(\ell) = \ell(v) \qquad \forall v \in V, \ell \in V^*,$$

and when $V$ is finite-dimensional this is an isomorphism. So, for finite-dimensional vector spaces, duality is like flipping a coin upside down: when you do it twice, you get back where you started—at least up to isomorphism.

Dualities are useful because they let you view the same situation in two different ways. Often dualities give an interesting description of the opposite of a familiar category $\mathsf{C}$. This is the category $\mathsf{C}^{\mathrm{op}}$ with the same objects as $\mathsf{C}$, but where a morphism $f : X \to Y$ is defined to be a morphism $f : Y \to X$ in $\mathsf{C}$, and the order of composition is reversed.

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Some dualities show a category is equivalent to its own opposite. For example, duality for vector spaces can be used to show the category of finite-dimensional vector spaces over any field is equivalent to its opposite. We're secretly using this whenever we take the transpose of a matrix. Similarly, Pontryagin duality says the category of locally compact abelian groups is equivalent to its own opposite. This duality sends each locally compact group $G$ to its "Pontryagin dual" $\widehat{G}$, and the Fourier transform of a function on $G$ is a function on $\widehat{G}$. For example, the Poincaré dual of the real line is the real line, while the Pontryagin dual of the circle is the integers.

More commonly, however, a category of mathematical objects is not equivalent to its opposite, and a duality relates two different categories. The opposite of a category of spaces is typically a category of commutative rings or algebras. For example, Gelfand–Naimark duality says the opposite of the category of compact Hausdorff spaces is the category of commutative $C^*$-algebras. In fact, to make the duality between spaces and commutative rings as nice as possible, Grothendieck *defined* a category of spaces called "affine schemes" to be the opposite of a category of commutative rings.

There are also dualities within category theory itself. The opposite of a category is itself a kind of a dual, and taking the opposite twice gives you back the category you started with:

$$(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathsf{C}.$$

But there is a subtler and very beautiful duality in category theory called "Isbell duality."

First, there is a map from any category $\mathsf{C}$ to the category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$, where objects are functors from $\mathsf{C}^{\mathrm{op}}$ to the category of sets and morphisms are natural transformations. This map takes any object $X \in \mathsf{C}$ to the functor

$$\hom(-, X) \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set},$$

which sends any object $A \in \mathsf{C}$ to the set $\hom(A, X)$ of all morphisms from $A$ to $X$. This map is called the **Yoneda embedding**, and is itself a functor:

$$\begin{array}{rccl} y\colon & \mathsf{C} & \to & [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}] \\ & X & \mapsto & \hom(-, X). \end{array}$$

The Yoneda embedding is fundamental in category theory. Philosophically it says that an object can be known by the behavior of the morphisms into it. It takes time to learn how to use it as a practical tool, but this is nicely explained in modern textbooks [3, 4]. One insight is this: just as we often take a set and form the vector space with that set as basis, it is often useful to treat a category $\mathsf{C}$ as sitting inside the larger category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$. The reason is that $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ has "colimits," which are analogous to linear combinations in a vector space. For example, we can sum $F, G \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ as follows:

$$(F + G)(X) = F(X) + G(X) \qquad \forall X \in \mathsf{C}^{\mathrm{op}}$$

where the sum at right is the usual disjoint union of sets. In fact $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ is the free category with colimits on the category $\mathsf{C}$.

But this whole story has a dual version! An object can also be known by the behavior of morphisms *out of* it. This fact is captured by the **co-Yoneda embedding**:

$$\begin{array}{rccl} z\colon & \mathsf{C} & \to & [\mathsf{C}, \mathsf{Set}]^{\mathrm{op}} \\ & X & \mapsto & \hom(X, -). \end{array}$$

The concept dual to colimit is "limit": just as colimits generalize sums, limits generalize products. Unsurprisingly, it turns out that $[\mathsf{C}, \mathsf{Set}]^{\mathrm{op}}$ is the free category with limits on the category $\mathsf{C}$.

In 1960, Isbell [2] noticed a wonderful link between the Yoneda and co-Yoneda embeddings, which has subsequently been clarified by many authors, as reviewed in [1]. Any functor $F \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ has an **Isbell conjugate** $F^* \colon \mathsf{C} \to \mathsf{Set}$, given by

$$F^*(X) = \hom(F, y(X)).$$

Similarly, any functor $G \colon \mathsf{C} \to \mathsf{Set}$ has an Isbell conjugate $G^* \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ given by

$$G^*(X) = \hom(z(X), G).$$

These two versions of Isbell conjugate give functors going back and forth like this:

$$\begin{tikzcd} {[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]} \arrow[r, bend left, "*"] & {[\mathsf{C}, \mathsf{Set}]^{\mathrm{op}}} \arrow[l, bend left, "*"] \end{tikzcd}$$

But these two functors are typically not inverses, not even up to natural isomorphism! Instead, Isbell duality says they are **adjoints**, meaning that $\hom(F^*, G)$ and $\hom(F, G^*)$ are naturally isomorphic for all $F \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ and $G \in [\mathsf{C}, \mathsf{Set}]^{\mathrm{op}}$. This is analogous to the situation for vector spaces that are not necessarily finite-dimensional: taking the dual defines adjoint functors going back and forth between $\mathsf{Vect}$ and $\mathsf{Vect}^{\mathrm{op}}$, where $\mathsf{Vect}$ is the category of *all* vector spaces over a given field.

Isbell duality sets the stage for a panoply of further developments [1]. For example, just as the vector spaces where $i \colon V \to V^{**}$ is an isomorphism are precisely the finite-dimensional ones, it is very interesting to study functors $F \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ such that the canonical map $i \colon F \to F^{**}$ is an isomorphism.

So far most applications of Isbell duality involve its generalization to enriched categories [5]. For example, this generalization gives a way to find, for any compact metric space $X$, the smallest compact metric space $Y$ containing a copy of $X$ with the property that for any $x \in X$ and $y \in Y$ there is a point $x' \in X$ such that

$$d(x, y) + d(y, x') = d(x, x').$$

However, it seems that Isbell duality still remains largely unexploited. Perhaps one problem is simply that this jewel of mathematics is not yet widely known.

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John C. Baez

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FROM THE AMS SECRETARY

# Calls for Nominations & Applications

## AMS Prizes & Awards

### Leroy P. Steele Prize for Lifetime Achievement

#### About this Prize

The Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students. The amount of this prize is US$10,000.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 March 2023

**Nomination Procedure:** `https://www.ams.org/steele-prize`

**Nominations can be submitted between February 1 and March 31.** Nominations for the Steele Prizes for Lifetime Achievement should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

### Leroy P. Steele Prize for Mathematical Exposition

#### About this Prize

The Steele Prize for Mathematical Exposition is awarded for a book or substantial survey or expository research paper. The amount of this prize is US$5,000.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 March 2023

**Nomination Procedure:** `https://www.ams.org/steele-prize`

**Nominations can be submitted between February 1 and March 31.** Nominations for the Steele Prizes for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

### Leroy P. Steele Prize for Seminal Contribution to Research

#### About this Prize

The Steele Prize for Seminal Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special Note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:

1. Open (2025)
2. Analysis/Probability (2026)
3. Algebra/Number Theory (2027)
4. Applied Mathematics (2028)
5. Geometry/Topology (2029)
6. **Discrete Mathematics/Logic (2024)**

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 March 2023

**Nomination Procedure:** `https://www.ams.org/steele-prize`

**Nominations can be submitted between February 1 and March 31.** Nominations for the Steele Prizes for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

# Chevalley Prize in Lie Theory

The Chevalley Prize is awarded for notable work in Lie theory published during the preceding six years; a recipient should be at most twenty-five years past the PhD.

### About this Prize

The Chevalley Prize was established in 2014 by George Lusztig to honor Claude Chevalley (1909–1984). Chevalley was a founding member of the Bourbaki group. He made fundamental contributions to class field theory, algebraic geometry, and group theory. His three-volume treatise on Lie groups served as standard reference for many decades. His classification of semisimple groups over an arbitrary algebraically closed field provides a link between Lie's theory of continuous groups and the theory of finite groups, to the enormous enrichment of both subjects.

The current prize amount is US$8,000, awarded in even-numbered years, without restriction on society membership, citizenship, or venue of publication.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** Submit a letter of nomination, complete bibliographic citations for the work being nominated, and a brief citation that might be used in the event that the nomination is successful.

To make a nomination go to `https://www.ams.org/chevalley-prize`.

# Frank Nelson Cole Prize in Algebra

This prize recognizes a notable research work in algebra that has appeared in the last six years. The work must be published in a recognized, peer-reviewed venue.

### About this Prize

This prize (and the Frank Nelson Cole Prize in Number Theory) was founded in honor of Professor Frank Nelson Cole upon his retirement after twenty-five years as secretary of the American Mathematical Society. Cole also served as editor-in-chief of the *Bulletin* for twenty-one years. The original fund was donated by Professor Cole from moneys presented to him on his retirement, and was augmented by contributions from members of the Society. The fund was later doubled by his son, Charles A. Cole, and supported by family members. It has been further supplemented by George Lusztig and by an anonymous donor.

The current prize amount is US$5,000, and the prize is awarded every three years.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

To make a nomination go to `https://www.ams.org/cole-prize-algebra`.

# Levi L. Conant Prize

This prize was established in 2000 in honor of Levi L. Conant to recognize the best expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* in the preceding five years.

### About this Prize

Levi L. Conant was a mathematician and educator who spent most of his career as a faculty member at Worcester Polytechnic Institute. He was head of the mathematics department from 1908 until his death and served as interim president of WPI from 1911 to 1913. Conant was noted as an outstanding teacher and an active scholar. He published a number of articles in scientific journals and wrote four textbooks. His will provided for funds to be donated to the AMS upon the death of his wife.

Prize winners are invited to present a public lecture at Worcester Polytechnic Institute as part of their Levi L. Conant Lecture Series, which was established in 2006.

The Conant Prize is awarded annually in the amount of US$1,000.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** Nominations with supporting information should be submitted online. Nominations should include a letter of nomination, a short description of the work that is the basis of the nomination, and a complete bibliographic citation for the article being nominated.

To make a nomination go to `https://www.ams.org/conant-prize`.

# Ulf Grenander Prize in Stochastic Theory and Modeling

The Grenander Prize recognizes exceptional theoretical and applied contributions in stochastic theory and modeling. It is awarded for seminal work, theoretical or applied, in the areas of probabilistic modeling, statistical inference, or related computational algorithms, especially for the analysis of complex or high-dimensional systems.

### About this Prize

This prize was established in 2016 by colleagues of Ulf Grenander (1923–2016). Professor Grenander was an influential scholar in stochastic processes, abstract inference, and pattern theory. He published landmark works throughout his career, notably his 1950 dissertation, *Stochastic Processes and Statistical Interference* at Stockholm University, *Abstract Inference*, his seminal *Pattern Theory: From representation to inference,* and *General Pattern Theory*. A long-time faculty member of Brown University's Division of Applied Mathematics, Grenander received many honors. He was a Fellow of the American Academy of Arts and Sciences and the National Academy of Sciences and was a member of the Royal Swedish Academy of Sciences.

The current prize amount is US$5,000, and the prize is awarded every three years.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** To make a nomination go to `https://www.ams.org/grenander-prize`.

# Bertrand Russell Prize

### About this Prize

The Bertrand Russell Prize of the AMS was established in 2016 by Thomas Hales. The prize looks beyond the confines of the profession to research or service contributions of mathematicians or related professionals to promoting good in the world. It recognizes the various ways that mathematics furthers fundamental human values. Mathematical contributions that further world health, our understanding of climate change, digital privacy, or education in developing countries are some examples of the type of work that might be considered for the prize.

The current prize amount is US$5,000, awarded every three years.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** Include a short description of the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included.

To make a nomination go to `https://www.ams.org/russell-prize`.

# Albert Leon Whiteman Memorial Prize

The Whiteman Prize recognizes notable exposition and exceptional scholarship in the history of mathematics.

### About this Prize

This prize was established in 1998 using funds donated by Mrs. Sally Whiteman in memory of her husband, Albert Leon Whiteman.

The US$5,000 prize is awarded every three years.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** Include a short description of the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included.

To make a nomination go to `https://www.ams.org/whiteman-prize`.

# Award for Distinguished Public Service

The Award for Distinguished Public Service recognizes a research mathematician who has made recent or sustained distinguished contributions to the mathematics profession through public service.

### About this Award

The AMS Council established this award in response to a recommendation from its Committee on Science Policy.

The US$4,000 award is presented every two years.

**Next Award:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** Submit a letter of nomination describing the candidate's accomplishments, a CV for the nominee, and a brief citation that explains why the work is important.

To make a nomination go to `https://www.ams.org/public-service-award`.

# Award for an Exemplary Program or Achievement in a Mathematics Department

This award recognizes a department which has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

### About this Award

This award was established in 2004. For the first three awards (2006–2008), the prize amount was US$1,200. The prize was endowed by an anonymous donor in 2008, and starting with the 2009 prize, the amount is US$5,000.

This US$5,000 prize is awarded annually. Departments of mathematical sciences in North America that offer at least a bachelor's degree in mathematical sciences are eligible.

**Next Award:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** A letter of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department is being nominated as well as the achievements which make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). Where possible, the letter and documentation should address how these successes 1) came about by systematic, reproducible changes in programs that might be implemented by others, and/or 2) have value outside the mathematical community. The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

To make a nomination go to `https://www.ams.org/department-award`.

# Award for Mathematics Programs that Make a Difference

The Award for Mathematics Programs that Make a Difference was established in 2005 by the AMS's Committee on the Profession to compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are potentially replicable models.

### About this Award

This award brings recognition to outstanding programs that have successfully addressed the issues of underrepresented groups in mathematics. Examples of such groups include racial and ethnic minorities, women, low-income students, and first-generation college students.

One program is selected each year by a Selection Committee appointed by the AMS President and is awarded US$1,000 provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity.

Preference is given to programs with significant participation by underrepresented minorities. Note that programs aimed at pre-college students are eligible only if there is a significant component of the program benefiting individuals from underrepresented groups at or beyond the undergraduate level. Nomination of one's own institution or program is permitted and encouraged.

**Next Award:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** The letter of nomination should describe the specific program being nominated and the achievements that make the program an outstanding success. It should include clear and current evidence of that success. A strong nomination typically includes a description of the program's activities and goals, a brief history of the program, evidence of its effectiveness, and statements from participants about its impact. The letter of nomination should not exceed two pages, with supporting documentation not to exceed three more pages. Up to three supporting letters may be included in addition to these five pages. Nomination of the writer's own institution or program is permitted. Non-winning nominations will automatically be reconsidered for the award for the next two years.

To make a nomination go to `https://www.ams.org/make-a-diff-award`.

## Award for Impact on the Teaching and Learning of Mathematics

This award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education.

Priorities of the award include recognition of:

(a) accomplished mathematicians who have worked directly with pre-college teachers to enhance teachers' impact on mathematics achievement for all students, or

(b) sustainable and replicable contributions by mathematicians to improving the mathematics education of students in the first two years of college.

### About this Award

The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education in 2013. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen.

The US$1,000 award is given annually.

**Next Award:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** Letters of nomination may be submitted by one or more individuals. The letter of nomination should describe the significant contributions made by the nominee(s) and provide evidence of the impact these contributions have made on the teaching and learning of mathematics. The letter of nomination should not exceed two pages, and may include supporting documentation not to exceed three additional pages. A brief curriculum vitae for each nominee should also be included. The non-winning nominations will automatically be reconsidered, without further updating, for the awards to be presented over the next two years.

To make a nomination go to `https://www.ams.org/impact`.

# Fellowships

## Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS Fellows Selection Committee, comprised of twelve members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at `https://www.ams.org/ams-fellows`.

# Joint Prizes

## George David Birkhoff Prize in Applied Mathematics (AMS-SIAM)

The Birkhoff Prize is awarded for an outstanding contribution to applied mathematics in the highest and broadest sense.

### About this Prize

The prize was established in 1967 in honor of Professor George David Birkhoff, with an initial endowment contributed by the Birkhoff family and subsequent additions by others. The American Mathematical Society (AMS) and the Society for Industrial and Applied Mathematics (SIAM) award the Birkhoff Prize jointly.

The current prize amount is US$5,000, awarded every three years to a member of AMS or SIAM.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** To make a nomination go to `https://www.ams.org/birkhoff-prize`.

## Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student (AMS-MAA-SIAM)

The Morgan Prize is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who was enrolled as an undergraduate in December at a college or university in the United States or its possessions, Canada, or Mexico is eligible for the prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate. Publication of research is not required.

### About this Prize

The prize was established in 1995. It is entirely endowed by a gift from Mrs. Frank (Brennie) Morgan. It is made jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

The current prize amount is US$1,200, awarded annually.

**Next Prize:** January 2024

**Nomination Period:** 1 February – 31 May 2023

**Nomination Procedure:** To nominate a student, submit a letter of nomination, a brief description of the work that is the basis of the nomination, and complete bibliographic citations (or copies of unpublished work). All submissions for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student's research.

To make a nomination go to `https://www.ams.org/morgan-prize`.

## JPBM Communications Award

This award is given each year to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to non-mathematical audiences.

### About this Award

This award was established by the Joint Policy Board for Mathematics (JPBM) in 1988. JPBM is a collaborative effort of the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and the American Statistical Association.

Up to two awards of US$2,000 are made annually. Both mathematicians and non-mathematicians are eligible.

**Next Prize:** January 2024

**Nomination Period:** open

**Nomination Procedure**: Nominations should be submitted on `mathprograms.org`. Note: Nominations collected before September 15th in year $N$ will be considered for an award in year $N$+2.

AMS COMMUNICATION

# An Interview with Ruth Charney

*Scott Hershberger*

Every other year, when a new AMS president takes office, the *Notices* publishes interviews with the outgoing and incoming presidents. Ruth Charney's two-year term as president will end on January 31, 2023. Charney is the Theodore and Evelyn Berenson Professor of Mathematics at Brandeis University. *Notices* contributing writer Scott Hershberger spoke with her in June 2022. An edited version of that interview follows.

***Notices:*** *What are you most proud of from your term as president?*

**Charney:** This has been a very challenging few years for the AMS, as well as the community as a whole, as a result of COVID, political controversy, and international conflicts. Much of my energy has been focused on navigating those challenges and getting people to work together in a productive way.

One place I feel I have made real progress is in bringing more diverse voices into the AMS. A major job of the president is to find volunteers to serve on AMS committees. The AMS has over 100 committees, and for a fair number of these, it's the purview of the president to appoint people to serve on them. One of the first steps we needed to take in addressing diversity issues was to make sure that the people serving on these committees represented a larger segment of the community. We have a Committee on Committees that helps the president generate names of potential candidates, and they've been fabulous. I've also been soliciting suggestions from many other sources for new people we might engage—young people and people from parts of the community we don't usually reach out to—and making sure that our committees have a nice balance between people with prior experience with the AMS and new voices. I feel like we've been pretty successful with that.

***Notices:*** *What did you hope to see happen during your term as president that didn't pan out?*

**Figure 1.** Ruth Charney is the outgoing AMS president.

*Scott Hershberger is a master's student in science communication at the University of Wisconsin–Madison. When he conducted this interview, he was the communications and outreach content specialist at the AMS. His email address is* scotthersh42@gmail.com.



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**Charney:** There are certain aspects of AMS governance that are a little klutzy, and I hoped to find a way to make things more efficient. Many good ideas emerge from the policy committees, but they take a multi-layer process to implement. We are considering changing the timing of meetings or having some remote meetings to try to make the process more efficient. I don't think this is something that people outside the governance structure see, but it can be difficult and frustrating at times. Of course, whatever it is one wants to do—and I'm saying this more generally, not just about the AMS—it always takes longer and is harder than one would like!

***Notices:*** *The Task Force on Understanding and Documenting the Historical Role of the AMS in Racial Discrimination released its report at the beginning of your term as president. What do you think has been the most important result of it so far?*

**Charney:** I would say that the most important outcomes of the Task Force report so far have been the creation of a new Director of Equity, Diversity, and Inclusion staff position and the hiring of Dr. Leona Harris for this position; progress toward including more diverse voices in our committees, as we discussed earlier; and publicizing the expertise of mathematicians of color through books, *Notices* articles, invited lectures, etc.

The AMS also created a new top-level policy committee, the Committee on Equity, Diversity, and Inclusion. While this committee was created independently of the Task Force, one of its main missions is to keep track of the progress being made on the Task Force recommendations and to report on this progress to the Council each year. It is important that we don't allow ourselves to stop paying attention to these issues, that we keep going and take the next step and the next step after that. I'm optimistic that the new staff position together with this new policy committee will assure that EDI is a continuing priority for the AMS.

***Notices:*** *What have you learned about the AMS and the math community during your time as president?*

**Charney:** The more I get involved with the AMS, the more impressed I am with the range and the impact of the society's activities. I really believe that the AMS is an essential pillar of the mathematics community, and it deserves people's appreciation and support.

For example, I think one of the most important things that the AMS does is government advocacy. We have an office in Washington, and our Director of Government Relations, Dr. Karen Saxe, talks to congressional members and staff frequently about issues relating to the math and science communities, making sure mathematics—including theoretical research mathematics—stays front and center in funding initiatives.

**Figure 2.** Ruth Charney in her backyard in Lexington, Massachusetts. Charney enjoys gardening in her spare time.

I've also become more aware of the breadth of viewpoints and concerns across the math community—people think in all different ways and have different priorities. That's part of what I find fascinating about doing these kinds of jobs. I get to know so many new people beyond my usual cohort of topologists and geometric group theorists (whom, of course, I like very much).

***Notices:*** *You led during a time of worsening discourse in society in general, and even in the math community in some ways. How did that affect your approach?*

**Charney:** It seems that whatever you do or say these days, somebody is angry about it. I'm the sort of person who wants to keep everybody happy. Well, it turns out that's not possible in this environment, and that's been frustrating for me.

I realized early on that it was going to be really important to listen to other people. With respect to diversity issues, I originally said to myself, "Oh, I went through this with bringing more females into the math community. I was president of the Association for Women in Mathematics, and I know how to do this." But I quickly became aware that the problems we face currently are just not the same. There are some aspects that are similar, but the more I listened to people's experiences, the more I realized that I didn't really know what the answers were, or even, in some cases, what the problems were.

While it's difficult to listen to people's anger, it's important to understand what it is that's upsetting them. That's been a learning process.

***Notices:*** *How do you view the AMS's relationships with other professional societies in math?*

**Charney:** We've been strengthening our interactions with other mathematics societies in recent years. We have worked with multiple societies to redesign meetings and institute new professional development programs. I hope that we continue to build on those relationships. Each society has its role to play and represents a certain aspect of the community, and that's wonderful—but it is also important that we work together toward advancing the profession.

***Notices:*** *The 2022 International Congress of Mathematicians was originally planned to be held in Russia—a very controversial choice from the start due to Russia's human rights abuses. What lessons should the AMS and the entire math community draw from this situation?*

**Charney**: It is really not the AMS who makes these decisions. That's the job of the International Mathematical Union. Of course, we're in touch with the people who are involved, and we can express opinions, but it's not up to us to make the decisions.

There's a somewhat related question about meetings we hold in the US: Should the AMS hold meetings in locations that either historically or currently are not welcoming to certain communities? This is a complicated question that is currently under discussion by the AMS policy committees. Regardless of the outcome of these discussions, the AMS is working to assure a welcoming environment at all our meetings.

***Notices:*** *What are your thoughts on the direction in which mathematics is going?*

**Charney:** I feel that there's a growing appreciation in the general public for the importance of mathematics in many different aspects of science and society. It's no longer viewed as a highly specialized, isolated field. We used to talk about pure math and applied math. In my opinion, the word "pure" is a poor choice. It suggests that we've got a wall around us. A better way to think of it is as theoretical and applied math, and the two merge into each other. The theoretical feeds into the applied, and the applied feeds back into the theoretical. Ideas from math, including the theoretical side, are now being used in all sorts of new ways. I think the future of mathematics is to play an ever-larger role in society—and that's good for us.

***Notices:*** *What challenges will the AMS face in the coming years?*

**Charney:** Almost all societies have been seeing a reduction in membership. People just don't join things anymore. Everything's available online, so why bother to pay a membership fee? But I claim that there is much to be gained from getting involved in professional societies. The AMS does a great deal for the mathematics community through publications, advocacy, meetings, professional development, etc., and it's also an opportunity to make all kinds of connections with other people. There are so many great reasons to join the AMS!

***Notices:*** *Have you given any advice to Bryna Kra about being president?*

**Charney:** We've certainly discussed how to move forward on various issues. Bryna has been very involved in the AMS for years, so she knows what she's getting into. We may have slightly different perspectives on some things, but I'm sure it will be a smooth transition and she will bring great energy to the job.

*An interview with Bryna Kra, the incoming AMS president, will appear in the February 2023 issue of* Notices of the AMS.

Scott Hershberger

## Credits

Figure 1 is courtesy of Mike Lovett for Brandeis University.
Figure 2 is courtesy of Ruth Charney.
Author photo is courtesy of Scott Hershberger.

# Polynomial Systems, Homotopy Continuation, and Applications

*Timothy Duff and Margaret Regan*

Systems of multivariate polynomial equations are ubiquitous throughout mathematics. They also appear prominently in scientific applications such as kinematics [20, 22], computer vision [11, 15], power flow engineering [18], and statistics [12]. Numerical homotopy continuation methods are a fundamental tool for both solving these systems and determining more refined information about their structure.

In this article, we offer a brief glimpse of polynomial homotopy continuation methods: the general theory, a few applications, and some software packages that implement these methods. Our aim is to spark the reader's interest in this exciting and broad area of research. We invite those looking to learn more to join us at the **AMS Short Course: Polynomial systems, homotopy continuation, and applications**, to be held January 2–3 at the 2023 Joint Mathematics Meetings in Boston.

## 1. Homotopy Continuation

Many types of homotopy continuation methods exist, but they all are based on the same strategy. A system of equations $g(z) = 0$ whose solutions are known, called the *start system*, can be continuously deformed into a system of equations $f(z) = 0$ whose solutions we would like to know, called the *target system*. The following example illustrates some of the key ideas.

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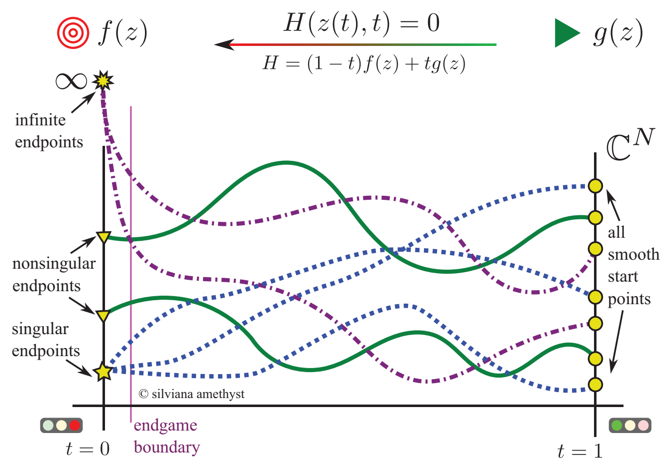


**Figure 1.** Graphical summary of homotopy continuation.

**Example 1.** Consider the target polynomial system

$$f(z) = \begin{bmatrix} 4(z_1^2 z_2 + z_1 z_2) - 3z_2 \\ z_2^2 - 2z_1 + 1 \end{bmatrix}.$$

The corresponding *total degree start system*

$$g(z) = \begin{bmatrix} z_1^3 - 1 \\ z_2^2 - 1 \end{bmatrix}$$

has 6 solutions of the form $(z_1, z_2) = (\omega, \pm 1)$, where $\omega$ is a third root of unity. These will serve as the smooth start points depicted in Figure 1.

Using the *straight-line homotopy*

$$H(z;t) = (1-t)f(z) + tg(z) = 0$$

to deform $g$ into $f$, each solution for $t \approx 1$ may be estimated from the start points by numerical predictor/corrector methods, also known as homotopy *path-tracking* [24, Ch. 2]. Iterating this procedure, we find for this example there are six smooth homotopy paths of the form $z(t) : (0,1] \to \mathbb{C}^2$, where $z(1)$ is one of the smooth start points. As $t \to 0^+$, one solution path will diverge towards infinity, and three will come together at the singular endpoint $(z_1, z_2) = (1/2, 0)$. The remaining two endpoints $(z_1, z_2) = (-1/2, \pm 2i)$ are finite and nonsingular. We see that $f$ has three solutions, five counting multiplicity.

Path-tracking methods are well-studied in numerical analysis, and are especially potent when applied to a parametrized polynomial system

$$f(z;p) = f(z_1, \dots, z_N; p_1, \dots, p_k) = \begin{bmatrix} f_1(z_1, \dots, z_N; p_1, \dots, p_k) \\ \vdots \\ f_n(z_1, \dots, z_N; p_1, \dots, p_k) \end{bmatrix} \quad (1)$$

where $z_1, \dots, z_N$ are *variables* representing unknown quantities and $p_1, \dots, p_k$ are *parameters* representing physical measurements. Section 2 gives few examples of such systems appearing in applications.

A general *parameter continuation theorem* [24, Theorem 7.1.1] is based on the fact that for *almost all* parameter values $p^* \in \mathbb{C}^k$, the system of equations $f(z;p^*) = 0$ has a finite number $d$ of solutions $z^* \in \mathbb{C}^N$ which are *nonsingular* in the sense that $J_z f(z^*; p^*)$, the $N \times N$ Jacobian matrix of $f$ with respect to $z$, is invertible. The number $d$ is sometimes called the *generic root count* of the system (1).

The essential observation of the parameter continuation theorem is that all isolated solutions can be computed via tailor-made homotopies which operate in a problem's natural parameter space. These *parameter homotopies* involve two phases, summarized below. See [5, Chap. 6] for more details.

*Ab initio phase.* The first step for a parameter homotopy is to fix parameter values $p^* \in \mathbb{C}^k$ and find $d$ nonsingular solutions to the system $f(z;p^*) = 0$. This can be accomplished with a straight-line homotopy as in Example 1. This has the advantage that the solutions of $H(z;1) = g(z) = 0$ are trivial to compute. More sophisticated methods allow us to track potentially fewer paths in this phase. These include multihomogeneous homotopies [24, Sec. 8.4.2], polyhedral homotopy [24, Sec. 8.5], and methods based on monodromy [24, Sec. 15.4].

*Parameter homotopy phase.* With the *ab initio* phase complete, the "online" parameter homotopy phase aims to solve $f(z;p) = 0$ for any sufficiently general choice of $p \in \mathbb{C}^k$. For this task, we utilize the parameter homotopy

$$H(z;t) = f(z; \tau(t) \cdot p^* + (1-\tau(t)) \cdot p) = 0 \quad \text{where} \quad \tau(t) = \frac{\gamma t}{1 + (\gamma - 1)t} \quad (2)$$

for all $t \in [0,1]$ and some fixed $\gamma \in \mathbb{C}$. In particular, $H(z;1) = f(z;p^*) = 0$ has known solutions $S$, computed in the *ab initio* phase, and one aims to compute the solutions to $H(z;0) = f(z;p) = 0$. For generic values of the constant $\gamma \in \mathbb{C}$, the arc $\tau(t) \cdot p^* + (1-\tau(t)) \cdot p$ for $t \in [0,1]$ connects $p^*$ to $p$ and avoids the complex discriminant locus. Thus, for $t \in [0,1]$, $H(z;t) = 0$ defines precisely $d$ solution paths connecting the $d$ points in $S$ with the $d$ solutions to $f(z;p) = 0$.

Our discussion of homotopy continuation methods in this section is necessarily incomplete. Here we list a few additional topics falling under the rubric of general methods. One important topic is *numerical algebraic geometry* [23], which allows us to study positive-dimensional algebraic varieties. In the opposite case of an *overdetermined* system, several techniques allow us to reduce to the case of a well-constrained parametrized system of the form (1); see [11] and the references therein. Lastly, we mention numerical certification methods which can prove that approximated solutions will converge to exact solutions (see, e.g., [13]), and deflation methods for regularizing systems with singular solutions [14, 16].

## 2. Applications

Polynomial homotopy continuation has been a key to advances in various applications. We summarize three that will be featured in our short course.

### 2.1. Kinematics.

Mechanical linkage systems of interest have constrained motions that are naturally modeled with systems of polynomial equations. Such polynomial formulations cover a wide breadth of mechanisms including planar, spherical, and spatial types.

For example, consider the 4-bar mechanism in Figure 2 where $A$ and $B$ are fixed pivots and $\ell_1$, $\ell_2$, and $\ell_3$ are the lengths of the moving links.

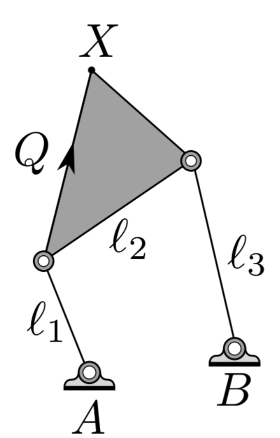


**Figure 2.** Schematic of a four-bar linkage.

The path synthesis problem associated with this mechanism seeks to find all possible linkages that meet certain design requirements. Wampler et al. [27] solved the exact path synthesis problem for 4-bars, also known as Alt's problem, which imposes that the coupler trace point $X$ passes through nine generic positions. Alt's problem amounts to solving a system with 8652 complex solutions. These solutions carry the extra structure of a 6-fold symmetry group, including the 3-fold *Roberts' cognate triplets.*

Homotopy continuation has been used for these exact synthesis problems by finding the roots of the corresponding system of polynomial equations [1, 8, 19–22]. These are large-scale (up to $10^6$) root-finding problems, where homotopy continuation is the only method capable of computing complete solution sets at such scales.

Other methods focus on approximate kinematic synthesis, relying on optimization techniques to accommodate any number of design specifications. For example, in [2], the approximate path synthesis problem using optimization yields about 303,249 ± 713 Roberts' cognate triples as critical points with 95% confidence for the 4-bar linkage. This offers an advantage over exact methods, with the downside being large computational effort. Numerical homotopy continuation methods were central to the use of monodromy loops that made this computation possible.

The use of homotopy continuation within optimization problems in kinematic design has also enabled the study of the configurations of the parallel 5-bar mechanism, which displays more nonlinearity that the serial 5-bar mechanism. Figure 3 shows this complicated configuration space. In [9], homotopy continuation is used to quantify transmission quality using the curves of input and output singularities. This enables developing a path that switches between non-neighboring output modes (i.e., solution sheets).

In general, homotopy continuation methods have led to the analysis and solving of much more complicated problems in kinematics.

**2.2. Algebraic statistics.** Maximum likelihood estimation (MLE) is a fundamental technique of statistical inference, in which the *likelihood* function associated to a data set is maximized over a space of all possible parameters that specifies a statistical model. A major theme in the field of *algebraic statistics* [25] is that the space of model parameters will often be an algebraic variety. In this case, homotopy continuation methods can be used for *global optimization* of the likelihood function. This complements the widely used *EM algorithm* for MLE, which has the advantage of being easy to implement, but is generally susceptible to local minima.

To make these ideas expressed above concrete, we consider a discrete statistical model from [12]. Fix positive

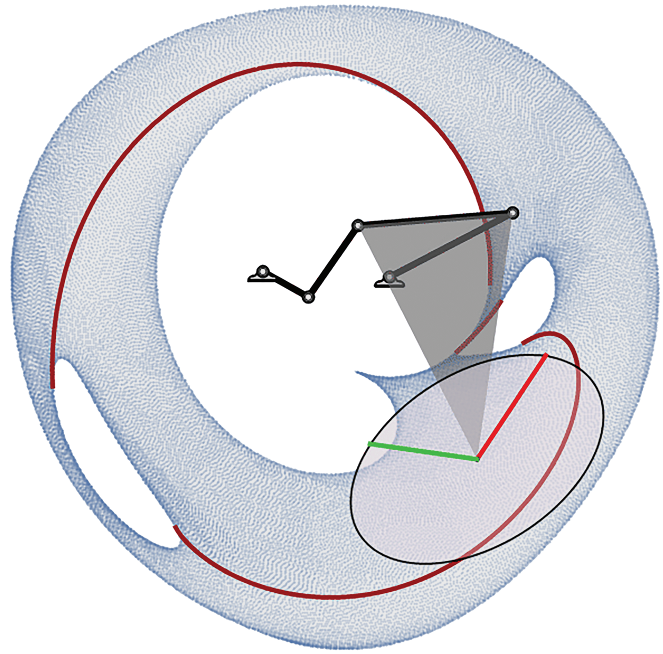

**Figure 3.** Workspace curve of input singularities and velocity ellipse for a five-bar linkage.

integers $m, n \geq 1$, and consider two discrete probability distributions,

$$p_X : \{1, \dots, m\} \to [0,1],$$
$$p_Y : \{1, \dots, n\} \to [0,1].$$

If we draw $k$ samples from each distribution, $X_1, \dots, X_k \sim p_X$, $Y_1, \dots, Y_k \sim p_Y$, we may record the frequency of all possible $mn$ outcomes into a $m \times n$ matrix of counts $U$. Let $P$ be the $m \times n$ matrix giving the joint distribution $p_{ij} = \Pr(X = i, Y = j)$. Our statistical model is the algebraic variety $\mathcal{V}_r \cap \Delta_{mn-1}$, where $\mathcal{V}_r$ is the variety of matrices of rank at most $r$, and $\Delta_{mn-1}$ is the probability simplex,

$$\Delta_{mn-1} = \{P \in \mathbb{R}^{m\times n} \mid p_{ij} \geq 0, \sum p_{ij} = 1\}.$$

Note that random variables $X$ and $Y$ are independent iff $P \in \mathcal{V}_r \cap \Delta_{mn-1}$. The MLE problem amounts to minimizing the *log-likelihood function*

$$\ell(P; U) = \log\left(\prod_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} p_{ij}^{u_{ij}}\right) - k \log\left(\left(\sum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} p_{ij}\right)\right). \qquad (3)$$

Calculating the partial derivatives $\frac{\partial \ell}{\partial p_{ij}}$ reveals that they are rational functions in $u_{ij}$ and $p_{ij}$, and this leads to a polynomial system of equations whose solutions are the critical points of (3) restricted to the model $\mathcal{V}_r \cap \Delta_{mn-1}$. Among these critical points is the maximum-likelihood estimate. The total number of critical points is known as the *ML degree* of the model. Using parameter homotopies, we can track exactly this number of homotopy paths to find all critical points. The ML degrees for small $m$, $n$, and $r$ are tabulated below (table adapted from [12].) Do you see any patterns?

| $(m,n) =$ | $(3,3)$ | $(3,4)$ | $(3,5)$ | $(4,4)$ | $(4,5)$ | $(4,6)$ |
|---|---|---|---|---|---|---|
| $r=1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $r=2$ | 10 | 26 | 58 | 191 | 843 | 3119 |
| $r=3$ | 1 | 1 | 1 | 191 | 843 | 3119 |
| $r=4$ | | | | 1 | 1 | 1 |

**2.3. Power flow systems.** Let $n \geq 2$ be an integer and consider a finite undirected graph $G$ on the vertices $\{1, \dots, n\}$. Fix $x_1 = 1$, $y_1 = 0$, and consider the system of $2(n-1)$ equations in $2(n-1)$ unknowns

$$\begin{aligned} x_i^2 + y_i^2 - 1 = 0 \quad & i = 2, \dots, n, \\ \sum_{(i,j)\in E(G)} b_{ij}(x_i y_j - x_j y_i) = 0 \quad & i = 2, \dots, n. \end{aligned} \tag{4}$$

This is one formulation of the *power flow equations* used to model a network of $n$ agents (also known as *buses*.) Each bus may represent a power station, customer, or some other entity within an electrical grid. The coefficients $b_{ij} \in \mathbb{R}$ for $(i,j) \in E(G)$ are called *susceptances* and are assumed to be known. The unknowns $(x_i, y_i)$ are the real and imaginary parts of the voltage at the $i$-th bus. The fixed values $(x_1, y_1) = (1, 0)$ determine the *reference bus*.

Solving the power flow equations plays an important role in operating and controlling electrical networks. It is common for engineers to approach this problem with local, iterative algorithms such as Newton's method, which will return a single real solution.

But what can we say about *all* solutions to (4)? Notice that there are $2^{n-1}$ "trivial" solutions obtained by fixing all non-reference buses $(x_i, y_i) = (\pm 1, 0)$ for $i = 2, \dots, n$. The number of "non-trivial" solutions turns out to depend on the topology of the graph $G$. At one extreme, for the complete graph $G = K_n$, these equations will have $\binom{2n-2}{n-1}$ solutions over the complex numbers, as long as the susceptances are sufficiently generic. For the $n$-cycle $G = C_n$, the generic complex root count is a more modest $n \cdot \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. In either case, there is a symmetry on these solutions that sends $(x_i, y_i) \mapsto (x_i, -y_i)$, allowing us to reduce the cost of homotopy path-tracking by a factor of 2. Of course, only the solutions where all $x_i$ and $y_i$ are *real* are of any practical interest. Figure 4 illustrates the distribution of real solutions as a subset of the susceptances vary for the 5-cycle $C_5$. The visible regions which are blue, red, green, purple, and yellow correspond to parameter values with 0, 2, 4, 6, and 8 solutions, respectively. We refer to the article [18] for detailed explanations and many other interesting experimental results obtained using homotopy continuation methods.

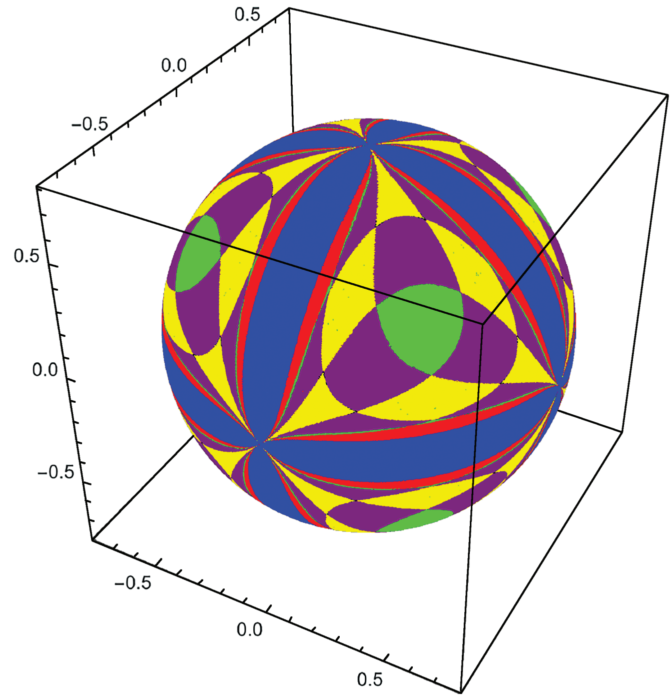


**Figure 4.** Distribution of real solutions to equations (4) for the cycle graph $G = C_5$, with three susceptances $b_{ij}$ drawn uniformly from the unit sphere in $\mathbb{R}^2$ and the other two fixed.

## 3. Software

A wide variety of software packages implementing polynomial homotopy continuation methods exists. Here we highlight three that will be used during the upcoming short course:

1. `Bertini` [4] is a standalone software package, whose functionality includes many of the standard homotopy methods for isolated solutions, as well as numerical irreducible decomposition for positive-dimensional solutions.
2. `HomotopyContinuation.jl` [6] is a software package written for the Julia language, a programming language designed for high-performance numerical computing.
3. `Macaulay2` [10] is a computer algebra system focused on computational commutative algebra and algebraic geometry. Primarily a tool for *symbolic* computation, it also has a growing number of *numerical* algebraic geometry packages [17].

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## Credits

Figure 1 is courtesy of Silviana Amethyst.
Figures 2 and 3 are courtesy of Jonathan D. Hauenstein.
Figure 4 is courtesy of Julia Lindberg.

# Math and Writing: Two Sides of the Same Coin?

*Anuraag Bukkuri*

"What do you want to be when you grow up?" was definitely not an easy question for me to answer when I was a kid. My quest for the answer has taken me on an exhilarating journey that has not yet come to an end. I was inspired by the words of Poe, the works of Euclid, and the adventures of Darwin. I wanted to do it all!

By the time I entered college, I decided to study mathematics and biology. I poured over books and papers on topics from ancient human fossils to cutting-edge medical research, from the esoteric realms of infinity-category theory to the practical applications of harmonic analysis. And I loved every minute of it.

My passion for writing, however, was relegated to a side-hobby at best, with a few miscellaneous poems scribbled in the margins of my organic chemistry notes. Something was missing. Something was incomplete.

As I started graduate school, I vowed to rekindle my writing activities. Amidst the never-ending deluge of seminars, papers, and conferences, I discovered the critical need for scientists to share their knowledge and excitement about science with the public. Realizing that I could use my communication skills to fill this need, I dove headfirst into the world of science writing.

I began writing for outlets such as *The Conversation*, explaining how ecology and evolution could help us tackle the biggest problems in cancer, and for the journal *Evolution*, summarizing and synthesizing the latest developments in evolutionary biology and ecology to a lay audience. It was a blast and the engagement I got from my audiences was addicting.

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When I learned about the AAAS Mass Media Science and Engineering Fellowship, I knew I had to apply. I was elated to discover that I had been chosen for the fellowship by the American Mathematical Society and placed at a local newspaper, *The Miami Herald*, where I'd be working as a health reporter for the summer. But I was wholly unprepared for what was to come.

The summer approached quickly, and I was soon immersed in the world of science journalism. I was calling local health officials about expanding vaccine and testing sites, analyzing the latest COVID and monkeypox trends, and speaking to medical experts and political pundits about the effects of the Roe v. Wade decision. But most importantly, I was reaching out to members of the South Florida community to understand their questions and concerns.

Many of the articles I wrote at the *Miami Herald* were in response to such queries: How can I track COVID trends in Florida? What are the symptoms of monkeypox and how can I get tested? Others were more detailed explainers about the impacts of the overturning of Roe v. Wade on medical care, the effects of the new COVID variant and the monkeypox virus, and what the government is doing about the monkeypox vaccine shortage. And yet others chronicled the stories of a trauma surgeon who went to work on war victims in Ukraine, or a summer camp for kids with cancer.

Each of these articles had a life of its own. Some were sad. Some were angry. Some were hopeful. But all of them were important for the members of the South Florida community. They gave the opportunity for a woman to open up, for the first time, about her lived experience with cancer. And for a man to express his frustration over the health inequities he's faced and the abrupt cancellation of his vaccine appointment.

As I return to graduate school, I will bring back with me many of the lessons I learned during my time as an AAAS/AMS Mass Media fellow. Namely, the ability to write quickly and effectively, construct compelling and thorough arguments, and find novel questions or unique angles on well-studied problems are all skills that will make me a better scientist. And although I plan to remain in academia, doing the work I love with colleagues and students that inspire me, science writing and communication will always be an integral part of who I am and what I do.

Anuraag Bukkuri

**Credits**

Photo of Anuraag Bukkuri is courtesy of Anuraag Bukkuri.

# Finding Meaning Outside Academia

*A. J. Stewart*

"Why did you decide to work in Congress?"

I get asked this question a lot. It seems a mathematician working in Congress is a novel enough event that it requires an explanation. It is only now, over a year after I accepted the AMS Congressional Fellowship, that I feel I can begin to accurately answer this question. My time on Capitol Hill showed me how to use mathematics outside academia. Whether it be as an integral part of a team working on large issues or committing to small actions that make a big difference to a single person, I saw that mathematics has a place in creating a brighter future.

For my fellowship, I was placed in Senator Raphael Warnock's office. I grew up in Florida, sometimes referred to as south of the South, and understood the challenges of being from a low-income family in the South. I wanted to utilize my mathematical skills on policy that would support a more equitable economy for all Americans. Senator Warnock's office was a perfect fit.

Within Senator Warnock's office I worked on the economic portfolio, which included housing, tax, trade, and financial services, with two other staffers. We were the hub for any content coming in or out of the office related to our portfolio. We met with constituents and stakeholders, created background documents, and lent our expertise on policy decisions. My background as a mathematician was immediately treated as an asset. It was assumed I would utilize my skills and training in any way to support the office. I felt trusted, valued, and part of a team. I was excited to be able to use mathematics to help working families and it was fulfilling to be part of an office committed to doing the same.

*A. J. Stewart was the 2021–2022 AMS Congressional Fellow. His email address is* `allenjstewart@protonmail.com`.



DOI: https://doi.org/10.1090/noti2608

However, no matter how committed a member of Congress may be, what gets done in Congress is completely reliant on the congressional body as a whole. If Congress is unable to reach sufficient consensus on any matter, then that matter won't be resolved. This legislative gridlock was not the overarching theme during my time on Capitol Hill. The 117th Congress was exceptionally productive by providing COVID-19 relief, necessary infrastructure funding, future investments in green technology to combat climate change, as well as much more.

Much of this work was a result of bipartisan cooperation, which was a common thread through all my work during the fellowship. I felt the importance of creating and supporting bipartisan coalitions. My mathematical skills lent themselves to this cooperative process. Since mathematics has a large foundation in problem solving and so much of problem solving is being able to view a problem in several different ways, this meant that I could use the same skills to approach a specific policy that I would use to approach a problem. This enabled me to view policy areas from multiple viewpoints, which is an important skill on Capitol Hill.

I always felt that I was able to lend my skills in any way possible even though I was only a small part of larger legislative actions. The passage of The Inflation Reduction Act required a process called "vote-a-rama" where numerous amendments are voted on during an extended period of time. This process can take over 24 hours to complete. As multiple amendments regarding complex tax law were filed (sometimes at 3 or 4 in the morning), I relied on deduction and inference to predict the exact changes the suggested amendment would have on the tax structure as well as who would feel its effects. I needed to make quick policy recommendations and it was my training as a mathematician that guided me.

Not every action on Capitol Hill involves staying up all night researching amendments. Some actions are much smaller in scope. The Economic Injury Disaster Loans (EIDL) are administered by the Small Business Administration (SBA) and provide emergency financial support for small businesses in disaster areas. As a result of the pandemic, specific COVID-19 EIDL loans were created. When the funding for a construction business in Georgia was delayed, I argued the business's case to the SBA. This meant that the business could stay afloat, and its employees could remain on payroll. It was a small action that helped a dozen or so working families. Being able to argue the merits of the case involved analyzing the required application documentation and using logic to persuade the SBA. Again, the skills I utilized were supported by the skills I use as a mathematician. It was mathematics that guided me as I supported and advocated for people in these small ways.

Now I return to the question of why I wanted to work in Congress or more specifically how I found a place outside academia. In the years leading up to 2020, I wanted to do more with mathematics. I felt that mathematics had so much to offer the world and should be used to address some of the hardest questions of our generation; wealth inequality, climate change, increased political polarization, etc. I wanted to work in Quantitative Justice. To apply my training and skills to support a more equitable and just world. I just didn't know how to do it or how I could be useful.

During my fellowship I realized mathematicians are desired for our quantitative training, logical skills, and problem-solving ability but we are also valued because of the unique perspective we bring. We naturally seek out truth and can see the center of systems and processes. Our education lends itself to finding more stable, equitable processes and so we possess necessary skills to address the inherent inequalities within our historical systems. We can be useful and the world outside academia is waiting for us to participate.

I know I won't solve the Riemann Hypothesis or the Hodge Conjecture, but I also know there is not a monopoly on worthy mathematical pursuits. Whether it be playing a small part of substantial climate change legislation or ensuring a construction business can pay its employees, mathematicians can make a difference in a multitude of ways. We need only step out into the world and commit to change.

The AMS Congressional Fellowship is a unique way to get involved in our government. For those interested in personally getting involved, the AMS funds one Congressional Fellow per year, with this year's application deadline of February 1, 2023, and there are multiple other types of science and technology policy fellowships available to mathematicians.

Learn about the AMS Congressional Fellowship at `https://www.ams.org/government/government/ams-congressional-fellowship`.

A. J. Stewart

### Credits

Photo of A. J. Stewart is courtesy of Jay Shepherd.

# AMS Updates

## Math Departments Coordinate Job Offer Deadlines

For the past 21 years, the American Mathematical Society has led the effort to gain broad endorsement for the following proposal:

That mathematics departments and institutes agree not to require a response prior to a certain date (usually around February 1 of a given year) to an offer of a postdoctoral position that begins in the fall of that year.

This proposal is linked to an agreement made by the National Science Foundation (NSF) that recipients of NSF Mathematical Sciences Postdoctoral Fellowships would be notified of their awards, at the latest, by the end of January.

This agreement ensures that our young colleagues entering the postdoctoral job market have as much information as possible about their options before making a decision. It also allows departmental hiring committees adequate time to review application files and make informed decisions. From our perspective, this agreement has worked well and has made the process more orderly. There have been very few negative comments. Last year, more than 180 mathematics and applied mathematics departments and institutes endorsed the agreement.

Therefore, we propose that mathematics departments again collectively enter into the same agreement for the upcoming cycle of recruiting, with the deadline set for Monday, February 6, 2023. The NSF's Division of Mathematical Sciences has already agreed that it will complete its review of applications and notify all applicants no later than Friday, January 27, 2023.

The American Mathematical Society facilitated the process by sending an email message to all doctoral-granting mathematics and applied mathematics departments and mathematics institutes. The list of departments and institutes endorsing this agreement was widely announced on the AMS website and is updated weekly until mid-January.

We ask that you view this year's formal agreement at `https://www.ams.org/employment/postdoc-offers.html` along with this year's list of adhering departments.

To streamline this year's process for all involved, we ask that you notify the AMS (`postdoc-deadline@ams.org`) **if and only if**: your department is not listed and you would like to be listed as part of the agreement; or your department is listed and you would like to withdraw from the agreement and be removed from the list.

Please feel free to email us with questions and concerns. Thank you for consideration of the proposal.

*—Catherine A. Roberts*
*AMS Executive Director*

*Torina Lewis*
*AMS Associate Executive Director*

## Inaugural Fall Graduate School Fair Draws More Than 300 Students

At the AMS's first Fall Graduate School Fair, held October 12, representatives of more than 40 graduate programs in the mathematical sciences hosted virtual tables to answer student questions.

The Gather platform provided a dynamic way to interact, replicating an in-person experience for more than 300 undergraduates and master's students. The students were provided with avatars that allowed them to navigate and interact within the virtual fair, much like an online video game.

In addition to the virtual tables, the Graduate School Fair kicked off with a panel of students and graduate chairs, who shared ways that students can explore graduate school options and can pose insightful questions of program staff.

"The panel provided exactly what I had hoped: an opportunity to empower students," said Sarah Bryant, AMS director of programs. "It also, I hope, prompted department representatives to think deeply about how they are recruiting to their programs." She added that the panel discussion was recorded and is available online on the AMS YouTube channel.

The AMS Fall Graduate School Fair will recur annually in addition to the Graduate School Fair at the Joint

Mathematics Meetings. "I wish this resource had been available when I was applying to graduate school," said panelist Christopher L. Cox, mathematics professor and department head, University of Tennessee at Chattanooga.

*—AMS Communications*

## New Version of MathJax Available

An updated version of MathJax is now available, reported its lead developer, Davide Cervone.

"MathJax 4.0.0-alpha.1 is the first release of a major update to MathJax," said Cervone, professor of mathematics at Union College. "It includes a number of significant new features that our sponsors and users have been waiting for."

For those unfamiliar, MathJax is an open-source tool for rendering mathematics on the web, designed for diverse audiences. Using MathJax does not require the viewer to download any software, and because it uses actual fonts, its output will scale and print better than math presented as images.

MathJax's four new features include:

- support for 10 different fonts, including the STIX2 font set, five Gyre math fonts, and a sans-serif font based on the Fira-Math font. "The original MathJax TeX font set remains available as an option," Cervone said.
- support for automatic and explicit line breaks in both display and in-line expressions, automatic breaking of text elements, better breaking in tables and arrays, more array column specifiers for the array preamble, and TeX macros to control line breaking, indentation, and alignment, as well as more commands for making fixed-size boxes in which line breaking can occur;
- support for HTML embedded in MathML and TeX expressions (e.g., you can insert form elements into the mathematical expressions);
- improvements to the assistive support by including the expression explorer in all combined components.

The release notes provide extensive information about the new features.

MathJax launched in 2010 as an open-source project under the sponsorship of the American Mathematical Society (AMS), the Society for Industrial and Applied Mathematics (SIAM), and Design Science, Inc., with additional support from StackExchange, the American Physical Society, Elsevier, the Optical Society of America, Project Euclid, WebAssign, and others.

When Design Science left the MathJax Consortium in 2013, the AMS assumed the role of Managing Partner and was responsible for day-to-day operations and management. In 2019, MathJax joined NUMFocus, a nonprofit organization that promotes open practices in research, data, and scientific computing by serving as a fiscal sponsor for open-source projects.

The AMS remains a founding partner of MathJax, making a substantial annual contribution and serving on its steering committee, said Thomas J. Blythe, AMS chief information officer.

"MathJax is vital to the AMS's commitment to making mathematics accessible on the internet," he said. "It's the cornerstone of our accessibility efforts for rendering mathematics in our journals and ePub versions of our ebooks, as well as in AMS MathViewer and MathSciNet®."

*—AMS Communications*

## Deaths of AMS Members

Andrew Bucki, of Edmond, Oklahoma, died on March 2, 2022. Born on March 29, 1946, he was a member of the Society for 38 years.

C. Carton-Lebrun, of Belgium, died on April 5, 2022. Born on January 8, 1942, she was a member of the Society for 50 years.

Ian Connell, of Canada, died on February 21, 2022. Born on December 20, 1934, he was a member of the Society for 60 years.

Ed Dubinsky, of Miami Springs, Florida, died on June 9, 2022. Born on February 7, 1935, he was a member of the Society for 57 years.

Bert Fristedt, of Bloomington, Minnesota, died on July 18, 2020. Born on April 8, 1937, he was a member of the Society for 57 years.

Betty B. Garrison, of San Diego, California, died on January 16, 2021. Born on July 1, 1932, she was a member of the Society for 66 years.

Eberhard Kaniuth, of Germany, died on April 27, 2017. Born on November 30, 1937, he was a member of the Society for 36 years.

J. Musielak, of Poland, died on October 11, 2020. Born on November 7, 1928, he was a member of the Society for 38 years.

Keith Phillips, of Boulder, Colorado, died on March 8, 2016. Born on June 11, 1937, he was a member of the Society for 53 years.

Helmut R. Salzmann, of Germany, died on March 8, 2022. Born on November 3, 1930, he was a member of the Society for 58 years.

Jerrold B. Tunnell, of Piscataway, New Jersey, died on April 1, 2022. Born on September 16, 1950, he was a member of the Society for 50 years.

# Mathematics People

## 2023 Breakthrough Prizes in Mathematics Announced


Daniel Spielman

AMS Member **Daniel A. Spielman** was awarded the 2023 Breakthrough Prize in Mathematics.

The world's largest science awards, each of five main Breakthrough prizes is $3 million.

Spielman is a Sterling Professor of Computer Science and a Professor of Statistics and Data Science and of Mathematics at Yale University. He was honored "for breakthrough contributions to theoretical computer science and mathematics, including to spectral graph theory, the Kadison-Singer problem, numerical linear algebra, optimization, and coding theory," the prize citation noted.

"In Mathematics, Daniel A. Spielman's insights and algorithms have been significant not only for mathematics, but for highly practical problems in computing, signal processing, engineering, and even the design of clinical trials," according to a news release.

Additionally, six New Horizons Prizes of $100,000 each were distributed between 11 early-career scientists and mathematicians who have already made a substantial impact on their fields. The 2023 New Horizons in Mathematics Prizes were awarded to:


Ana Caraiani

- **Ana Caraiani**, Imperial College London and University of Bonn, for diverse transformative contributions to the Langlands program, and in particular for work with Peter Scholze on the Hodge–Tate period map for Shimura varieties and its applications. Caraiani is an AMS Fellow.
- **Ronen Eldan**, Weizmann Institute of Science and Microsoft Research, for the creation of the stochastic localization method, which has led to significant progress in several open problems in high-dimensional geometry and probability, including Jean Bourgain's slicing problem and the KLS conjecture.
- **James Maynard**, Oxford University and Institute for Advanced Study, for multiple contributions to analytic number theory, and in particular to the distribution of prime numbers.

Three Maryam Mirzakhani New Frontiers Prizes of $50,000 each were awarded to women mathematicians who have recently completed their PhDs and who have produced important results:

- **Maggie Miller**, Stanford University and Clay Mathematics Institute, for work on fibered ribbon knots and surfaces in 4-dimensional manifolds.
- **Jinyoung Park**, Stanford University, for contributions to the resolution of several major conjectures on thresholds and selector processes.
- **Vera Traub**, University of Bonn, for advances in approximation results in classical combinatorial optimization problems, including the traveling salesman problem and network design.

The 2023 Breakthrough Prize laureates in Fundamental Physics, Life Sciences, and Mathematics were announced on September 22, 2022 by the Breakthrough Prize Foundation and its founding sponsors—Sergey Brin, Priscilla Chan and Mark Zuckerberg, Julia and Yuri Milner, and Anne Wojcicki.

*—From a Breakthrough Prize announcement*

## 2023 ICIAM Prizes Announced

The International Council for Industrial and Applied Mathematics (ICIAM) announced the winners of 2023 ICIAM Prizes, which will be awarded during the opening ceremony of the International Congress for Industrial and Applied Mathematics, to be held in Tokyo on August 20–25, 2023.

**Maria Colombo** of EPFL Lausanne, Switzerland, was awarded the ICIAM Collatz Prize for fundamental contributions to the regularity theory and the analysis of singularities in elliptic PDEs, geometric variational problems, transport equations, and incompressible fluid dynamics.

**Alfio Quarteroni** of Politecnico di Milano, Italy, was awarded the ICIAM Lagrange Prize for groundbreaking work in finite element and spectral methods, domain decomposition methods, discontinuous Galerkin methods, numerical solution of incompressible Navier–Stokes equations, multiphysics and multiscale modeling—with application to fluid dynamics, geophysics, the human heart and circulatory system, the COVID-19 epidemic, and improvement of sports performance for the America's Cup sailing competition.

**Weinan E** of Peking University, China, and Princeton University was awarded the ICIAM Maxwell Prize for seminal contributions to applied mathematics and in particular on analysis and application of machine learning algorithms, multi-scale modeling, the modeling of rare events and stochastic partial differential equations.

**Leslie Greengard** of New York University has been awarded the ICIAM Pioneer Prize for groundbreaking work on fast algorithms, including the fast multipole method (one of the top 10 algorithms of the 20th century), fast Gauss transform, and fast direct solvers; and for the development of innovative high-order, automatically adaptive algorithms for differential and integral equations.

**Jose Mario Martinez Perez** of the University of Campinas, Brazil, was awarded the ICIAM Su Buchin Prize for outstanding achievements in research—a combination of theory, practice, software, and applications for solving large-scale optimization problems—and in fostering the development of the optimization and applied mathematics communities in Latin America.

**Cleve B. Moler** of Math Works, Inc., was awarded the ICIAM Industry Prize for his outstanding contributions to the development of mathematical and computational tools and methods for the solution of science and engineering problems and his invention of MATLAB, which allows industrial users to harness efficient and reliable numerical methods to execute numerical simulations in ever-expanding domains of science and engineering.

The 2023 ICIAM Prize Committee was chaired by ICIAM President Ya-xiang Yuan. Chairs of the prize subcommittees were Gang Bao (Maxwell Prize), Alfredo Bermudez (Pioneer Prize), Nira Chamberlain (Industry Prize), Leah Edelstein-Keshet (Lagrange Prize), Lois Curfman McInnes (Su Buchin Prize), and Kim-Chuan Toh (Collatz Prize).

*—From an ICIAM announcement*

# Yunqing Tang Receives SASTRA Ramanujan Prize

Yunqing Tang

The 2022 SASTRA Ramanujan Prize was awarded to **Yunqing Tang**, an assistant professor at the University of California, Berkeley, who has been described as "one of the best young number theorists to emerge in recent years worldwide."

Tang was honored for "having established, by herself and in collaboration, a number of striking results on some central problems in arithmetic geometry and number theory," the prize citation noted. "… Her works display a remarkable combination of sophisticated techniques, in which the arithmetic and geometry of modular curves and of Shimura varieties play a central role, and have strong links with the discoveries of Srinivasa Ramanujan in the area of modular equations."

Tang earned her PhD from Harvard University in 2016 under supervision of Mark Kisin. A native of China, she has been a member of the Institute for Advanced Study in Princeton, NJ; a junior researcher (chargée de recherche) at CNRS/Université Paris-Sud; and an instructor and assistant professor at Princeton University. "She is one of the deepest and most creative mathematicians of her age, and her wide-ranging contributions are bound to have impact in the decades ahead," the citation noted.

The annual $10,000 prize is for outstanding contributions by individuals not exceeding the age of 32 in areas of mathematics influenced by Srinivasa Ramanujan in a broad sense. "The age limit has been set at 32 because Ramanujan achieved so much in his brief life of 32 years," according to the citation.

*—From the SASTRA Ramanujan Prize*

# Hendricks Awarded AWM Joan and Joseph Birman Research Prize

The Association for Women in Mathematics (AWM) has awarded the fifth AWM Joan & Joseph Birman Research Prize in Topology and Geometry to **Kristen Hendricks**, associate professor of mathematics at Rutgers University.

Hendricks is being honored for highly influential work on equivariant aspects of Floer homology theories. Her work "in low-dimensional and symplectic topology has revolutionized the understanding of equivariant aspects of

Floer theories, allowing powerful equivariant techniques to be used to solve classical, non-equivariant problems," according to a press release. "Hendricks' pioneering work on involutive Heegaard Floer homology has had wide-ranging applications, particularly to questions that straddle the border between 3- and 4-dimensional topology. The impact of her contributions to the understanding of homology cobordism groups, and to the closely related subject of knot concordance, has been profound. Hendricks' work has also opened new doors in the realm of symplectic topology, where her work with collaborators introduced one of the first general constructions of equivariant Floer homology."

Hendricks, who received her PhD in 2013 from Columbia University, was a Hedrick Assistant Adjunct Professor at the University of California, Los Angeles and an Assistant Professor at Michigan State University before joining the faculty at Rutgers.

"Joan Birman was a great inspiration to me while I was fortunate enough to interact with her as a graduate student at Columbia, and my appreciation and respect for her achievements has only increased as my perspective has matured," Hendricks said. "I'm also delighted to have my name on the same list as the previous prize winners, all of whom I hold in great esteem.

"I am greatly indebted to many excellent mentors, most especially my first undergraduate professor, Tom Coates; my primary graduate adviser, Robert Lipshitz; and my postdoctoral supervisor, Ciprian Manolescu," she said. "I am also grateful to both my former colleagues at Michigan State and my current colleagues at Rutgers for their unfailing supportiveness."

Established in 2012, the AWM Joan & Joseph Birman Research Prize highlights exceptional research in topology/geometry by a woman early in her career. The award is made possible by a generous contribution from Joan Birman, whose work has been in low-dimensional topology, and her husband, Joseph, who was a theoretical physicist specializing in applications of group theory to solid state physics. Hendricks will receive her award at the 2023 Joint Mathematics Meetings in Boston.

—*From the Association for Women in Mathematics*

# Sonnleitner Wins 2022 Joseph F. Traub Information-Based Complexity Young Researcher Award

**Mathias Sonnleitner**, University of Passau, Germany, has received the 2022 Joseph F. Traub Information-Based Complexity Young Researcher Award from the *Journal of Complexity*. The annual award is given for significant contributions to information-based complexity by a young researcher who has not reached their 35th birthday by September 30 in the year of the award. The award consists of $1,000 and a plaque, which was presented at the conference "Approximation and Geometry in High Dimensions," held in Będlewo, Poland, in October 2022.

—*From* Journal of Complexity

### Credits

Photos of Daniel Spielman and Ana Caraiani are courtesy of the Breakthrough Prize.

Photo of Yunqing Tang is courtesy of Professor George Bergman of the University of California, Berkeley math department.

# Classified Advertising
## *Employment Opportunities*

### MASSACHUSETTS

**Northeastern University**

The College of Science at Northeastern University invites applications for positions at all ranks (Assistant Professor, Associate Professor, or Professor), beginning in academic year 2023–2024 in the fields of Mathematical Modeling and Computation, broadly defined. Primary appointments will be in Mathematics with joint appointments in other departments including Physics and/or other colleges including Khoury College of Computer Sciences, the College of Engineering, and Bouvé College of Health Sciences. Appointments will have the opportunity to collaborate in cross-disciplinary teams across the University and will complement existing strengths.

Mathematical Modeling and Computation are at the heart of multiple areas of great societal impact. The College of Science is looking for exceptionally qualified individuals to fill faculty roles in these fields. Candidates will be considered from all areas concerned with the Mathematical Foundations of Modeling and Optimization in the Sciences and Engineering, or the Mathematics of Computation, with some emphasis on data-driven research. Relevant areas of study include Applied Analysis, Partial Differential Equations, Optimization, Discrete Mathematics, Quantum Computing, Algebraic Geometry, Data Science, High Dimensional Statistics, Probability, Complexity Theory, Security, and Cryptography.

Our tenure and promotion process values collaborative research and teamwork. Hires will be mentored for success, with mentoring teams and group guidance. In addition, a strong and effective faculty development strategy is part of the Northeastern institutional mission.

At Northeastern University, we embrace a culture of respect, where each person is valued for their contribution and is treated fairly. We oppose all forms of racism. We support a culture that does not tolerate any form of discrimination and where each person may belong. We strive to have a diverse membership, one where each person is trained and mentored to promote their success. See our website for more information about the College and its Leadership Team.

#### Responsibilities:

Potential hires are expected to develop vigorous research programs cross cutting the fields of Mathematics, Physics, Computer Science, Engineering, or Health Sciences. Faculty at Northeastern are expected to develop independent research programs that attract external funding; teach courses at the graduate and undergraduate level; supervise students and postdocs in their area of research; and participate in service to the department, university, and discipline. Qualified candidates must have excellence in, or a demonstrated commitment to, working with diverse student populations and/or in a culturally diverse work and educational environment.

#### Qualifications:

Applicants must have a PhD in Mathematics, Physics, Computer Science, Engineering, or a related field by the appointment start date. We encourage applicants from a wide range of backgrounds, including academia and industry. All applicants should have a strong record of scholarly accomplishment that demonstrates the ability to build a strong research program. Candidates seeking appointment at the Associate or Full Professor level should have substantial research productivity and an established history of grant support and academic service. Research excellence is the top-most priority. Depending on the research profile and expertise, a joint or affiliate appointment in another department(s) within the University is possible.

## Additional Information:

Interested candidates should view the complete position description and apply at: Northeastern University Careers: (`https://northeastern.wd1.myworkdayjobs.com/en-US/careers/details/Open-Rank--Assistant-Associate-Professor--Mathematical-Modeling-and-Computation_R109613?q=%20math%20modeling`) or at MathJobs: (`https://www.mathjobs.org/jobs/list/20881`) with a curriculum vita that includes a list of publications, research statement, teaching statement, an equity statement, and names and contact information for at least three professional references. Applications will be reviewed beginning on November 30, 2022.

Northeastern University is an equal opportunity employer, seeking to recruit and support a broadly diverse community of faculty and staff. Northeastern values and celebrates diversity in all its forms and strives to foster an inclusive culture built on respect that affirms inter-group relations and builds cohesion.

All qualified applicants are encouraged to apply and will receive consideration for employment without regard to race, religion, color, national origin, age, sex, sexual orientation, disability status, or any other characteristic protected by applicable law.

To learn more about Northeastern University's commitment and support of diversity and inclusion, please see `www.northeastern.edu/diversity`.

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## CHINA

**Tianjin University, China**
**Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics**

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to `mathjobs@tju.edu.cn`.

For more information, please visit `cam.tju.edu.cn` or contact Mr. Albert Liu at `mathjobs@tju.edu.cn`, telephone: 86-22-2740-6039.

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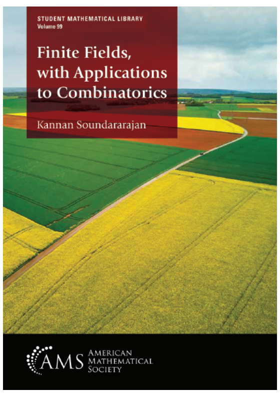
# New Books Offered by the AMS

## Algebra and Algebraic Geometry




### Finite Fields, with Applications to Combinatorics

**Kannan Soundararajan**, *Stanford University, CA*

This book uses finite field theory as a hook to introduce the reader to a range of ideas from algebra and number theory. It constructs all finite fields from scratch and shows that they are unique up to isomorphism. As a payoff, several combinatorial applications of finite fields are given: Sidon sets and perfect difference sets, de Bruijn sequences and a magic trick of Persi Diaconis, and the polynomial time algorithm for primality testing due to Agrawal, Kayal and Saxena.

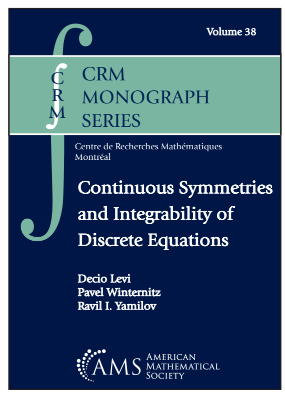
The book forms the basis for a one term intensive course with students meeting weekly for multiple lectures and a discussion session. Readers can expect to develop familiarity with ideas in algebra (groups, rings and fields), and elementary number theory, which would help with later classes where these are developed in greater detail. And they will enjoy seeing the AKS primality test application tying together the many disparate topics from the book. The pre-requisites for reading this book are minimal: familiarity with proof writing, some linear algebra, and one variable calculus is assumed. This book is aimed at incoming undergraduate students with a strong interest in mathematics or computer science.

*This item will also be of interest to those working in discrete mathematics and combinatorics and number theory.*

**Student Mathematical Library**, Volume 99
February 2023, 170 pages, Softcover, ISBN: 978-1-4704-6930-6, LC 2022037518, 2010 *Mathematics Subject Classification*: 11–01, 05–01, 12–01, 11A07, 11A51, 05B10, 12E20, List US$59, AMS Institutional member US$47.20, **All Individuals US$47.20**, Order code STML/99

bookstore.ams.org/stml-99

## Analysis




### Continuous Symmetries and Integrability of Discrete Equations

**Decio Levi**, *Roma Tre University, Rome, Italy, and INFN, Roma Tre Section, Rome, Italy*, **Pavel Winternitz**, *Université de Montréal, QC, Canada*, and **Ravil I. Yamilov**, *UFA Federal Research Center of the Russian Academy of Science, Russia*

This book on integrable systems and symmetries presents new results on applications of symmetries and integrability techniques to the case of equations defined on the lattice. This relatively new field has many applications, for example, in describing the evolution of crystals and molecular systems defined on lattices, and in finding numerical approximations for differential equations preserving their symmetries.

The book contains three chapters and five appendices. The first chapter is an introduction to the general ideas about symmetries, lattices, differential difference and partial difference equations and Lie point symmetries defined on them. Chapter 2 deals with integrable and linearizable systems in two dimensions. The authors start from the prototype of integrable and linearizable partial differential equations, the Korteweg de Vries and the Burgers equations. Then they consider the best known integrable differential difference and partial difference equations. Chapter 3 considers generalized symmetries and conserved densities as integrability criteria. The appendices provide details which may help the readers' understanding of the subjects presented in Chapters 2 and 3.

This book is written for PhD students and early researchers, both in theoretical physics and in applied mathematics, who are interested in the study of symmetries and integrability of difference equations.

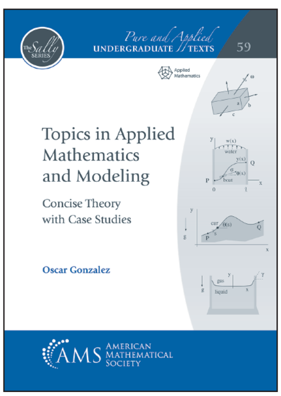
*This item will also be of interest to those working in differential equations and mathematical physics.*

Titles in this series are co-published with the Centre de Recherches Mathématiques.

**CRM Monograph Series**, Volume 38
January 2023, approximately 509 pages, Hardcover, ISBN: 978-0-8218-4354-3, 2010 *Mathematics Subject Classification*: 34–XX, 35–XX, 35Cxx, 35Pxx, 37Kxx, 39–XX, 39Axx; 17B67, 22E65, 34M55, 34C14, 34K04, 34K08, 34K17, 34L25, 35A22, 35B06, 35Q53, 37J35, 37K40, 37K06, 37K10, 37K15, 37K30, 37K35, 39A06, 39A14, 39A36, List US$125, **AMS members US$100**, **MAA members US$112.50**, Order code CRMM/38

`bookstore.ams.org/crmm-38`

## Applications




### Topics in Applied Mathematics and Modeling

#### Concise Theory with Case Studies

**Oscar Gonzalez**, *University of Texas at Austin, TX*

The analysis and interpretation of mathematical models is an essential part of the modern scientific process. *Topics in Applied Mathematics and Modeling* is designed for a one-semester course in this area aimed at a wide undergraduate audience in the mathematical sciences. The prerequisite for access is exposure to the central ideas of linear algebra and ordinary differential equations.

The subjects explored in the book are dimensional analysis and scaling, dynamical systems, perturbation methods, and calculus of variations. These are immense subjects of wide applicability and a fertile ground for critical thinking and quantitative reasoning, in which every student of mathematics should have some experience. Students who use this book will enhance their understanding of mathematics, acquire tools to explore meaningful scientific problems, and increase their preparedness for future research and advanced studies.

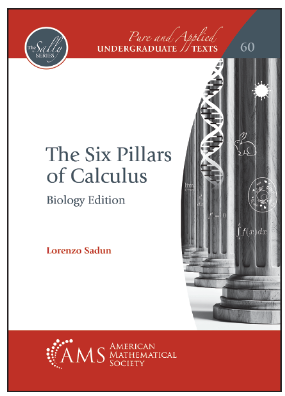
The highlights of the book are case studies and mini-projects, which illustrate the mathematics in action. The book also contains a wealth of examples, figures, and regular exercises to support teaching and learning. The book includes opportunities for computer-aided explorations, and each chapter contains a bibliography with references covering further details of the material.

*This item will also be of interest to those working in analysis and differential equations.*

**Pure and Applied Undergraduate Texts**, Volume 59
February 2023, approximately 219 pages, Softcover, ISBN: 978-1-4704-6991-7, 2010 *Mathematics Subject Classification*: 00A69, 34A26, 34E10, 37N99, 41A58, 49K15, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code AMSTEXT/59

`bookstore.ams.org/amstext-59`

## Calculus




### The Six Pillars of Calculus

#### Biology Edition

**Lorenzo Sadun**, *University of Texas at Austin, TX*

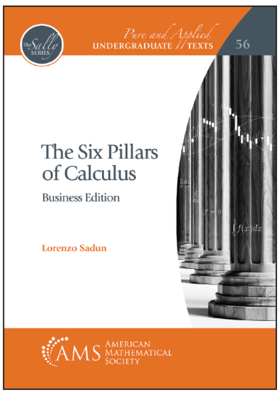
*The Six Pillars of Calculus: Biology Edition* is a conceptual and practical introduction to differential and integral calculus for use in a one- or two-semester course. By boiling calculus down to six common-sense ideas, the text invites students to make calculus an integral part of how they view the world. Each pillar is introduced by tackling and solving a challenging, realistic problem. This engaging process of discovery encourages students to wrestle with the material and understand the reasoning behind the techniques they are learning—to focus on *when* and *why* to use the tools of calculus, not just on how to apply formulas.

Modeling and differential equations are front and center. Solutions begin with numerical approximations; derivatives and integrals emerge naturally as refinements of those approximations. Students use and modify computer programs to reinforce their understanding of each algorithm.

The Biology Edition of the Six Pillars series has been extensively field-tested at the University of Texas. It features hundreds of examples and problems specifically designed for students in the life sciences. The core ideas are introduced by modeling the spread of disease, tracking changes in the amount of $CO_2$ in the atmosphere, and optimizing blood flow in the body. Along the way, students learn about optimal drug delivery, population dynamics, chemical equilibria, and probability.

**Pure and Applied Undergraduate Texts**, Volume 60
February 2023, approximately 383 pages, Softcover, ISBN: 978-1-4704-6996-2, LC 2022038762, 2010 *Mathematics Subject Classification*: 00–01, 26A06, 92–10, 92D30, 92D25, 00A71, 26–01, List US$99, **AMS members US$79.20**, **MAA members US$89.10**, Order code AMSTEXT/60

bookstore.ams.org/amstext-60




### The Six Pillars of Calculus
Business Edition

**Lorenzo Sadun**, *University of Texas at Austin, TX*

*The Six Pillars of Calculus: Business Edition* is a conceptual and practical introduction to differential and integral calculus for use in a one- or two-semester course. By boiling calculus down to six common-sense ideas, the text invites students to make calculus an integral part of how they view the world. Each pillar is introduced by tackling and solving a challenging, realistic problem. This engaging process of discovery encourages students to wrestle with the material and understand the reasoning behind the techniques they are learning—to focus on *when* and *why* to use the tools of calculus, not just on how to apply formulas.

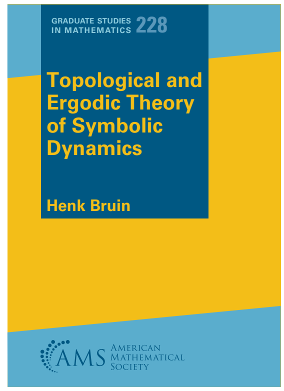
Modeling and differential equations are front and center. Solutions begin with numerical approximations; derivatives and integrals emerge naturally as refinements of those approximations. Students use and modify computer programs to reinforce their understanding of each algorithm.

The Business Edition of the Six Pillars series has been extensively field-tested at the University of Texas. It features hundreds of examples and problems designed specifically for business students. The core ideas are introduced by modeling market penetration of a new product, tracking changes in the national debt, and maximizing the profit of a business. Along the way, students learn about present value, consumer and producer surplus, amortization, and probability.

**Pure and Applied Undergraduate Texts**, Volume 56
November 2022, approximately 381 pages, Softcover, ISBN: 978-1-4704-6995-5, LC 2022028500, 2010 *Mathematics Subject Classification*: 00–01, 26A06, 01–01, 00A71, 26–01, List US$99, **AMS members US$79.20**, **MAA members US$89.10**, Order code AMSTEXT/56

bookstore.ams.org/amstext-56

# Differential Equations




### Topological and Ergodic Theory of Symbolic Dynamics

**Henk Bruin**, *University of Vienna, Austria*

Symbolic dynamics is essential in the study of dynamical systems of various types and is connected to many other fields such as stochastic processes, ergodic theory, representation of numbers, information and coding, etc. This graduate text introduces symbolic dynamics from a perspective of topological dynamical systems and presents a vast variety of important examples.

After introducing symbolic and topological dynamics, the core of the book consists of discussions of various subshifts of positive entropy, of zero entropy, other non-shift minimal action on the Cantor set, and a study of the ergodic properties of these systems. The author presents recent developments such as spacing shifts, square-free shifts, density shifts, $\mathcal{B}$-free shifts, Bratteli-Vershik systems, enumeration scales, amorphic complexity, and a modern and complete treatment of kneading theory. Later, he provides an overview of automata and linguistic complexity (Chomsky's hierarchy).

The necessary background for the book varies, but for most of it a solid knowledge of real analysis and linear algebra and first courses in probability and measure theory, metric spaces, number theory, topology, and set theory suffice. Most of the exercises have solutions in the back of the book.

*This item will also be of interest to those working in discrete mathematics and combinatorics and applications.*

**Graduate Studies in Mathematics**, Volume 228
February 2023, 460 pages, Hardcover, ISBN: 978-1-4704-6984-9, LC 2022034733, 2010 *Mathematics Subject Classification*: 37B10; 37B05, 28D05, 11J70, 68R15, List US$125, **AMS members US$100**, **MAA members US$112.50**, Order code GSM/228

bookstore.ams.org/gsm-228

# New in Memoirs of the AMS

## Algebra and Algebraic Geometry

### Mackey Profunctors

**D. Kaledin**, *Steklov Mathematical Institute, Moscow, Russia, and National Research University Higher School of Economics, Moscow, Russia*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1385
December 2022, 90 pages, Softcover, ISBN: 978-1-4704-5536-1, 2010 *Mathematics Subject Classification*: 18G99, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/280/1385

`bookstore.ams.org/memo-280-1385`

## Analysis

### One-Dimensional Dyadic Wavelets

**Peter M. Luthy**, *College of Saint Vincent, Riverdale, NY*, **Hrvoje Šikić**, *University of Zagreb, Croatia*, **Fernando Soria**, *Universidad Autónoma de Madrid, Spain*, **Guido L. Weiss**, *Washington University in St. Louis, MO*, and **Edward N. Wilson**, *Washington University in St. Louis, MO*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1378
December 2022, 152 pages, Softcover, ISBN: 978-1-4704-5374-9, 2010 *Mathematics Subject Classification*: 42C40, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/280/1378

`bookstore.ams.org/memo-280-1378`

## Applications

### A Probabilistic Approach to Classical Solutions of the Master Equation for Large Population Equilibria

**Jean-François Chassagneux**, *Université Paris Cité, France*, **Dan Crisan**, *Imperial College London, United Kingdom*, and **François Delarue**, *Université Côte d'Azur, Nice, France*

*This item will also be of interest to those working in probability and statistics.*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1379
December 2022, 123 pages, Softcover, ISBN: 978-1-4704-5375-6, 2010 *Mathematics Subject Classification*: 93E20; 60H30, 60K35, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/280/1379

`bookstore.ams.org/memo-280-1379`

## Differential Equations

### Asymptotic Spreading for General Heterogeneous Fisher-KPP Type Equations

**Henri Berestycki**, *École des Hautes en Sciences Sociales, Paris, France* and **Grégoire Nadin**, *Laboratoire Jacques-Louis Lions, Paris, France*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1381
December 2022, 100 pages, Softcover, ISBN: 978-1-4704-5429-6, 2010 *Mathematics Subject Classification*: 35B40, 35B27, 35K57; 35B50, 35K10, 35P05, 47B65, 49L25, List US$85, **AMS members US$68**, MAA members US$76.50, Order code MEMO/280/1381

`bookstore.ams.org/memo-280-1381`

### Adiabatic Evolution and Shape Resonances

**Michael Hitrik**, *University of California, Los Angeles, CA*, **Andrea Mantile**, *Université de Reims, France*, and **Johannes Sjöstrand**, *Université de Bourgogne, Dijon, France*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1380
December 2022, 90 pages, Softcover, ISBN: 978-1-4704-5421-0, 2010 *Mathematics Subject Classification*: 35B34, 35J10, 35P20, 35S05, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/280/1380

`bookstore.ams.org/memo-280-1380`

## Geometry and Topology

### Horocycle Dynamics: New Invariants and Eigenform Loci in the Stratum $\mathcal{H}(1,1)$

**Matthew Bainbridge**, *University of Indiana, Bloomington, IN*, **John Smillie**, *University of Warwick, Coventry, United Kingdom*, and **Barak Weiss**, *Tel Aviv University, Israel*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1384
December 2022, 100 pages, Softcover, ISBN: 978-1-4704-5539-2, 2010 *Mathematics Subject Classification*: 37D40, 30F30, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/280/1384

`bookstore.ams.org/memo-280-1384`

### Partial Compactification of Monopoles and Metric Asymptotics

**Chris Kottke**, *New College of Florida, Sarasota, FL* and **Michael Singer**, *University College London, United Kingdom*

*This item will also be of interest to those working in mathematical physics.*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1383
December 2022, 110 pages, Softcover, ISBN: 978-1-4704-5541-5, 2010 *Mathematics Subject Classification*: 53C07; 58J99, 81T13, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/280/1383

`bookstore.ams.org/memo-280-1383`

## Number Theory

### Hypergeometric Functions Over Finite Fields

**Jenny Fuselier**, *High Point University, NC*, **Ling Long**, *Louisiana State University, Baton Rouge, LA*, **Ravi Ramakrishna**, *Cornell University, Ithaca, NY*, **Holly Swisher**, *Oregon State University, Corvallis, OR*, and **Fang-Ting Tu**, *Louisiana State University, Baton Rouge, LA*

**Memoirs of the American Mathematical Society**, Volume 280, Number 1382
December 2022, 120 pages, Softcover, ISBN: 978-1-4704-5433-3, 2010 *Mathematics Subject Classification*: 11T23, 11T24, 33C05, 33C20, 11F80, 11S40, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/280/1382

`bookstore.ams.org/memo-280-1382`

# New AMS-Distributed Publications

## Differential Equations


### Constructive and Destructive Interferences in Nonlinear Hyperbolic Equations

**R. Carles**, *Université de Rennes 1, France* and **C. Cheverry**, *Université de Rennes 1, France*

This book introduces a physically realistic model for explaining how electromagnetic waves can be internally generated, propagate and interact in strongly magnetized plasmas or in nuclear magnetic resonance experiments. It studies high frequency solutions of nonlinear hyperbolic equations for time scales at which dispersive and nonlinear effects can be present in the leading term of the solutions. It explains how the produced waves can accumulate during long times to produce constructive and destructive interferences which, in the above contexts, are part of turbulent effects.

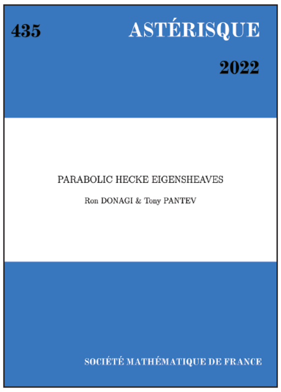
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Mémoires de la Société Mathématique de France**, Number 174
October 2022, 110 pages, Softcover, ISBN: 978-2-85629-946-3, 2010 *Mathematics Subject Classification*: 35Lxx, 35Qxx, 42B20, 70K30, 78Axx, 35Bxx, List US$63, **AMS members US$50.40**, Order code SMFMEM/174

`bookstore.ams.org/smfmem-174`

NEW BOOKS

# Geometry and Topology




### Parabolic Hecke Eigensheaves

**Ron Donagi**, *Department of Mathematics, University of Pennsylvania David Rittenhouse Lab, Philadelphia, PA* and **Tony Pantev**, *Department of Mathematics, University of Pennsylvania David Rittenhouse Lab, Philadelphia, PA*

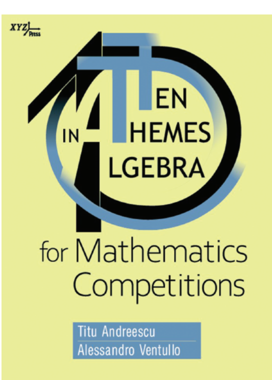
The authors study the Geometric Langlands Conjecture (GLC) for rank two flat bundles on the projective line $C$ with tame ramification at five points $p1,p2,p3,p4,p5$. In particular, they construct the automorphic $\mathcal{D}$-modules predicted by GLC on the moduli space of rank two parabolic bundles on $(C,p1,p2,p3,p4,p5)$. The construction uses non-abelian Hodge theory and a Fourier-Mukai transform along the fibers of the Hitchin fibration to reduce the problem to one in classical projective geometry on the intersection of two quadrics in $\mathbb{P}^4$.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Astérisque**, Number 435
October 2022, 192 pages, Softcover, ISBN: 978-2-85629-960-9, 2010 *Mathematics Subject Classification*: 14D24, 22E57, 14F10, 14A30, 14F08, 14H60, 14D23, List US$75, **AMS members US$60**, Order code AST/435

`bookstore.ams.org/ast-435`

# Math Education




### Ten Themes in Algebra for Mathematics Competitions

**Titu Andreescu**, *University of Texas at Dallas, TX* and **Alessandro Ventullo**, *University of Milan, Italy*

This book contains ten frequently recurring themes in algebraic problems. Each chapter starts with a brief introduction that includes examples useful for the reader to grasp the main ideas needed to solve the proposed problems. The first chapter deals with quadratic functions and underscores the use of the discriminant and the relations involving the roots of a quadratic trinomial and its coefficients.

The second chapter emphasizes that every square of a real number is non-negative. This simple property leads to numerous applications also encountered in subsequent chapters. Chapter 3 focuses on several inequalities, including the most famous inequality in the world of mathematical Olympiads: the Cauchy-Schwarz Inequality. Chapter 4 is devoted to problems related to minima and maxima of algebraic expressions. These problems can also be approached using the techniques studied in the previous chapter.

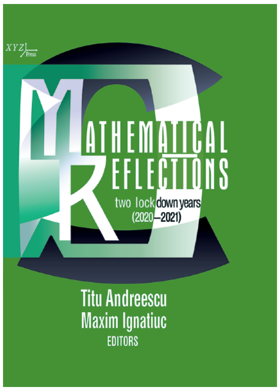
The fifth chapter is about a beautiful identity involving the cubes of three numbers and the triple of their product and you will see that this identity has numerous interesting applications. Chapter 6 deals with complex numbers. Some definitions and useful results are given to assist the reader in solving the proposed problems. The seventh chapter features Lagrange's Identity, which has various unexpected applications, including those involving problems related to number theory. Chapter 8 focuses on the so-called Sophie Germain's Identity. Here, too, you will find problems in which the application of this identity will be anything but obvious. Chapter 9 looks at expressions of the form $t+k/t$ and meaningful applications. Finally, the last chapter is about the fifth-degree polynomials $x^5+x\pm1$ and assorted non-routine problems. Solutions to all proposed problems are provided in the second part of the book: there is a corresponding solution chapter for each of the ten chapters in the first part.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

**XYZ Series**, Volume 46
January 2022, 307 pages, Softcover, ISBN: 978-1-7358315-5-8, 2010 *Mathematics Subject Classification*: 00A05, 00A07, 97U40, 97D50, List US$59.95, **AMS members US$47.96**, Order code XYZ/46

`bookstore.ams.org/xyz-46`

## NEW BOOKS




### Mathematical Reflections
#### Two Lockdown Years (2020–2021)

**Titu Andreescu**, *University of Texas at Dallas, TX* and **Maxim Ignatiuc**, *University of Texas at Dallas, TX*, Editors

*Mathematical Reflections: Two Lockdown Years* is a compilation and revision of the 2020 and 2021 volumes from the online journal of the same name. This book is aimed at high school students, participants in math competitions, undergraduates, and anyone who has a fire for mathematics. Passionate readers submitted many of the problems, solutions, and articles, and all require creativity, experience, and comprehensive mathematical knowledge. This book is a great resource for students training for advanced national and international mathematics competitions such as USAMO and IMO.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

**XYZ Series**, Volume 45
September 2022, 736 pages, Hardcover, ISBN: 978-1-7358315-7-2, 2010 *Mathematics Subject Classification*: 00A05, 00A07, 97U40, 97D50, List US$79.95, **AMS members US$63.96**, Order code XYZ/45

`bookstore.ams.org/xyz-45`

清華大学 求真书院
Qiuzhen College, Tsinghua University

# QIUZHEN COLLEGE, TSINGHUA UNIVERSITY ANNOUNCES THE ESTABLISHMENT OF A PH.D PROGRAM IN MATHEMATICS

QIUZHEN COLLEGE, TSINGHUA UNIVERSITY

*TOP 14 IN THE 2023 QS WORLD UNIVERSITY RANKINGS*
*TOP 24 IN THE 2022 QS MATHEMATICS WORLD UNIVERSITY RANKING*

Founded in 1911, Tsinghua University is one of the most prestigious higher education institutions in the world. Its founding mission is to nurture talented students and promoting advanced scientific research. Qiuzhen College was established at Tsinghua University in 2021 with the goal of cultivating world-class mathematicians. We are now accepting PhD applications from students worldwide.

@THU_QZC

@Qiuzhen Tsinghua

## WORLD-RENOWNED MATHEMATICIANS

Graduate students of Qiuzhen college can expect to receive professional and tailor-made guidance from our distinguished faculty consisting of first-class international researchers, including at least two Fields Medalists. Graduate students can also get mentorship from a great number of young scholars and early career mathematicians. Among the mathematicians who have joined us, Shing-Tung Yau, 1982 Fields Mdedalist, is a world-renowned mathematician, and the dean of Qiuzhen College. Caucher Birkar, 2018 Fields Medalist, is a prominent scholar of algebraic geometry. Nicolai Reshetikhin, Fellow of the American Mathematical Society, is a well-known researcher in Low-dimensional topology. Donald Bruce Rubin, emeritus John L. Loeb, was the professor of statistics at Harvard University, and a member of the American Academy of Arts and Sciences. Six main research fields represented in Qiuzhen College include: Analysis and Differential Equations/ Algebra and Number Theory/ Geometry and Topology/ Probability and Statistics/ Theoretical Physics/ Computational and Applied Mathematics.

## FINANCIAL SUPPORT

Qiuzhen College has set up the Qiuzhen Outstanding Doctoral Students Program. Eligible applicants will receive 200,000 RMB (about US$30,000) a year during their first two years of doctoral studies, renewable up to a total of 5 years based on applicant's performance.

## INTERNATIONAL COLLABORATIONS

Qiuzhen College has established exchange programs with multiple world-renowned international institutions, including Harvard, UC Berkeley, U Chicago, Oxford, etc. PhD candidates of the Qiuzhen program are encouraged to engage in short-term research visits with support during their studies.

MORE DETAILS ABOUT QIUZHEN COLLEGE CAN BE FOUND ON OUR WEBSITE: https://qzc.tsinghua.edu.cn/en/
AND FOR INFORMATION SPECIFIC TO THE PHD PROGRAM, PLEASE VISIT: https://qzc.tsinghua.edu.cn/en/Admissions/graduate/Application1.htm

WE WILL ORGANIZE AN "**ONLINE OPEN HOUSE**" FOR INTERESTED APPLICANTS. FOR FURTHER QUESTIONS, PLEASE CONTACT US:
Email: qzsy@tsinghua.edu.cn Tel: +86-10-62780940 / +86-10-62780524

# Meetings & Conferences of the AMS
## January Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at `www.ams.org/meetings`.

**Important Information About AMS Meetings:** Potential organizers, speakers, and hosts should refer to `https://www.ams.org/meetings/meetings-general` for general information regarding participation in AMS meetings and conferences.

**Abstracts:** Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

### Associate Secretaries of the AMS

**Central Section:** Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: `stovall@math.wisc.edu`; telephone: (608) 262-2933.

**Eastern Section:** Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: (610) 758-3717.

**Southeastern Section:** Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: (706) 542-2547.

**Western Section:** Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: `mamanes@hawaii.edu`; telephone: (808) 956-4679.

## Meetings in this Issue



*The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at* `https://www.ams.org/welcoming-environment-policy`.

# Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

**New: Sectional Meetings Require Registration to Submit Abstracts.** In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

## Boston, Massachusetts

*John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel*

### January 4–7, 2023

*Wednesday – Saturday*

### Meeting #1183

Associate Secretary for the AMS: Steve Weintraub
Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 44, Issue 1

### Deadlines

For organizers: Expired
For abstracts: Expired

*The scientific information listed below may be dated. For the latest information, see* `https://www.ams.org/amsmtgs/national.html`.

### Joint Invited Addresses

**Laura G. DeMarco**, Harvard University, *Rigidity and uniformity in algebraic dynamics* (AWM-AMS Noether Lecture).

**Jordan S. Ellenberg**, University of Wisconsin-Madison, *Outward-facing mathematics* (JPBM Communications Award Lecture).

**Philip Maini**, University of Oxford, *Are we there yet? Modelling collective cell motion in biology and medicine* (AAAS-AMS Invited Address).

**Omayra Ortega**, Sonoma State University, *Who are we serving with our scholarship: a Covid model case study* (MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Lecture).

**Grant Sanderson**, 3blue1brown, *Raising the ceiling and lowering the floor of math exposition* (JPBM Communications Award Lecture).

**Bernd Sturmfels**, University of California, Berkeley, *The Quadratic Formula Revisited* (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

**Talithia Williams**, Harvey Mudd College, *The power of talk: engaging the public in mathematics* (JPBM Communications Award Lecture).

## AMS Invited Addresses

**Rodrigo Banuelos**, Purdue University, *Sharp inequalities in probability and harmonic analysis*.

**Richard G. Baraniuk**, Rice University, *The Mathematics of Deep Learning* (AMS Josiah Willard Gibbs Lecture).

**Eugenia Cheng**, School of the Art Institute of Chicago, *Associativity, Commutativity and Units: a Higher-dimensional ballet* (AMS Erdős Memorial Lecture).

**Camillo De Lellis**, Institute For Advanced Study, Princeton, *Flows of nonsmooth vector fields* (AMS Colloquium Lecture I).

**Camillo De Lellis**, Institute For Advanced Study, Princeton, *Flows of nonsmooth vector fields* (AMS Colloquium Lecture II).

**Camillo De Lellis**, Institute For Advanced Study, Princeton, *Flows of nonsmooth vector fields* (AMS Colloquium Lecture III).

**Wilfrid Gangbo**, UCLA, *Recent Progress on Master equations in Mean Field Games*.

**Ling Long**, Louisiana State University, *A Stroll in the Garden of Hypergeometric Functions* (AMS Maryam Mirzakhani Lecture).

**Chris Rasmussen**, Center for Research in Math and Science Education, *Three Models of Successful Department Change Approaches for Infusing Active Learning in Introductory Mathematics Courses* (AMS Lecture on Education).

**Nikhil Srivastava**, University of California, Berkeley, *Four Ways to Diagonalize a Matrix* (von Neumann Lecture).

**Rekha Rachel Thomas**, University of Washington, *Ideals and Varieties of the Pinhole Camera*.

## Invited Addresses of Other JMM Partners

**Nathan N. Alexander**, Morehouse College, *Histories of African Americans Connecting Mathematics and Society* (NAM Cox-Talbot Address).

**Jeremy David Avigad**, Carnegie Mellon University, *The promise of formal mathematics* (ASL Invited Address).

**Peter Cholak**, University of Notre Dame, *Ramsey like theorems on the rationals* (ASL Invited Address).

**Sylvester James Gates, Jr**, Clark Leadership Chair in Science, University of Maryland; past president of American Physical Society, National Medal of Science, *What challenges does data science present to mathematics education?* (TPSE Invited Address).

**Edray Goins**, Pomona College, *Distance Makes the Math Grow Deeper: Rational Distance Sets, Nate Dean, and Me* (PME J. Sutherland Frame Lecture).

**Ryan Hynd**, University of Pennsylvania, *The Blaschke–Lebesgue theorem revisited* (NAM Claytor-Woodard Lecture).

**Franziska Jahnke**, University of Münster, *Model theory of perfectoid fields* (ASL Invited Address).

**Apoorva Khare**, Indian Institute of Science, *Analysis applications of Schur polynomials* (ILAS Invited Address).

**Stephen S. Kudla**, University of Toronto, *Modularity of generating series of divisors on unitary Shimura varieties* (AIM Alexanderson Award Lecture).

**Luis Antonio Leyva**, Vanderbilt-Peabody College, *Undergraduate Mathematics Education as a White, Cisheteropatriarchal Space and Opportunities for Structural Disruption to Advance Queer of Color Justice* (Spectra Lavender Lecture).

**Sandra Müller**, Technical University of Vienna, *Universally Baire sets, determinacy and inner models* (ASL Invited Address).

**Mason Porter**, University of California, Los Angeles, *Bounded-confidence models of opinion dynamics on networks* (SIAM Invited Address).

**Robert Santos**, US Census Bureau, *To be announced* (ASA Committee of Presidents of Statistical Societies Lecture).

**Lynn Scow**, California State University, San Bernardino, *Semi-retractions and the Ramsey Property* (ASL Invited Address).

**Assaf Shani**, Harvard University, *Classifying invariants for Borel equivalence relations* (ASL Invited Address).

**Erik Donal Walsberg**, University of California Irvine, *Model theory of large fields* (ASL Invited Address).

## Invited Addresses of Other Organizations

**Estrella Johnson**, Virginia Tech, *What the Research Says about Active Learning – and What it Doesn't* (Project NExT Lecture on Teaching and Learning).

**Russell Marcus**, Hamilton College, *A Philosophical Account of Mathematics that Won't Make You Hate Philosophers* (SIGMAA in the Philosophy of Mathematics Guest Lecture).

## AMS Special Sessions

*Advances and Applications in Integral and Differential Equations*, **Jeffrey W. Lyons**, The Citadel, **Sougata Dhar**, The University of Connecticut, and **Jeffrey T. Neugebauer**, Eastern Kentucky University.

*Advances in Markov Models: Gambler's Ruin, Duality and Queueing Applications*, **Alan Krinik**, California State Polytechnic University, Pomona, and **Randall James Swift**, California State Polytechnic University.

*Advances in Modeling Mosquito-borne Disease Dynamics and Control Methods*, **Zhuolin Qu**, University of Texas at San Antonio, and **Michael A. Robert**, Virginia Tech.

*Advances in Nonlinear Boundary Value Problems*, **Nsoki Mavinga**, Swarthmore College, **Maya Chhetri**, UNC Greensboro, and **R. Pardo**, Complutense University of Madrid.

*Advances in Operator Algebras*, **Sarah Browne**, University of Kansas, **Priyanga Ganesan**, Texas A&M University, and **David Jekel**, University of California, San Diego.

*Advances in Partial Differential Equations, Numerical Analysis, and their Applications*, **Andrew Miller**, Bridgewater State University, and **Joshua L. Flynn**, McGill University.

*Advances in Qualitative Theory and Applications to Life Sciences of Differential, Difference, and Dynamic Equations*, **Elvan Akin**, Missouri University S&T, and **Naveen K. Vaidya**, San Diego State University.

*Analysis and Differential Equations at Undergraduate Institutions*, **Ryan Alvarado**, Amherst College, and **Lyudmila Korobenko**, Reed College.

*Applications of Riemann Surfaces*, **Mark Syd Bennett** and **Bernard Deconinck**, University of Washington, **Charles Wang**, Harvard University, and **Turku Ozlum Celik**, Bogazici University.

*Applications of Tensors in Computer Science*, **Harm Derksen**, Northeastern University, **Neriman Tokcan**, Broad Institute, and **Benjamin Lovitz**, University of Waterloo.

*Applied Category Theory (a Mathematics Research Communities session)*, **Charlotte Aten**, University of Denver, **Layla H.M. Sorkatti**, Southern Illinois University, and **Abigail Hickok**, University of California, Los Angeles.

*Applied Enumerative Geometry*, **Frank Sottile**, Texas A&M University, and **Taylor Brysiewicz**, Max Planck Institute For Mathematics In the Sciences.

*Applied Topology: Theory and Implementation*, **Nikolas Schonsheck**, **Chad Giusti**, **Melinda Kleczynski**, and **Jerome Roehm**, University of Delaware.

*Arithmetic Geometry Informed by Computation*, **Jennifer Balakrishnan**, Boston University, and **Bjorn Poonen** and **Andrew V. Sutherland**, Massachusetts Institute of Technology.

*Arithmetic Statistics*, **Allechar Serrano Lopez**, Harvard University, and **Robert James Lemke Oliver**, Tufts University.

*Automorphic Forms and Representation Theory*, **Spencer Leslie**, Duke University, and **Solomon Friedberg**, Boston College.

*Coding Theory for Modern Applications*, **Allison Beemer**, University of Wisconsin-Eau Claire, **Hiram H. Lopez**, Cleveland State University, and **Rafael D'Oliveira**, Clemson University.

*Complex and Arithmetic Dynamical Systems* (AMS-AWM), **Laura G. DeMarco** and **Niki Myrto Mavraki**, Harvard University, and **Max Weinreich**, Brown University.

*Complexity and Topology in Computational Algebraic Geometry*, **Ali Mohammad Nezhad** and **Saugata Basu**, Purdue University.

*Complex Systems in the Life Sciences*, **Xiang-Sheng Wang**, University of Louisiana at Lafayette, **Zhisheng Shuai**, University of Central Florida, and **Gail S. Wolkowicz**, McMaster University.

*Current Directions in the Philosophy of Mathematics* (AMS-SIGMAA), **Bonnie Gold**, Monmouth University, and **Kevin Iga**, Pepperdine University.

*Current Progress in Computational Biomedicine*, **Nektarios Valous**, National Center for Tumor Diseases Heidelberg, German Cancer Research Center, Heidelberg, Germany, **Anna Konstorum**, Center for Computing Sciences, Institute for Defense Analyses, MD, **Heiko Enderling**, Department of Integrated Mathematical Oncology, H. Lee Moffitt Cancer Center & Research Institute, Tampa, FL, USA, and **Dirk Jäger**, National Center for Tumor Diseases, German Cancer Research Center, Heidelberg, Germany.

*Data Science at the Crossroads of Analysis, Geometry, and Topology (a Mathematics Research Communities session)*, **Hitesh Gakhar**, University of Oklahoma, **Harlin Lee**, University of California, Los Angeles, and **Josué Tonelli-Cueto**, The University of Texas at San Antonio.

*Definability, Computability, and Model Theory: A Special Session dedicated to Gerald E. Sacks*, **Nathanael Leedom Ackerman**, Harvard University, **Ted Slaman**, University of California, Berkeley, and **Cameron E. Freer**, Massachusetts Institute of Technology.

*Discrete and Hybrid Dynamical Systems: Time Scales and Fractional Approaches*, **Billy Jackson**, University of Wisconsin Madison.

*Distance Problems in Continuous, Discrete and Finite Field Settings*, **Abdul Basit**, Johns Hopkins University, **Eyvindur Ari Palsson**, Virginia Tech University, and **Steven Joel Miller**, Williams College.

*Dynamics, Geometry & Group Actions*, **Kathryn Lindsey**, Boston College, and **Boris Hasselblatt**, Tufts University.

*Dynamics of PDEs on Heterogeneous Domains: Theory & Applications*, **Denis Daniel Patterson**, Princeton University, **Ryan Nolan Goh**, Boston University, and **Jonathan Touboul**, Brandeis University.

*Ecological and Evolutionary Dynamics in Life and Social Sciences*, **Sabrina H. Streipert**, McMaster University, and **Yun Kang** and **Lucero Rodriguez Rodriguez**, Arizona State University.

*Excursions in Arithmetic Geometry*, **Tony Shaska**, Research Institute of Science and Technology.

*Financial Mathematics*, **Sixian Jin** and **Stephan Strum**, Worcester Polytechnic Institute.

*Geometric PDEs*, **Theodora Bourni**, University of Tennessee, Knoxville, and **Brett Kotschwar**, Arizona State University.

*Geometry and Dynamics in Moduli Spaces of Abelian Differentials*, **Chris Johnson**, Western Carolina University, **Martin J. Schmoll**, Clemson University, **Chris Martin Judge**, Indiana University, and **Jane Wang**, Indiana University Bloomington.

*Homotopy Theory: Connections and Applications*, **Elden Elmanto**, Harvard University, and **Daniel C. Isaksen**, Wayne State University.

*If You Build It They Will Come: Presentations by Scholars in the National Alliance for Doctoral Studies in the Mathematical Sciences*, **Rolando de Santiago**, Purdue University, and **David Goldberg**, Math Alliance/Purdue University.

*Integrable Systems and Symplectic Group Actions*, **Joseph Palmer** and **Susan Tolman**, University of Illinois Urbana-Champaign.

*Integral Equations and Applications*, **Irina Mitrea**, Temple University, and **Shari Moskow**, Drexel University.

*Kicks, Shocks, Recovery and Resilience: Impulsive Models in Ecology and Socio-Economic Systems*, **Punit Gandhi**, Virginia Commonwealth University, and **Sarah Iams**, Harvard University.

*Langlands Program*, **Shanna Dobson**, University of California, Riverside.

*Lessons Learned from Successful Departmental Efforts to Transform Precalculus and Calculus*, **Chris Rasmussen**, Center for Research in Math and Science Education.

*Math Circle Activities as a Gateway into Mathematics*, **Lauren L. Rose**, Bard College, **Brandy S. Wiegers**, Central Washington University, **Gabriella A. Pinter**, University of Wisconsin, Milwaukee, and **Nick Rauh**, Julia Robinson Math Festivals.

*Mathematical Foundations of Democracy*, **Stanley Chang**, **Andrew Schultz**, and **Ismar Volic**, Wellesley College.

*Mathematical Methods in Machine Learning and Optimization*, **Carlos M. Ortiz-Marrero**, Pacific Northwest National Laboratory, and **Ryan W. Murray**, North Carolina State University.

*Mathematical Modeling of Ecology and Evolution: From Infectious Disease to the Evolution of Cooperation*, **Daniel Brendan Cooney**, University of Pennsylvania, **Chadi M. Saad-Roy**, Princeton University, **Olivia Jessica Chu**, Dartmouth College, and **Benjamin Allen**, Emmanuel College, Boston, MA.

*Mathematics and Fiber Arts*, **sarah-marie belcastro**, MathILy and Smith College, and **Carolyn Ann Yackel**, Mercer University.

*Mathematics and the Arts*, **Karl M. Kattchee**, University of Wisconsin-La Crosse, **Doug Norton**, Villanova University, and **Anil Venkatesh**, Adelphi University.

*Mathematics Standards, Equity, Policy, and Politics* (AMS-SIGMAA), **Yvonne Lai**, University of Nebraska-Lincoln, **Tyler Kloefkorn**, American Mathematical Society, **Dave Kung**, Charles A. Dana Center, The University of Texas at Austin, and **Blain Patterson**, Virginia Military Institute.

*Modeling Collective Behavior in Biology*, **Alexandria Volkening**, Purdue University, and **Philip Maini**, University of Oxford.

*Models and Methods for Sparse (Hyper) Network Science (a Mathematics Research Communities session)*, **Sarah Tymochko**, Michigan State University, **Jessalyn Bolkema**, California State University, Dominguez Hills, **Himanshu Gupta**, University of Delaware, **Fangfei Lan**, University of Utah, and **Nicholas W. Landry**, University of Colorado Boulder.

*Modular Forms, Hypergeometric Functions, Character Sums and Galois Representations*, **Ling Long**, Louisiana State University, **Wen-Ching Winnie Li**, Pennsylvania State University, **William Yun Chen**, Institute for Advanced Study, and **Holly Swisher**, Oregon State University.

*New Developments in Differential Geometry and Topology*, **Megan M. Kerr**, Wellesley College, and **Catherine Searle**, Wichita State University.

*Nonlinear Evolution Equations and Their Applications*, **Guoping Zhang**, **Gaston Mandata N'Guerekata**, **Xuming Xie**, **Mingchao Cai**, and **Jemal S. Mohammed-Awel**, Morgan State University.

*Nonlocal Frameworks in Analysis and Mathematical Modeling*, **Nicole Buczkowski**, University of Nebraska, Lincoln, and **Petronela Radu** and **Anh Vo**, University of Nebraska-Lincoln.

*Number Theory at Non-PhD Granting Institutions*, **Steven Joel Miller**, Williams College, **Naomi Tanabe**, Bowdoin College, **Harris Daniels**, Amherst College, **Enrique Treviño**, Lake Forest College, and **Alia Hamieh**, University of Northern British Columbia.

*Orthogonal Polynomials and their Applications*, **Ahmad Barhoumi**, University of Michigan, **Roozbeh Gharakhloo**, Colorado State University, and **Andrei Martinez-Finkelshtein**, Baylor University.

*Partial Differential Equations and Complex Variables*, **Hyun-Kyoung Kwon**, University At Albany, SUNY, **Bingyuan Liu**, The University of Texas Rio Grande Valley, and **Qi Han**, Texas A & M University San Antonio.

*Perspectives on Eigenvalue Computation*, **Nikhil Srivastava**, University of California, Berkeley, **Peter Buergisser**, Technische Universität Berlin Institut Für Mathematik, and **James Demmel**, University of California, Berkeley.

*Polymath Jr: Mentoring and Learning*, **Steven Joel Miller**, Williams College, **Johanna Franklin**, Hofstra University, **Adam Sheffer**, Baruch College, CUNY, and **Yunus E. Zeytuncu**, University of Michigan - Dearborn.

*Polynomial Systems, Homotopy Continuation and Applications*, **Margaret Regan**, Duke University, and **Timothy Duff**, University of Washington.

*Promoting Equity Through Active Learning in Undergraduate Mathematics: Precalculus*, **Jose Maria Menendez**, Pima Community College, **Ksenija Simic-Muller**, Pacific Lutheran University, and **Anthony Fernandes**, University of North Carolina - Charlotte.

*Quadratic Forms, Modular Forms, and Applications*, **Fang-Ting Tu**, Louisiana State University, **Gene S. Kopp**, Purdue University, and **Jingbo Liu**, Texas A&M University-San Antonio.

*Quaternions*, **Johannes Familton**, Borough of Manhattan Community College, CUNY, **Chris McCarthy**, BMCC, City University of New York, and **Terrence Richard Blackman**, Medgar Evers Community College, CUNY.

*Recent Advances in Arithmetic Dynamics*, **Joseph H. Silverman**, Brown University, **Jacqueline Anderson**, Bridgewater State University, and **John R. Doyle**, Oklahoma State University.

*Recent Advances in Nonlinear Partial Differential Equations and their Applications*, **Qi Han**, Texas A&M University-San Antonio, and **Jing Tian**, Towson University.

*Recent Development in Partial Differential Equations Related to Geometric and Harmonic Analysis*, **Meijun Zhu**, University of Oklahoma, and **Xiaodong Wang**, Michigan State University.

*Recent Developments in Geometric Measure Theory*, **Camillo De Lellis**, Institute For Advanced Study, Princeton, **Antonio De Rosa**, University of Maryland, and **Luca Spolaor**, University of California, San Diego.

*Recent Developments in Numerical Methods for PDEs*, **Leo G. Rebholz**, Clemson University, and **Michael Neilan**, University of Pittsburgh.

*Recent Trends in Discrete-Time Ecological and Epidemiological Models*, **Mustafa R. Kulenovic**, University of Rhode Island, and **Abdul-Aziz Yakubu**, Howard University.

*Research Community in Algebraic Combinatorics*, **Rosa C. Orellana** and **Nadia Lafrenière**, Dartmouth College.

*Research from the Graduate Research Workshop in Combinatorics (GRWC)*, **Steve Butler**, Iowa State University, **Xavier Perez-Gimenez**, University of Nebraska-Lincoln, and **Puck Rombach**, University of Vermont.

*Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs* (AMS-SIAM), **Darren A. Narayan**, Rochester Institute of Technology, **Khang Tran**, California State University, Fresno, **Mark Daniel Ward**, Purdue University, **John C. Wierman**, Johns Hopkins University, and **Christopher O'Neil**, San Diego State University.

*Resolutions of Singularities and Cohomology in Geometry and Representation Theory*, **Iva Halacheva**, Northeastern University, **Roman Bezrukavnikov**, Massachusetts Institute of Technology, **Peter Crooks**, Utah State University, and **Valerio Toledano Laredo**, Northeastern University.

*Rethinking Number Theory*, **Allechar Serrano Lopez**, Harvard University, **Lea Beneish**, University of California, Berkeley, and **Soumya Sankar**, Ohio State University.

*Riemannian Manifolds with Lower Scalar Curvature Bounds*, **Brian Allen**, University of Hartford, and **Demetre Kazaras**, Duke University.

*Scholarship on Teaching and Learning Introductory Statistics*, **Jennifer McNally**, **Laura Kyser Callis**, and **Steven LeMay**, Curry College.

*Spatial Ecology Applications Using Reaction Diffusion Models*, **Jerome Goddard II**, Auburn University Montgomery, and **Ratnasingham Shivaji**, University of North Carolina at Greensboro.

*Statistics and Data Science Curriculum in a Mathematics Department*, **Qing Wang** and **Anny-Claude Joseph**, Wellesley College.

*Stimulating Student Engagement in Differential Equations through Modeling Activities*, **Kyle T. Allaire**, Worcester State University, **Lisa Naples**, Macalester College, and **Timothy Antonelli**, Worcester State University.

*Stochastic Analysis and Applications*, **Parisa Fatheddin**, Ohio State University, Marion, and **Michael A. Salins**, Boston University.

*Tensor Representation, Completion, Modeling and Analytics of Complex Data*, **Ivo D. Dinov** and **Joshua Welch**, University of Michigan.

*The Combinatorics and Geometry of Jordan Type and Lefschetz Properties*, **A. Iarrobino**, Northeastern University, and **Leila Khatami**, Union College.

*The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors*, **Laurel Ohm**, Princeton University, **Shanise Walker**, Clark Atlanta University, and **Ziva Myer**, Duke University.

*The History of Mathematics*, **Jemma Lorenat**, Pitzer College, **Adrian Rice**, Randolph-Macon College, **Deborah Kent**, University of St. Andrews, and **Daniel E. Otero**, Xavier University.

*The Math and Art of Mathemalchemy*, **Samantha Pezzimenti**, Penn State Brandywine, **Carolyn Ann Yackel**, Mercer University, and **Edmund O. Harriss**, University of Arkansas.

*The Mathematics of RNA and DNA*, **Chris McCarthy**, BMCC, City University of New York, and **Johannes Familton**, Borough of Manhattan Community College, CUNY.

*The Scholarship of Teaching and Learning: Past, Present, and Future*, **Jacqueline M. Dewar**, Loyola Marymount University, **Thomas F. Banchoff**, Brown University, **Curtis D. Bennett** and **Brian P. Katz**, California State University, Long Beach, **Lewis D. Ludwig**, Denison University, and **Larissa Schroeder**, University of Nebraska Omaha.

*The Teaching and Learning of Undergraduate Ordinary Differential Equations: An Interdisciplinary Approach*, **Viktoria Savatorova**, Central Connecticut State University, **Itai Seggev**, Wolfram Research, **Iordanka Panayotova**, Christopher Newport University, and **Beverly H. West**, Cornell University.

*Topological and Combinatorial Methods in Commutative Algebra*, **Augustine O'Keefe**, Connecticut College, and **Jennifer Biermann**, Hobart and William Smith Colleges.

*Topology, Algebra, and Geometry in the Mathematics of Data Science*, **Henry Kvinge**, **Tim Doster**, and **Tegan Emerson**, Pacific Northwest National Laboratory.

*Topology, Structure and Symmetry in Graph Theory*, **Mark Ellingham**, Vanderbilt University, and **Lowell Abrams**, George Washington University.

*Trees in Many Contexts (a Mathematics Research Communities session)*, **Ann Wells Clifton**, Louisiana Tech University, **Fadekemi Janet Osaye**, Alabama State University, **Lora Bailey**, Grand Valley State University, **Alex Wiedemann**, Randolph-Macon College, and **Reem Mahmoud**, Virginia Commonwealth University.

*Undergraduate Research Activities in Mathematical and Computational Biology*, **Timothy D. Comar**, Benedictine University, **Hannah Callender Highlander**, University of Portland, and **Anne E. Yust**, University of Pittsburgh.

*Understanding COVID-19: Three Years of Mathematical Models to Address the Global Pandemic*, **Hwayeon Ryu**, Elon University, **Lauren M. Childs**, Virginia Tech, and **Kamila Larripa**, Humboldt State University.

*Variational Methods, Optimal Control and Hamilton-Jacobi Equations*, **Wilfrid Gangbo**, UCLA, **Andrzej Swiech**, Georgia Tech, **Alpar Meszaros**, University of Durham, and **Chenchen Mou**, City University of Hong Kong.

*Women in Automorphic Forms*, **Mathilde Gerbelli-Gauthier**, Institute for Advanced Study, **Maria Fox**, University of Oregon, and **Manami Roy**, Fordham University.

## AIM Special Sessions

*Automorphic Forms and Special Cycles*, **Tonghai Yang**, University of Wisconsin, Madison, **Stephen S. Kudla**, University of Toronto, and **Jan Hendrik Bruinier**, Technical University of Darmstadt.

*Little School Dynamics: Cool Dynamics Research by Researchers at PUIs*, **Kimberly Ayers**, California State University, San Marcos, **Ami Radunskaya**, Pomona College, **Han Li**, Wesleyan University, **David M. McClendon**, Ferris State University, and **Andrew Parrish**, Eastern Illinois University.

## ASL Special Sessions

*Model-theoretic and "Higher Infinite" Methods in Descriptive Set Theory and Related Areas*, **Rehana Patel**, AIMS-Senegal, **Alexander Kechris**, California Institute of Technology, **Alejandro Poveda**, Hebrew University of Jerusalem, and **Assaf Shani**, Harvard University.

*Tame Geometry and Applications to Analysis*, **Alexi Block Gorman** and **Elliot Kaplan**, McMaster University, and **Daniel Miller**, Emporia State University.

## AWM Special Sessions

*AWM Workshop: Women in Commutative Algebra (WiCA)*, **Claudia Miller**, Syracuse University, and **Janet Striuli**, Fairfield University.

*Celebrating the Mathematical Contributions of the AWM*, **Michelle Ann Manes**, University of Hawaii, **Kathryn E. Leonard**, Occidental College, **Donatella Danielli**, Arizona State University, and **Ami Radunskaya**, Pomona College.

*Recent Developments in the Analysis of Local and Nonlocal PDEs*, **Alaa Haj Ali** and **Donatella Danielli**, Arizona State University.

*Women, Art, and Mathematics: Mathematics in the Literary Arts and Pedagogy in Creative Settings*, **Shanna Dobson**, University of California, Riverside, **Stephanie Lewkiewicz**, Temple University, and **Elizabeth Donovan**, Murray State University.

*Women in Graph Theory*, **Karen L. Collins**, Wesleyan University, **Sandra R. Kingan**, Brooklyn College and the Graduate Center, CUNY, **Brigitte Servatius**, Worcester Polytechnic Institute, and **Ann N. Trenk**, Wellesley College.

## COMAP Special Sessions

*COMAP's Modeling Contests: Engaging Students and Faculty in Mathematical Modeling*, **Amanda I. Beecher**, Ramapo College of New Jersey, **Steve Horton**, US Military Academy (Emeritus), and **Kayla Blyman**, Saint Martin's University.

## ILAS Special Sessions

*Innovative and Effective Ways to Teach Linear Algebra*, **David M. Strong**, Pepperdine University, **Gil Strang**, MIT, **Sepideh Stewart**, University of Oklahoma, and **Megan Wawro**, Virginia Tech.

*Matrices and Operators*, **Mohsen Aliabadi**, Iowa State University, and **Tin-Yau Tam** and **Pan-Shun Lau**, University of Nevada, Reno.

*Matrix Analysis and Applications*, **Hugo Woerdeman**, Drexel University, and **Edward Poon**, Embry-Riddle Aeronautical University.

*The Inverse Eigenvalue Problem for a Graph and Zero Forcing* (ILAS-AIM), **Mary Flagg**, University of St. Thomas, and **Bryan A Curtis**, Iowa State University.

## MSRI/SLMath Special Sessions

*African Diaspora Joint Mathematics Working Groups (ADJOINT)*, **Edray Herber Goins**, Pomona College, and **Anisah Nabilah Nu'Man**, Spelman College.

*Summer Research in Mathematics (SRiM): Analytic Number Theory*, **Ayla Gafni**, University of Mississippi, **Amita Malik**, Max Planck Institute, Bonn, and **Sneha Chaubey**, Indian Institute of Information Technology Delhi.

*Summer Research in Mathematics (SRiM): Applied and Computational Mathematics*, **Yunan Yang**, ETH Zurich, **Jingwei Hu**, University of Washington, and **Yifei Lou**, The University of Texas at Dallas.

*Summer Research in Mathematics (SRiM): Cluster Algebras and Related Topics*, **Sunita Chepuri**, University of Michigan, **Elizabeth Kelley**, University of Illinois at Urbana-Champaign, and **Esther Banaian**, University of Minnesota.

*Summer Research in Mathematics (SRiM): Differential and Metric Geometry*, **Catherine Searle**, Wichita State University, **Lee T. Kennard**, Syracuse University, and **Elahe Khalili Samani**, University of Notre Dame.

*Summer Research in Mathematics (SRiM): Dynamics and Operator Algebras*, **Sarah Reznikoff**, Kansas State University, **Sarah Browne**, University of Kansas, **Elizabeth Anne Gillaspy**, University of Montana, and **Lauren Chase Ruth**, Mercy College.

*Summer Research in Mathematics (SRiM): Geometric and Topological Combinatorics*, **Margaret M. Bayer**, University of Kansas, **Marija Jelic Milutinovic**, University of Belgrade, and **Julianne Vega**, Kennesaw State University.

*Summer Research in Mathematics (SRiM) : Mathematical Modeling and Analysis in Eye Research*, **Atanaska Dobreva**, Augusta University, and **Erika Camacho**, Arizona State University.

*Summer Research in Mathematics (SRiM): Unknotting Operations*, **Hannah Turner**, Georgia Institute of Technology, and **Samantha Allen**, Duquesne University.

*The MSRI Undergraduate Program (MSRI-UP)*, **Federico Ardila**, San Francisco State University.

## NSF Special Sessions

*NSF Session on Outcomes and Innovations from NSF Undergraduate Education Programs in the Mathematical Sciences*, **Michael Ferrara** and **Mindy Capaldi**, Division of Undergraduate Education, National Science Foundation, **John R. Haddock**, National Science Foundation, **Elise Nicole Lockwood**, Division of Undergraduate Education, National Science Foundation, and **Lee L. Zia**, National Science Foundation.

## PMA Special Sessions

*BSM Special Session: Mathematical Research in Budapest for Students and Faculty*, **Kristina Cole Garrett**, Budapest Semesters in Mathematics.

## SIAM Minisymposium

*SIAM ED Session on Education as Research and Research as Education to include a panel discussion on the benefits and challenges of integrating research and teaching.*, **Benjamin Galluzzo** and **Kathleen Kavanagh**, Clarkson University.

*SIAM Minisymposium on Applications of the Maslov Index*, **Christopher K. R. T. Jones** and **Emmanuel Fleurantin**, University of North Carolina.

*SIAM Minisymposium on Combinatorial Optimization*, **Annie Raymond**, University of Massachusetts.

*SIAM Minisymposium on Fractional Dynamics*, **Lukasz Plociniczak**, Wroclaw University of Science and Technology, and **Krzysztof Burnecki**, Wrocław University of Science and Technology.

*SIAM Minisymposium on Imaging and Inverse Problems*, **Andrea Arnold**, Worcester Polytechnic Institute.

*SIAM Minisymposium on Numerical Linear Algebra: Algorithms, Computations, and Application*, **James Nagy** and **Elizabeth Newman**, Emory University.

*SIAM Minisymposium on Quantitative Justice (a NAM-SIAM Joint Session)*, **Ron Buckmire**, Occidental College, **Omayra Ortega**, Sonoma State University, and **Carrie Diaz Eaton**, Bates College.

*SIAM Minisymposium on Quantum Algorithms*, **Dong An**, University of Maryland, and **Di Fang** and **Lin Lin**, University of California, Berkeley.

### SPECTRA Special Sessions

*Research by LGBTQ+ Mathematicians*, **Juliette Emmy Bruce**, University of California, Berkeley, **Christopher Goff**, University of the Pacific, and **Rebecca R.G.**, George Mason University.

# Atlanta, Georgia

*Georgia Institute of Technology*

### March 18–19, 2023

*Saturday – Sunday*

### Meeting #1184

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 2

### Deadlines

For organizers: Expired
For abstracts: January 17, 2023

*The scientific information listed below may be dated. For the latest information, see* https://www.ams.org/amsmtgs/sectional.html.

### Invited Addresses

**Betsy Stovall**, University of Wisconsin-Madison, *Title to be announced*.
**Blair Dowling Sullivan**, University of Utah, *Title to be announced*.
**Yusu Wang**, University of California San Diego, *Title to be announced*.
**Amie Wilkinson**, University of Chicago, *Title to be announced* (Erdős Memorial Lecture).

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* https://www.ams.org/cgi-bin/abstracts/abstract.pl.

*Advanced Topics in Graph Theory and Combinatorics* (Code: SS 1A), **Songling Shan**, Illinois State University, and **Guangming Jing**, Augusta University.

*Advances in Applied Dynamical Systems and Mathematical Biology* (Code: SS 25A), **Chunhua Shan**, The University of Toledo, and **Guihong Fan**, Columbus State University.

*Advances in Mathematical Finance and Optimization* (Code: SS 24A), **Ibrahim Ekren**, **Arash Fahim**, and **Lingjiong Zhu**, Florida State University.

*Algebraic Methods in Algorithms* (Code: SS 16A), **Kevin Shu** and **Mehrdad Ghadiri**, Georgia Institute of Technology.

*Combinatorial Matrix Theory* (Code: SS 22A), **Zhongshan Li**, **Marina Arav**, and **Hein Van der Holst**, Georgia State University.

*Combinatorics, Probability and Computation in Molecular Biology* (Code: SS 38A), **Christine Heitsch** and **Brandon Legried**, Georgia Institute of Technology.

*Commutative Algebra and its Interactions with Algebraic Geometry* (Code: SS 10A), **Michael Brown** and **Henry K. Schenck**, Auburn University.

*Contact and Symplectic Topology in Dimensions 3 and 4* (Code: SS 34A), **Akram Alishahi**, **Peter Lambert-Cole**, and **Gordana Matic**, University of Georgia.

*Discrete Analysis* (Code: SS 31A), **Giorgis Petridis**, **Neil Lyall**, and **Akos Magyar**, University of Georgia.

*Disordered and Periodic Quantum Systems* (Code: SS 26A), **Rodrigo Bezerra de Matos** and **Wencai Liu**, Texas A&M University, and **Xiaowen Zhu**, University of Washington.

*Diversity in Mathematical Biology* (Code: SS 36A), **Daniel Alejandro Cruz**, University of Florida, and **Margherita Maria Ferrari**, University of Manitoba.

*Dynamics of Partial Differential Equations* (Code: SS 18A), **Gong Chen**, Georgia Institute of Technology, **Hao Jia**, University of Minnesota, and **Dallas Albritton**, Princeton University.

*Fractal Geometry and Dynamical Systems* (Code: SS 11A), **Mrinal Kanti Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, and **Scott Kaschner**, Butler University.

*Geometric and Combinatorial Aspects of Lie Theory* (Code: SS 40A), **William Graham**, University of Georgia, **Amber Russell**, Butler University, and **Scott Larson**, University of Georgia.

*Geometric Group Theory* (Code: SS 4A), **Ryan Dickmann**, Georgia Institute of Technology, **Sahana H. Balasubramanya**, University of Münster, and **Abdoul Karim Sane** and **Dan Margalit**, Georgia Institute of Technology.

*Harmonic Analysis* (Code: SS 14A), **Betsy Stovall**, University of Wisconsin-Madison, **Benjamin Jaye**, Georgia Tech, and **Manasa Vempati**.

*High-dimensional Convexity and Probability* (Code: SS 13A), **Galyna Livshyts** and **Orli Herscovici**, Georgia Institute of Technology, and **Dan Mikulincer**, MIT.

*Knots, Skein Modules and Categorification* (Code: SS 8A), **Marithania Silvero-Casanova**, Universidad De Sevilla, **Rhea Palak Bakshi**, The George Washington University, **Jozef Henryk Przytycki**, George Washington University, and **Radmila Sazdanovic**, North Carolina State University.

*Logic, Combinatorics, and their Interactions* (Code: SS 27A), **Anton Bernshteyn**, Georgia Institute of Technology, and **Robin Tucker-Drob**, University of Florida.

*Macdonald Theory at the Intersection of Combinatorics, Algebra, and Geometry* (Code: SS 37A), **Olya Mandelshtam**, University of Waterloo, **Sean Griffin**, UC Davis, and **Andy Wilson**, Kennesaw State University.

*Mathematical Modeling and Simulation Techniques in Fluid Structure Interaction Problems* (Code: SS 17A), **Pejman Sanaei**, Georgia State University.

*Mathematical Modeling of Populations and Diseases Transmissions* (Code: SS 33A), **Yang Li**, Georgia State University, **Jia Li**, University of Alabama in Huntsville, and **Xiang-Sheng Wang**, University of Louisiana at Lafayette.

*Multiscale Approaches to Modeling Ecological and Evolutionary Dynamics* (Code: SS 28A), **Daniel Brendan Cooney**, University of Pennsylvania, **Denis Daniel Patterson**, Princeton University, **Olivia Chu**, Dartmouth College, and **Chadi M Saad-Roy**, University of California, Berkeley.

*Qualitative Aspects of Nonlinear PDEs: Well-posedness and Asymptotics* (Code: SS 23A), **Atanas G. Stefanov**, University of Alabama Birmingham, **Fazel Hadaifard**, University of California - Riverside, and **Jiahong Wu**, Oklahoma State University.

*Quasi-periodic Schrödinger Operators and Quantum Graphs* (Code: SS 35A), **Fan Yang**, Louisiana State University, **Matthew Powell**, UCI, and **Burak Hatinoglu**, UC Santa Cruz.

*Recent Advances and Applications in Imaging Sciences* (Code: SS 39A), **Carmeliza Luna Navasca**, University of Alabama at Birmingham, **Fatou Sanogo**, Bates College, and **Elizabeth Newman**, Emory University.

*Recent Development in Advanced Numerical Methods for Partial Differential Equations* (Code: SS 21A), **Seulip Lee** and **Lin Mu**, University of Georgia.

*Recent Developments in Commutative Algebra* (Code: SS 5A), **Thomas Polstra**, University of Alabama, and **Florian Enescu**, Georgia State University.

*Recent Developments in Graph Theory* (Code: SS 32A), **Guantao Chen**, Georgia State University, **Zhiyu Wang**, Georgia Institute of Technology, and **Xingxing Yu**, Georgia Tech.

*Recent Developments in Mathematical Aspects of Inverse Problems and Imaging* (Code: SS 20A), **Yimin Zhong** and **Junshan Lin**, Auburn University.

*Recent Developments on Analysis and Computation for Inverse Problems for PDEs* (Code: SS 2A), **Dinh-Liem Nguyen**, Kansas State University, **Loc Nguyen**, UNC Charlotte, and **Khoa Vo**, Florida A&M University.

*Recent Trends in Structural and Extremal Graph Theory* (Code: SS 29A), **Joseph Guy Briggs** and **Jessica McDonald**, Auburn University.

*Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS)* (Code: SS 6A), **Daniel K. Nakano**, University of Georgia, **Chun-Ju Lai**, Institute of Mathematics, Academia Sinica, Taipei 10617 Taiwan, and **Weiqiang Wang**, University of Virginia.

*Singer-Hopf Conjecture in Geometry and Topology* (Code: SS 9A), **Luca Di Cerbo**, University of Florida, and **Laurentiu Maxim**, University of Wisconsin-Madison.

*Spectral Theory* (Code: SS 19A), **Rudi Weikard**, University of Alabama at Birmingham, and **Stephen P. Shipman**, Louisiana State University.

*Stochastic Analysis and its Applications* (Code: SS 7A), **Parisa Fatheddin**, Ohio State University, Marion, and **Kazuo Yamazaki**, Texas Tech University.

*Stochastic Processes and Related Topics* (Code: SS 15A), **Ngartelbaye Guerngar**, University of North Alabama, and **Le Chen**, **Erkan Nane**, and **Jerzy Szulga**, Auburn University.

*Topological Persistence: Theory, Algorithms, and Applications* (Code: SS 12A), **Luis Scoccola**, Northeastern University, **Hitesh Gakhar**, University of Oklahoma, and **Ling Zhou**, The Ohio State University.

*Topology and Geometry of 3- and 4-Manifolds* (Code: SS 3A), **Miriam Kuzbary**, Georgia Institute of Technology, **David T. Gay**, University of Georgia, **Jon Simone**, Georgia Institute of Technology, and **Nur Saglam**, Georgia Tech.

*Undergraduate Mathematics and Statistics Research* (Code: SS 30A), **Leslie Julianna Meadows**, Georgia State University, **Tsz Ho Chan** and **Asma Azizi**, Kennesaw State University, and **Mark Grinshpon**, Georgia State University.

# Spring Eastern Virtual Sectional Meeting

*Meeting virtually, hosted by the American Mathematical Society*

## April 1–2, 2023

*Saturday – Sunday*

## Meeting #1185

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 2

## Deadlines

For organizers: Expired
For abstracts: January 30, 2023

*The scientific information listed below may be dated. For the latest information, see* `https://www.ams.org/amsmtgs/sectional.html`.

## Invited Addresses

**Kirsten Eisentraeger**, Pennsylvania State University, *Title to be announced*.
**Jason Manning**, Cornell University, *Title to be announced*.
**Jennifer L Mueller**, Colorado State University, *Title to be announced*.

## Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Analysis and Differential Equations at Undergraduate Institutions* (Code: SS 3A), **William R. Green**, Rose-Hulman Institute of Technology, and **Katharine A. Ott**, Bates College.

*Analysis of Markov, Gaussian and Stationary Stochastic Processes* (Code: SS 4A), **Alan C Krinik**, California State Polytechnic University, Pomona, and **Randall J. Swift**, Cal Poly Pomona.

*Cybersecurity and Cryptography* (Code: SS 10A), **Lubjana Beshaj**, Army Cyber Institute, **Shekeba Monshref**, IBM, and **Angela Robinson**, NIST.

*Fractal Geometry and Dynamical Systems* (Code: SS 6A), **Mrinal Kanti Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, **Sangita Jha**, Department of Mathematics, National Institute of Technology Rourkela, India, and **Saurabh Verma**, Indian Institute of Information Technology Allahabad.

*Gauge Theory, Geometric Analysis, and Low-Dimensional Topology* (Code: SS 7A), **Paul M. N Feehan**, Rutgers University, New Brunswick, and **Thomas Gibbs Leness**, Florida International University.

*Hypergeometric Functions, q-series and Generalizations* (Code: SS 5A), **Howard Saul Cohl**, National Institute of Standards and Technology, **Robert Maier**, University of Arizona, and **Roberto Costas-Santos**, Universidad Loyola de Andalucía.

*Modeling, Analysis, and Control of Populations Impacted by Disease and Invasion* (Code: SS 1A), **Rachel Natalie Leander** and **Wandi Ding**, Middle Tennessee State University.

*Quasiconformal Analysis and Geometry on Metric Spaces* (Code: SS 8A), **Dimitrios Ntalampekos**, Stony Brook University, and **Hrant Hakobyan**, Kansas State University.

*Recent Advances in Differential Geometry* (Code: SS 2A), **Bogdan D. Suceava**, California State University Fullerton, **Adara M. Blaga**, West University of Timişoara, Romania, **Cezar Oniciuc**, "Al.I.Cuza" University of Iaşi, Romania, **Marian Ioan**

**Munteanu**, "Al.I.Cuza" University of Iaşi, Iaşi, Romania, **Shoo Seto**, California State University, Fullerton, and **Lihan Wang**, California State University, Long Beach.

*Recent Advances in Infinite-Dimensional Stochastic Analysis* (Code: SS 9A), **Vincent R. Martinez**, Hunter College (CUNY), **Hung Nguyen**, UCLA, and **Nathan E. Glatt-Holtz**, Tulane University.

*Recent Advances in Ion Channel Models and Poisson-Nernst-Planck Systems* (Code: SS 11A), **Zilong Song**, Utah State University, and **Xiang-Sheng Wang**, University of Louisiana at Lafayette.

*Recent Progress in Chromatic Graph Theory* (Code: SS 12A), **Hemanshu Kaul** and **Samantha Dahlberg**, Illinois Institute of Technology.

# Cincinnati, Ohio

*University of Cincinnati*

### April 15–16, 2023

*Saturday – Sunday*

### Meeting #1186

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 2

### Deadlines

For organizers: Expired
For abstracts: February 13, 2023

*The scientific information listed below may be dated. For the latest information, see* `https://www.ams.org/amsmtgs/sectional.html`.

### Invited Addresses

**Johnny Guzman**, Brown University, *Title to be announced*.
**Lisa Piccirillo**, MIT, *Title to be announced*.
**Krystal Taylor**, The Ohio State University, Department of Mathematics, *Title to be announced*.
**Nathaniel Whitaker**, University of Massachusetts, *Title to be announced* (Einstein Public Lecture in Mathematics).

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Advances in Dispersive Partial Differential Equations* (Code: SS 13A), **William R. Green**, Rose-Hulman Institute of Technology, **Mehmet Burak Erdogan**, University of Illinois at Urbana Champaign, and **Michael J. Goldberg**, University of Cincinnati.

*Advances in Radial Basis Functions for Numerical Simulation* (Code: SS 7A), **Jonah A. Reeger**, Air Force Institute of Technology, and **Cecile Piret**, Michigan Technological University.

*Algorithms, Number Theory, and Cryptography* (Code: SS 8A), **Jonathan P. Sorenson** and **Jonathan Webster**, Butler University.

*Arithmetic Statistics* (Code: SS 19A), **Brandon Alberts**, Eastern Michigan University, and **Soumya Sankar**, Ohio State University.

*Brauer Groups in Algebraic Geometry and Arithmetic* (Code: SS 25A), **Jack Petok** and **Sarah Frei**, Dartmouth College.

*Cluster Algebras, Positivity and Related Topics* (Code: SS 28A), **Eric Bucher**, Xavier University, **John Machacek**, The University of Oregon, and **Nicholas Ovenhouse**, Yale University.

*Combinatorial and Geometric Knot Theory* (Code: SS 11A), **Micah Chrisman**, The Ohio State University, **Sujoy Mukherjee**, University of Denver, and **Robert G Todd**, Mount Mercy University.

*Commutative Algebra with Connections to Combinatorics and Geometry* (Code: SS 24A), **Aleksandra C. Sobieska**, University of Wisconsin - Madison, and **Jay Yang**, Washington University in St. Louis.

*Ends and Boundaries of Groups: On the Occasion of Mike Mihalik's 70th Birthday* (Code: SS 2A), **Craig R. Guilbault**, University of Wisconsin-Milwaukee, and **Kim E. Ruane**, Tufts University.

*Extremal Graph Theory* (Code: SS 18A), **Neal Bushaw**, Virginia Commonwealth University, **Puck Rombach** and **Calum Buchanan**, University of Vermont, and **Vic Bednar**, Virginia Commonwealth University.

*Geometric and Analytic Methods in PDE* (Code: SS 20A), **Dennis Kriventsov**, Rutgers University, **Mariana Smit Vega Garcia**, Western Washington University, and **Mark Allen**, Brigham Young University.

*Growth Models, Random Media, and Limit Theorems* (Code: SS 6A), **Magda Peligrad**, **Wlodek Bryc**, and **Xiaoqin Guo**, University of Cincinnati.

*Harmonic Analysis and its Applications to Signals and Information* (Code: SS 12A), **Dustin G. Mixon**, The Ohio State University, and **Matthew Fickus**, Air Force Institute of Technology.

*Homological Methods in Commutative Algebra* (Code: SS 27A), **Michael DeBellevue**, Syracuse University, and **Josh Pollitz**, University of Utah.

*Inequalities in Harmonic Analysis* (Code: SS 26A), **Ryan Gibara**, University of Cincinnati, **Kabe Moen**, University of Alabama, and **Leonid Slavin**, University of Cincinnati.

*Interactions between Analysis, PDE, and Probability in Non-smooth Spaces* (Code: SS 1A), **Nageswari Shanmugalingam**, University of Cincinnati, **Luca Capogna**, Smith College, and **Jeremy T. Tyson**, National Science Foundation.

*Interactions between Noncommutative Ring Theory and Algebraic Geometry* (Code: SS 3A), **Jason Gaddis**, Miami University, and **Robert Won**, George Washington University.

*Mathematical Modeling in Biosciences* (Code: SS 29A), **Sookkyung Lim**, University of Cincinnati, **Jeungeun Park**, SUNY at New Paltz, **Yanyu Xiao**, University of Cincinnati, **Hem R. Joshi**, Xavier University, Cincinnati, and **David Gerberry**, Xavier University.

*Modern Trends in Numerical PDEs* (Code: SS 4A), **Johnny Guzman**, Brown University, and **Michael Neilan**, University of Pittsburgh.

*Nonlinear Partial Differential Equations from Variational Problems and Fluid Dynamics* (Code: SS 16A), **Tao Huang**, Wayne State University, **Hengrong Du**, Vanderbilt University, and **Changyou Wang**, Purdue University.

*Probabilistic and Extremal Combinatorics* (Code: SS 14A), **Jozsef Balogh**, University of Illinois at Urbana-Champaign, and **Tao Jiang**, Miami University.

*Quantitative Aspects of Symplectic Topology* (Code: SS 15A), **Jun Li**, University of Dayton, **Olguta Buse**, IUPUI, and **Richard Keith Hind**, University of Notre Dame.

*Recent Advances in Finite Element Methods: Theory and Applications* (Code: SS 21A), **Tamas L. Horvath**, Oakland University, and **Giselle Sosa Jones**, University of Houston.

*Recent Developments in the Study of Fluid Flows, Turbulence, and its Applications* (Code: SS 30A), **Vincent Martinez**, CUNY Hunter College & Graduate Center, and **Samuel Punshon-Smith**, Tulane University.

*Recent Trends in Graph Theory* (Code: SS 23A), **Adam Blumenthal**, Westminster College, and **Katherine Perry**, University of Soka.

*Recent Trends in Integrable Systems and Applications* (Code: SS 5A), **Deniz Bilman** and **Robert J. Buckingham**, University of Cincinnati.

*Representation Theory, Geometry and Mathematical Physics* (Code: SS 22A), **Daniele Rosso**, Indiana University Northwest, and **Jonas T. Hartwig**, Iowa State University.

*Stochastic Analysis and its Applications* (Code: SS 9A), **Po-Han Hsu**, University of Cincinnati, **Tai-Ho Wang**, Baruch College, CUNY, and **Ju-Yi Yen**, University Of Cincinnati.

*The Interface of Geometric Measure Theory and Harmonic Analysis* (Code: SS 10A), **Eyvindur Ari Palsson**, Virginia Tech, and **Krystal Taylor**, The Ohio State University, Department of Mathematics.

*Topological and Geometric Methods in Combinatorics* (Code: SS 17A), **Zoe Wellner** and **R. Amzi Jeffs**, Carnegie Mellon University.

# Fresno, California

*California State University, Fresno*

### May 6–7, 2023

*Saturday – Sunday*

### Meeting #1187

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: March 16, 2023
Issue of *Abstracts*: Not applicable

### Deadlines

For organizers: Expired
For abstracts: March 7, 2023

## MEETINGS & CONFERENCES

*The scientific information listed below may be dated. For the latest information, see* `https://www.ams.org/amsmtgs/sectional.html`.

### Invited Addresses

**Sami Assaf**, University of Southern California, *Title to be announced*.
**Natalia Komarova**, University of California, Irvine, *Title to be announced*.
**Joseph Teran**, University of California, Los Angeles, *Title to be announced*.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Advances by Scholars in the Pacific Math Alliance*, **Andrea Arauza Rivera**, California State University, East Bay, **Mario Banuelos**, California State University, Fresno, and **Jessica De Silva**, California State University, Stanislaus.

*Advances in Functional Analysis and Operator Theory*, **Michel L. Lapidus**, University of California, Riverside, **Marat V. Markin**, California State University, Fresno, and **Igor Nikolaev**, St. John's University.

*Algebraic Structures in Knot Theory*, **Carmen Caprau**, California State University, Fresno, **Sam Nelson**, Claremont McKenna College, and **Neslihan Gügümcu**, Izmir Institute of Technology in Turkey.

*Algorithms in the Study of Hyperbolic 3-manifolds*, **Robert Haraway, III** and **Maria Trnkova**, University of California, Davis.

*Analysis of Fractional Differential and Difference Equations with its Application*, **Bhuvaneswari Sambandham**, Dixie State University, and **Aghalaya S. Vatsala**, University of Louisiana at Lafayette.

*Artin-Schelter Regular Algebras and Related Topics*, **Ellen Kirkman**, Wake Forest University, and **James Zhang**, University of Washington.

*Combinatorics and Representation Theory (associated with the Invited Address by Sami Assaf)*, **Sami Assaf**, University of Southern California, and **Nicolle Gonzalez**, University of California, Berkeley.

*Complexity in Low-Dimensional Topology*, **Jennifer Schultens**, University of California, Davis, and **Eric Sedgwick**, DePaul University.

*Data Analysis and Predictive Modeling*, **Earvin Balderama**, California State University, Fresno, and **Adriano Zambom**, California State University, Northridge.

*Inverse Problems*, **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

*Math Circle Games and Puzzles that Teach Deep Mathematics*, **Maria Nogin**, **Agnes Tuska**, **Yaomingxin Lu**, and **Gábor Molnár-Sáska**, California State University, Fresno.

*Mathematical Biology: Confronting Models with Data*, **Erica Rutter**, University of California, Merced.

*Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Komarova)*, **Natalia Komarova** and **Jesse Kreger**, University of California, Irvine.

*Methods in Non-Semisimple Representation Categories*, **Eric Friedlander**, University of Southern California, Los Angeles, **Julia Pevtsova**, University of Washington, Seattle, and **Paul Sobaje**, Georgia Southern University, Statesboro.

*Research in Mathematics by Early Career Graduate Students*, **Doreen De Leon**, **Marat Markin**, and **Khang Tran**, California State University, Fresno.

*Scientific Computing*, **Changho Kim**, University of California, Merced, and **Roummel Marcia**.

*The Use of Computational Tools and New Augmented Methods in Networked Collective Problem Solving*, **Mario Banuelos**, California State University, Fresno, **Andrew G. Benedek**, Research Centre for the Humanities, Hungary, and **Agnes Tuska**, California State University, Fresno.

*Women in Mathematics*, **Doreen De Leon**, **Katherine Kelm**, and **Oscar Vega**, California State University, Fresno.

*Zero Distribution of Entire Functions*, **Tamás Forgács** and **Khang Tran**, California State University, Fresno.

# Buffalo, New York

*University at Buffalo (SUNY)*

### September 9–10, 2023
*Saturday – Sunday*

### Meeting #1188
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: July 27, 2023
Issue of *Abstracts*: To be announced

### Deadlines
For organizers: February 9, 2023
For abstracts: July 18, 2023

*The scientific information listed below may be dated. For the latest information, see* `https://www.ams.org/amsmtgs/sectional.html`.

### Invited Addresses
**Jennifer Balakrishnan**, Boston University, *Title to be announced*.
**Sigal Gottlieb**, University of Massachusetts, Dartmouth, *Title to be announced*.
**Samuel Payne**, University of Texas, *Title to be announced*.

# Omaha, Nebraska

*Creighton University*

### October 7–8, 2023
*Saturday – Sunday*

### Meeting #1190
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison

Program first available on AMS website: August 17, 2023
Issue of *Abstracts*: To be announced

### Deadlines
For organizers: March 7, 2023
For abstracts: August 8, 2023

# Mobile, Alabama

*University of South Alabama*

### October 13–15, 2023
*Friday – Sunday*

### Meeting #1189
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: August 24, 2023
Issue of *Abstracts*: To be announced

### Deadlines
For organizers: March 13, 2023
For abstracts: August 15, 2023

*The scientific information listed below may be dated. For the latest information, see* `https://www.ams.org/amsmtgs/sectional.html`.

### Invited Addresses
**Theresa Anderson**, Carnegie Mellon University, *Title to be announced*.
**Laura Miller**, University of Arizona, *Title to be announced*.
**Cornelius Pillen**, University of South Alabama, *Title to be announced*.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Mathematical Modeling of Problems in Biological Fluid Dynamics* (Code: SS 1A), **Laura Miller**, University of Arizona, and **Nick Battista**, The College of New Jersey.

# Albuquerque, New Mexico

*University of New Mexico*

### October 21–22, 2023

*Saturday – Sunday*

### Meeting #1191

Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: August 31, 2023
Issue of *Abstracts*: Not applicable

### Deadlines

For organizers: March 21, 2023
For abstracts: August 22, 2023

# Auckland, New Zealand

### December 4–8, 2023

*Monday – Friday*
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# San Francisco, California

*Moscone West Convention Center*

### January 3–6, 2024

*Wednesday – Saturday*
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Tallahassee, Florida

*Florida State University in Tallahassee*

### March 23–24, 2024

*Saturday – Sunday*
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Washington, District of Columbia

*Howard University*

### April 6–7, 2024

*Saturday – Sunday*
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# San Francisco, California

*San Francisco State University*

### May 4–5, 2024

*Saturday – Sunday*
Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable

Issue of *Abstracts*: Not applicable

### Deadlines

For organizers: To be announced
For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see* `https://www.ams.org/amsmtgs/sectional.html`.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Recent Advances in Differential Geometry*, **Zhiqin Lu**, University of California, **Shoo Seto** and **Bogdan Suceavă**, California State University, Fullerton, and **Lihan Wang**, California State University, Long Beach.

# Palermo, Italy

### July 23–26, 2024

*Tuesday – Friday*
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Riverside, California

*University of California, Riverside*

### October 26–27, 2024

*Saturday – Sunday*
Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable

Issue of *Abstracts*: Not applicable

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Seattle, Washington

*Washington State Convention Center and the Sheraton Seattle Hotel*

### January 8–11, 2025

*Wednesday – Saturday*
Associate Secretary for the AMS: Brian Boe
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Washington, District of Columbia

*Walter E. Washington Convention Center and Marriott Marquis Washington DC*

### January 4–7, 2026

*Sunday – Wednesday*
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

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