

GLOBAL STEADY SUBSONIC FLOWS THROUGH INFINITELY LONG NOZZLES FOR THE FULL EULER EQUATIONS

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ABSTRACT. We are concerned with global steady subsonic flows through general infinitely long nozzles for the full Euler equations. The problem is formulated as a boundary value problem in the unbounded domain for a nonlinear elliptic equation of second order in terms of the stream function. It is established that, when the oscillation of the entropy and Bernoulli functions at the upstream is sufficiently small in $C^{1,1}$ and the mass flux is in a suitable regime, there exists a unique global subsonic solution in a suitable class of general nozzles. The assumptions are required to prevent from the occurrence of supersonic bubbles inside the nozzles. The asymptotic behavior of subsonic flows at the downstream and upstream, as well as the critical mass flux, has been clarified.

1. INTRODUCTION

We are concerned with global steady subsonic flows through general infinitely long nozzles for the full Euler equations (without the isentropic and irrotational requirement). The two-dimensional steady full Euler equations take the following form:

$$(\rho u)_{x_1} + (\rho v)_{x_2} = 0, \tag{1.1}$$

$$(\rho u^2)_{x_1} + (\rho uv)_{x_2} + p_{x_1} = 0, \tag{1.2}$$

$$(\rho uv)_{x_1} + (\rho v^2)_{x_2} + p_{x_2} = 0, \tag{1.3}$$

$$(\rho u(E + \frac{p}{\rho}))_{x_1} + (\rho v(E + \frac{p}{\rho}))_{x_2} = 0, \tag{1.4}$$

where ρ , (u, v) , p , and E denote the density, velocity, pressure, and total energy respectively. Moreover,

$$E = \frac{1}{2}(u^2 + v^2) + \frac{\gamma p}{(\gamma - 1)\rho} \tag{1.5}$$

with adiabatic exponent $\gamma > 1$.

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Consider flows through an infinitely long nozzle given by

$$\Omega = \{(x_1, x_2) : f_1(x_1) < x_2 < f_2(x_1), -\infty < x_1 < \infty\},$$

with the nozzle wall $\partial\Omega := W_1 \cup W_2$, where

$$W_i = \{(x_1, x_2) : x_2 = f_i(x_1), -\infty < x_1 < \infty\}, \quad i = 1, 2,$$

as in Fig 1.

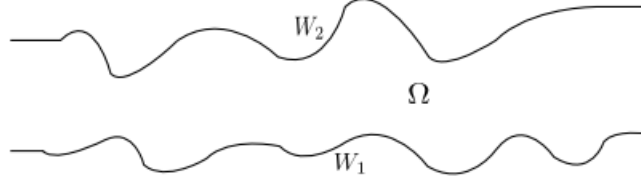


FIGURE 1. Infinite nozzle

Suppose that W_1 and W_2 satisfy

$$f_2(x_1) > f_1(x_1) \quad \text{for } x_1 \in (-\infty, \infty), \quad (1.6)$$

$$f_1(x_1) \rightarrow 0, \quad f_2(x_1) \rightarrow 1 \quad \text{as } x_1 \rightarrow -\infty \quad \text{in } C^{2,\alpha}, \quad (1.7)$$

$$f_1(x_1) \rightarrow a, \quad f_2(x_1) \rightarrow b > a \quad \text{as } x_1 \rightarrow \infty \quad \text{in } C^{2,\alpha}, \quad (1.8)$$

and there exists $\alpha > 0$ such that

$$\|f_i\|_{C^{2,\alpha}(\mathbb{R})} \leq C, \quad i = 1, 2, \quad (1.9)$$

for some positive constant C . It follows that Ω satisfies the uniform exterior sphere condition with some uniform radius $r > 0$.

Suppose that the nozzle walls are solid so that the flow satisfies the slip boundary condition:

$$(u, v) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (1.10)$$

where \mathbf{n} is the unit outward normal to the nozzle wall $\partial\Omega$. It follows from (1.1) and (1.10) that

$$\int_{\ell} (\rho u, \rho v) \cdot \mathbf{n} \, dl \equiv m \quad (1.11)$$

for some constant m , where ℓ is any curve transversal to the x_1 -direction, and \mathbf{n} is the normal of ℓ in the positive x_1 -direction.

If the flow is away from the vacuum state, it follows from (1.2)–(1.4) that

$$(u, v) \cdot \nabla (\ln p - \gamma \ln \rho) = 0, \quad (1.12)$$

which implies that $\frac{p}{\rho^\gamma}$ is a constant along each streamline, provided that the solution is C^1 -smooth. We assume that the entropy function is given in the upstream, i.e.,

$$\frac{\gamma p}{(\gamma - 1)\rho^\gamma} \rightarrow S(x_2) \quad \text{as } x_1 \rightarrow -\infty, \quad (1.13)$$

where $S(x_2)$, defined on $[0, 1]$, is the entropy. The sonic speed of the flow is defined by

$$c = \sqrt{\frac{\gamma p}{\rho}}. \quad (1.14)$$

By (1.1) and (1.4), we obtain

$$(u, v) \cdot \nabla \left(\frac{1}{2}(u^2 + v^2) + \frac{\gamma p}{(\gamma - 1)\rho} \right) = 0. \quad (1.15)$$

This implies that $\frac{1}{2}(u^2 + v^2) + \frac{\gamma p}{(\gamma - 1)\rho}$, which is called the Bernoulli function, is a constant along each streamline. We assume that the Bernoulli function is given in the upstream, i.e.,

$$\frac{(u^2 + v^2)}{2} + \frac{\gamma p}{(\gamma - 1)\rho} \rightarrow B(x_2) \quad \text{as } x_1 \rightarrow -\infty, \quad (1.16)$$

where $B(x_2)$ is defined on $[0, 1]$.

Problem 1. *Solve the full Euler system (1.1)–(1.4) with the boundary condition (1.10), the mass flux condition (1.11), and the asymptotic conditions (1.13) and (1.16).*

Set

$$\underline{S} = \inf_{x_2 \in [0, 1]} S(x_2), \quad \underline{B} = \inf_{x_2 \in [0, 1]} B(x_2).$$

The main results of this paper are the following.

Theorem 1.1 (Main Theorem). *Let the nozzle walls $\partial\Omega$ satisfy (1.6)–(1.9), and let $\underline{S} > 0$ and $\underline{B} > 0$. Then there exists $\delta_0 > 0$ such that, if*

$$\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0, 1])} \leq \delta \quad \text{for } 0 < \delta \leq \delta_0, \quad (1.17)$$

and

$$(SB^{-\gamma})'(0) \geq 0, \quad (SB^{-\gamma})'(1) \leq 0, \quad (1.18)$$

there exists $\hat{m} \geq 2\delta_0^{1/8}$ such that, for any $m \in (\delta^{1/4}, \hat{m})$, there exists a global solution (i.e. a full Euler flow) $(\rho, u, v, p) \in C^{1, \alpha}(\bar{\Omega})$ of Problem 1 such that

- (i) *Subsonicity and positivity of the horizontal velocity: The flow is uniformly subsonic globally with positive horizontal velocity in the whole nozzle, i.e.,*

$$\sup_{\bar{\Omega}} (u^2 + v^2 - c^2) < 0, \quad u > 0 \quad \text{in } \bar{\Omega}; \quad (1.19)$$

- (ii) *Far field behavior: The flow satisfies the following asymptotic behavior in the far fields:*

- (a) *As $x_1 \rightarrow -\infty$,*

$$p \rightarrow p_0 > 0, \quad u \rightarrow u_0(x_2) > 0, \quad (v, \rho) \rightarrow (0, \rho_0(x_2; p_0)), \quad (1.20)$$

$$\nabla p \rightarrow 0, \quad \nabla u \rightarrow (0, u'_0(x_2)), \quad \nabla v \rightarrow 0, \quad \nabla \rho \rightarrow (0, \rho'_0(x_2; p_0)) \quad (1.21)$$

uniformly for $x_2 \in K_1 \Subset (0, 1)$;

(b) As $x_1 \rightarrow \infty$,

$$p \rightarrow p_1 > 0, \quad u \rightarrow u_1(x_2) > 0, \quad (v, \rho) \rightarrow (0, \rho_1(x_2; p_1)), \quad (1.22)$$

$$\nabla p \rightarrow 0, \quad \nabla u \rightarrow (0, u'_1(x_2)), \quad \nabla v \rightarrow 0, \quad \nabla \rho \rightarrow (0, \rho'_1(x_2; p_1)) \quad (1.23)$$

uniformly for $x_2 \in K_2 \Subset (a, b)$, where p_0 and p_1 are both positive constants,

$$\rho_0(x_2; p_0) = \left(\frac{\gamma p_0}{(\gamma - 1)S(x_2)} \right)^{\frac{1}{\gamma}}, \quad \rho_1(x_2; p_1) = \left(\frac{\gamma p_1}{(\gamma - 1)S(x_2)} \right)^{\frac{1}{\gamma}},$$

and $p_0, p_1, u_0(x_2)$ and $u_1(x_2)$ can be determined by $m, S(x_2), B(x_2)$, and $b - a$ uniquely;

- (iii) *Uniqueness: The full Euler flow of Problem 1 satisfying (1.19) and the asymptotic behavior (1.20)–(1.23) is unique.*
- (iv) *Critical mass flux: \hat{m} is the upper critical mass flux for the existence of subsonic flow in the following sense: either*

$$\sup_{\bar{\Omega}} (u^2 + v^2 - c^2) \rightarrow 0 \quad \text{as } m \rightarrow \hat{m}, \quad (1.24)$$

or there is no $\sigma > 0$ such that, for all $m \in (\hat{m}, \hat{m} + \sigma)$, there are full Euler flows of Problem 1 satisfying (1.19), the asymptotic behavior (1.20)–(1.23), and

$$\sup_{m \in (\hat{m}, \hat{m} + \sigma)} \sup_{\bar{\Omega}} (c^2 - (u^2 + v^2)) > 0. \quad (1.25)$$

The assumptions in Theorem 1.1 are required to prevent from the occurrence of supersonic bubbles inside the infinitely long nozzles.

There has been some literature on the analysis of the infinitely nozzle problems. For potential flows, Chen-Feldman [5, 6] established the existence and stability of multidimensional transonic flows through an infinite nozzle of arbitrary cross-sections; also see Chen-Dafermos-Slemrod-Wang [8] and Kim [14]. Xie-Xin [16] established the existence of global subsonic isentropic flows and obtained the critical upper bound of mass flux under the assumption that the derivative of the Bernoulli function equals to zero on the two boundaries. For the steady full Euler equations, Chen-Chen-Feldmann [7] established the first existence of global transonic flows in two-dimensional infinite nozzles of slowly varying cross-sections; also see Chen [9]. Motivated by the earlier results, the focus of this paper is on the full Euler equations for the infinitely nozzle problem with general varying cross-sections by developing some useful new techniques. Some further related results can be found in Bae-Feldman [1], Canic-Keyfitz-Lieberman [4], Glimm-Ji-Li-Zhang-Zheng [13], Serre [15], Yuan [17], and the references cited therein.

We remark that the main difference between our results and those in [16] is that our results allow the varying entropy function, so that the far behavior of the density and the equation for the stream function is not only determined by the Bernoulli function, but also by the entropy function. Thus, it is not clear whether one can directly use the implicit function theorem to obtain the density with respect to the Bernoulli function at the upstream, which is the starting point of our study of this

problem. Furthermore, it is not direct to see how the maximum principle can be employed to locate these solutions of the stream function in the physical interval and then to extend the existence for small enough momenta which is obtained by the standard energy estimates to the critical mass flux only by simply making the assumption on the Bernoulli function on the boundary as in [16]. In this paper, for the far field behavior, we introduce the ratio of the entropy and Bernoulli function; then, by carefully defining the upper and lower bounds for the pressure, we find the far field behavior of the pressure with respect to the others. In order to use the maximum principle, we extend the entropy function via a special form, under which we find a condition on the ratio of the entropy and Bernoulli function with some power which looks like but not exactly the condition on the change of the momenta on the boundary at the upstream. With the uniform estimates from the maximum principle, we extend the existence of solutions to the critical mass flux.

The organization of the paper is as follows. In Section 2, we reformulate the problem as *Problem 2* by deriving the governing equation and boundary conditions for full Euler flows in terms of the stream function, provided that the Euler flow has a simple topological structure and satisfies the asymptotic behavior (1.20)–(1.23). In Section 3, the existence of solutions to a modified elliptic problem is established. Subsequently, in Section 4, we analyze the asymptotic behavior of solutions in a larger class and show the uniqueness of the solution to the boundary value problem. This yields the existence of solutions of the boundary value problem for the stream functions. In Section 5, the existence and uniqueness of solutions of Problem 2 are established, and, in Section 6, some refined estimates for the stream function is derived. The proof of the main theorem (Theorem 1.1) except the existence part of the critical mass flux is provided based on the results of Sections 2–5. In Section 8, we obtain the critical mass flux. Combining these estimates with the asymptotic behavior obtained in Section 4 yield the existence of full Euler flows which satisfy all the properties in Theorem 1.1.

2. REFORMULATION OF THE PROBLEM FOR STREAM FUNCTIONS

In this section we introduce the stream functions for the two-dimensional steady compressible full Euler flows and derive an equivalent formulation for the full Euler flows in the nozzles.

2.1. Equations. It follows from (1.1) that there exists a stream function ψ such that

$$\psi_{x_1} = -\rho v, \quad \psi_{x_2} = \rho u. \quad (2.1)$$

Furthermore, from (1.12), we have

$$p = \frac{\gamma - 1}{\gamma} \mathcal{S}(\psi) \rho^\gamma. \quad (2.2)$$

By (1.15), the Bernoulli law can be also written as

$$\frac{1}{2} |\nabla \psi|^2 + \mathcal{S}(\psi) \rho^{\gamma+1} = \mathcal{B}(\psi) \rho^2. \quad (2.3)$$

In the subsonic region, we have

$$|\nabla\psi|^2 < c^2\rho^2 = (\gamma - 1)\mathcal{S}(\psi)\rho^{\gamma+1},$$

which implies

$$\rho^{\gamma-1} > \frac{2\mathcal{B}}{(\gamma+1)\mathcal{S}}.$$

Let $\chi = \frac{1}{2}|\nabla\psi|^2$ and $g(\rho, \psi) = \mathcal{B}(\psi)\rho^2 - \mathcal{S}(\psi)\rho^{\gamma+1}$. We obtain

$$\frac{\partial g}{\partial \rho} = \left(\frac{2\mathcal{B}}{\mathcal{S}} - (\gamma+1)\rho^{\gamma-1} \right) \rho \mathcal{S} < 0$$

in the subsonic region. Hence, by the implicit function theorem, there exists a unique $\rho = \rho(\chi, \psi)$ such that

$$\chi = g(\rho, \psi) = \mathcal{B}(\psi)\rho^2 - \mathcal{S}(\psi)\rho^{\gamma+1}. \quad (2.4)$$

From (2.4), we have

$$\rho_\chi = -\frac{1}{(\gamma+1)\mathcal{S}\rho^\gamma - 2\mathcal{B}\rho}, \quad \rho_\psi = \frac{\mathcal{B}'\rho - \mathcal{S}'\rho^\gamma}{(\gamma+1)\mathcal{S}\rho^{\gamma-1} - 2\mathcal{B}}.$$

Then we have

$$\begin{aligned} \rho_{x_1} &= \rho_\chi(\psi_{x_1}\psi_{x_1x_1} + \psi_{x_2}\psi_{x_2x_1}) + \rho_\psi\psi_{x_1} \\ &= \frac{-\psi_{x_1}\psi_{x_1x_1} - \psi_{x_2}\psi_{x_2x_1} + \psi_{x_1}(\mathcal{B}'\rho^2 - \mathcal{S}'\rho^{\gamma+1})}{(\gamma+1)\mathcal{S}\rho^\gamma - 2\mathcal{B}\rho}, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \rho_{x_2} &= \rho_\chi(\psi_{x_1}\psi_{x_1x_2} + \psi_{x_2}\psi_{x_2x_2}) + \rho_\psi\psi_{x_2} \\ &= \frac{-\psi_{x_1}\psi_{x_1x_2} - \psi_{x_2}\psi_{x_2x_2} + \psi_{x_2}(\mathcal{B}'\rho^2 - \mathcal{S}'\rho^{\gamma+1})}{(\gamma+1)\mathcal{S}\rho^\gamma - 2\mathcal{B}\rho}. \end{aligned} \quad (2.6)$$

Now we reduce the Euler system into a second-order nonlinear equation. Multiplying equation (1.3) by $(\gamma+1)\mathcal{S}\rho^\gamma - 2\mathcal{B}\rho$, using expressions (2.5)–(2.6), and making algebraic manipulations, we obtain

$$\psi_{x_2}(a_{ij}(\psi, \nabla\psi)\psi_{x_ix_j} - F(\psi, \nabla\psi)) = 0, \quad (2.7)$$

where

$$\begin{aligned} a_{11}(\psi, \nabla\psi) &= (\gamma-1)\mathcal{S}\rho^{\gamma+1} - \psi_{x_2}^2, \\ a_{12}(\psi, \nabla\psi) &= a_{21}(\psi, \nabla\psi) = \psi_{x_1}\psi_{x_2}, \\ a_{22}(\psi, \nabla\psi) &= (\gamma-1)\mathcal{S}\rho^{\gamma+1} - \psi_{x_1}^2, \\ F(\psi, \nabla\psi) &= \frac{\gamma-1}{\gamma}\rho^{\gamma+3}(\gamma\mathcal{S}\mathcal{B}' - 2\mathcal{S}'\mathcal{B} + \mathcal{S}\mathcal{S}'\rho^{\gamma-1}). \end{aligned}$$

If $u > 0$ in Ω , then $\psi_{x_2} > 0$. Thus we have

$$a_{ij}(\psi, \nabla\psi)\psi_{x_ix_j} = F(\psi, \nabla\psi). \quad (2.8)$$

Multiplying (2.8) by $((\gamma + 1)\mathcal{S}\rho^{\gamma+3} - 2\mathcal{B}\rho^4)^{-1}$, we obtain

$$\nabla \cdot \left(\frac{\nabla \psi}{\rho} \right) = \mathcal{B}'\rho - \frac{1}{\gamma} \mathcal{S}'\rho^\gamma. \quad (2.9)$$

In summary, we have the following proposition.

Proposition 2.1. *For any smooth flow away from the vacuum state in the nozzle Ω satisfying (1.7)–(1.9), if the flow is globally subsonic and*

$$u > 0 \quad \text{in } \Omega. \quad (2.10)$$

Then the new system formed by (2.1)–(2.3) and (2.9) is equivalent to the original Euler equations (1.1)–(1.4).

The previous derivation is obviously invertible for the subsonic flow, so we omit the details of the proof for Proposition 2.1. In order to establish the existence of solutions to system (1.1)–(1.4), it suffices to establish the existence of solutions to system (2.1)–(2.3) and (2.9) satisfying (2.10).

2.2. Relations between $\mathcal{S}(\psi)$, $\mathcal{B}(\psi)$, and the Asymptotic Behavior of ψ at $x_1 \rightarrow \pm\infty$. First, it follows from (1.10) that the nozzle walls are streamlines, so ψ is constant on each wall. By (1.11) and the fact that $\psi_{x_2} > 0$ since $u > 0$, we have

$$0 < \psi < m \text{ in } \Omega, \quad \psi = 0 \text{ on } W_1, \quad \psi = m \text{ on } W_2. \quad (2.11)$$

Then we study the density-speed relation by using the entropy relation (2.2) and the Bernoulli law (2.3). Here, unlike isentropic flow which does not need to study the entropy relation, i.e., the entropy function $S(x_2)$, we start from the ratio of these two functions B and S as follows.

Let $D(x_2) = (BS^{-\frac{1}{\gamma}})(x_2)$. For any $s > 0$, $\bar{p}(s) = \frac{\gamma-1}{\gamma} s^{\frac{\gamma}{\gamma-1}} > 0$ is the unique solution of

$$\left(\frac{\gamma \bar{p}(s)}{\gamma - 1} \right)^{\frac{\gamma-1}{\gamma}} = s.$$

Moreover, from (2.3), the speed

$$q(p, x_2; s) = \sqrt{2S^{\frac{1}{\gamma}}(x_2) \left(s - \left(\frac{\gamma p}{\gamma - 1} \right)^{\frac{\gamma-1}{\gamma}} \right)}.$$

Hence, for fixed s and $x_2 \in [0, 1]$, q is a strictly decreasing function of p on $[0, \bar{p}(s)]$. By the definition of $\bar{p}(s)$, one has

$$q(\bar{p}(s), x_2; s) = 0 < c(\bar{p}(s), x_2).$$

Now we claim that $q(0, x_2; s) > c(0, x_2)$. Indeed,

$$q(p, x_2; s) \rightarrow \sqrt{2S^{\frac{1}{\gamma}}(x_2)s} > 0 \quad \text{as } p \rightarrow 0,$$

and, by the definition of sonic speed, $c(0, x_2) = 0$. Thus, $q(0, x_2) > 0 = c(0, x_2)$. This completes the claim.

Since $c^2(p, x_2) = (\gamma - 1)S(x_2)\rho^{\gamma-1} = (\gamma^{\gamma-1}(\gamma - 1)S(x_2))^{\frac{1}{\gamma}} p^{\frac{\gamma-1}{\gamma}}$ is an increasing function of p , there exists a unique $\mathfrak{p}(s) \in [0, \bar{\mathfrak{p}}(s)]$ such that

$$c^2(\mathfrak{p}(s), x_2) = q^2(\mathfrak{p}(s), x_2; s).$$

More precisely,

$$\mathfrak{p}(s) = \frac{\gamma - 1}{\gamma} \left(\frac{2s}{\gamma + 1} \right)^{\frac{\gamma}{\gamma-1}}.$$

In summary, we have

Lemma 2.2. *There exist $\bar{\mathfrak{p}} = \bar{\mathfrak{p}}(s)$, $\mathfrak{p} = \mathfrak{p}(s)$, and $\Gamma = \Gamma(s, x_2)$ such that*

$$S^{\frac{1}{\gamma}}(x_2) \left(\frac{\gamma \bar{\mathfrak{p}}(D(x_2))}{\gamma - 1} \right)^{\frac{\gamma-1}{\gamma}} = B(x_2), \quad (2.12)$$

$$S^{\frac{1}{\gamma}}(x_2) \left(\frac{\gamma \mathfrak{p}(s)}{\gamma - 1} \right)^{\frac{\gamma-1}{\gamma}} + \frac{\Gamma^2(s, x_2)}{2} = B(x_2), \quad (2.13)$$

$$c^2(\mathfrak{p}(s), x_2) = \Gamma^2(s, x_2), \quad (2.14)$$

where $\bar{\mathfrak{p}}(s)$, $\mathfrak{p}(s)$, and $\Gamma(s, x_2)$ are the maximum pressure, critical pressure, and critical speed, respectively, for the fixed ratio s of the Bernoulli function and the entropy function.

Then direct calculations show that

$$\frac{d\bar{\mathfrak{p}}}{ds} > 0, \quad \frac{d\mathfrak{p}}{ds} > 0.$$

Clearly, $\mathfrak{p}(s) < \bar{\mathfrak{p}}(s)$ for $s > 0$. By the continuity and monotonicity of $\mathfrak{p}(s)$ and $\bar{\mathfrak{p}}(s)$, there exists a unique $\underline{\delta} > 0$ such that

$$\mathfrak{p}(\underline{D} + \underline{\delta}) = \bar{\mathfrak{p}}(\underline{D}). \quad (2.15)$$

where $\underline{D} = \inf_{x_2 \in [0,1]} D(x_2)$.

Moreover, it follows from (2.15) that there exists a uniform constant $C > 0$ such that

$$\begin{cases} C^{-1} \leq \mathfrak{p}(\underline{D}) < \bar{\mathfrak{p}}(\underline{D}) = \mathfrak{p}(\underline{D} + \underline{\delta}) \leq C, \\ C^{-1} \leq \mathfrak{p}'(s) \leq C, \quad C^{-1} \leq \bar{\mathfrak{p}}'(s) \leq C & \text{if } s \in (\underline{D}, \underline{D} + \underline{\delta}), \\ C^{-1} \leq S^{\frac{1}{\gamma}}(x_2) \left(\frac{\gamma p}{\gamma - 1} \right)^{\frac{\gamma-1}{\gamma}} \leq C & \text{if } p \in (\mathfrak{p}(\underline{D}), \bar{\mathfrak{p}}(\underline{D} + \underline{\delta})). \end{cases} \quad (2.16)$$

Hereafter, C denotes a generic constant which depends only essentially on S and B .

If $S(x_2)$ and $B(x_2)$ satisfy

$$\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0,1])} \leq \delta, \quad (2.17)$$

then

$$\bar{D} = \sup_{x_2 \in [0,1]} \frac{B(x_2)}{S^{1/\gamma}(x_2)} \leq \underline{D} + C\delta. \quad (2.18)$$

Finally, we study the behavior of \mathcal{S} and \mathcal{B} at the upstream and downstream in the far fields of the nozzle where the flow may have certain simple structure. Indeed, for

the flows satisfying (1.20)–(1.25), one can determine p_0 , p_1 , $\rho_0(x_2)$, $\rho_1(x_2)$, $u_0(x_2)$, and $u_1(x_2)$ first.

If the flow satisfies (1.20), then

$$\frac{u_0^2(x_2)}{2} + S(x_2)\rho_0^{\gamma-1}(x_2; p_0) = B(x_2), \quad u_0(x_2) > 0, \quad \rho_0(x_2; p_0) = \left(\frac{\gamma p_0}{(\gamma-1)S(x_2)} \right)^{\frac{1}{\gamma}} \quad (2.19)$$

and

$$\int_0^1 \rho_0(x_2; p_0) u_0(x_2) dx_2 = m, \quad (2.20)$$

which imply that

$$u_0(x_2) = \sqrt{2(B(x_2) - S(x_2)\rho_0^{\gamma-1}(x_2; p_0))}, \quad (2.21)$$

and

$$m = \int_0^1 \rho_0(x_2; p_0) \sqrt{2(B(x_2) - S(x_2)\rho_0^{\gamma-1}(x_2; p_0))} dx_2. \quad (2.22)$$

Lemma 2.3. *Let $\delta \leq \frac{\delta}{2}$. It follows from (2.15) that $\mathfrak{p}(D(x_2)) \leq \mathfrak{p}(\bar{D}) < \bar{\mathfrak{p}}(\underline{D})$. Then we have*

- (i) *For given $S(x_2)$, $B(x_2)$, and $m > 0$, (2.22) has a solution p_0 satisfying $p_0 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$ such that $\rho_0(x_2; p_0)$ and $u_0(x_2) > 0$ satisfy (2.19) for $x_2 \in [0, 1]$;*
- (ii) *If $\underline{S} > 0$ and $\|B - \underline{B}\|_{C^{1,1}([0,1])} = \delta \leq \hat{\delta}_0$ for some small $\hat{\delta}_0$, then there is a positive constant C such that*

$$\begin{cases} C^{-1}\delta^{2\beta} \leq \bar{\mathfrak{p}}(\underline{D}) - p_0 \leq C, \\ C^{-1}\delta^\beta \leq u_0(x_2) \leq C, \\ |u'_0(x_2)| = \frac{|B'(x_2)| + \frac{1}{\gamma}|S'(x_2)|\rho_0^{\gamma-1}(x_2; p_0)}{u_0(x_2)} \leq C\delta^{1-\beta}. \end{cases} \quad (2.23)$$

The proof of this lemma is as follows.

Result (i) is for obtaining a global subsonic flow in the nozzle. Clearly, from (2.19)

$$\frac{d}{dp_0} \left(\int_0^1 \rho_0(x_2; p_0) \sqrt{2(B(x_2) - S(x_2)\rho_0^{\gamma-1}(x_2; p_0))} dx_2 \right) < 0, \quad \frac{d\rho_0(x_2; p_0)}{dp_0} > 0$$

for $p_0 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$. Let

$$\varrho(D; x_2) := \left(\frac{\gamma \mathfrak{p}(D)}{(\gamma-1)S(x_2)} \right)^{\frac{1}{\gamma}} \quad \text{and} \quad \bar{\varrho}(D; x_2) := \left(\frac{\gamma \bar{\mathfrak{p}}(D)}{(\gamma-1)S(x_2)} \right)^{\frac{1}{\gamma}}.$$

It follows from (2.16) and (2.17) that

$$\begin{aligned}
& \int_0^1 \bar{\varrho}(\underline{D}; x_2) \sqrt{2S^{1/\gamma}(x_2) (D(x_2) - (S^{1/\gamma} \bar{\varrho}(\underline{D}; x_2))^{\gamma-1})} dx_2 \\
&= \int_0^1 \bar{\varrho}(\underline{D}; x_2) \sqrt{2S^{1/\gamma}(x_2) (D(x_2) - \underline{D})} dx_2 \\
&\leq C\delta^{1/2}.
\end{aligned}$$

In addition,

$$\begin{aligned}
& \int_0^1 \varrho(\bar{D}; x_2) \sqrt{2S^{1/\gamma}(x_2) (D(x_2) - (S^{1/\gamma} \varrho(\bar{D}; x_2))^{\gamma-1})} dx_2 \\
&\geq \int_0^1 \varrho(\bar{D}; x_2) \sqrt{2S^{1/\gamma}(x_2) (\underline{D} - (S^{1/\gamma} \varrho(\bar{D}; x_2))^{\gamma-1})} dx_2 \\
&= \int_0^1 \varrho(\bar{D}; x_2) \sqrt{2S(x_2) (\bar{\varrho}(\underline{D}; x_2)^{\gamma-1} - \varrho(\bar{D}; x_2)^{\gamma-1})} dx_2 \\
&= \int_0^1 \varrho(\bar{D}; x_2) \sqrt{2S(x_2) (\varrho(\underline{D} + \underline{\delta}; x_2)^{\gamma-1} - \varrho(\bar{D}; x_2)^{\gamma-1})} dx_2 \\
&\geq \int_0^1 \varrho(\bar{D}; x_2) \sqrt{2S(x_2) (\varrho(\underline{D} + \underline{\delta}; x_2)^{\gamma-1} - \varrho(\underline{D} + \underline{\delta}/2; x_2)^{\gamma-1})} dx_2 \\
&\geq C^{-1} \underline{\delta}^{1/2}.
\end{aligned}$$

Therefore, for any $\beta \in (0, 1/3)$, there exists $\tilde{\delta}_0 \in (0, \underline{\delta}/2)$ such that (2.22) has a unique solution $p_0 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$, if $0 \leq \delta \leq \tilde{\delta}_0$ and $m \in (\delta^\beta, m_1)$, where m_1 satisfies $C^{-1} \underline{\delta}^{1/2} \geq m_1 \geq 2\tilde{\delta}_0^\beta > C\delta^{1/2}$. Later on, for simplicity, we will choose $\beta = 1/4$. However, all the results hold for $\beta \in (0, 1/3)$.

By virtue of (2.22), one has

$$\begin{aligned}
m &= \int_0^1 \rho_0(x_2; p_0) \sqrt{2S^{1/\gamma}(x_2) (D(x_2) - (S^{1/\gamma}(x_2) \rho_0(x_2; p_0))^{\gamma-1})} dx_2 \\
&= \int_0^1 \rho_0(x_2; p_0) \sqrt{2S^{1/\gamma}(x_2) (D(x_2) - \underline{D} + \underline{D} - (S^{1/\gamma}(x_2) \rho_0(x_2; p_0))^{\gamma-1})} dx_2 \\
&\leq C \int_0^1 \rho_0(x_2; p_0) \sqrt{\delta + (S^{1/\gamma}(x_2) \bar{\varrho}(\underline{D}; x_2))^{\gamma-1} - (S^{1/\gamma}(x_2) \rho_0(x_2; p_0))^{\gamma-1}} dx_2.
\end{aligned}$$

Thus, we have

$$\delta + (S^{1/\gamma}(x_2) \bar{\varrho}(\underline{D}; x_2))^{\gamma-1} - (S^{1/\gamma}(x_2) \rho_0(x_2; p_0))^{\gamma-1} \geq C^{-1} \delta^{2\beta}.$$

Since $\beta < 1/3$, then there exists $\hat{\delta}_0 \in (0, \tilde{\delta}_0)$ such that, if $0 < \delta \leq \hat{\delta}_0$, then

$$\bar{\varrho}(\underline{D}; x_2)^{\gamma-1} - \rho_0^{\gamma-1}(x_2; p_0) \geq C^{-1} \delta^{2\beta}.$$

Consequently, if $\underline{S} > 0$ and $\|B - \underline{B}\|_{C^{1,1}([0,1])} = \delta \leq \hat{\delta}_0$, there exists a constant $C > 0$ such that (2.23) holds. This completes the proof of Lemma 2.3.

Next, to determine the asymptotic states in the downstream, we parameterize the streamlines in the downstream by their positions in the upstream. Using (1.20),

(1.22), and (2.10), we define

$$y = y(s) \quad \text{for } s \in [0, 1], \quad (2.24)$$

such that

$$S(s)\rho_0^{\gamma-1}(s; p_0) + \frac{u_0^2(s)}{2} = S(s)\rho_1^{\gamma-1}(s; p_0) + \frac{u_1^2(y(s))}{2}, \quad u_1(y(s)) > 0, \quad (2.25)$$

$$\int_0^s \rho_0(t; p_0)u_0(t)dt = \int_0^{y(s)} \rho_1(t; p_1)u_1(t)dt, \quad \rho_i(s; p_i) := \left(\frac{\gamma p_i}{(\gamma-1)S(s)} \right)^{\frac{1}{\gamma}}, \quad (2.26)$$

$$y(0) = a, \quad y(1) = b. \quad (2.27)$$

Then the streamline which starts at $(-\infty, s)$ ends at $(\infty, y(s))$.

The next procedure is similar as before, where we consider ρ_i instead of p_i for the monotone relationship between them and for simplicity by recalling

$$\varrho(D; x_2) := \left(\frac{\gamma \mathfrak{p}(D)}{(\gamma-1)S(x_2)} \right)^{\frac{1}{\gamma}} \quad \text{and} \quad \bar{\varrho}(D; x_2) := \left(\frac{\gamma \bar{\mathfrak{p}}(D)}{(\gamma-1)S(x_2)} \right)^{\frac{1}{\gamma}}.$$

The mapping in (2.24) is well-defined due to condition (2.25) and (2.26). In fact, (2.26) deduces that

$$\rho_0(s; p_0)u_0(s) = \rho_1(s; p_1)u_1(y(s))y'(s). \quad (2.28)$$

If $\rho_1 < \bar{\varrho}(\underline{D}; x_2) \leq \bar{\varrho}(D; x_2)$, then

$$S(s)(\rho_0^{\gamma-1}(s; p_0) - \rho_1^{\gamma-1}(s; p_1)) + \frac{u_0^2(s)}{2} = B(s) - S(s)\rho_1^{\gamma-1}(s) > B(s) - S^{1/\gamma}(s)D(s) = 0.$$

Then we have

$$\begin{cases} \frac{dy}{ds} = \frac{\rho_0(s; p_0)u_0(s)}{\rho_1(s; p_1)\sqrt{2S(s)(\rho_0^{\gamma-1}(s; p_0) - \rho_1^{\gamma-1}(s; p_1)) + u_0^2(s)}}, \\ y(0) = a, \end{cases} \quad (2.29)$$

where the pressure in the downstream p_1 satisfies

$$\int_0^1 \frac{\rho_0(s; p_0)u_0(s)}{\rho_1(s; p_1)\sqrt{2S(s)(\rho_0^{\gamma-1}(s; p_0) - \rho_1^{\gamma-1}(s; p_1)) + u_0^2(s)}} ds = b - a. \quad (2.30)$$

It remains to show that there exists $p_1 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$ satisfying (2.30). As in the proof to Lemma 2.3, we find that, for $p_1 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$,

$$\frac{d}{dp_1} \int_0^1 \frac{\rho_0(s; p_0)u_0(s)}{\rho_1(s; p_1)\sqrt{2S(s)(\rho_0^{\gamma-1}(s; p_0) - \rho_1^{\gamma-1}(s; p_1)) + u_0^2(s)}} ds > 0.$$

On one hand, there exists $\bar{\delta}_0 \in (0, \tilde{\delta}_0)$ such that, if $\delta \leq \bar{\delta}_0$, then

$$\begin{aligned} & \int_0^1 \frac{\rho_0(s; p_0)u_0(s)}{\bar{\varrho}(D; s)\sqrt{2S(s)(\rho_0^{\gamma-1}(s; p_0) - \bar{\varrho}(D; s)^{\gamma-1}) + u_0^2(s)}} ds \\ &= \int_0^1 \frac{\rho_0(s; p_0)u_0(s)}{\bar{\varrho}(D; s)\sqrt{2S^{1/\gamma}(s)(D(s) - \underline{D})}} ds \\ &\geq C\delta^{(2\beta-1)/2} > b - a. \end{aligned}$$

On the other hand,

$$\begin{aligned}
& \int_0^1 \frac{\rho_0(s; p_0) u_0(s)}{\varrho(\bar{D}; s) \sqrt{2S(s) (\rho_0^{\gamma-1}(s; p_0) - \varrho(\bar{D}; s)^{\gamma-1}) + u_0^2(s)}} ds \\
&= \int_0^1 \frac{\rho_0(s; p_0) u_0(s)}{\varrho(\bar{D}; s) \sqrt{2S^{1/\gamma}(s) (D(s) - (S^{1/\gamma} \varrho(\bar{D}))^{\gamma-1})}} ds \\
&\leq \left(\frac{\gamma-1}{\gamma \mathfrak{p}(\bar{D})} \right)^{\frac{1}{\gamma}} \int_0^1 \frac{S^{\frac{1}{2\gamma}}(s) \rho_0(s; p_0) u_0(s)}{\sqrt{2(D - (S^{1/\gamma} \varrho(\bar{D}))^{\gamma-1})}} ds \\
&= \left(\frac{\gamma-1}{\gamma \mathfrak{p}(\bar{D})} \right)^{\frac{1}{\gamma}} \int_0^1 \frac{S^{\frac{1}{2\gamma}}(s) \rho_0(s; p_0) u_0(s)}{\sqrt{2 \left((\frac{\gamma \mathfrak{p}(\bar{D})}{\gamma-1})^{\frac{\gamma-1}{\gamma}} - (\frac{\gamma \mathfrak{p}(D)}{\gamma-1})^{\frac{\gamma-1}{\gamma}} \right)}} ds \\
&\leq \left(\frac{\gamma-1}{\gamma \mathfrak{p}(\bar{D})} \right)^{\frac{1}{\gamma}} \frac{m}{\sqrt{2 \left((\frac{\gamma \mathfrak{p}(D+\delta)}{\gamma-1})^{\frac{\gamma-1}{\gamma}} - (\frac{\gamma \mathfrak{p}(D)}{\gamma-1})^{\frac{\gamma-1}{\gamma}} \right)}} \max_{x_2 \in [0,1]} (S^{\frac{1}{2\gamma}}(x_2)) \\
&= \left(\frac{\gamma-1}{\gamma \mathfrak{p}(\bar{D})} \right)^{\frac{1}{\gamma}} \frac{m}{\sqrt{\frac{4}{\gamma+1} (D+\delta-D)}} \max_{x_2 \in [0,1]} (S^{\frac{1}{2\gamma}}(x_2)) \\
&\leq \frac{Cm}{\delta^{1/2}} < b - a.
\end{aligned} \tag{2.31}$$

Thus, there exists a unique $p_1 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$ such that (2.30) holds, provided that $0 \leq \delta \leq \bar{\delta}_0$ and $m \in (\delta^\beta, m_2)$ for some $\bar{\delta}_0$ small enough and $2\bar{\delta}_0^\beta \leq m_2 \leq \min\{m_1, C^{-1}(b-a)\delta^{1/2}\}$. Once p_1 determined, $y(s)$, $\rho_1(s; p_1)$, and $u_1(s)$ can be obtained from (2.25), (2.26), and (2.28). Therefore, the above calculations yield the following proposition.

Proposition 2.4. *Let $\underline{S}, \underline{B} > 0$, and D be the ratio of $S^{1/\gamma}$ and B . There exists $\bar{\delta}_0 > 0$ such that, for any $S, B \in C^{1,1}([0, 1])$ satisfying (2.17) with $\delta \leq \bar{\delta}_0$ respectively, there exists $\bar{m} \geq 2\bar{\delta}_0^\beta$, $\beta \in (0, \frac{1}{3})$, such that*

- (i) *Existence: There exists solutions (ρ_0, u_0, p_0) to (2.19)–(2.20) and (ρ_1, u_1, p_1) to (2.25)–(2.27) if $m \in (\delta^\beta, \bar{m})$ with $\rho_j^\gamma(y_j; p_j) = \frac{\gamma}{\gamma-1} \frac{p_j}{S(y_j)}$, $j = 0, 1$, $y_0 = x_2$ and $y_1 = y(s)$;*
- (ii) *Subsonicity: $p_0, p_1 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$;*
- (iii) *Limiting behaviors: Either $p_0 \rightarrow \mathfrak{p}(\bar{D})$ or $p_1 \rightarrow \mathfrak{p}(\bar{D})$ as $m \rightarrow \bar{m}$,*

where $\bar{D} = \sup_{x_2 \in [0,1]} D(x_2)$ and

$$\bar{m} = \sup\{s : m \in (\delta^\beta, s) \text{ such that there exist } p_0, p_1 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))\}. \tag{2.32}$$

Proof. Results (i)–(ii) are direct corollaries of Lemmas 2.2–2.3. It suffices to verify (iii).

For $m \in (\delta^\beta, m_2)$, $p_0, p_1 \in (\mathfrak{p}(\bar{D}), \bar{\mathfrak{p}}(\underline{D}))$. For fixed $S(x_2)$ and $B(x_2)$, p_0 decreases as m increases. If

$$m \rightarrow \tilde{m} = \int_0^1 \varrho(\bar{D}; x_2) \sqrt{2S^{1/\gamma}(x_2) (D(x_2) - (S^{1/\gamma} \varrho(\bar{D}))^{\gamma-1})} dx_2,$$

then

$$p_0 \rightarrow \mathfrak{p}(\bar{D}).$$

For \bar{m} defined in (2.32), $\bar{m} \in [m_2, \tilde{m}]$. Note that both p_0 and p_1 are uniformly away from $\bar{\mathbf{p}}(\underline{D})$. If neither p_0 nor p_1 approaches to $\bar{\mathbf{p}}(\underline{D})$ as $m \rightarrow \bar{m}$, then there always exist $p_0, p_1 \in (\bar{\mathbf{p}}(\underline{D}), \bar{\mathbf{p}}(\underline{D}))$ for $m \in (\delta^\beta, \bar{m} + \epsilon)$ for some small positive ϵ , which contradicts with the definition of \bar{m} . This completes the proof. \square

2.3. Reformulation of Problem 1: Problem 2. Let X_2 be the coordinate in the upstream. Since $\rho_0(X_2; p_0)u_0(X_2) > 0$ for $X_2 \in [0, 1]$, ψ is an increasing function of X_2 . Thus, we can represent X_2 as a function of ψ , which is defined by

$$X_2 = \kappa(\psi), \quad 0 \leq \psi \leq m,$$

so that

$$\psi(X_2) = \int_0^{X_2} \rho_0(s; p_0)u_0(s)ds.$$

It follows from Proposition 2.1 that, if (2.10) holds in Ω , through each point $(x_1, x_2) \in \Omega$, there exists a unique streamline which starts from the upstream. Along each streamline, the stream function is a constant by the definition. Therefore, through any (x_1, x_2) in the nozzle, there exists a unique streamline from $(-\infty, \kappa(\psi))$ with $\psi = \psi(x_1, x_2)$. Thus, we denote

$$\mathcal{S} = \mathcal{S}(\kappa(\psi)), \quad \mathcal{B} = \mathcal{B}(\kappa(\psi)) \quad \text{for } 0 \leq \psi \leq m.$$

Then our main task in the rest of the paper is to solve the following problem:

Problem 2 (Reformulation of Problem 1). *Seek a solution of the boundary value problem:*

$$\begin{cases} \nabla \cdot \left(\frac{\nabla \psi}{\rho(|\nabla \psi|^2, \psi)} \right) = \mathcal{B}'(\psi)\rho(|\nabla \psi|^2, \psi) - \frac{1}{\gamma}\mathcal{S}'(\psi)\rho^\gamma(|\nabla \psi|^2, \psi) & \text{in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}m & \text{on } \partial\Omega, \end{cases} \quad (2.33)$$

such that

(i) *The flow field induced by*

$$\rho = \rho(|\nabla \psi|^2, \psi), \quad u = \frac{\psi_{x_2}}{\rho}, \quad v = -\frac{\psi_{x_1}}{\rho}, \quad p = \frac{\gamma - 1}{\gamma}\mathcal{S}(\psi)\rho^\gamma$$

satisfies (1.20)–(1.25);

(ii) *In the upstream, we have*

$$\psi(X_2) = \int_0^{X_2} \rho_0(s; p_0)u_0(s)ds, \quad 0 \leq \psi \leq m. \quad (2.34)$$

3. EXISTENCE OF SOLUTIONS OF A MODIFIED BOUNDARY VALUE PROBLEM

There are three main difficulties to solve the boundary value problem (2.33). The first is that equation (2.33) may degenerate at the sonic states. The second is that, although the entropy and Bernoulli function are well-defined on $[0, m]$, the density ρ is not well-defined for arbitrary ψ and $|\nabla \psi|$. The last is that the problem is in an unbounded domain. Our basic strategy is to extend the definition of $\mathcal{S}(\psi)$ and $\mathcal{B}(\psi)$ appropriately, introduce the elliptic cut-off to truncate $|\nabla \psi|$ in $\rho(|\nabla \psi|^2, \psi)$ in

a suitable way, and use a sequence of bounded domains and solve the problems on it to approximate the original one.

In this section we first introduce a modified problem and then solve it, which can be indeed used to solve the original problem with the asymptotic behavior in §4.

Set

$$a(s) = \begin{cases} \mathcal{S}'(s) & \text{if } 0 \leq s \leq m, \\ \mathcal{S}'(m) \frac{2m-s}{m} & \text{if } m \leq s \leq 2m, \\ \mathcal{S}'(0) \frac{s+m}{m} & \text{if } -m \leq s \leq 0, \\ 0, & \text{if } s \geq 2m \text{ or } s \leq -m, \end{cases}$$

and

$$b(s) = \begin{cases} (\frac{\mathcal{B}}{\mathcal{S}^\gamma})'(s) & \text{if } 0 \leq s \leq m, \\ (\frac{\mathcal{B}}{\mathcal{S}^\gamma})'(m) \frac{2m-s}{m} & \text{if } m \leq s \leq 2m, \\ (\frac{\mathcal{B}}{\mathcal{S}^\gamma})'(0) \frac{s+m}{m} & \text{if } -m \leq s \leq 0, \\ 0, & \text{if } s \geq 2m, \text{ or } s \leq -m. \end{cases}$$

We define

$$\tilde{\mathcal{S}}(s) = \mathcal{S}(0) + \int_0^s a(t)dt, \quad \tilde{\mathcal{B}}(s) = \tilde{\mathcal{S}}^\gamma(s) \left(\frac{\mathcal{B}(0)}{\mathcal{S}^\gamma(0)} + \int_0^s b(t)dt \right). \quad (3.1)$$

Then $(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}) \in C^{1,1}(\mathbb{R})$. We remark here that the definition of $b(s)$ in this particular form instead of B' itself is for some technical reason, roughly speaking, due to the maximum principle. Moreover, since $m > \delta^\beta$, there exists a suitably small $\bar{\delta}_1$ such that, when $\delta < \bar{\delta}_1$,

$$0 < \underline{B} - C\delta \leq \frac{1}{2} \tilde{u}_0^2(s) + \tilde{\mathcal{S}}(s) \rho_0^{\gamma-1} \leq \sup_{x_2 \in [0,1]} B(x_2) + C\delta, \quad \tilde{u}_0(s) > 0$$

for some $C > 0$, where $\|(\tilde{\mathcal{S}} - \tilde{\mathcal{S}}(0), \tilde{\mathcal{B}} - \tilde{\mathcal{B}}(0))\|_{C^{1,1}(\mathbb{R})} \leq \delta^{1-\beta}$.

In the rest of the paper, we will always use the following notations:

$$\rho_1(|\nabla\psi|^2, \psi) = \frac{\partial \rho(|\nabla\psi|^2, \psi)}{\partial |\nabla\psi|^2}, \quad \rho_2(|\nabla\psi|^2, \psi) = \frac{\partial \rho(|\nabla\psi|^2, \psi)}{\partial \psi}.$$

It is easy to see that

$$\rho_1(|\nabla\psi|^2, \psi) = -\frac{1}{2\rho(c^2 - |\nabla\psi|^2/\rho^2)}$$

goes to $-\infty$ when the flow approaches the sonic state from the subsonic states. To avoid it, we introduce the following cut-off function. For $\epsilon > 0$, let

$$\zeta_0(s) = \begin{cases} s & \text{if } s < -2\epsilon, \\ -\frac{3}{2}\epsilon & \text{if } s \geq -\epsilon \end{cases} \quad (3.2)$$

be a smooth increasing function such that $|\zeta'_0| \leq 1$. We define

$$\tilde{\Delta}^2(|\nabla\psi|^2, \psi) := \zeta_0(|\nabla\psi|^2 - (\gamma - 1)\tilde{\mathcal{S}}(\psi)\tilde{\rho}^{\gamma+1}) + (\gamma - 1)\tilde{\mathcal{S}}(\psi)\tilde{\rho}^{\gamma+1}, \quad (3.3)$$

where

$$\frac{1}{2}\tilde{\Delta}^2(|\nabla\psi|^2, \psi) + \tilde{\mathcal{S}}(\psi)\tilde{\rho}^{\gamma+1} = \tilde{\mathcal{B}}(\psi)\tilde{\rho}^2. \quad (3.4)$$

A direct calculation shows

$$\tilde{\mathcal{S}}_{ij}(q, z) = \tilde{\rho}(|q|^2, z)\delta_{ij} - 2\tilde{\rho}_1(|q|^2, z)\xi_i\xi_j, \quad (3.5)$$

and

$$\tilde{\rho}_1(|\nabla\psi|^2, \psi) = \frac{\zeta'_0\tilde{\rho}}{4\tilde{\mathcal{B}}\tilde{\rho}^2 - (\gamma + 1)^2\tilde{\mathcal{S}}\tilde{\rho}^{\gamma+1} + (\gamma^2 - 1)\zeta'_0\tilde{\mathcal{S}}\tilde{\rho}^{\gamma+1}} < 0.$$

Obviously, there exist two positive constants $\lambda(\epsilon)$ and $\Lambda(\epsilon)$ such that

$$\lambda|\xi|^2 \leq \tilde{\mathcal{S}}_{ij}(q, z)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad (3.6)$$

for any $z \in \mathbb{R}$, $q \in \mathbb{R}^2$, and $\xi \in \mathbb{R}^2$, which means that the modified equation is uniformly elliptic. Thus, instead of solving *Problem 2*, we first solve the following problem:

Problem 3 (Modified Problem). *Seek a solution to the boundary value problem:*

$$\begin{cases} \nabla \cdot \left(\frac{\nabla\psi}{\tilde{\rho}} \right) = \tilde{B}'\tilde{\rho} - \frac{1}{\gamma}\tilde{\mathcal{S}}'\tilde{\rho}^\gamma & \text{in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}m & \text{on } \partial\Omega \end{cases} \quad (3.7)$$

such that $\|\psi\|_{C^{1,1}}$ has a uniform upper bound.

Proposition 3.1. *Let the boundary $\partial\Omega$ satisfy (1.6)–(1.9). Then there exists $0 < \delta_1 \leq \min\{\bar{\delta}_0, \bar{\delta}_1\}$ such that, if $\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0,1])} \leq \delta$ with $0 < \delta \leq \delta_1$ and $m \in (\delta^\beta, m_1)$ with $m_1 = 2\delta^{\beta/2} \leq \bar{m}$, where \bar{m} is defined in Proposition 2.2, then Problem 3 has a solution $\psi \in C^{2,\alpha}(\bar{\Omega})$ satisfying*

$$|\psi| \leq C(\epsilon, \delta), \quad |\nabla\psi|^2 \leq (\gamma - 1)\mathcal{S}\rho^{\gamma+1} - 2\epsilon \quad (3.8)$$

for some $\epsilon > 0$, so $|\nabla\psi|^2 \leq \underline{\Sigma}(\epsilon) - 2\epsilon$ with $\underline{\Sigma}(\epsilon) := (\gamma + 1)(\underline{S} + \delta + \epsilon)\left(\frac{2(B + \delta - \epsilon)}{(\gamma + 1)(\underline{S} + \epsilon)}\right)^{\frac{\gamma+1}{\gamma-1}}$.

Proof. The proof of the existence part is standard via approximation by the corresponding problems on bounded domains, while inequality (3.8) is crucial here, since $\underline{\Sigma}(\epsilon)$ depends not only on B but also on S for the non-isentropic flows. We divide the proof into four steps.

1. First, we use a sequence of boundary value problems on bounded domains to approximate *Problem 3* on the unbounded domain. Since the key point is to obtain estimate (3.8), we focus on the following boundary value problem:

$$\begin{cases} \nabla \cdot \left(\frac{\nabla\psi}{\tilde{\rho}} \right) = \tilde{B}'\tilde{\rho} - \frac{1}{\gamma}\tilde{\mathcal{S}}'\tilde{\rho}^\gamma & \text{in } \Omega_L, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}m & \text{on } \partial\Omega_L, \end{cases} \quad (3.9)$$

where Ω_L satisfies

$\{(x_1, x_2) : (x_1, x_2) \in \Omega, |x_1| < L\} \subset \Omega_L \subset \{(x_1, x_2) : (x_1, x_2) \in \Omega, |x_1| < 4L\}$
for all positive constants $L > L_0 > 0$, with L_0 sufficiently large, and $\partial\Omega_L \in C^{2,\alpha_1}$, $0 < \alpha_1 < \alpha$, satisfies the uniform exterior sphere condition with uniform radius r_0 , $0 < r_0 < r$.

2. Equation (3.7) can be written as

$$\tilde{A}_{ij}\partial_{ij}\psi - \tilde{\rho}_2|\nabla\psi|^2 = (\tilde{B}' - \frac{1}{\gamma}\tilde{S}'\tilde{\rho}^{\gamma-1})\tilde{\rho}^3, \quad (3.10)$$

where the repeated index is the summation with respect to the index from now on and

$$\tilde{\rho}_2 = \frac{-2\tilde{B}'\tilde{\rho} - (\gamma-1)\zeta'_0\tilde{S}'\tilde{\rho}^\gamma + (\gamma+1)\tilde{S}'\tilde{\rho}^\gamma}{4\tilde{B} - (\gamma+1)^2\mathcal{S}\rho^{\gamma-1} + (\gamma^2-1)\zeta'_0\tilde{S}\rho^{\gamma-1}}. \quad (3.11)$$

Therefore, (3.10) becomes

$$\tilde{S}_{ij}(\nabla\psi, \psi)\partial_{ij}\psi = \mathcal{F}(\nabla\psi, \psi), \quad (3.12)$$

where

$$\mathcal{F}(\nabla\psi, \psi) = (\tilde{B}' - \frac{1}{\gamma}\tilde{S}'\tilde{\rho}^{\gamma-1})\tilde{\rho}^3 + \tilde{\rho}_2|\nabla\psi|^2.$$

Instead of (3.9), we first solve the following problem:

$$\begin{cases} \tilde{A}_{ij}(\nabla\psi, \psi)\partial_{ij}\psi = \tilde{\mathcal{F}}(\nabla\psi, \psi) & \text{in } \Omega_L, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}m & \text{on } \partial\Omega_L, \end{cases} \quad (3.13)$$

where $\tilde{\mathcal{F}}(\nabla\psi, \psi) = (\tilde{B}' - \frac{1}{\gamma}\tilde{S}'\tilde{\rho}^{\gamma-1})\tilde{\rho}^3 + \tilde{\rho}_2\tilde{\Delta}^2$ for dealing with the fact that $\tilde{\mathcal{F}}$ has quadratic growth in $|\nabla\psi|$. By the definition of ζ , \tilde{S} , and \tilde{B} , we have

$$|\tilde{\mathcal{F}}(\nabla\psi, \psi)| \leq C\delta. \quad (3.14)$$

3. Then, by the standard existence theory of elliptic equations, there exists a solution ψ_L to (3.13). Furthermore, writing $\psi_L^- = \min\{\psi_L, 0\}$ and $\psi_L^+ = \max\{\psi_L, 0\}$, by the maximum principle with the source term (cf. Theorem 3.7 in [12]),

$$\min_{\partial\Omega_L} \psi_L^- - \frac{C}{\lambda} \sup_{\Omega_L} |\tilde{\mathcal{F}}| \leq \psi_L \leq \sup_{\partial\Omega_L} \psi_L^+ + \frac{C}{\lambda} \sup_{\Omega_L} |\tilde{\mathcal{F}}|, \quad (3.15)$$

where $C = e^d - 1$ with $d = \sup\{f_2(x_1) - f_1(x_1)\}$. Then we have

$$-C\delta^{1-2\beta} - C\delta^{1-\beta} \leq \psi_k \leq m + C\delta^{1-2\beta} + C\delta^{1-\beta} \quad \text{for } k \text{ sufficiently large.}$$

Moreover, one can obtain some other estimates for ψ_k . In fact, we can use the following more precise form with the same notations and symbols as those in Chapter 12 in [12],

$$[u]_{1,\alpha} \leq C(\gamma, \Omega) \left(1 + \|\nabla u\|_0 + \frac{\|f\|_0}{\lambda} \right). \quad (3.16)$$

Here, $C(\gamma, \Omega)$ depends only on $\text{diam}(\Omega)$ and the C^2 -norm of $\partial\Omega$.

Applying estimate (3.16) to problem (3.13) deduces that there exists $\mu = \mu(\frac{\Lambda}{\lambda}) > 0$ such that, for any $x_0 \in \bar{\Omega}_L$ and for ψ_k with $k \geq 4L$, we have

$$[\psi_k]_{1,\mu;B_1(x_0)\cap\Omega_L} \leq C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2) \left(1 + \|\nabla\psi_k\|_{0;B_1(x_0)\cap\Omega_L} + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right). \quad (3.17)$$

Furthermore, using the interpolation inequality and the maximum principle (3.15), we obtain

$$\|\psi_k\|_{1;B_1(x_0)\cap\Omega_L} \leq \eta C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2) \left(1 + \|\nabla\psi_k\|_{0;B_1(x_0)\cap\Omega_L} + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right) + C_\eta \left(m + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right),$$

where $C > 0$ is the same constants as that in (3.15). Taking η_0 sufficiently small so that $\eta C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2) \leq \frac{1}{2}$ if $\eta \leq \eta_0$, then

$$\|\psi_k\|_{1;B_1(x_0)\cap\Omega_L} \leq \eta C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2) \left(1 + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right) + C_\eta \left(m + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right). \quad (3.18)$$

Thus, the Hölder estimate (3.17) becomes

$$\begin{aligned} & \|\psi_k\|_{1,\mu;B_1(x_0)\cap\Omega_L} \\ &= \|\psi_k\|_{1;B_1(x_0)\cap\Omega_L} + [\psi_k]_{1,\mu;B_1(x_0)\cap\Omega_L} \\ &\leq \left(1 + C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2)\right) \|\psi_k\|_{1;B_1(x_0)\cap\Omega_L} + C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2) \left(m + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right) \\ &\leq C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2) \left(1 + m + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right). \end{aligned} \quad (3.19)$$

Since, for any $x, y \in \bar{\Omega}_L$,

$$\frac{|\nabla\psi_k(x) - \nabla\psi_k(y)|}{|x - y|^\mu} \leq \begin{cases} \|\psi_k\|_{1,\mu;B_1(x_0)\cap\Omega_L} & \text{if } y \in B_1(x_0) \cap \Omega_L, \\ 2\|\psi_k\|_{1;B_1(x_0)\cap\Omega_L} & \text{if } y \notin B_1(x_0) \cap \Omega_L, \end{cases}$$

which, together with (3.18) and (3.19), yields the following Hölder estimate:

$$[\psi_k]_{1,\mu;\Omega_L} \leq C(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2) \left(1 + m + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right). \quad (3.20)$$

Thus, it follows from the standard Schauder estimate that

$$\|\psi_k\|_{2,\alpha;B_{1/2}(x_0)\cap\Omega_L} \leq C(\frac{\Lambda}{\lambda}, \|f_1\|_{C^{2,\alpha}}, \|f_2\|_{C^{2,\alpha}}, m, \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}).$$

Thus,

$$\|\psi_k\|_{2,\alpha;\Omega_L} \leq C(\frac{\Lambda}{\lambda}, \|f_1\|_{2,\alpha}, \|f_2\|_{2,\alpha}, m, \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}). \quad (3.21)$$

4. Using the Arzela-Ascoli lemma and a diagonal procedure, we see that there exists a subsequence ψ_{k_l} such that

$$\psi_{k_l} \rightarrow \psi \text{ in } C^{2,\vartheta}(K) \quad \text{for any compact set } K \subset \bar{\Omega} \text{ and } \vartheta < \alpha.$$

Here, ψ satisfies the following problem:

$$\begin{cases} \tilde{A}_{ij}(\nabla\psi, \psi)\partial_{ij}\psi = \tilde{\mathcal{F}}(\nabla\psi, \psi) & \text{in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}m & \text{on } \partial\Omega, \end{cases}$$

with the estimate

$$\begin{aligned} \|\psi\|_{1;\Omega} &\leq \eta C(\lambda, \|f_1\|_2, \|f_2\|_2) \left(1 + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right) + C_\eta \left(m + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda}\right) \\ &\leq \eta C(\lambda, \|f_1\|_2, \|f_2\|_2) (1 + C\delta) + C_\eta (m + C\delta), \end{aligned} \quad (3.22)$$

where $\eta \in (0, \eta_0)$ and C depends only on $\bar{\delta}_0$, \bar{m} , Λ , and λ . Next, we prove that

$$|\nabla\psi|^2 \leq (\gamma - 1)\mathcal{S}\rho^{\gamma+1} - 2\epsilon.$$

Otherwise,

$$\begin{aligned} (\gamma - 1)\mathcal{S}\rho^{\gamma+1} &< |\nabla\psi|^2 + 2\epsilon \\ &\leq \eta C(\lambda, \|f_i\|_2) (1 + C\delta) + C_\eta (m + C\delta) \\ &\leq (\gamma - 1)\mathcal{S} \left(\frac{2\mathcal{B}}{(\gamma + 1)\mathcal{S}}\right)^{\frac{\gamma+1}{\gamma-1}}. \end{aligned}$$

Thus,

$$\rho < \left(\frac{2\mathcal{B}}{(\gamma + 1)\mathcal{S}}\right)^{\frac{1}{\gamma-1}}$$

and

$$2\mathcal{B}\rho^2 - (\gamma + 1)\mathcal{S}\rho^{\gamma+1} \geq 0,$$

which contradict with the fact that

$$2\mathcal{B}\rho^2 - (\gamma + 1)\mathcal{S}\rho^{\gamma+1} = \zeta_0(|\nabla\psi|^2 - (\gamma - 1)\mathcal{S}\rho^{\gamma+1}) < 0.$$

Thus, the solution ψ satisfies

$$|\nabla\psi|^2 \leq \underline{\Sigma}(\epsilon) - 2\epsilon \quad (3.23)$$

for any $\delta \in (0, \delta_1)$ and $m \in (\delta^\beta, 2\delta_1^{\beta/2})$. Then (3.8) follows from (3.22) and (3.23).

Furthermore, (3.20) and (3.21) yield the following higher order estimates

$$\|\psi\|_{1,\mu;\bar{\Omega}} \leq C\left(\frac{\Lambda}{\lambda}, \|f_1\|_2, \|f_2\|_2\right) \left(1 + m + \frac{\|\mathcal{F}\|_0}{\lambda}\right), \quad (3.24)$$

and

$$\|\psi\|_{2,\bar{\Omega}} \leq C\left(\frac{\Lambda}{\lambda}, \|f_1\|_{2,\alpha}, \|f_2\|_{2,\alpha}, m, \frac{\|\mathcal{F}\|_0}{\lambda}\right). \quad (3.25)$$

This completes the proof. \square

Remark 3.2. Estimate (3.8) in Proposition 3.1 implies that the cut-off function introduced in (3.2) and (3.3) can be removed.

4. FAR FIELD BEHAVIOR OF SOLUTIONS OF **Problem 3**

In this section, we study the far field behavior of solutions to *Problem 3*. We now show that the solutions to *Problem 3* satisfy the asymptotic behavior (1.20)–(1.25), and $0 \leq \psi \leq m$. From this, we can remove both the extension and the elliptic cut-off (3.7). Therefore, these solutions solve *Problem 2*. In addition, the stream function formulation is consistent with the formulation of *Problem 1* for the non-isentropic Euler system in the infinitely long nozzle, as long as the flow induced by a solution to *Problem 2* satisfies (1.20)–(1.25) and (2.10). Furthermore, the far field behavior is crucial also for the consequent result of the uniqueness of the solutions. First we have

Lemma 4.1. *For $\epsilon > 0$, there exists $\delta_2 \in (0, \bar{\delta}_0]$ such that, if*

- (i) $\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}} \leq \delta \leq \delta_2$;
- (ii) $m \in (\delta^\beta, \bar{m})$, where \bar{m} is defined as *Proposition 2.4*,

then there exists a function $\bar{\psi}$ that satisfies

$$\psi \rightarrow \bar{\psi} \quad \text{as } x_1 \rightarrow -\infty,$$

and

$$\bar{\psi}(x_1, x_2) = \bar{\psi}(x_2) = \int_0^{x_2} \rho_0(s; p_0) u_0(s) ds, \quad (4.1)$$

where ρ_0 and u_0 are uniquely determined by S , B , and m as §2, so $\bar{\psi}$ is independent of x_1 .

Proof. The proof is based on the blowup argument in combination with the energy estimate, which consists of three parts: The first is for the existence of $\bar{\psi}$, the second is for the independence of $\bar{\psi}$ of x_1 , and the third is the explicit form (4.1) for $\bar{\psi}$.

1. *Existence of the far field function.* It is convenient to introduce a new coordinate to flatten the boundary walls of the nozzle, as follows:

$$\begin{cases} t_1(x_1, x_2) = x_1, \\ t_2(x_1, x_2) = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}, \end{cases}$$

then the nozzle becomes $(-\infty, \infty) \times [0, 1]$. Obviously, the coordinate transform is reversible, since

$$\det \begin{bmatrix} \frac{\partial t_1}{\partial x_1} & \frac{\partial t_1}{\partial x_2} \\ \frac{\partial t_2}{\partial x_1} & \frac{\partial t_2}{\partial x_2} \end{bmatrix} = \left| \begin{bmatrix} 1 & 0 \\ *, \frac{1}{f_2(x_1) - f_1(x_1)} \end{bmatrix} \right| = \frac{1}{f_2(x_1) - f_1(x_1)} \neq 0.$$

In addition, we remark here that the equation does not change the type of ellipticity under the coordinate transformation, since

$$\begin{aligned}
& a_{ij} \partial_{x_i x_j} \psi \\
&= a_{11} \partial_{t_1 t_1} \psi - 2 \left(\frac{a_{11} (x_2 - f_1(x_1)) (f_2'(x_1) - f_1'(x_1))}{(f_2(x_1) - f_1(x_1))^2} - \frac{a_{12}}{(f_2(x_1) - f_1(x_1))} \right) \partial_{t_1 t_2} \psi \\
&+ \left(\frac{a_{11} (x_2 - f_1(x_1))^2 (f_2'(x_1) - f_1'(x_1))^2}{(f_2(x_1) - f_1(x_1))^4} - \frac{2a_{12} (x_2 - f_1(x_1)) (f_2'(x_1) - f_1'(x_1))}{(f_2(x_1) - f_1(x_1))^3} \right. \\
&\quad \left. + \frac{a_{22}}{(f_2(x_1) - f_1(x_1))^2} \right) \partial_{t_2 t_2} \psi \\
&+ \text{lower terms (involving } \partial_{t_i} \psi \text{ and } \psi).
\end{aligned}$$

In the new coordinates, define

$$\psi^{(n)} = \psi(t_1(x_1 - n, x_2), t_2(x_1 - n, x_2)).$$

For any compact set $K \Subset (-\infty, \infty) \times [0, 1]$, it follows from (3.25) and the $C^{2,\alpha}$ -bounds of the walls f_1 and f_2 that

$$\|\psi^{(n)}\|_{C^{2,\alpha}(K)} \leq C \quad \text{for } n \text{ sufficiently large.}$$

Then, as in Step 4 of the proof to Proposition 3.1, there exists a subsequence $\psi^{(n_k)}$ such that

$$\psi^{(n_k)} \rightarrow \bar{\psi} \quad \text{in } C^{2,\vartheta}(K) \quad (4.2)$$

for any $K \Subset (-\infty, \infty) \times [0, 1]$ and any $\vartheta \in (0, \alpha)$. From (1.6)–(1.9) and (3.25), and the facts that $f_1(x_1) \rightarrow 0$ and $f_2(x_1) \rightarrow 1$ in $C^{2,\alpha}$ as $x_1 \rightarrow -\infty$, which also means that $f_i'(x_1) \rightarrow 0$ in $C^{1,\alpha}$ as $x_1 \rightarrow -\infty$, then $\bar{\psi}$ satisfies

$$\begin{cases} \nabla \cdot \left(\frac{\nabla \bar{\psi}}{\bar{\rho}(|\nabla \bar{\psi}|^2, \bar{\psi})} \right) = (\tilde{B}' \bar{\rho} - \frac{1}{\gamma} \tilde{S}' \bar{\rho} \gamma) (|\nabla \bar{\psi}|^2, \bar{\psi}) & \text{in } D, \\ \bar{\psi} = 0 & \text{on } x_2 = 0, \\ \bar{\psi} = m & \text{on } x_2 = 1, \end{cases} \quad (4.3)$$

where $D = (-\infty, \infty) \times (0, 1)$, and $\bar{\psi}$ also satisfies

$$|\bar{\psi}| \leq C(\epsilon, \delta), \quad |\nabla \bar{\psi}|^2 \leq \Sigma(\epsilon) - 2\epsilon. \quad (4.4)$$

Thus, by similar arguments as in §3, on any compact set $E \subset (-\infty, \infty) \times [0, 1]$,

$$\|\bar{\psi}\|_{C^{1,\mu}(E)} \leq C(\epsilon, \delta),$$

and

$$\|\bar{\psi}\|_{C^{2,\alpha}(E)} \leq C(\epsilon, \delta). \quad (4.5)$$

Thus, $\bar{\psi} \in C^{2,\alpha}(\bar{D})$. This completes the first part.

2. Differentiate the equation in (4.3) with respect to x_1 and set $\omega = \bar{\psi}_{x_1}$. Then

$$\partial_i \left(\frac{\tilde{A}_{ij}(\nabla \bar{\psi}, \bar{\psi})}{\tilde{\rho}^2(|\nabla \bar{\psi}|^2, \bar{\psi})} \partial_j \omega \right) - \partial_i \left(\frac{\tilde{\rho}_2(|\nabla \bar{\psi}|^2, \bar{\psi}) \partial_i \bar{\psi}}{\tilde{\rho}^2(|\nabla \bar{\psi}|^2, \bar{\psi})} \omega \right) = \tilde{\Theta}(|\nabla \bar{\psi}|^2, \bar{\psi}) \omega + \tilde{\vartheta}(|\nabla \bar{\psi}|^2, \bar{\psi}) \partial_i \bar{\psi} \partial_i \omega, \quad (4.6)$$

where $\tilde{A}_{ij}(q, z)$, $\tilde{\Theta}(q, z)$, and $\tilde{\vartheta}(q, z)$ satisfy

$$\begin{aligned} \tilde{A}_{ij}(q, z) &= \tilde{\rho}(|q|^2, z) \delta_{ij} - 2\tilde{\rho}_1(|q|^2, z) q_i q_j, \\ \tilde{\Theta}(s, z) &= \tilde{\mathcal{B}}''(z) \tilde{\rho}(s, z) - \frac{1}{\gamma} \tilde{\mathcal{S}}''(z) \tilde{\rho}^\gamma + (\tilde{\mathcal{B}}'(z) - \tilde{\mathcal{S}}'(z) \tilde{\rho}^{\gamma-1}(s, z)) \tilde{\rho}_2(s, z), \\ \tilde{\vartheta}(s, z) &= 2(\tilde{\mathcal{B}}'(z) - \tilde{\mathcal{S}}'(z) \tilde{\rho}^{\gamma-1}(s, z)) \tilde{\rho}_1(s, z), \end{aligned}$$

for $q \in \mathbb{R}^2$, $s \geq 0$, and $z \in \mathbb{R}$, where $(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}) \in C^{1,1}(\mathbb{R})$. Since it is unknown whether $\bar{\psi} \in C^3(D)$, equation (4.6) holds in the weak sense. It follows from (4.4) that

$$|\tilde{A}_{ij}(\nabla \bar{\psi}, \bar{\psi})| \leq \Lambda(\epsilon),$$

where Λ depends only on ϵ . Furthermore, ω satisfies the following boundary conditions:

$$\omega = 0 \quad \text{on } x_2 = 0, 1.$$

As usual for energy estimates, let η be a C_0^∞ -function satisfying

$$\eta = 1 \text{ for } |s| < L, \quad \eta = 0 \text{ for } |s| > L + 1, \quad |\eta'(s)| \leq 2. \quad (4.7)$$

Multiply $\eta^2(x_1)\omega$ and integrate it on both sides of (4.6), then integrate the left side, and plug the explicit forms of \tilde{A}_{ij} , $\tilde{\rho}_1(|\nabla \bar{\psi}|^2, \bar{\psi})$, and $\tilde{\rho}_2(|\nabla \bar{\psi}|^2, \bar{\psi})$ into it. We obtain

$$\int \int_D \frac{\eta^2 |\nabla \omega|^2}{\tilde{\rho}(|\nabla \bar{\psi}|^2, \bar{\psi})} dx_1 dx_2 = \sum_{i=1}^6 I_i, \quad (4.8)$$

where

$$\begin{aligned} I_1 &= - \int \int_D \frac{|\nabla \bar{\psi} \cdot \nabla \omega|^2 \eta^2}{\tilde{\rho}(|\nabla \bar{\psi}|^2, \bar{\psi}) (\tilde{\rho}^2(|\nabla \bar{\psi}|^2, \bar{\psi}) c^2 - |\nabla \bar{\psi}|^2)} dx_1 dx_2, \\ I_2 &= -2 \int \int_D \frac{\tilde{A}_{ij}(\nabla \bar{\psi}, \bar{\psi})}{\tilde{\rho}^2(|\nabla \bar{\psi}|^2, \bar{\psi})} \eta \omega \partial_j \omega \partial_i \eta dx_1 dx_2, \\ I_3 &= \int \int_D \frac{\tilde{\rho}_2(|\nabla \bar{\psi}|^2, \bar{\psi}) \nabla \bar{\psi} \cdot \nabla \eta}{\tilde{\rho}^2(|\nabla \bar{\psi}|^2, \bar{\psi})} \eta \omega^2 dx_1 dx_2, \\ I_4 &= 2 \int \int_D \frac{(\tilde{\mathcal{B}}'(\bar{\psi}) - \tilde{\mathcal{S}}'(\bar{\psi}) \tilde{\rho}^{\gamma-1}(s, \bar{\psi})) \nabla \bar{\psi} \cdot \nabla \eta}{\tilde{\rho}^2(|\nabla \bar{\psi}|^2, \bar{\psi}) c^2 - |\nabla \bar{\psi}|^2} \eta \omega^2 dx_1 dx_2, \\ I_5 &= - \int \int_D \left(\tilde{\mathcal{B}}''(\bar{\psi}) \tilde{\rho}(s, \bar{\psi}) - \frac{1}{\gamma} \tilde{\mathcal{S}}''(\bar{\psi}) \tilde{\rho}^\gamma \right) \eta^2 \omega^2 dx_1 dx_2, \\ I_6 &= - \int \int_D \frac{\tilde{\rho}^2(\tilde{\mathcal{B}}'(\bar{\psi}) - \tilde{\mathcal{S}}'(\bar{\psi}) \tilde{\rho}^{\gamma-1}(|\nabla \bar{\psi}|^2, \bar{\psi}))^2}{\tilde{\rho}^2(|\nabla \bar{\psi}|^2, \bar{\psi}) c^2 - |\nabla \bar{\psi}|^2} \tilde{\rho}_2(|\nabla \bar{\psi}|^2, \bar{\psi}) \eta^2 \omega^2 dx_1 dx_2. \end{aligned}$$

Now we make the estimates. First, by the Hölder inequality, it is easy to see that

$$I_1 + I_4 + I_6 \leq 0.$$

Second, since $\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0,1])} \leq \delta$ and $m \in (\delta^\beta, \bar{m})$, we have

$$\|(\tilde{\mathcal{S}} - \tilde{\mathcal{S}}(0), \tilde{\mathcal{B}} - \tilde{\mathcal{B}}(0))\|_{C^{1,1}(\mathbb{R})} \leq \delta^{1-\beta}.$$

Thus,

$$|I_5| \leq C\delta^{1-\beta} \int_{-L-1}^{L+1} \int_0^1 \omega^2 dx_1 dx_2, \quad (4.9)$$

and $\tilde{\rho} \leq \bar{\varrho}(\bar{D}; x_2)$, where C is independent of ϵ . Thus, from (4.8) and the definition of η , if δ_2 is sufficiently small, we obtain

$$\begin{aligned} & \int_{-L}^L \int_0^1 |\nabla \omega|^2 dx_2 dx_1 \\ & \leq |I_2 + I_3| + |I_5| \\ & \leq C(\epsilon) \left(\int_{-L-1}^{-L} + \int_L^{L+1} \right) \left(\int_0^1 (|\nabla \omega|^2 + |\nabla \omega| + \omega^2) dx_2 \right) dx_1 \\ & \quad + C\delta_2^{1-\beta} \int_{-L}^L \int_0^1 |\nabla \omega|^2 dx_2 dx_1 \\ & \leq C(\epsilon) \left(\int_{-L-1}^{-L} + \int_L^{L+1} \right) \left(\int_0^1 (|\nabla \omega|^2 + \omega^2) dx_2 \right) dx_1 \\ & \quad + \frac{1}{2} \int_{-L}^L \int_0^1 |\nabla \omega|^2 dx_2 dx_1. \end{aligned}$$

Notice that $\omega = 0$ on $x_2 = 0, 1$. It follows from the Poincaré inequality that there exists a constant C independent of l such that

$$\int_{-L}^L \int_0^1 |\nabla \omega|^2 dx_2 dx_1 \leq C \left(\int_{-L-1}^{-L} + \int_L^{L+1} \right) \left(\int_0^1 |\nabla \omega|^2 dx_2 \right) dx_1 \quad (4.10)$$

for large L . It follows from (4.5) that

$$\int_{-L}^L \int_0^1 |\nabla \omega|^2 dx_2 dx_1 \leq \left(\int_{-L-1}^{-L} + \int_L^{L+1} \right) \left(\int_0^1 |\nabla \omega|^2 dx_2 \right) dx_1 \leq C$$

for some uniform constant C independent of L and for some constant C . Passing the limit $L \rightarrow \infty$ yields

$$\int_{-\infty}^{\infty} \int_0^1 |\nabla \omega|^2 dx_2 dx_1 \leq C.$$

Hence,

$$\left(\int_{-L-1}^{-L} + \int_L^{L+1} \right) \left(\int_0^1 |\nabla \omega|^2 dx_2 \right) dx_1 \rightarrow 0 \quad \text{as } L \rightarrow \infty. \quad (4.11)$$

Using (4.10) by passing the limit $l \rightarrow \infty$ as before again, we obtain

$$\int_{-\infty}^{\infty} \int_0^1 |\nabla \omega|^2 dx_2 dx_1 = 0,$$

which implies $\omega = 0$. Therefore,

$$\bar{\psi} = \bar{\psi}(x_2),$$

which solves the following boundary value problem:

$$\begin{cases} \frac{d}{dx_2} \left(\frac{\nabla \bar{\psi}}{\bar{\rho}(|\nabla \bar{\psi}|^2, \bar{\psi})} \right) = \tilde{G}(\nabla \bar{\psi}, \bar{\psi}) & \text{in } D, \\ \bar{\psi}(0) = 0, \\ \bar{\psi}(1) = m, \end{cases} \quad (4.12)$$

which completes the first part.

3. *Explicit form of $\bar{\psi}(x_2)$.* Suppose that there are two solutions $\bar{\psi}_1$ and $\bar{\psi}_2$ to (4.12). Let $\bar{\phi} = \bar{\psi}_1 - \bar{\psi}_2$. Then $\bar{\phi}$ satisfies

$$\begin{cases} (\bar{a}\bar{\phi}' + \bar{b}\bar{\phi})' = \bar{c}\bar{\phi}' + \bar{d}\bar{\phi}, \\ \bar{\phi}(0) = \bar{\phi}(1) = 0, \end{cases} \quad (4.13)$$

where

$$\begin{aligned} \bar{a} &= \int_0^1 \frac{\tilde{\rho}(|\tilde{\psi}'|^2, \tilde{\psi}) - 2\tilde{\rho}_1(|\tilde{\psi}'|^2, \tilde{\psi})|\tilde{\psi}'|^2}{\tilde{\rho}^2(|\tilde{\psi}'|^2, \tilde{\psi})} ds, & \bar{b} &= \int_0^1 \frac{-\tilde{\rho}_2(|\tilde{\psi}'|^2, \tilde{\psi})\tilde{\psi}'}{\tilde{\rho}^2(|\tilde{\psi}'|^2, \tilde{\psi})} ds, \\ \bar{c} &= \int_0^1 \tilde{\vartheta}(|\tilde{\psi}'|^2, \tilde{\psi}')\tilde{\psi}' ds, & \bar{d} &= \int_0^1 \tilde{\Theta}(|\tilde{\psi}'|^2, \tilde{\psi}') ds, \end{aligned}$$

with $\tilde{\psi} = s\bar{\psi}_1 + (1-s)\bar{\psi}_2$, where $\tilde{\vartheta}$ and $\tilde{\Theta}$ are defined in (4.6). Multiplying $\bar{\phi}$ on both sides of equation (4.13) and integrating it over $[0, 1]$, we have

$$-\int_0^1 (\bar{a}\bar{\phi}'^2 + \bar{b}\bar{\phi}'\bar{\phi}) dx_2 = \int_0^1 (\bar{c}\bar{\phi}'\bar{\phi} + \bar{d}\bar{\phi}^2) dx_2.$$

Thus, we have

$$\begin{aligned} & \int_0^1 \int_0^1 \frac{\bar{\phi}'^2}{\tilde{\rho}^2(|\tilde{\psi}'|^2, \tilde{\psi})} ds dx_2 \\ &= - \int_0^1 \int_0^1 (\tilde{B}''(\tilde{\psi})\tilde{\rho}(|\tilde{\psi}'|^2, \tilde{\psi}) - \frac{1}{\gamma}\tilde{S}''(\tilde{\psi})\tilde{\rho}^\gamma(|\tilde{\psi}'|^2, \tilde{\psi}))\bar{\phi}^2 ds dx_2 \\ & \quad + \int_0^1 \int_0^1 \frac{\tilde{\rho}^2|\tilde{\psi}'|^2\bar{\phi}'^2}{2\tilde{B}\tilde{\rho} - (\gamma+1)\tilde{S}\tilde{\rho}^\gamma} ds dx_2 + \int_0^1 \int_0^1 \frac{(\tilde{B}'\tilde{\rho} - \tilde{S}'\tilde{\rho}^\gamma)^2\bar{\phi}^2}{2\tilde{B}\tilde{\rho} - (\gamma+1)\tilde{S}\tilde{\rho}^\gamma} ds dx_2 \\ & \quad - \int_0^1 \int_0^1 \frac{2(\tilde{B}'\tilde{\rho} - \tilde{S}'\tilde{\rho}^\gamma)\tilde{\psi}'\bar{\phi}'\bar{\phi}}{2\tilde{B}\tilde{\rho}^2 - (\gamma+1)\tilde{S}\tilde{\rho}^{\gamma+1}} ds dx_2. \end{aligned}$$

The sum of the last three terms is negative. By the smallness of δ and the Poincaré inequality as in Step 2, we have

$$\int_0^1 |\bar{\phi}'|^2 dx_2 \leq 0,$$

which yields $\bar{\phi} = 0$. Thus, the solution to (4.12) is unique. Obviously, we know that

$$\bar{\psi} = \bar{\psi}(x_2) = \int_0^{x_2} \rho_0(s; p_0) u_0(s) ds.$$

is a solution to the boundary value problem (4.12). In fact, from (2.19),

$$\frac{u_0^2(x_2)}{2} + \mathcal{S}(\bar{\psi}(x_2)) \rho_0^{\gamma-1}(x_2; p_0) = \mathcal{B}(\bar{\psi}(x_2)), \quad \rho_0(x_2; p_0) = \left(\frac{\gamma p_0}{(\gamma-1) \mathcal{S}(\bar{\psi}(x_2))} \right)^{\frac{1}{\gamma}},$$

we have

$$\left(\frac{\bar{\psi}_{x_2}(x_2)}{\rho_0(x_2; p_0)} \right)_{x_2} = u_0'(x_2) = \mathcal{B}'(\bar{\psi}) \rho_0(x_2; p_0) - \frac{1}{\gamma} \mathcal{S}'(\bar{\psi}) \rho_0^\gamma(x_2; p_0).$$

This completes the proof. \square

It follows from Lemma 4.1 that the flow induced by the stream function ψ satisfies (1.20)–(1.21) in the upstream. The similar properties in the downstream can be obtained in the same way.

As indicated at the beginning of this section, an important maximum estimate for the stream function can be yielded as a consequence of the far field behaviors, and one could see the reason for the way defining $\tilde{\mathcal{B}}$ in (3.1).

Proposition 4.2. *Suppose that (1.17) holds with $\delta \leq \min\{\delta_1, \delta_2\}$, and $(S(x_2), B(x_2))$ satisfy (1.18). Then the solution to Problem 3 satisfies*

$$0 \leq \psi \leq m \quad \text{in } \Omega. \quad (4.14)$$

Proof. It follows from Proposition 4.1 that

$$\psi(x_1, x_2) \rightarrow \int_0^{x_2} \rho_0(s; p_0) u_0(s) ds \quad \text{uniformly as } x_1 \rightarrow -\infty,$$

and

$$\psi(x_1, x_2) \rightarrow \int_0^{x_2} \rho_1(s; p_1) u_1(s) ds \quad \text{uniformly as } x_1 \rightarrow \infty.$$

Therefore, for any $\epsilon > 0$, there exists $L > 0$ such that

$$-\epsilon \leq \psi(x_1, x_2) < m + \epsilon \quad \text{if } |x_1| \geq L. \quad (4.15)$$

We claim that

$$-\epsilon \leq \psi(x_1, x_2) < m + \epsilon \quad \text{if } |x_1| \leq L. \quad (4.16)$$

We show our claim by contradiction. More precisely, suppose that there exists a point $X_{\max} = (x_{10}, x_{20})$ with $|x_{10}| \leq L$ such that

$$\psi(X_{\max}) = \max_{X \in \{|x_1| \leq L\}} \psi(x_1, x_2) \geq m + \epsilon.$$

Let $\hat{\rho} = \tilde{\rho}(\psi(X_{\max}))$. We have

$$\begin{aligned}\tilde{A}_{ij}(0, \psi(X_{\max}))\partial_{ij}\psi(X_{\max}) &= \hat{\rho}(\mathcal{B}' - \frac{1}{\gamma}\mathcal{S}'\hat{\rho}^{\gamma-1}) \\ &= \hat{\rho}(\mathcal{B}' - \frac{1}{\gamma}\frac{\mathcal{B}}{\mathcal{S}}\mathcal{S}')(\psi(X_{\max})) \\ &= \frac{\hat{\rho}}{\mathcal{B}(\psi(X_{\max}))}(\ln(\mathcal{S}^{-\gamma}\mathcal{B}))'(\psi(X_{\max})) \geq 0,\end{aligned}$$

where we have used the fact that $\nabla\psi(X_{\max}) = 0$, hence $\hat{\rho}^{\gamma-1} = \frac{\mathcal{B}}{\mathcal{S}}(\psi(X_{\max}))$. Thus, we have

$$0 \leq \tilde{A}_{ij}(0, \psi(X_{\max}))\partial_{ij}\psi(X_{\max}) < 0,$$

which is a contradiction. That is,

$$-\epsilon \leq \psi(x_1, x_2) < m + \epsilon \quad \text{in } \{\psi \geq m\} \cap \{|x_1| \leq L\}.$$

Since $\frac{d}{d\psi} \ln(\mathcal{S}^{-\gamma}\mathcal{B}) \leq 0$ in the domain $\{\psi \leq 0\}$, we can similarly show that

$$-\epsilon \leq \psi(x_1, x_2) < m + \epsilon \quad \text{in } \{\psi \leq 0\} \cap \{|x_1| \leq L\}.$$

Combining these estimates together, we obtain

$$-\epsilon \leq \psi(x_1, x_2) < m + \epsilon \quad \text{in } \Omega.$$

Since ϵ is arbitrary, we have

$$0 \leq \psi(x_1, x_2) \leq m \quad \text{in } \Omega.$$

This completes the proof. \square

5. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF **Problem 2**

Propositions 3.1 and 4.2 imply that the solutions established in Proposition 3.1 are the solutions of the boundary value problem (2.33), *Problem 2*.

Proposition 5.1. *Let the boundary $\partial\Omega$ satisfy (1.6)–(1.9). Let (1.17) hold with $\delta \leq \min\{\delta_1, \delta_2\}$, and $(S(x_2), B(x_2))$ satisfy (1.18). Then there exists $0 < \delta_1 \leq \min\{\bar{\delta}_0, \bar{\delta}_1\}$ such that, if $\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0,1])} \leq \delta$ with $0 < \delta \leq \delta_1$ and $m \in (\delta^\beta, m_1)$ with $m_1 = 2\delta^{\beta/2} \leq \bar{m}$, where \bar{m} is defined in Proposition 2.2, then Problem 2 (i.e. (2.33)) has a uniformly subsonic solution $\psi \in C^{2,\alpha}(\bar{\Omega})$ satisfying*

$$0 \leq \psi \leq m \quad \text{in } \Omega.$$

Next, we will use the energy estimates again to show that uniformly subsonic solutions of *Problem 2* are unique.

Proposition 5.2. *Let the boundary $\partial\Omega$ satisfy (1.6)–(1.9). Then there exists $\delta_3 \in (0, \bar{\delta}_0]$ such that, if*

$$(i) \quad \|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0,1])} \leq \delta \text{ with } 0 < \delta \leq \delta_3,$$

$$(ii) \quad m \in (\delta^\beta, \bar{m}),$$

then there exists at most one solution ψ of Problem 2 satisfying

$$0 \leq \psi(x_1, x_2) \leq m, \quad |\nabla\psi|^2 \leq \underline{\Sigma}(\epsilon) - 2\epsilon \text{ for some } \epsilon > 0. \quad (5.1)$$

Proof. As before, let ψ_1 and ψ_2 be two solutions to (2.33). Set $\hat{\psi} = \psi_1 - \psi_2$. Then $\hat{\psi}$ satisfies

$$\begin{cases} \partial_i(a_{ij}\partial_j\hat{\psi}) + \partial_i(b_i\hat{\psi}) = c_i\partial_i\hat{\psi} + d\hat{\psi} & \text{in } \Omega, \\ \hat{\psi} = 0 & \text{on } W_1 \cup W_2, \end{cases} \quad (5.2)$$

where

$$\begin{aligned} a_{ij} &= \int_0^1 \frac{A_{ij}(D\tilde{\psi}, \tilde{\psi})}{\rho^2(|\nabla\tilde{\psi}|^2, \tilde{\psi})} ds, & b_i &= - \int_0^1 \frac{\rho_2(|\nabla\tilde{\psi}|^2, \tilde{\psi})\partial_i\tilde{\psi}}{\rho^2(|\nabla\tilde{\psi}|^2, \tilde{\psi})} ds, \\ c_i &= \int_0^1 \vartheta(|\nabla\tilde{\psi}|^2, \tilde{\psi})\partial_i\tilde{\psi} ds, & d &= \int_0^1 \Theta(|\nabla\tilde{\psi}|^2, \tilde{\psi}) ds, \end{aligned}$$

$\tilde{\psi} = s\psi_1 + (1-s)\psi_2$, A_{ij} , Θ and ϑ are defined as (4.6), except we replace $(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}, \tilde{\rho})$ by $(\mathcal{S}, \mathcal{B}, \rho)$.

Multiplying $\eta^2\hat{\psi}^+$ and integrating on both sides of (5.2), where η is defined in (4.7) and $\hat{\psi}^+ = \max\{\hat{\psi}(x), 0\}$, then, similar to the proof of Lemma 4.1, we have

$$\int \int_{\Omega \cap \{|x_1| \leq l\} \cap \{\hat{\psi} \geq 0\}} |\nabla\hat{\psi}|^2 dx_1 dx_2 \leq C(\underline{B}, \epsilon) \int \int_{\Omega \cap \{l \leq |x_1| \leq l+1\} \cap \{\hat{\psi} \geq 0\}} |\nabla\hat{\psi}|^2 dx_1 dx_2.$$

Since the solutions ψ_1 and ψ_2 have the same far field behavior, and note that $|\hat{\psi}|$ and $|\nabla\hat{\psi}| \rightarrow 0$ as $|x_1| \rightarrow \infty$, we have

$$\int \int_{\Omega \cap \{\hat{\psi} \geq 0\}} |\nabla\hat{\psi}|^2 dx_1 dx_2 = 0,$$

Similarly, we can show that

$$\int \int_{\Omega \cap \{\hat{\psi} \leq 0\}} |\nabla\hat{\psi}|^2 dx_1 dx_2 = 0,$$

which implies that $\hat{\psi} = 0$. This completes the proof. \square

6. REFINED PROPERTIES OF STREAM FUNCTIONS FOR **Problem 1**

In this section, we derive some refined properties for solutions to the boundary value problem (2.33), *Problem 2*. More precisely, it is shown that ψ_{x_2} is always positive, together with the asymptotic behavior and the estimates obtained in §3–5, yields that (ρ, u, v, p) induced by ψ satisfies the original Euler equations, the boundary conditions, the constraints on the mass flux, the Bernoulli constant, and the entropy equation.

Lemma 6.1. *Let the boundary $\partial\Omega$ satisfies (1.6)–(1.9). Then there exists $\delta_4 \in (0, \bar{\delta}_0]$ such that, if*

- (i) $\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0,1])} \leq \delta$ with $0 < \delta \leq \delta_4$,
- (ii) $m \in (\delta^\beta, \bar{m})$,
- (iii) ψ satisfies (5.1) and solves Problem 2,

then ψ satisfies

$$0 < \psi < m \quad \text{in } \Omega, \quad (6.1)$$

and

$$\psi_{x_2} > 0 \quad \text{in } \bar{\Omega}. \quad (6.2)$$

Proof. We divide the proof into four steps.

1. *Equation and the boundary condition.* From (5.1) and the boundary conditions: $\psi(0) = 0$ and $\psi(1) = m$, we have

$$\psi_{x_2} \geq 0 \quad \text{on } \partial\Omega. \quad (6.3)$$

Let $w = \psi_{x_2}$. From Lemma 4.1, w satisfies

$$\begin{aligned} & \partial_i \left(\frac{A_{ij}(\nabla\psi, \psi)}{\rho^2(|\nabla\psi|^2, \psi)} \partial_j w \right) - \partial_i \left(\frac{\rho_2(|\nabla\psi|^2, \psi) \partial_i \psi}{\rho^2(|\nabla\psi|^2, \psi)} w \right) \\ &= \Theta(|\nabla\psi|^2, \psi) w + \vartheta(|\nabla\psi|^2, \psi) \partial_i \psi \partial_i w \end{aligned} \quad (6.4)$$

in the weak sense, where A_{ij} , Θ , and ϑ are defined in (4.6), except replacing $(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}, \tilde{\rho})$ by $(\mathcal{S}, \mathcal{B}, \rho)$.

2. *Positivity in Ω :* That is,

$$w \geq 0 \quad \text{in } \Omega. \quad (6.5)$$

First, the far field behavior of ψ implies that $\psi_{x_2} \rightarrow \rho_i u_i > 0$ when $(-1)^{i+1} x_1 \rightarrow \infty, i = 0, 1$, which implies that $w(x_1, x_2) > 0$ for $|x_1| > L$ with L sufficiently large. As before, multiplying (6.4) by $w^- = \min\{w, 0\}$, integrating it on both sides and noticing (6.3), we have

$$\begin{aligned} & \int \int_{\{U \leq 0\}} \frac{|\nabla w|}{\rho^2(|\nabla\psi|^2, \psi)} dx_1 dx_2 \\ & \leq - \int \int_{\{w \leq 0\}} \left(\mathcal{B}''(\psi) - \frac{1}{\gamma} \mathcal{S}''(\psi) \rho^{\gamma-1}(|\nabla\psi|^2, \psi) \right) w^2 dx_1 dx_2 \\ & \leq C\delta \int \int_{\{U \leq 0\}} w^2 dx_1 dx_2. \end{aligned}$$

For each x_1 , we define an open set:

$$K_{x_1} := \{x_2 : f(x_1) \leq x_2 \leq f(x_2), w(x_1, x_2) < 0\} = \cup_i I_{x_1}^i,$$

where each $I_{x_1}^i$ is a connected open component of K_{x_1} . Then for every $x_2 \in I_{x_1}^i$,

$$w(x_1, x_2) = \int_{\min I_{x_1}^i}^{x_2} \partial_{x_2} w(x_1, s) ds.$$

Therefore,

$$\begin{aligned}
\int \int_{\{w \leq 0\}} w^2 dx_1 dx_2 &= \int_{-l}^l dx_1 \sum_i \int_{I^i} w^2(x_1, x_2) dx_1 dx_2 \\
&= \int_{-l}^l dx_1 \sum_i \int_{I^i} \left(\int_{\min I_{x_1}^i}^{x_2} \partial_{x_2} w(x_1, s) ds \right)^2 dx_2 \\
&\leq \int_{-l}^l \sum_i \int_{I^i} \int_{\min I^i}^{\max I^i} (\partial_{x_2} w(x_1, s))^2 ds (\max I^i - \min I^i)^2 dx_2 \\
&\leq \max_{x_1 \in \mathcal{R}} |f(x_2) - f(x_1)|^2 \int \int_{\{w \leq 0\}} |\nabla w|^2 dx_1 dx_2.
\end{aligned}$$

Hence,

$$\int \int_{\{w \leq 0\}} \frac{|\nabla w|}{\rho^2(|\nabla \psi|^2, \psi)} dx_1 dx_2 \leq C\delta \int \int_{\{w \leq 0\}} |\nabla w|^2 dx_1 dx_2,$$

which means

$$\int \int_{\{w \leq 0\}} |\nabla w|^2 dx_1 dx_2 \leq 0,$$

Thus, (6.5) must hold.

3. *Strict positivity in Ω :* That is,

$$\psi_{x_2} = w > 0 \quad \text{in } \Omega \quad (6.6)$$

for any weak solutions w to (6.4). Denote $\tilde{w} := e^{-\sigma x_2} w$, which is a nonnegative weak solution to

$$\partial_i \left(\frac{A_{ij}}{\rho^2} e^{\sigma x_2} \partial_j \tilde{w} \right) + \left(\frac{A_{i2}}{\rho^2} \sigma - \frac{\rho_2 \partial_i \psi}{\rho^2} - \vartheta(|\nabla \psi|^2, \psi) \partial_i \psi \right) e^{\sigma x_2} \partial_i \tilde{w} + G e^{\sigma x_2} \tilde{w} = 0,$$

where A_{ij} and ϑ are defined in (4.6), and

$$G = \frac{A_{22}}{\rho^2} \sigma^2 + \left(\partial_i \left(\frac{A_{i2}}{\rho^2} \right) - \frac{\rho_2 \partial_2 \psi}{\rho^2} - \vartheta(|\nabla \psi|^2, \psi) \partial_2 \psi \right) \sigma - \partial_i \left(\frac{\rho_2 \partial_i \psi}{\rho^2} \right) - \Theta(|\nabla \psi|^2, \psi)$$

with Θ also defined in (4.6). Choosing $\sigma > 0$ sufficiently large so that $G > 0$, then

$$\partial_i \left(\frac{A_{ij}}{H^2} e^{\sigma x_2} \partial_j \tilde{w} \right) + \left(\frac{A_{i2}}{\rho^2} \sigma - \frac{\rho_2 \partial_i \psi}{\rho^2} - \vartheta(|\nabla \psi|^2, \psi) \partial_i \psi \right) e^{\sigma x_2} \partial_i \tilde{w} \leq 0.$$

This implies that (6.6) holds, so does inequality (6.1).

4. *Positivity on boundary.* We now show that $\psi_{x_2} > 0$ at $W_1 \cup W_2$ in this step. Without loss of generality, we prove it on W_2 .

First, if $(\mathcal{SB}^{-\gamma})'(m) < 0$, since $\psi = m$ on W_2 , then, for any $(x_1^0, f_2(x_1^0)) \in W_2$, there exists a small disk $\mathcal{N} \subset \Omega$ satisfying $\overline{\mathcal{N}} \cap \overline{\Omega} = (x_1^0, f_2(x_1^0))$ such that $\frac{d \ln(\mathcal{S}^{-\gamma} \mathcal{B})}{d\psi} \geq 0$ in \mathcal{N} , which implies

$$A_{ij}(\nabla \psi, \psi) \partial_{ij} \psi > 0 \quad \text{in } \mathcal{N}.$$

Combining this with the fact that $\psi < m$, by the Hopf lemma, we have

$$\psi_{x_2}(x_1^0, f_2(x_1^0)) > 0.$$

The remaining case is $(\mathcal{SB}^{-\gamma})'(m) = 0$. It is easy to see that ψ satisfies

$$A_{ij}(\nabla\psi, \psi)\partial_{ij}(\psi - m) - \rho_2|\nabla(\psi - m)|^2 + R(\psi - m) = 0$$

with $R = -\frac{\rho^2(B'\rho - \frac{1}{\gamma}S'\rho^{\gamma-1})}{\psi - m}$. By the Hopf lemma again, we have

$$\partial_{x_2}\psi > 0 \quad \text{in } W_2.$$

Similarly, we can show that $\psi_{x_2} > 0$ on W_1 . This completes the proof. \square

7. PROOF OF THEOREM 1.1 EXCEPT THE CRITICAL MASS FLUX

We now prove Theorem 1.1 (Main Theorem of this paper), except the existence part of the critical mass flux which will be shown in Section 8 below.

Let $\delta_0 := \min\{\delta_1, \delta_2, \delta_3, \delta_4\} > 0$. If $\|(S - \underline{S}, B - \underline{B})\|_{C^{1,1}([0,1])} \leq \delta$ with $0 < \delta \leq \delta_0$, for any $m \in (\delta^\beta, 2\delta_0^{\beta/2})$, there exists a solution of *Problem 2*. It follows from Lemmas 4.1 and 6.1 that the flow field induced by ψ satisfies (2.10), and hence Proposition 3.1 guarantees the existence of Euler flows. Furthermore, *Propositions 3.1 and 5.2* imply the uniqueness of Euler flows with asymptotic (1.13) and (1.16), the mass flux condition (1.11), and the asymptotic behavior determined by (1.20)–(1.25).

This completes the proof.

8. EXISTENCE OF THE CRITICAL MASS FLUX

In §5–7, we have shown that, for the given Bernoulli function and the entropy function in the upstream satisfying (1.17)–(1.18), there exists a Euler flow, as long as $m \in (\delta^\beta, 2\delta_0^{\beta/2})$. In this section, we find the critical mass flux, which can be obtained by following the arguments as in [2, 3, 16]. For self-containedness, we give the proof in this section.

Proposition 8.1. *Let the boundary $\partial\Omega$ satisfy (1.6)–(1.9), $S(x_2)$ and $B(x_2)$ satisfy the asymptotic condition (1.13) and (1.16) for $x_2 \in [0, 1]$ respectively, and let (1.18) hold. Then there exists $\hat{m} \leq \bar{m}$ such that, if $m \in (\delta^\beta, \hat{m})$, there exists a unique ψ of Problem 2 satisfying*

$$0 < \psi < m \quad \text{in } \Omega, \tag{8.1}$$

$$M(m) := \sup_{\Omega} \{|\nabla\psi|^2 - (\gamma - 1)\mathcal{S}(\psi)\rho^{\gamma+1}\} < 0, \tag{8.2}$$

where $\mathcal{B}(\psi) = \frac{u_0^2(\psi)}{2} + \mathcal{S}(\psi)\rho_0^{\gamma-1}(\psi)$. Furthermore, either $M(m) \rightarrow 0$ as $m \rightarrow \hat{m}$ or there does not exist $\sigma > 0$ such that (2.33) has solutions for all $m \in (\hat{m}, \hat{m} + \sigma)$ and

$$\sup_{m \in (\hat{m}, \hat{m} + \sigma)} M(m) < 0. \tag{8.3}$$

Proof. For the given entropy function S and Bernoulli function B in the upstream satisfying (1.13) and (1.16) and any $m \in (\delta^\beta, \bar{m})$, one can define ρ_0 and $u_0(x_2)$ as in §2. Note that ρ_0 and u_0 depend on m by definition; thus in this section we denote them by $\rho_0(m)$ and $u_0(\psi; m)$, respectively.

Let $\{\varepsilon_n\}_{n=1}^\infty$ be a strictly decreasing sequence of positive numbers such that $\varepsilon_1 \leq \varepsilon_0/4$ and $\varepsilon_n \downarrow 0$. We introduce

$$\zeta_n(s) = \begin{cases} s & \text{if } s < -2\varepsilon_n, \\ -\frac{3}{2}\varepsilon_n & \text{if } s \geq -\varepsilon_n. \end{cases}$$

Then ζ_n is an increasing smooth function. We define

$$\begin{aligned} & \tilde{\Delta}_n(|\nabla\psi|^2, \psi; m) \\ & := \zeta_n(|\nabla\psi|^2 - (\gamma - 1)\tilde{\mathcal{S}}(\psi)\tilde{\rho}^{\gamma+1}(|\nabla\psi|, \psi; m)) + (\gamma - 1)\tilde{\mathcal{S}}(\psi)\tilde{\rho}^{\gamma+1}(|\nabla\psi|, \psi; m). \end{aligned}$$

Then there exist two positive constants $\lambda(n)$ and $\Lambda(n)$ such that

$$\lambda(n)|\xi|^2 \leq \tilde{A}_{ij}^n(q, z; m)\xi_i\xi_j \leq \Lambda(n)|\xi|^2$$

for any $z \in \mathbb{R}$, $q \in \mathbb{R}^2$, and $\xi \in \mathbb{R}^2$, where

$$\tilde{A}_{ij}^{(n)}(q, z; m) = \tilde{\rho}^n(|q|^2, z; m)\delta_{ij} - 2\tilde{\rho}_1^n(|q|^2, z; m)\xi_i\xi_j.$$

Thus, for any $m \in (\delta^\beta, \bar{m})$, there exists a solution $\psi^n(x; m)$ to the problem:

$$\begin{cases} \tilde{A}_{ij}^{(n)}(q, z; m)\partial_{ij}\psi = \mathcal{F}_n(\nabla\psi, \psi; m) & \text{in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}m & \text{on } \partial\Omega, \end{cases} \quad (8.4)$$

where

$$\mathcal{F}_n(\nabla\psi, \psi; m) = (\tilde{B}' - \frac{1}{\gamma}\tilde{\mathcal{S}}'(\tilde{\rho}^n)^{\gamma-1})(\tilde{\rho}^n)^2 + \tilde{\rho}_2^{(n)}|\nabla\psi|^2.$$

Moreover, if

$$|\nabla\psi^{(n)}|^2 - (\gamma - 1)\tilde{\mathcal{S}}(\psi)\tilde{\rho}^{\gamma+1}(|\nabla\psi|, \psi; m) \leq -2\varepsilon_n, \quad (8.5)$$

then $\zeta'_n = 1$. Similar to §3, we have

$$0 \leq \psi^n(x; m) \leq m.$$

By the definition of $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{B}}$, we can estimate I_5 in (4.9), which is independent of ε_n . Furthermore, it follows from Lemma 4.1 that the solution in (8.4) satisfying (8.5) has the far field behavior as (4.1). In addition, by Proposition 5.2, such a solution is unique in the class of solutions satisfying (4.1).

Note that, in general, we do not know the uniqueness of solutions to the boundary value problem (8.4). Set

$$S_n(m) = \{\psi^n(x; m) : \psi^n(x; m) \text{ solves problem (8.4)}\}. \quad (8.6)$$

Define

$$M_n(m) = \inf_{\psi^n \in S_n(m)} \sup_{\bar{\Omega}} \{|\nabla\psi^{(n)}|^2 - (\gamma - 1)\tilde{\mathcal{S}}(\psi)\tilde{\rho}_n^{\gamma+1}(|\nabla\psi|, \psi; m)\}, \quad (8.7)$$

and

$$T_n = \{s : \delta^\beta \leq s, M_n(m) \leq -4\varepsilon_n \text{ if } m \in (\delta^\beta, s)\}.$$

It follows from Proposition 3.1, Lemma 4.1, and Proposition 4.2 that

$$[\delta_0^\beta, 2\delta_0^{\beta/2}] \subset T_n,$$

hence $T_n \neq \emptyset$. We define $m_n = \sup T_n$.

The sequence $\{m_n\}$ has the following properties:

1. $M_n(m)$ is left continuous for $m \in (\delta^\beta, m_n]$. Indeed, let $\{m_n^{(k)} \in (\delta^\beta, m_n)\}$ and $m_n^{(k)} \uparrow m$. Since $M_n(m_n^{(k)}) \leq -4\varepsilon_n$, we have

$$\|\psi^{(n)}(x; m_n^{(k)})\|_{C^{2,\alpha}} \leq C(n).$$

Therefore, there exists a subsequence $\psi^{(n)}(x; m_n^{(k_l)})$ such that

$$\psi^{(n)}(x; m_n^{(k_l)}) \rightarrow \psi.$$

Moreover, ψ solves (8.4), and $M_n(m) \leq \lim \psi^{(n)}(x; m_n^{(k_l)})$. Thus,

$$M_n(m) \leq -4\varepsilon_n.$$

Note that all these solutions satisfy the far field behavior as (4.1), by uniqueness of solutions in this class,

$$M_n(m) = \lim \psi^{(n)}(x; m_n^{(k)}).$$

2. $m_n \leq \bar{m}$: If this were not true, by the definition of m_n , $\bar{m} \in T_n$. It follows from the left-continuity of $M_n(m)$ that

$$M_n(\bar{m}) \leq -4\varepsilon_n.$$

Thus, by means of the proof of Lemma 4.1, $\psi^n(x; \bar{m})$ has far field behavior as in (4.1). However, it follows from the definition of \bar{m} that

$$\begin{aligned} & \sup_{x \in \bar{\Omega}} \{|\nabla \psi^{(n)}|^2 - (\gamma - 1)\tilde{\mathcal{S}}(\psi)\tilde{\rho}_n^{\gamma+1}(|\nabla \psi^{(n)}|, \psi^{(n)}; m)\} \\ & \geq \sup_{s \in [0,1]} \max \left\{ \begin{aligned} & |\rho_0(s; \bar{m})u_0(s; \bar{m})|^2 - (\gamma - 1)S(s)\rho_0^{\gamma+1}(s; \bar{m}), \\ & |\rho_1(s; \bar{m})u_1(s; \bar{m})|^2 - (\gamma - 1)S(s)\rho_1^{\gamma+1}(s; \bar{m}) \end{aligned} \right\} \\ & \geq \sup_{s \in [0,1]} \max \left\{ \begin{aligned} & 2B(s)\rho_0^2(s; \bar{m}) - (\gamma + 1)S(s)\rho_0^{\gamma+1}(s; \bar{m}), \\ & 2B(s)\rho_1^2(s; \bar{m}) - (\gamma + 1)S(s)\rho_1^{\gamma+1}(s; \bar{m}) \end{aligned} \right\} \\ & \geq \sup_{s \in [0,1]} \varrho^2(\bar{D}; s)S^{\frac{1}{\gamma}}(s) \left(2D(x_2) - (\gamma + 1) \left(\frac{\gamma \mathfrak{p}(\bar{D})}{\gamma - 1} \right)^{\frac{\gamma-1}{\gamma}} \right) \\ & \geq \sup_{s \in [0,1]} 2\varrho^2(\bar{D}; s)S^{\frac{1}{\gamma}}(s)(D(x_2) - \bar{D}) \\ & = 0, \end{aligned}$$

where $\rho_1(s; \bar{m}) = \rho_1(y(s); \bar{m})$, $u_1(s; \bar{m}) = u_1(y(s); \bar{m})$, and $y(s)$ is the function defined in (2.24). Thus, $M_n(\bar{m}) \geq 0$. This is a contradiction. Therefore, $m_n \leq \bar{m}$.

Finally, $\{m_n\}$ is an increasing sequence, which follows from the definition of $\{m_n\}$ directly. Define

$$\hat{m} = \lim_{n \rightarrow \infty} m_n.$$

Then \hat{m} is well-defined and $\hat{m} \leq \bar{m}$. Note that, for any $m \in (\delta^\beta, \hat{m})$, there exists $m_n > m$ such that $M_n(m) \leq -4\varepsilon_n$. Thus,

$$\phi = \psi^{(n)}(x; m)$$

solves (2.33) and

$$\sup_{\bar{\Omega}} \{|\nabla \psi|^2 - (\gamma - 1)\mathcal{S}(\psi)\tilde{\rho}_n^{\gamma+1}(|\nabla \psi^{(n)}|, \psi^{(n)}; m)\} = M_n(m) \leq -4\varepsilon_n.$$

If $\sup_{m \in (\delta^\beta, \hat{m})} M_n(m) < 0$, then there exists n such that

$$\sup_{m \in (\delta^\beta, \hat{m})} M_n(m) < -4\varepsilon_n.$$

Then the same argument as the proof for the left continuity of $M_n(m)$ on $(\delta^\beta, m_n]$ yields

$$M_n(\hat{m}) < -4\varepsilon_n.$$

Suppose that there exists $\sigma > 0$ such that (2.33) always has a solution ψ for $m \in (\hat{m}, \hat{m} + \sigma)$, and

$$\sup_{m \in (\hat{m}, \hat{m} + \sigma)} M(m) < 0. \quad (8.8)$$

Then there exists $k > 0$ such that

$$\sup_{m \in (\hat{m}, \hat{m} + \sigma)} M(m) = \sup_{m \in (\hat{m}, \hat{m} + \sigma)} \{|\nabla \psi|^2 - (\gamma - 1)\mathcal{S}(\psi)\tilde{\rho}^{\gamma+1}(|\nabla \psi|, \psi; m)\} < -4\varepsilon_{n+k}.$$

This yields that $m_{n+k} \geq \hat{m} + \sigma$, which is a contradiction. Thus, either $M(m) \rightarrow 0$ or there does not exist $\sigma > 0$ such that (2.33) has a solution for all $m \in (\hat{m}, \hat{m} + \sigma)$ and (8.3) holds. This completes the proof. \square

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