The Tricomi Equation

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The *Tricomi equation* is a second-order partial differential equation of mixed elliptic-hyperbolic type for $u(x, y)$ with the form:

$$u_{xx} + xu_{yy} = 0.$$

It was first analyzed in the work by Francesco Giacomo Tricomi (1923) on the well-posedness of a boundary value problem. The equation is hyperbolic in the half plane $x < 0$, elliptic in the half plane $x > 0$, and degenerates on the line $x = 0$. Its characteristic equation is

$$dy^2 + xdx^2 = 0,$$

whose solutions are

$$y \pm \frac{2}{3}(-x)^{\frac{3}{2}} = C$$

for any constant C, which are real for $x < 0$. The characteristics comprise two families of semicubical parabolas lying in the half plane $x < 0$, with cusps on the line $x = 0$. This is of hyperbolic degeneracy, for which the two characteristic families coincide, *perpendicularly* to the line $x = 0$.

For $\pm x > 0$, set $\tau = \frac{2}{3}(\pm x)^{\frac{3}{2}}$. Then the Tricomi equation becomes the classical elliptic or hyperbolic *Euler-Poisson-Darboux equation*:

$$u_{\tau\tau} \pm u_{yy} + \frac{\beta}{\tau}u_\tau = 0.$$

The index $\beta = \frac{1}{3}$ determines the singularity of solutions near $\tau = 0$, equivalently, $x = 0$.

Many important problems in fluid mechanics and differential geometry can be reduced to corresponding problems for the Tricomi equation, particularly *transonic flow problems* and *isometric embedding problems*. The Tricomi equation is a prototype of the *generalized Tricomi equation*:

$$u_{xx} + K(x)u_{yy} = 0.$$

For a steady-state transonic flow in $\mathbb{R}^2$, $u(x, y)$ is the stream function of the flow, $K(x)$ and x are functions of the velocity, which are positive at subsonic and negative at supersonic speeds, and y is the angle of inclination of the velocity. The solutions $u(x, y)$ also serve as entropy generators for entropy pairs of the potential flow system for the velocity. For the isometric embedding problem of two-dimensional Riemannian manifolds into $\mathbb{R}^3$, the function $K(x)$ has the same sign as the Gaussian curvature.

A closely related partial differential equation is the *Keldysh equation*:

$$xu_{xx} + u_{yy} = 0.$$

It is hyperbolic when $x < 0$, elliptic when $x > 0$, and degenerates on the line $x = 0$. Its characteristics are

$$y \pm \frac{1}{2}(-x)^{\frac{1}{2}} = C$$

for any constant C, which are real for $x < 0$. The two characteristic families are (quadratic) parabolas lying in the half plane $x < 0$ and coincide, *tangentially* to the degenerate line $x = 0$, which is of parabolic degeneracy. For $\pm x > 0$, the Keldysh equation becomes the elliptic or hyperbolic *Euler-Poisson-Darboux equation* with index $\beta = -\frac{1}{4}$, by setting $\tau = \frac{1}{2}(\pm x)^{\frac{1}{2}}$. Many important problems in continuum mechanics can be also reduced to corresponding problems for the Keldysh equation, particularly shock reflection-diffraction problems in gas dynamics.

Further Reading

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3. Chen, G.-Q., Slemrod, M. and Wang, D. 2008. *Vanishing viscosity method for transonic flow*, Arch. Ration. Mech. Anal. **189**, 159–188.
4. Han, Q. and Hong, J.-X. 2006. *Isometric Embedding of Riemannian Manifolds in Euclidean Spaces*, American Mathematical Society, Providence.
5. Morawetz, C. M. 2004. Mixed equations and transonic flow, *J. Hyper. Diff. Eqs.* **1**, 1–26.