LOCAL UNIQUENESS OF STEADY SPHERICAL TRANSONIC SHOCK-FRONTS FOR THE THREE–DIMENSIONAL FULL EULER EQUATIONS

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ABSTRACT. We establish the local uniqueness of steady transonic shock solutions with spherical symmetry for the three-dimensional full Euler equations. These transonic shock-fronts are important for understanding transonic shock phenomena in divergent nozzles. From mathematical point of view, we show the uniqueness of solutions of a free boundary problem for a multidimensional quasilinear system of mixed-composite elliptic–hyperbolic type. To this end, we develop a decomposition of the Euler system which works in a general Riemannian manifold, a method to study a Venttsel problem of nonclassical nonlocal elliptic operators, and an iteration mapping which possesses locally a unique fixed point. The approach reveals an intrinsic structure of the steady Euler system and subtle interactions of its elliptic and hyperbolic part.

1. INTRODUCTION

The study of the Euler equations for compressible fluids is one of the central topics in the mathematical fluid dynamics, and the analysis of solutions to the system is of particular interest in applications. In particular, in the recent years, important progress has been made in the analysis of transonic shock solutions of the steady potential flow equation and the steady Euler system in multidimensions (cf. [4, 5, 6, 7, 8, 9, 15, 21, 22] and the references cited therein).

In this paper, we are concerned with the local uniqueness of transonic shock solutions with spherical symmetry for the three-dimensional, steady, full Euler system of polytropic gases. Such a study not only helps us to understand transonic shock phenomena occurred in divergent nozzles, which have many important applications, but also provides new insights for the theory of free boundary problems of partial differential equations of composite–mixed elliptic–hyperbolic type. This problem can be formulated as a free boundary problem for the Euler system in a spherical shell, with the transonic shock-front as a free boundary, which is a graph of a function defined on $\mathbf{S}^2$ (the unit 2–sphere in $\mathbb{R}^3$). Therefore, for such a problem, although there exists a system of global Descartes coordinates, it is more convenient to use the local spherical coordinates and the terminology of differential geometry (see Appendix A).

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Let $r^0 < r^1$ be two positive constants. A spherical shell in $\mathbb{R}^3$ centered at the origin is a Riemannian manifold $\mathcal{M} = [r^0, r^1] \times \mathbf{S^2}$ with a metric $G = G_{ij} dx^i \otimes dx^j$ [†] in local spherical coordinates (see Appendix A). Its boundary $\partial\mathcal{M}$ is $\Sigma^0 \cup \Sigma^1$ with

$$\Sigma^i := \{(x^0, x) \in \mathcal{M} : x^0 = r^i, x = (x^1, x^2) \in \mathbf{S}^2\}, \qquad i = 0, 1,$$

denoting respectively the entry and exit of $\mathcal{M}$.

Let p, ρ, and S in $\mathcal{M}$ represent the pressure, density, and entropy of gas flow in the manifold respectively. For polytropic gases, $p = A(S)\rho^\gamma$ with the adiabatic exponent $\gamma > 1$, and the sonic speed is

$$c = \sqrt{\gamma p/\rho}.$$

Let u be the velocity of fluid flow, which is a vector field in $\mathcal{M}$ whose integral curves are called fluid trajectories in $\mathcal{M}$. Then the steady, full Euler equations for compressible fluids in $\mathcal{M}$ are (cf. §16.5 in [18]):

$$\varphi := \operatorname{div}(\rho u \otimes u) + \operatorname{grad} p = 0, \tag{1.1}$$

$$\varphi_1 := \operatorname{div}(\rho u) = 0, \tag{1.2}$$

$$\varphi_2 := \operatorname{div}(\rho E u) = 0, \tag{1.3}$$

where div and grad are respectively the divergence and gradient operator in $\mathcal{M}$, and

$$E := \frac{1}{2} G(u, u) + \frac{c^2}{\gamma - 1}.$$

We use $U = (p, \rho, u)$ to denote the state of fluid flow. If U depends only on x^0 and $u = u^0(x^0)\partial_0$, then (1.1)–(1.3) can be reduced to the following differential equations:

$$\frac{\mathrm{d}u^0}{\mathrm{d}x^0} = \frac{2c^2 u^0}{x^0((u^0)^2 - c^2)}, \tag{1.4}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}x^0} = -\frac{2\rho (u^0)^2}{x^0((u^0)^2 - c^2)}, \tag{1.5}$$

$$\frac{\mathrm{d}p}{\mathrm{d}x^0} = -\frac{2\rho c^2 (u^0)^2}{x^0((u^0)^2 - c^2)}. \tag{1.6}$$

It has been shown in Yuan [22] that, for equations (1.4)–(1.6), given supersonic data $U_b^-(r^0) = (p_b^-(r^0), \rho_b^-(r^0), u_b^-(r^0))$ on the entry Σ^0, there exists an interval I such that, if the back pressure $p_b^+(r^1) \in I$, then there exists a unique $r_b \in (r^0, r^1)$ so that

$$S_b = \{(x^0, x^1, x^2) \in \mathcal{M} \,:\, x^0 = r_b, (x^1, x^2) \in \mathbf{S}^2\}$$

is a transonic shock-front, determined by the Rankine-Hugoniot jump conditions: The flow $U_b^-(x^0)$, $x^0 \in (r^0, r_b)$, ahead of S_b is supersonic; $U_b^+(x^0), x^0 \in (r_b, r^1)$, behind of S_b is subsonic; and the physical entropy condition $p_b^+(r_b) > p_b^-(r_b)$ holds on S_b. We call such a spherical transonic shock solution $U_b := (U_b^-, U_b^+; S_b)$ to the Euler equations as a *background solution*. The objective of this paper is to study the uniqueness of transonic shock solutions, if it exists, in a neighborhood of a class of background solutions under the three-dimensional perturbations of the upcoming

[†]In this paper we always use the Einstein summation convention for the Roman indices from 0 to 2 and for the Greek indices from 1 to 2.

supersonic flow U_b^- at the entry Σ^0. This class of background solutions especially include those background solutions satisfying the *S-Condition* defined in §2. In particular, we prove that, for a given back pressure, for sufficiently large upstream pressure and Mach number, the background transonic shock-front itself is locally unique.

The main theorem of this paper is the following:

Theorem 1.1 (Main Theorem)**.** *Let U_b and γ satisfy the S-condition (see Definition* 2.3 *in* §2*). Then there exist ε_0 and C_0 depending only on U_b and γ such that, if the upcoming supersonic flow U^- on Σ^0 satisfies*

$$\left\|U^- - U_b^-\right\|_{C^{3,\alpha}(\Sigma^0)} < \varepsilon \leq \varepsilon_0 \tag{1.7}$$

for some $\alpha \in (0,1)$ and there exists a transonic shock solution $U = (U^-, U^+; S^\psi)$ of (1.1)–(1.3) *in $\mathcal{M}$ satisfying Property* (A) *below, then this solution is unique. Here Property* (A) *consists of the three conditions:*

(i) *$S^\psi = \{(x^0, x^1, x^2) \in \mathcal{M} : x^0 = \psi(x^1, x^2), (x^1, x^2) \in \mathbf{S}^2\}$ is the shock-front; U^- is the supersonic flow ahead of S^ψ; and U^+ is the subsonic flow behind of S^ψ. The physical entropy condition: $p^+|_{S^\psi} > p^-|_{S^\psi}$ holds on S^ψ. Moreover, $\psi \in C^{4,\alpha}(\mathbf{S}^2)$ satisfies*

$$\|\psi - r_b\|_{C^{4,\alpha}(\mathbf{S}^2)} \leq C_0\varepsilon; \tag{1.8}$$

(ii) *The back pressure is*

$$p(r^1, x^1, x^2) = p_b^+(r^1); \tag{1.9}$$

(iii) *For $\mathcal{M}_\psi^\pm = \{(x^0, x^1, x^2) \in \mathcal{M} : \pm(x^0 - \psi(x^1, x^2)) \geq 0, (x^1, x^2) \in \mathbf{S}^2\}$,*

$$\left\|U^\pm - U_b^\pm\right\|_{C^{3,\alpha}(\mathcal{M}_\psi^\pm)} \leq C_0\varepsilon, \tag{1.10}$$

with

$$\|U\|_{C^{k,\alpha}(\mathcal{M})} := \|(p,\rho)\|_{C^{k,\alpha}(\mathcal{M})} + \|\bar{u}\|_{C_1^{k,\alpha}(\mathcal{M})},$$

where $\bar{u}$ is the 1-form corresponding to the vector field u via the Riemannian metric of $\mathcal{M}$, and $C_r^{k,\alpha}$ denotes the space of r-forms in $\mathcal{M}$ (i.e., $A^r(\mathcal{M})$) with $C^{k,\alpha}$ components in local coordinates and the norms are defined in the usual way by partition of unity in the manifold.

As explained in [10, 22], the background solutions coincide with the solutions of the steady quasi-one-dimensional model of flows in divergent nozzles. Therefore, the above uniqueness result will help to understand transonic shock phenomena in divergent nozzles, as well as the effectiveness of the quasi-one-dimensional model [19]. Also see [20] for an explanation of the quasi-one-dimensional model from the viewpoint of flows in Riemannian manifolds and [15] for the stability result of transonic shock-fronts for the two-dimensional case.

Apart from the physical implications, the approach by considering the Euler equations in a Riemannian manifold is of interest itself in mathematics. We note that some studies have been made for conservation laws in general Riemannian manifolds (cf. [1, 2, 17] and the references cited therein). This approach via differential geometry reveals some intrinsic structures of the steady Euler system, which are valid in general Riemannian manifolds.

Finally, we remark that the existence and uniqueness of supersonic flow U^- in $\mathcal{M}$ subject to the initial data $U^-|_{\Sigma^0}$ satisfying (1.7) follow directly from the theory of semi-global classical solutions of the Cauchy problem of quasilinear symmetric hyperbolic systems if ε_0 is sufficiently small (cf. [14, 18]). Furthermore, one can obtain

$$\left\|U^- - U_b^-\right\|_{C^{3,\alpha}(\mathcal{M})} \leq C_1\varepsilon, \tag{1.11}$$

where $C_1 > 0$ and $\varepsilon_0 > 0$ depend solely on $U_b^-(r^0)$ and r^1. Thus, we focus on showing the uniqueness of ψ and U^+ below. Indeed, what we obtain here is much more than this: We design an iteration mapping and show that it always has a unique fixed point, and any solution to (1.1)–(1.3) must be a fixed point of this iteration mapping. Therefore, the solution to (1.1)–(1.3) is unique, and then Theorem 1.1 is proved. However, we have not known whether the fixed point of the iteration mapping is a solution to the original problem (1.1)–(1.3), therefore, the existence problem for solutions to (1.1)–(1.3) is still open, though we believe that the ideas and approaches developed here will be useful to establish an existence theorem which is out of the scope of this paper.

For simplicity, we write U^+ as U from now on. We emphasize here that the supersonic flow U^- is defined in the whole $\mathcal{M}$, while, by Proposition 9 in [22], U_b^+ can be extended to $[r_b - h_b, r^1] \times \mathbf{S}^2$ and still obeys the Euler system, with $h_b > 0$ depending only on $U_b^-(r^0)$.

The rest of this paper is organized as follows. In §2, we derive some elliptic or transport equations, as well as certain equations of exterior differential forms in $\mathcal{M}$ and on the shock-front, from the Euler equations (1.1)–(1.3) and the Rankine–Hugoniot jump conditions. Most of the formulas obtained here are also valid in general Riemannian manifolds. Based on these decompositions, in §3, we present an iteration mapping and establish the existence of a unique fixed point, which yields Theorem 1.1. Some facts and notations of differential geometry are shown and described in Appendix A.

We remark in passing that the analysis and results developed here should be straightforward extended to the higher dimensional case, even general Riemannian manifolds for most of them.

2. Reduction of the Euler system and Rankine-Hugoniot Jump Conditions

In this section we introduce a reduction of the Euler system and analyze the Rankine-Hugoniot jump conditions.

2.1. The Euler Equations in $\mathcal{M}$.

We use d to denote the exterior differential operator and $D_u\omega$ to denote the covariant derivative of a tensor field ω with respect to a vector field u in $\mathcal{M}$; while $\nabla_u\omega$ is the covariant derivative on $\mathbf{S}^2$. $\mathcal{L}_u\omega$ is the Lie derivative of ω with respect to u in $\mathcal{M}$. The symbol Δ represents the Laplacian of forms in $\mathcal{M}$, and Δ' is the Laplacian of forms on $\mathbf{S}^2$, which are both positive operators (cf. (A.10)). Note that $\psi \in C^{4,\alpha}(\mathbf{S}^2)$ defines a mapping from $\mathbf{S}^2$ to $\mathcal{M}$ by $(x^1, x^2) \mapsto (\psi(x^1, x^2), x^1, x^2)$. We use ψ^* to denote the pull back of forms and functions induced by this mapping; for example, for a function $p \in A^0(\mathcal{M})$, $\psi^* p = p|_{S^\psi}$. The volume 2-form of $\mathbf{S}^2$ is written as vol, and vol^3 is the volume 3-form of $\mathcal{M}$.

In the following, we derive some well-known equations from the Euler system (1.1)–(1.3) which are valid only for C^1 flows (cf. [18]). Since these involve differentiations of (1.1)–(1.3), the solution of the reduced equations might not be a solution to the Euler system (1.1)–(1.3). One point below is to express the relations between these derived equations and the original Euler equations, which may be useful in the future to verify that the solution of these derived equations satisfies the Euler system indeed.

The conservation of mass (1.2) can be written (equivalently for C^1 flows) as

$$\varphi_1 = D_u\rho + \rho \operatorname{div} u = 0. \tag{2.1}$$

By the identity

$$\operatorname{div}(u\otimes v) = (\operatorname{div} v)u + D_v u$$

for two vector fields u and v, the momentum equations (1.1) become

$$\varphi_0 := \frac{1}{\rho}(\varphi - \varphi_1 u) = D_u u + \frac{1}{\rho}\operatorname{grad} p = 0. \tag{2.2}$$

Similarly, the conservation law of energy (1.3) may be written as

$$\varphi_3 := \frac{1}{\rho}(\varphi_2 - E\varphi_1) = D_u E = 0. \tag{2.3}$$

This is exactly the well-known Bernoulli law.

Since

$$\frac{1}{2}D_u\big(|u|^2\big) = G(D_u u, u) = G(\varphi_0 - \frac{1}{\rho}\operatorname{grad} p,\, u) = G(\varphi_0, u) - \frac{1}{\rho}D_u p,$$

we have

$$\begin{aligned}\varphi_4 \quad &:= \quad \frac{1}{\rho^{\gamma-1}}\big(\varphi_3 - G(\varphi_0,u)\big) = \frac{1}{\rho^{\gamma-1}}\Big(\frac{\gamma}{\gamma-1}D_u\big(A(S)\rho^{\gamma-1}\big) - \frac{1}{\rho}D_u\big(A(S)\rho^{\gamma}\big)\Big)\\ &= \quad D_u A(S) = 0,\end{aligned} \tag{2.4}$$

i.e., the invariance of entropy along the flow trajectories for C^1 flows.

For a vector field $u = u^i\partial_i$ in $\mathcal{M}$, we always use $\bar{u} = u^j G_{ij} dx^i$ to denote its corresponding 1-form with respect to the metric G of $\mathcal{M}$. Then (2.2) is equivalent to

$$\bar{\varphi}_0 = D_u\bar{u} + \frac{dp}{\rho} = \mathcal{L}_u\bar{u} - d(\frac{|u|^2}{2}) + \frac{dp}{\rho} = 0, \tag{2.5}$$

since $\mathcal{L}_u\bar{u} = D_u\bar{u} + d(\frac{|u|^2}{2})$. This implies

$$\mathcal{L}_u d\bar{u} = -d\big(\frac{1}{\rho}\big)\wedge dp + d\bar{\varphi}_0, \tag{2.6}$$

which is a transport equation of vorticity. Moreover, $d\bar{u}|_{S^\psi}$, the initial value of vorticity on S^ψ, expressed in local spherical coordinates, is

$$\begin{aligned} d\bar{u}|_{S^\psi} \quad &= \quad d\big(\psi^*(u^i G_{ij})\big)\wedge dx^j - \big(\psi^*\partial_0(u^i G_{ij})\big)d\psi\wedge dx^j + \big(\psi g_{\alpha\beta}\psi^* u^\beta - \psi^*(\frac{\partial_\alpha p}{\rho u^0})\big)dx^0\wedge dx^\alpha\\ &\quad + \psi^2 g_{\alpha\beta}\psi^*(\frac{dx^\alpha(\varphi_0)}{u^0})dx^0\wedge dx^\beta, \end{aligned} \tag{2.7}$$

where $g = g_{\alpha\beta}dx^\alpha\otimes dx^\beta$ is the standard metric of $\mathbf{S}^2$.

Let d^* be the codifferential operator in $\mathcal{M}$. Using the identity $d^*\bar{u} = -\operatorname{div} u$, we obtain

$$\varphi_1 = -d^*(\rho\bar{u}) = 0. \tag{2.8}$$

Let $\langle\cdot,\,\cdot\rangle$ be the inner product in $\mathcal{M}$ of forms and let $*$ be the Hodge star operator, which imply

$$*\mathrm{vol}^3 = 1, \qquad *1 = \mathrm{vol}^3, \qquad d(*\bar{u}) = (\operatorname{div} u)\,\mathrm{vol}^3, \qquad \alpha\wedge *\beta = \langle\alpha,\beta\rangle\,\mathrm{vol}^3.$$

Then (2.8) is

$$\varphi_1 = -\rho d^* \bar{u} + \langle d\rho, \bar{u}\rangle. \tag{2.9}$$

Note that, by the equation of state $p = A(S)\rho^\gamma$, we have

$$\langle d\rho, \bar{u}\rangle = D_u\rho = \frac{D_u p}{c^2} - \frac{\partial p}{\partial A}\,\frac{D_u A(S)}{c^2}.$$

Thus, by setting

$$\bar{\varphi}_1 := \frac{\varphi_1}{\rho} + \frac{1}{\rho c^2}\big(\varphi_2 - E\varphi_1 - \rho G(\varphi_0, u)\big), \tag{2.10}$$

the equation of conservation of mass may be written as

$$\bar{\varphi}_1 = -d^*\bar{u} + \frac{D_u p}{\gamma p} = \operatorname{div} u + \frac{D_u p}{\gamma p} = 0. \tag{2.11}$$

By this equation and the definition of the Laplacian of forms $\Delta = dd^* + d^*d$, we also have

$$\Delta\bar{u} = d^*(d\bar{u}) + d(\frac{D_u p}{\gamma p}) - d\bar{\varphi}_1. \tag{2.12}$$

2.2. **The Rankine–Hugoniot Jump Conditions on a Shock-Front.** A shock-front S^ψ is a hyper-surface in $\mathcal{M}$ across which the physical variables have a jump. In our case, it can be expressed as the graph of a mapping $\psi : \mathbf{S}^2 \to \mathcal{M}$. In local spherical coordinates, we may write

$$S^\psi = \{(x^0, x^1, x^2) \in \mathcal{M} \,:\, x^0 = \psi(x^1, x^2),\ (x^1, x^2) \in \mathbf{S}^2\}. \tag{2.13}$$

The normal vector field and the corresponding normal 1-form of S^ψ with respect to $\mathcal{M}$ are

$$n = (\partial_\alpha\psi G^{\alpha\beta}\partial_\beta - G^{00}\partial_0)|_{S^\psi} \qquad \text{and} \qquad \bar{n} = (d\psi - dx^0)|_{S^\psi}.$$

From (1.1)–(1.3), the Rankine–Hugoniot jump conditions (i.e., the R–H conditions) on S^ψ are

$$\lceil G(u,n)\rho u + pn\rfloor|_{S^\psi} = 0 \quad\Longleftrightarrow\quad \lceil \bar{n}(u)\rho\bar{u} + p\bar{n}\rfloor|_{S^\psi} = 0, \tag{2.14}$$

$$\lceil G(u,n)\rho\rfloor|_{S^\psi} = 0 \quad\Longleftrightarrow\quad \lceil \bar{n}(u)\rho\rfloor|_{S^\psi} = 0, \tag{2.15}$$

$$\lceil G(u,n)\rho E\rfloor|_{S^\psi} = 0 \quad\Longleftrightarrow\quad \lceil \bar{n}(u)\rho E\rfloor|_{S^\psi} = 0, \tag{2.16}$$

where $\lceil\cdot\rfloor$ denotes the jump of a quantity across S^ψ. It is well-known (see [10, 11]) that a piecewise C^1 state $U = (U^-, U^+; S^\psi)$ is a weak entropy solution of (1.1)–(1.3) if and only if U satisfies these equations in $\mathcal{M}^\pm_\psi$ in the classical sense, the R–H conditions along S^ψ, as well as the physical entropy condition $p^+|_{S^\psi} > p^-|_{S^\psi}$ on S^ψ.

Set

$$u = u_0 + u_1 := u^0\,\partial_0 + u^\alpha\,\partial_\alpha, \tag{2.17}$$

$$\bar{u} = \bar{u}^0 + \bar{u}^1 := u^0 G_{00}\,dx^0 + u^\alpha G_{\alpha\beta}\,dx^\beta. \tag{2.18}$$

Then $\bar{n}(u) = d\psi(u_1) - u^0$, and (2.14) can be separated into

$$\psi^*\lceil pd\psi - g\bar{u}^1\rfloor = 0, \tag{2.19}$$

$$\psi^*\lceil gu^0 + p\rfloor = 0, \tag{2.20}$$

where $g = g(U, \psi, D\psi) := (\rho u^0 - d\psi(\rho u_1))|_{S^\psi}$ is a function on $\mathbf{S}^2$, with $d\psi \in A^1(\mathbf{S}^2)$ being considered as a 1-form in $\mathcal{M}$; and the expression $g(U, \psi, D\psi)$ means that g depends on U, ψ, and the first-order derivatives of ψ. Then (2.15)–(2.16) become

$$\psi^*\lceil g \rfloor = 0, \qquad \psi^*\lceil E \rfloor = 0. \tag{2.21}$$

Equations (2.3) and (2.21) indicate that E is a constant along the same trajectory even across the shock-front. Therefore, we may write $E|_{S^\psi} = E_0(x), x \in \mathbf{S}^2$, with $E_0(x)$ a given function depending only on the supersonic data at the entry.

Now, if $\psi^*\lceil p \rfloor \neq 0$ (which is guaranteed later by the physical entropy condition of the background solution), (2.19) yields

$$d\psi = \omega := \frac{g\psi^*(\lceil \bar{u}^1 \rfloor)}{\psi^*\lceil p \rfloor} = \mu_0 \psi^*(\bar{u}^1) + g_0(U, U^-, \psi, D\psi) \in A^1(\mathbf{S}^2) \tag{2.22}$$

by using the first equation in (2.21), with

$$\mu_0 := \left.\frac{(\rho u^0)_b^+}{p_b^+ - p_b^-}\right|_{S_b} = \left.\frac{\gamma+1}{2} \frac{\left((u^0)_b^+\right)^2}{\left(c^2 - (u^0)^2\right)_b^+}\right|_{S_b} > 0,$$

and g_0 the higher order term defined below (see Definition 2.2)[1], which contains U, U^-, ψ, and $D\psi$ (the first-order derivatives of ψ). We note that equations (2.20)–(2.22) are equivalent to (2.14)–(2.16). Equation (2.22) also indicates $d\omega = 0$. Therefore, we have

$$d(\psi^* \bar{u}^1) = \chi(U, U^-, \psi) := -\frac{dg_0}{\mu_0}. \tag{2.23}$$

Definition 2.1. A constant r^p is called the *position* of the surface S^ψ defined by (2.13) provided that $\int_{\mathbf{S}^2} (\psi - r^p) \,\mathrm{vol} = 0$. The function $\psi^p := \psi - r^p$ is called the *profile* of S^ψ.

Remark 2.1. The reason why we distinguish the "position" and the "profile" is that they are determined by different mechanisms: The "profile" is determined by the R–H conditions, while the "position" is determined by the solvability conditions closely related to the conservation of mass.

Definition 2.2. Let $\hat{U} = U^+ - U_b^+$. A higher order term is an expression that contains either

(i) $U^- - U_b^-$ and its first-order derivatives;

or

(ii) the products of ψ^p, $r^p - r_b, \hat{U}$, and their derivatives $D\hat{U}, D^2\hat{U}, D\psi, D^2\psi$, and $D^3\psi$, where $D^k u$ are the k^{th}-derivatives of u in local coordinates.

Next, we linearize the R–H conditions (2.20)–(2.21). We write them equivalently as

$$G_i(\psi^* U, \psi^* U^-) = \Psi_i(\psi^* U, \psi^* U^-, D\psi), \qquad i = 1, 2, 3,$$

with

$$G_1 = \psi^*\lceil \rho (u^0)^2 + p \rfloor, \qquad \Psi_1 = \psi^*(\lceil d\psi(\rho u^0 u_1) \rfloor), \tag{2.24}$$

$$G_2 = \psi^*\lceil \rho u^0 \rfloor, \qquad \Psi_2 = \psi^*(\lceil d\psi(\rho u_1) \rfloor), \tag{2.25}$$

$$G_3 = \psi^*\lceil E \rfloor, \qquad \Psi_3 = 0. \tag{2.26}$$

[1]From now on, we always use g_i to denote the higher order terms on S^ψ, and f_i to denote the higher order terms in $\mathcal{M}_\psi^+$.

As in [15], since $G_i(U_b^+(r_b,x),U_b^-(r_b,x))=0$ for $x=(x^1,x^2)\in\mathbf{S^2}$, we have

$$\begin{aligned}
&\partial_+G_i(U_b^+(r_b,x),U_b^-(r_b,x))\bullet\big(U(\psi(x),x)-U_b^+(\psi(x),x)\big) \qquad (2.27)\\
&=\Big\{-\big(\partial_+G_i(U_b^+(\psi(x),x),U_b^-(\psi(x),x))-\partial_+G_i(U_b^+(r_b,x),U_b^-(r_b,x))\big)\\
&\qquad\qquad\bullet\big(U(\psi(x),x)-U_b^+(\psi(x),x)\big)\\
&\qquad+\partial_+G_i(U_b^+(\psi(x),x),U_b^-(\psi(x),x))\bullet\big(U(\psi(x),x)-U_b^+(\psi(x),x)\big)\\
&\qquad-\big(G_i(U(\psi(x),x),U^-(\psi(x),x))-G_i(U_b^+(\psi(x),x),U^-(\psi(x),x))\big)\\
&\qquad-\big(G_i(U_b^+(\psi(x),x),U^-(\psi(x),x))-G_i(U_b^+(\psi(x),x),U_b^-(\psi(x),x))\big)+\Psi_i\Big\}\\
&\quad-\Big\{G_i(U_b^+(\psi(x),x),U_b^-(\psi(x),x))-G_i(U_b^+(r_b,x),U_b^-(r_b,x))\Big\}\\
&=:\mathrm{I}_i+\mathrm{II}_i,
\end{aligned}$$

where we use "$\bullet$" as the scalar product of vectors in the phase (Euclidean) space, and $\partial_+G_i(U,U^-)$ and $\partial_-G_i(U,U^-)$ as the gradient of $G_i(U,U^-)$ with respect to the variables U and U^-, respectively.

By the Taylor expansion, the terms in I_i are of higher order. However,

$$\mathrm{II}_1=\frac{2}{r_b}\big(p_b^+(r_b)-p_b^-(r_b)\big)(\psi^p+r^p-r_b)+O(|\psi-r_b|^2),\qquad \mathrm{II}_j=0,\quad j=2,3.$$

The Landau symbol $O(|\psi|^2)$ means the terms of order at least to be two of ψ. One can also obtain

$$\begin{aligned}
d_0 \quad&:=\quad \det\Big(\frac{\partial(G_1,G_2,G_3)}{\partial(u^0,p,\rho)}\Big)\Big|_{(U_b^-,U_b^+;S_b)}\\
&=\quad \det\begin{pmatrix} 2\rho_b^+(u^0)_b^+ & 1 & ((u^0)_b^+)^2\\ \rho_b^+ & 0 & (u^0)_b^+\\ (u^0)_b^+ & \frac{\gamma}{\gamma-1}\frac{1}{\rho_b^+} & -\frac{(c^2)_b^+}{\gamma-1}\frac{1}{\rho_b^+}\end{pmatrix}\Bigg|_{x^0=r_b}\\
&=\quad \frac{(c^2-(u^0)^2)_b^+(r_b)}{\gamma-1}>0.
\end{aligned}$$

Thus, (2.27) is equal to

$$\begin{aligned}
&\begin{pmatrix} 2\rho_b^+(u^0)_b^+ & 1 & ((u^0)_b^+)^2\\ \rho_b^+ & 0 & (u^0)_b^+\\ (u^0)_b^+ & \frac{\gamma}{\gamma-1}\frac{1}{\rho_b^+} & -\frac{(c^2)_b^+}{\gamma-1}\frac{1}{\rho_b^+}\end{pmatrix}\Bigg|_{x^0=r_b}\begin{pmatrix}\psi^*(\widehat{u^0})\\ \psi^*\hat p\\ \psi^*\hat\rho\end{pmatrix}\\
&=\begin{pmatrix} -\frac{2}{r_b}\big(p_b^+(r_b)-p_b^-(r_b)\big)\\ 0\\ 0\end{pmatrix}(\psi-r_b)+\text{h.o.t.},
\end{aligned}$$

where *h.o.t.* represents the *higher order terms* for short.

We can solve this linear system to obtain

$$\psi^*(\widehat{u^0})=\mu_1\,(\psi^p+r^p-r_b)+g_1(U,U^-,\psi,D\psi), \tag{2.28}$$

$$\psi^*(\hat p)=\mu_2\,(\psi^p+r^p-r_b)+g_2(U,U^-,\psi,D\psi), \tag{2.29}$$

$$\psi^*(\hat\rho)=\mu_3\,(\psi^p+r^p-r_b)+g_3(U,U^-,\psi,D\psi). \tag{2.30}$$

Using $A(S)=p\rho^{-\gamma}$, we also obtain

$$\psi^*(\widehat{A(S)})=\mu_4\left(\psi^p+r^p-r_b\right)+g_4(U,U^-,\psi,D\psi), \tag{2.31}$$

where

$$\mu_1=\frac{4\gamma(u^0)_b^+(r_b)}{(\gamma+1)r_b}>0,\qquad \mu_2=-\frac{4\rho_b^+(r_b)}{(\gamma+1)r_b}\big((\gamma-1)(u^0)^2+c^2\big)_b^+(r_b)<0,$$
$$\mu_3=-\frac{4\gamma\rho_b^+(r_b)}{(\gamma+1)r_b}<0,\qquad \mu_4=\frac{4(\gamma-1)}{(\gamma+1)r_b(\rho_b^+(r_b))^{\gamma-1}}(c^2-(u^0)^2)_b^+(r_b)>0.$$

2.3. **Restriction of the Conservation of Mass and Momentum on the Shock-Front.** We now calculate $d^*(\psi^*\bar{u}_1)$ by restricting the equations of conservation of mass and momentum on the shock-front S^ψ and obtain a second-order elliptic equation for ψ^p.

In local spherical coordinates, we have

$$\begin{aligned}
d^*(\psi^*\bar{u}^1) &= -\mathrm{div}(\psi^*(u^\alpha G_{\alpha\beta})g^{\beta\gamma}\partial_\gamma)\\
&= -\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}\psi^2\psi^*u^\alpha)\\
&= -\psi^2\psi^*\big(\frac{1}{\sqrt{G}}\partial_\alpha(\sqrt{G}u^\alpha)\big)-\psi^*\partial_0\big((x^0)^2d\psi(u_1)\big)\\
&= -\psi^2\psi^*\big(\mathrm{div}\,u-\frac{1}{\sqrt{G}}\partial_0(\sqrt{G}u^0)\big)-\psi^*\partial_0\big((x^0)^2d\psi(u_1)\big)\\
&= \psi^2\psi^*(d^*\bar{u})+\psi^2\psi^*\big(\frac{1}{\sqrt{G}}\partial_0(\sqrt{G}u^0)\big)-\psi^*\partial_0\big((x^0)^2d\psi(u_1)\big)\\
&= \psi^2\psi^*\big(\frac{D_up}{\gamma p}-\bar{\varphi}_1\big)+\psi^*\partial_0\big((x^0)^2u^0\big)-\psi^*\partial_0\big((x^0)^2d\psi(u_1)\big).
\end{aligned}$$

On the other hand, from the conservation of momentum,

$$\begin{aligned}
\psi^*\big(dx^0(\varphi_0)\big) &= \psi^*\big(u^0\partial_0u^0\big)+\psi^*\big(\frac{\partial_0p}{\rho}\big)+\psi^*\big(u^\alpha\partial_\alpha u^0+u^\alpha u^\beta\Gamma^0_{\alpha\beta}\big)\\
&= \psi^*\big(u^0\partial_0u^0\big)+\psi^*\big(\frac{\partial_0p}{\rho}\big)+\nabla_{\psi^*u_1}(\psi^*u^0)\\
&\quad -d\psi(\psi^*u_1)\big(\psi^*\partial_0u^0\big)-\psi\, g(\psi^*u_1,\psi^*u_1),
\end{aligned}$$

where we have set $\psi^*u_1:=(\psi^*u^\alpha)\partial_\alpha$ which is a well-defined vector field on $\mathbf{S}^2$, $\nabla_{\psi^*u_1}(\psi^*u^0)$ is a higher order term, and $\Gamma^0_{\alpha\beta}$ are the Christoffel symbols (see Section A.1). Then we obtain

$$d^*(\psi^*\bar{u}^1)=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

with

$$
\begin{aligned}
\mathrm{I} &= \psi^2\psi^*\Big(\frac{dx^0(\varphi_0)}{u^0}-\bar\varphi_1\Big)=\psi^2\psi^*\Big(\frac{dx^0(\varphi_0)}{(u^0)^+_b}-\bar\varphi_1\Big)+\text{h.o.t.},\\
\mathrm{II} &= \psi^2\psi^*\Big(\frac{(u^0)^2-c^2}{\gamma p u^0}\partial_0 p\Big)+2\psi\psi^*(u^0),\\
\mathrm{III} &= \frac{\psi^2}{\psi^* u^0}\big(\psi\,|\psi^* u_1|^2+d\psi(\psi^* u_1)\,(\psi^*\partial_0 u^0)-\nabla_{\psi^* u_1}(\psi^* u^0)\big)\\
&\quad -\psi^*\partial_0\big((x^0)^2 d\psi(u_1)\big)+\frac{\psi^2}{\gamma p}\big(\nabla_{\psi^* u_1}(\psi^* p)-d\psi(\psi^* u_1)(\psi^*\partial_0 p)\big),
\end{aligned}
$$

where, in the term I, we used that $\varphi_0=\varphi_0-(\varphi_0)^+_b$ is small so that $dx^0(\varphi_0)(\frac{1}{u^0}-\frac{1}{u^0_b})$ is a higher order term.

Note that, for the background solution, II $=0$. Then the Taylor expansion and the boundary conditions (2.28)–(2.31) yield

$$
\mathrm{II}=\mu_5\,\psi^*(\partial_0\hat p)+\mu_6\,\psi^p+\mu_6\,(r^p-r_b)+O(|(\psi^p,r^p-r_b,\psi^*(\hat U))|^2), \tag{2.32}
$$

with

$$
\begin{aligned}
\mu_5 &= \frac{r_b^2((u^0)^2-c^2)^+_b}{\gamma p^+_b (u^0)^+_b}\bigg|_{r=r_b}<0,\\
\mu_6 &= \frac{8\gamma(u^0)^+_b(r_b)}{(\gamma+1)(1-t_s)}\big((\gamma-1)t_s^2+t_s+1\big)>0,
\end{aligned}
$$

where $t_s=t(r_b)=(M^+_b)^2(r_b)\in(0,1)$, and $M^+_b(r_b)=\frac{(u^0)^+_b(r_b)}{c^+_b(r_b)}$ is the Mach number of the flow behind the transonic shock-front of the background solution.

We note that the term III consists of higher order terms. Then we obtain

$$
\begin{aligned}
d^*(\psi^*\bar u^1) \;&=\; \mu_5\,\psi^*(\partial_0\hat p)+\mu_6\,\psi^p+\mu_6\,(r^p-r_b)\\
&\quad +g_5(U,U^-,\psi,DU,D\psi)+\psi^2\psi^*\Big(\frac{dx^0(\varphi_0)}{(u^0)^+_b}-\bar\varphi_1\Big).
\end{aligned} \tag{2.33}
$$

Therefore, by (2.22), we have

$$
\begin{aligned}
\Delta'\psi^p+\mu_7\psi^p \;&=\; \mu_0\mu_6(r^p-r_b)+\mu_0\mu_5\,\psi^*\partial_0\hat p\\
&\quad +g_6(U,U^-,\psi,DU,D\psi,D^2\psi)+\mu_0\psi^2\psi^*\Big(\frac{dx^0(\varphi_0)}{(u^0)^+_b}-\bar\varphi_1\Big),
\end{aligned} \tag{2.34}
$$

with $g_6=\mu_0 g_5+d^*g_0$ and $\mu_7=-\mu_0\mu_6<0$.

By (2.33) and the divergence theorem, we choose

$$
r^p-r_b=-\frac{1}{4\pi\mu_6}\int_{\mathbf{S}^2}\Big(\mu_5\,\psi^*(\partial_0\hat p)+g_5(U,U^-,\psi,DU,D\psi)\Big)\,\mathrm{vol}. \tag{2.35}
$$

Substituting this into (2.29), we obtain

$$
\psi^p=\frac{1}{\mu_2}\left(\psi^*\hat p-\mu_8\int_{\mathbf{S}^2}\psi^*(\partial_0\hat p)\,\mathrm{vol}+g_7(U,U^-,\psi,D\psi)\right), \tag{2.36}
$$

with $\mu_8 = -\frac{\mu_2\mu_5}{4\pi\mu_6} < 0$ and $g_7 = \frac{\mu_2}{4\pi\mu_6}\int_{\mathbf{S}^2} g_5 \,\mathrm{vol} - g_2$. Therefore, from (2.34), we also obtain an equation for the pressure on S^ψ:

$$\begin{aligned}&\Delta'(\psi^*\hat{p}) + \mu_7(\psi^*\hat{p}) + \mu_9(\psi^*\partial_0\hat{p})\\ &= g_8(U, U^-, \psi, DU, D\psi, D^2U, D^2\psi, D^3\psi) + \mu_2\mu_0\psi^2\psi^*\big(\frac{dx^0(\varphi_0)}{(u^0)_b^+} - \bar{\varphi}_1\big),\end{aligned} \tag{2.37}$$

where $\mu_9 = -\mu_0\mu_2\mu_5 < 0$ and $g_8 = \Delta' g_2 + \mu_7 g_2 + \mu_0\mu_2 g_5 + \mu_2 d^* g_0$.

2.4. **An Elliptic Equation for the Pressure in $\mathcal{M}_\psi^+$.** First, we note the following tensor identity:

$$\operatorname{div}(D_u u) - D_u(\operatorname{div} u) = C_1^1 C_2^1(Du \otimes Du) + \operatorname{Ric}(u, u),$$

where Du is the covariant differential of the vector field u in a Riemannian manifold, $\operatorname{Ric}(\cdot,\cdot)$ is the Ricci curvature tensor, and $C_j^i(T)$ is the contraction on the upper i and lower j indices of a tensor T. In our case, since $\mathcal{M}$ is flat, $\operatorname{Ric}(u, u) \equiv 0$. From the Euler equations, we have

$$\operatorname{div}(D_u u) - D_u(\operatorname{div} u) = \operatorname{div}\varphi_0 - D_u\bar{\varphi}_1 + D_u\big(\frac{D_u p}{\gamma p}\big) - \operatorname{div}\big(\frac{\operatorname{grad} p}{\rho}\big),$$

while direct calculation yields that, for $\partial_0^2 = \partial_0\partial_0$,

$$\begin{aligned} D_u\big(\frac{D_u p}{\gamma p}\big) - \operatorname{div}\big(\frac{\operatorname{grad} p}{\rho}\big) &= \frac{1}{\gamma p}\Big(((u^0)^2 - c^2)\partial_0^2 p - \frac{2c^2}{x^0}\partial_0 p + \frac{c^2}{(x^0)^2}\Delta' p\Big)\\ &\quad + \frac{1}{\gamma p^2}\Big((c^2 - (u^0)^2)(\partial_0 p)^2 + p\partial_0 p\,\partial_0\big(\frac{(u^0)^2}{2} - c^2\big)\Big)\\ &\quad + f_1(U, DU, D^2U),\\ C_1^1C_2^1(Du\otimes Du) - \frac{2u^0}{x^0}\bar{\varphi}_1 &= (\partial_0 u^0)^2 - \frac{2}{(x^0)^2}(u^0)^2 - \frac{1}{x^0}\partial_0(u^0)^2 - \frac{2}{x^0}\,\frac{(u^0)^2}{\gamma p}\partial_0 p\\ &\quad + f_2(U, DU, D^2U).\end{aligned}$$

Now the point is that the above right-hand sides may be expressed as functions of $\partial_0^2 p, \partial_0 p, p$, $\psi^* A(S)$, and some higher order terms.

Indeed, due to the conservation of momentum,

$$\partial_0 u^0 = \frac{1}{u^0}dx^0(\varphi_0) - \frac{1}{\rho u^0}\partial_0 p + f_3(U, DU),$$

where $f_3 = \frac{1}{x^0u^0}G(u_1, u_1) - \frac{1}{u^0}D_{u_1}u^0$ is a higher order term. From the Bernoulli law,

$$\partial_0(u^0)^2 = -\frac{2}{\gamma - 1}\partial_0(c^2) + \frac{2\varphi_3}{u^0} + f_4(U, DU).$$

By the equation of state,

$$\partial_0(c^2) = (\gamma - 1)p^{-\frac{1}{\gamma}}A(S)^{\frac{1}{\gamma}}\partial_0 p + p^{\frac{\gamma-1}{\gamma}}A(S)^{\frac{1-\gamma}{\gamma}}\partial_0 A(S).$$

However, the invariance of entropy implies that

$$\partial_0 A(S) = \frac{\varphi_4}{u^0} + f_5(U, DU).$$

Therefore, we have

$$A(S)(x^0,x)=\psi^*A(S)+\int_{\psi(x)}^{x^0}\frac{\varphi_4}{u^0}(s,x)\,\mathrm{d}s+f_6(U,\psi,DU), \tag{2.38}$$

and

$$\begin{aligned}
&c^2=\gamma p^{\frac{\gamma-1}{\gamma}}\big(\psi^*A(S)\big)^{\frac{1}{\gamma}}+L_1(\varphi_4)+f_7(U,\psi,DU),\\
&(u^0)^2=2E_0(x)-\frac{2\gamma}{\gamma-1}p^{\frac{\gamma-1}{\gamma}}\big(\psi^*A(S)\big)^{\frac{1}{\gamma}}+L_2(\varphi_3,\varphi_4)+f_8(U,\psi,DU),
\end{aligned}$$

where L_1 and L_2 do not involve the derivatives of φ_i, and $L_1(0)=L_2(0,0)=0$. We then have

$$\begin{aligned}
&\rho x_0^2\Big(\mathrm{div}\varphi_0-D_u\bar{\varphi}_1-\frac{2u^0}{x^0}\bar{\varphi}_1+L_3(dx^0(\varphi_0),\varphi_3,\varphi_4)\Big)\\
&=x_0^2\big(1-\frac{(u^0)^2}{c^2}\big)\partial_0^2p-\Delta'p+2x_0\big(2-\frac{(u^0)^2}{c^2}\big)\partial_0p\\
&\quad+\frac{x_0^2}{p}\big(\frac{(u^0)^2}{c^2}+\frac{c^2}{\gamma(u^0)^2}\big)(\partial_0p)^2+\frac{4\gamma}{\gamma-1}p-4E_0(x)p^{\frac{1}{\gamma}}\big(\psi^*A(S)\big)^{-\frac{1}{\gamma}}\\
&=e_1\,\partial_0^2\hat{p}-\Delta'\hat{p}+e_2\,\partial_0\hat{p}+e_3\,\hat{p}+e_4\,\psi^*\hat{p}+f_9(U,U^-,\psi,DU,D\psi,D^2U)
\end{aligned}\tag{2.39}$$

by (2.29) and (2.31), where $L_3(0,0,0)=0$ and $e_i=e_i(x^0)$, $i=1,\cdots,4$, are known functions determined by the background solution:

$$\begin{aligned}
e_1&=(x^0)^2(1-t)>0,\\
e_2&=\frac{2x^0}{1-t}\big((1+2\gamma)t^2-3t+4\big)>0,\\
e_3&=\frac{-2}{(t-1)^3}\big(6-19t-7t^2(-2+\gamma)+t^4\gamma(1+2\gamma)+t^3(-3+2\gamma-4\gamma^2)\big),\\
e_4&=\frac{\mu_4}{\mu_2}\,\frac{2\rho^\gamma(2+(\gamma-1)t)}{(\gamma-1)(t-1)^3}\,\big((2\gamma-3)t^2+8t-3\big)\\
&=\frac{1-t_s}{\big(\rho_b^+(r_b)\big)^\gamma(1+(\gamma-1)t_s)}\,\frac{2\rho^\gamma(2+(\gamma-1)t)}{(1-t)^3}\,\big((2\gamma-3)t^2+8t-3\big),
\end{aligned}$$

where $t=t(x^0)=(M_b^+)^2(x^0)\in(0,1)$ and $t_s=t(r_b)=\Big(\frac{(u^0)_b^+(r_b)}{c_b^+(r_b)}\Big)^2=(M_b^+)^2(r_b)\in(0,1)$ with $M_b^+(x^0)=\frac{(u^0)_b^+(x^0)}{c_b^+(x^0)}$. Since $e_1>0$, (2.39) is *an elliptic equation* for $\hat{p}$.

We also recall that $t(x^0)$ is monotonically decreasing for the background solution and satisfies the following differential equation (cf. [22]):

$$\frac{\mathrm{d}t}{\mathrm{d}x^0}=\frac{2t}{x^0}\,\frac{2+(\gamma-1)t}{t-1}. \tag{2.40}$$

2.5. **The Normalization of $\mathcal{M}_\psi^+$ and Reduced Equations.** The above equations and boundary conditions are obtained in $\mathcal{M}_\psi^+$ for the given subsonic flow U and the shock-front ψ satisfying Theorem 1.1. The computations are relatively easy for the sake of rather simple metric G. However, to show the uniqueness of the transonic shock-front and the subsonic flow behind it, we need to set up an iteration mapping to find a new $\hat{\psi}\in\mathcal{K}_\sigma$:

$$\mathcal{K}_\sigma := \Big\{\psi \in C^{4,\alpha}(\mathbf{S}^2) \,:\, \|\psi - r_b\|_{C^{4,\alpha}(\mathbf{S}^2)} \le \sigma \le \sigma_0\Big\} \tag{2.41}$$

for a positive constant σ_0 to be specified later, by solving several boundary value problems in $\mathcal{M}_\psi^+$ for any $\psi \in \mathcal{K}_\sigma$, and then show that there is a fixed point that is unique. Therefore, it is convenient to introduce a $C^{4,\alpha}$–homeomorphism $\Psi : (x^0, x) \in \mathcal{M}_\psi^+ \mapsto (y^0, y) \in \Omega := [0,1] \times \mathbf{S}^2$ by

$$y^0 = \frac{x^0 - \psi(x)}{r^1 - \psi(x)}, \qquad y = (y^1, y^2) = (x^1, x^2), \tag{2.42}$$

to normalize $\mathcal{M}_\psi^+$ to Ω. We set $\Omega^j = \{j\} \times \mathbf{S}^2, j = 0, 1$. Then $\partial\Omega = \Omega^0 \cup \Omega^1$. We will use i to denote the embedding of Ω^0 in Ω. We also define the metric of Ω to be

$$\tilde{G} = (r^1 - r_b)^2 dy^0 \otimes dy^0 + \big((r^1 - r_b)y^0 + r_b\big)^2 g,$$

which differs from the metric induced by Ψ only those terms involving $D\psi$ or $\psi - r_b$. Therefore, according to (2.18), we define, for a vector field u in Ω, the corresponding $\bar{u}^1 = u^\alpha \tilde{G}_{\alpha\beta} dy^\beta$. In the following, $\partial_i = \frac{\partial}{\partial y^i}$ in Ω for short. Then (2.7) can be written on Ω^0 as

$$\begin{aligned} d\bar{u}|_{\Omega^0} \;\; &= \;\; d(\bar{u}|_{\Omega^0}) + E_1(U, \psi, DU, D\psi, D^2\psi)|_{\Omega^0} \wedge d\psi \\ &\quad + (r^1 - r_b)\Big((r^1 - r_b) r_b g_{\alpha\beta} u^\beta|_{\Omega^0} - \frac{\partial_\alpha p}{\rho_b^+ (u^0)_b^+}\Big|_{\Omega^0}\Big) dy^0 \wedge dy^\alpha \\ &\quad + g_9(U, \psi, DU, D\psi) + (r^1 - r_b)\, \psi^2 g_{\alpha\beta} \frac{dy^\alpha(\varphi_0)}{u^0}\Big|_{\Omega^0} dy^0 \wedge dy^\beta, \end{aligned} \tag{2.43}$$

where E_1 is a 1-form in Ω depending smoothly on $U, \psi, DU, D\psi$, and $D^2\psi$. Equations (2.22)–(2.23) on Ω^0 are respectively

$$d\psi = \omega := \mu_0 i^*(\bar{u}^1) + \bar{g}_0(U, U^-, \psi, D\psi) \in A^1(\mathbf{S}^2), \tag{2.44}$$

$$d(i^*\bar{u}^1) = \chi(U, U^-, \psi) := -\frac{d\bar{g}_0}{\mu_0}. \tag{2.45}$$

Similarly, (2.28)–(2.31) are transferred to

$$i^*(\widehat{u^0}) = \bar{\mu}_1\,(\psi^p + r^p - r_b) + \bar{g}_1(U, U^-, \psi, D\psi), \tag{2.46}$$

$$i^*(\hat{p}) = \mu_2\,(\psi^p + r^p - r_b) + \bar{g}_2(U, U^-, \psi, D\psi), \tag{2.47}$$

$$i^*(\hat{\rho}) = \mu_3\,(\psi^p + r^p - r_b) + \bar{g}_3(U, U^-, \psi, D\psi), \tag{2.48}$$

$$i^*(\widehat{A(S)}) = \mu_4\,(\psi^p + r^p - r_b) + \bar{g}_4(U, U^-, \psi, D\psi), \tag{2.49}$$

where $\bar\mu_1 = \frac{\mu_1}{r^1-r_b} > 0$. In addition, we have

$$(2.33) \iff d^*(i^*\bar u^1) = \frac{\mu_5}{r^1-r_b} i^*(\partial_0\hat p) + \mu_6\,\psi^p + \mu_6\,(r^p - r_b)$$
$$+\bar g_5(U, U^-, \psi, DU, D\psi) + \psi^2 i^*\Big((r^1-r_b)\frac{dy^0(\varphi_0)}{(u^0)_b^+} - \bar\varphi_1\Big), \tag{2.50}$$

$$(2.34) \iff \Delta'\psi^p + \mu_7\psi^p = \mu_0\mu_6(r^p - r_b) + \frac{\mu_0\mu_5}{r^1-r_b} i^*(\partial_0\hat p) + \bar g_6(U, U^-, \psi, DU, D\psi, D^2\psi)$$
$$+\mu_0\psi^2 i^*\big((r_1-r_b)\frac{dy^0(\varphi_0)}{(u^0)_b^+} - \bar\varphi_1\big), \tag{2.51}$$

$$(2.35) \iff r^p - r_b = -\frac{1}{4\pi\mu_6}\int_{\mathbf{S}^2}\Big(\frac{\mu_5}{r^1-r_b} i^*(\partial_0\hat p) + \bar g_5(U, U^-, \psi, DU, D\psi)\Big)\,\mathrm{vol}, \tag{2.52}$$

$$(2.36) \iff \psi^p = \frac{1}{\mu_2}\Big(i^*\hat p - \frac{\mu_8}{r^1-r_b}\int_{\mathbf{S}^2} i^*(\partial_0\hat p)\,\mathrm{vol} + \bar g_7(U, U^-, \psi, D\psi)\Big), \tag{2.53}$$

$$(2.37) \iff \Delta'(i^*\hat p) + \mu_7(i^*\hat p) + \frac{\mu_9}{r^1-r_b}\, i^*(\partial_0\hat p) = \bar g_8(U, U^-, \psi, DU, D\psi, D^2U, D^2\psi, D^3\psi)$$
$$+\mu_2\mu_0\psi^2 i^*\big((r^1-r_b)\frac{dy^0(\varphi_0)}{(u^0)_b^+} - \bar\varphi_1\big), \tag{2.54}$$

$$(2.39) \iff (r^1-\psi)^2\rho x_0^2\Big(\mathrm{div}\varphi_0 - D_u\bar\varphi_1 - \frac{2(r^1-r_b)(u^0)_b^+}{(r^1-r_b)y^0+r_b}\,\bar\varphi_1 + L_3((r^1-r_b)\,dy^0(\varphi_0), \varphi_3, \varphi_4)\Big)$$
$$= e_1\,\partial_0^2\hat p - (r^1-r_b)^2\Delta'\hat p + (r^1-r_b)e_2\,\partial_0\hat p + (r^1-r_b)^2 e_3\,\hat p + (r^1-r_b)^2 e_4\, i^*\hat p$$
$$-\bar f_9(U, U^-, \psi, DU, D\psi, D^2U), \tag{2.55}$$

where $\bar f_j, \bar g_j$ differ from f_j, g_j by some higher order terms due to the facts that $\tilde G$ differs from $(\Psi^{-1})^*G$ by the terms involving $D\psi$ or $\psi - r_b$ and that $\hat U$ is small. We also note that $e_i = e_i((r^1-r_b)y^0 + r_b)$ in (2.55).

Remark 2.2. An important observation is that, by (2.4), if $\varphi_4 = 0$, we may write

$$\varphi_3 = G(\varphi_0, u) = (r^1-r_b)\cdot u_b^+((r^1-r_b)y^0 + r_b)\cdot dy^0(\varphi_0) + \text{h.o.t.},$$

so we may write $L_3((r^1-r_b)\,dy^0(\varphi_0), \varphi_3, 0)$ in (2.55) as an expression of $dy^0(\varphi_0)$ with coefficients depending only on y^0 by adjusting the higher order term $\bar f_9$.

2.6. **The S-Condition.** We now state the *S-condition* assumed in our main theorem.

Consider the following boundary value problem:

$$e_1 v'' + (r^1-r_b)e_2 v' + (r^1-r_b)^2(e_3 - \lambda_n)v = -(r^1-r_b)^2 e_4, \tag{2.56}$$
$$v(0) = 1, \qquad v(1) = 0, \tag{2.57}$$
$$v'(0) = -\frac{\lambda_n + \mu_7}{\mu_9}(r^1-r_b), \tag{2.58}$$

where $\lambda_n = n(n+1), n = 0, 1, 2, \cdots$, and $e_i = e_i(t(y^0))$ with $t(y^0) = (r^1-r_b)y^0 + r_b$ are considered as functions of y^0 on $[0, 1]$.

Definition 2.3. A background solution U_b satisfies the *S-Condition* if, for each $n = 0, 1, 2, \cdots$, problem (2.56)–(2.58) does not have a solution.

The following lemmas show that there exist certain background solutions which satisfy the S-Condition.

Lemma 2.1. *For given $\gamma > 1$, let U_b be a background solution determined by the supersonic upstream data $p_b^-(r^0), \rho_b^-(r^0), t_b^-(r^0) = (M_b^-)^2(r^0)$, and the back pressure $p_b^+(r^1)$, where $M_b^-(r^0)$ is the Mach number of the upstream supersonic flow. Then, for given $\rho_b^-(r^0)$ and the back pressure $p_b^+(r^1)$, when $p_b^-(r^0)$ and $M_b^-(r^0)$ are sufficiently large, U_b satisfies the S-Condition.*

Proof. We divide the proof into three steps.

Step 1. By an analysis of the background solution in [22], it suffices to show that, if $\kappa = r^1 - r_b > 0$ and $\sigma = t_b^+(r_b) = (M_b^+)^2(r_b)$ are rather small, then U_b satisfies the S-Condition. We will prove by contradiction: we will first assume that (2.56)–(2.58) has a solution and then lead to a contradiction.

We note that, once $t_b^-(r^0)$ is given, we can solve $t(x^0)$ for all $x^0 \in [r^0, r^1]$ (i.e., it is independent of $p_b^-(r^0), \rho_b^-(r^0)$, and $p_b^+(r^1)$). This can be seen from (2.40) and the Rankine-Hugoniot condition of the Mach number:

$$\Big(\frac{1}{\sqrt{t_b^-(r_b)}} + \frac{\gamma-1}{2}\Big)\Big(\frac{1}{\sqrt{t_b^+(r_b)}} + \frac{\gamma-1}{2}\Big) = \frac{(\gamma+1)^2}{4}.$$

Moreover, since $p_b^+(r^1)$ is given, we have the estimate:

$$1 \le \frac{p_b^+(x^0)}{p_b^+(r_b)} \le C \qquad \text{for any} \quad x^0 \in (r_b, r^1),$$

with C depending only on $p_b^+(r^1)$, r^0, and r^1.

Step 2. We choose σ_0 such that, for $t \le \sigma_0$, $e_4(t) \le 0$. Hence, for any $t_b^+(r_b) \le \sigma_0$ (this requires $t_b^-(r^0)$ large), by (2.40),

$$e_4(t(y^0)) \le 0 \qquad \text{for } y^0 \in (0,1).$$

In addition, if $n \ge 3$, we see that $e_3 - \lambda_n < 0$ and $\frac{(r^1-r_b)(\lambda_n+\mu_7)}{\mu_9} < 0$. So, by the Hopf maximum principle, we infer a contradiction if $v(1) = 0$ for a solution v to (2.56)–(2.58).

For $n = 0, 1, 2$, we will utilize an energy estimate below to obtain a contradiction if κ is also small.

Step 3. We first reformulate (2.56)–(2.58). Let $h_n = -\frac{\mu_7+\lambda_n}{\mu_9}\kappa$. For simplicity, we write the independent variable y^0 as y. Then, by multiplying

$$p_n(y) = \exp\Big(\int_0^y \big(2h_n + \kappa\frac{e_2(t(s))}{e_1(t(s))}\big)\,\mathrm{d}s\Big)$$

to (2.56), we see that $w = e^{-h_n y}v$ satisfies

$$p_n(y)\frac{\mathrm{d}(p_n(y)\frac{\mathrm{d}w}{\mathrm{d}y})}{\mathrm{d}y} + \kappa^2\alpha_n(y)w(y) = \kappa^2\beta_n(y),$$

where

$$\alpha_n(y) = p_n(y)^2\Big(\frac{(\lambda_n+\mu_7)^2}{\mu_9^2} - \frac{\lambda_n+\mu_7}{\mu_9}\,\frac{e_2(t(y))}{e_1(t(y))} + \frac{e_3(t(y))-\lambda_n}{e_1(t(y))}\Big),$$

$$\beta_n(y) = -\big(p_n(y)\big)^2 e^{-h_n y}\frac{e_4(t(y))}{e_1(t(y))}.$$

By a change of the independent variable $y \to z_n$:

$$z_n = \int_0^y \frac{\mathrm{d}s}{p_n(s)} \qquad \text{with} \quad z_n^* = \int_0^1 \frac{\mathrm{d}s}{p_n(s)},$$

the above equation becomes

$$w'' + \kappa^2 \alpha_n w = \kappa^2 \beta_n, \tag{2.59}$$

where $'$ is the derivative with respect to z_n, $\alpha_n = \alpha_n(y(z_n))$, and $\beta_n = \beta_n(y(z_n))$. The boundary conditions are

$$w(0) = 1, \qquad w'(0) = 0, \qquad w(z_n^*) = 0. \tag{2.60}$$

Now, multiplying w to (2.59) and integrating on $[0, z_n^*]$ yield

$$\int_0^{z_n^*} (w')^2 \,\mathrm{d}z = -\kappa^2 \int_0^{z_n^*} \beta_n w \,\mathrm{d}z + \kappa^2 \int_0^{z_n^*} \alpha_n w^2 \,\mathrm{d}z. \tag{2.61}$$

Note here that, since $n < 3$, and $|\beta_n|, |\alpha_n|$, and z_n^* are bounded by a constant C,

$$\Big| \int_0^{z_n^*} \beta_n w(z) \,\mathrm{d}z \Big| \le C \int_0^{z_n^*} \Big| \int_{z_n^*}^{z} w'(s) \,\mathrm{d}s \Big| \,\mathrm{d}z \le 2C \sqrt{z_n^*} \sqrt{\int_0^{z_n^*} (w')^2 \,\mathrm{d}z},$$

$$\Big| \int_0^{z_n^*} \alpha_n w(z)^2 \,\mathrm{d}z \Big| \le C \int_0^{z_n^*} \Big(\int_{z_n^*}^{z} w'(s) \,\mathrm{d}s \Big)^2 \,\mathrm{d}z \le C \frac{z_n^*}{2} \int_0^{z_n^*} (w')^2 \,\mathrm{d}z.$$

By (2.60), $\int_0^{z_n^*} (w')^2 \,\mathrm{d}z \ne 0$. Then, from (2.61), we have

$$\int_0^{z_n^*} (w')^2 \,\mathrm{d}z \le \frac{C' \kappa^2}{1 - C' \kappa^2} \le C'' \kappa^2.$$

On the other hand,

$$1 = \Big| \int_0^{z_n^*} w' \,\mathrm{d}z \Big|^2 \le z_n^* \int_0^{z_n^*} (w')^2 \,\mathrm{d}z \le C''' \kappa^2.$$

This reaches a contradiction when $\kappa < \min\{\frac{1}{\sqrt{C'''}}, \frac{1}{\sqrt{2C'}}\}$. Note that C''' depends only on r^0, r^1, $t_b^-(r^0)$, n, and $p_b^+(r^1)$. $\square$

Remark 2.3. In Lemma 2.1, for the case that $\kappa = r^1 - r_b > 0$ is small, we require only $p_b^-(r_0)$ to be large (see [22]).

In the following, we provide some other results on the existence of background solutions that satisfy the S-Condition. For given $[r^0, r^1]$, we note that a background solution U_b^+ is determined by the five parameters $(\gamma, r_b, p_b^+(r_b), \rho_b^+(r_b), t(r_b))$ with $t(r_b) = (M_b^+)^2(r_b)$, $\gamma > 1$, $r_b \in (r^0, r^1)$, $p_b^+(r_b) > 0$, $\rho_b^+(r_b) > 0$, and $t(r_b) \in (0, 1)$.

Lemma 2.2. *For given $\gamma > 1$, $\rho_b^+(r_b) > 0$, and $\sigma_0 \in (0,1)$, there exist a set $S_1 \subset [0, \sigma_0]$ of at most countable infinite points and a set $S_2 \in [r^0, r^1]$ of at most finite points such that the background solution determined by $\gamma > 1$, $r_b \in (r^0, r^1) \setminus S_2$, $p_b^+(r_b) > 0$, $\rho_b^+(r_b) > 0$, and $t(r_b) \in (0, \sigma_0) \setminus S_1$ satisfies the S-Condition.*

Proof. Let $\sigma = t(r_b) \in [0, \sigma_0]$. Then, by (2.40), we know that $t = t(x^0, \sigma)$ is analytical with respect to σ. Also, by the theory of ordinary differential equations, $v = v(y^0, \sigma)$ is an analytical function of σ. Hence, we establish an analytical mapping $f_n : [0, \sigma_0] \to \mathbb{R}$ by

$$f_n(\sigma) = v(1, \sigma), \qquad n = 0, 1, 2, \cdots \tag{2.62}$$

Now, if $f_n^{-1}(\{0\})$ has infinite points, then the zeros of f_n have an accumulate point in $[0, \sigma_0]$, which implies $f_n \equiv 0$, especially, $f_n(0) = 0$.

Note that $t(x^0, 0) \equiv 0$. Thus, in this case, (2.56)–(2.58) become

$$\begin{aligned} &((r^1 - r_b)y^0 + r_b)^2 v'' + 8(r^1 - r_b)\big((r^1 - r_b)y^0 + r_b\big)v' + (r^1 - r_b)^2(12 - \lambda_n)v \\ &\qquad = 12(r^1 - r_b)^2 \frac{\rho_b^+(x^0)^\gamma}{\rho_b^+(r_b)^\gamma} \geq 0, \end{aligned} \tag{2.63}$$

$$v(0) = 1, \tag{2.64}$$

$$v'(0) = \frac{(r^1 - r_b)\lambda_n}{2r_b} \geq 0, \tag{2.65}$$

$$v(1) = 0. \tag{2.66}$$

For $n \geq 3$, $(r^1 - r_b)^2(12 - \lambda_n) \leq 0$. By the Hopf maximum principle, we infer a contradiction at $y^0 = 0$. Therefore, in these cases, $f_n^{-1}(\{0\})$ consists of finite points, which implies that $\cup_{n=3}^{\infty} f_n^{-1}(\{0\})$ is countable.

For $n = 0, 1, 2$, we also recognize that the solution v of problem (2.63)–(2.65) is an analytical function of the parameter r_b. Hence, we have an analytical mapping $g_n : [r^0, r^1] \to \mathbb{R}$ defined by $g_n(r_b) = v(1, r_b)$. We claim that $\cup_{n=0}^{2} g_n^{-1}(\{0\})$ has only finite points. Indeed, if $g_n^{-1}(\{0\})$ contains infinite points, then $g_n \equiv 0$, especially $g_n(r^1) = 0$. However, in this case, (2.63)–(2.66) are reduced to

$$\begin{cases} v'' = 0, \\ v(0) = 1, \quad v(1) = 0, \quad v'(0) = 0. \end{cases} \tag{2.67}$$

Obviously, there is no solution to this problem. □

Lemma 2.3. *For given $\gamma > 1, \rho_b^+(r_b) > 0$, and $t(r_b) \in (0, 1)$, there exists a set $S_3 \subset [r^0, r^1]$ of at most countable infinite numbers of points such that the background solution determined by γ, $r_b \in (r^0, r^1) \setminus S_3$, $p_b^+(r_b) > 0, \rho_b^+(r_b)$, and $t(r_b)$ satisfies the S-Condition.*

Proof. As in the above proof, consider $v(1)$ as an analytical function of r_b for each n. If the preimage of the zero has infinite points for some n, then the function is identically zero and, by (2.67), there is a contradiction. □

Lemma 2.4. *There exists $\sigma_0 \in (0, 1)$ such that, for given $\gamma > 1, p_b^+(r_b) > 0, \rho_b^+(r_b) > 0$, and $t(r_b) = (M_b^+)^2(r_b) \in (0, \sigma_0)$, there exists $r_* \in (r^0, r^1)$ so that the background solution, determined by $\gamma > 1, r_b \in (r_*, r^1), p_b^+(r_b), \rho_b^+(r_b)$, and $t(r_b)$, satisfies the S-Condition.*

Proof. We choose σ_0 such that, for $t \leq \sigma_0$, $e_4(t) \leq 0$. Hence, for any $t(r_b) \leq \sigma_0$,

$$e_4(t(y^0)) \leq 0 \qquad \text{for } y^0 \in (0, 1)$$

by (2.40). In addition, if $n \geq 3$, we see that

$$e_3 - \lambda_n < 0, \qquad \frac{(r^1 - r_b)(\lambda_n + \mu_7)}{\mu_9} < 0.$$

Thus, by the Hopf maximum principle, we infer a contradiction if $v(1) = 0$.

Now, for $n = 0, 1, 2$, consider $K = \cup_{n=0}^{2} g_n^{-1}(\{0\})$ which have at most finite points. Note that $r^1 \notin K$. Thus, let $r_* = \sup K < r^1$ (if $K = \emptyset$, let $r_* = r^0$). Then the lemma is proved. □

This lemma improves somewhat the results of Lemma 2.1, but we do not have an estimate of $r^1 - r_*$ here as that in Lemma 2.1.

3. An Iteration Mapping and Decomposition of the Euler System

In this section we set up an iteration mapping and show that it has a unique fixed point. By the derivations in §2, it is easy to see that any transonic shock solution to (1.1)–(1.3) satisfying those requirements in Theorem 1.1 must be a fixed point of the iteration mapping, which implies its uniqueness claimed in Theorem 1.1. Another motivation to introduce the iteration mapping is for constructing approximate solutions to show the existence of global transonic shock solutions, which require further exploration.

3.1. **The Iteration Set.** For given $\psi \in \mathcal{K}_\sigma$, its position r^p and profile ψ^p satisfy

$$\|\psi^p\|_{C^{4,\alpha}(\mathbf{S}^2)} \leq 2\sigma, \quad |r^p - r_b| \leq \sigma.$$

We solve the candidate subsonic flow in $\mathcal{M}_\psi^+ := \{(x^0, x) \in \mathcal{M} : x^0 \geq \psi(x), x \in \mathbf{S}^2\}$. By (2.42), we write the set of possible variations of the subsonic flows as

$$O_\delta \quad := \quad \left\{ \check{U} = (\check{p}, \check{\rho}, \check{u}) \,:\, \|(\check{p}, \check{\rho})\|_{C^{3,\alpha}(\Omega)} + \|\bar{\check{u}}\|_{C_1^{3,\alpha}(\Omega)} \leq \delta \leq \delta_0 \right\} \tag{3.1}$$

with constant δ_0 to be chosen later. Given U^- satisfying (1.11), for any $\psi \in \mathcal{K}_\sigma$ and $\check{U} \in O_\delta$, we construct a mapping $\mathcal{K}_\sigma \times O_\delta \to \mathcal{K}_\sigma \times O_\delta$ denoted as

$$\mathcal{T}(\psi, \check{U}) = (\hat{\psi}, \hat{U})$$

by the following iteration process. Then we show that $\mathcal{T}$ has a unique fixed point in $\mathcal{K}_\sigma \times O_\delta$.

3.2. **A Nonlocal Venttsel Problem for the Candidate Pressure in Ω.** We first choose ε_0, σ_0, and δ_0 small enough such that the formulations in §2 valid. For any $\psi \in \mathcal{K}_\sigma$, $\check{U} \in O_\delta$, and U^- satisfying (1.11), we may express the higher order terms $\bar{f}_i$ and $\bar{g}_i$ in terms of $U = U_b^+ + \check{U}$.

By (1.9) and (2.54)–(2.55), we solve $\hat{p}$ from the following linear nonlocal Venttsel problem:

$$\begin{aligned}
&e_1\partial_0^2\hat{p} - (r^1 - r_b)^2\Delta'\hat{p} + (r^1 - r_b)e_2\partial_0\hat{p} + (r^1 - r_b)^2 e_3\hat{p} + (r^1 - r_b)^2 e_4\, \hat{p}|_{\Omega^0} \\
&\qquad = \bar{f}_9(U, U^-, \psi, DU, D\psi, D^2U) \qquad\qquad \text{in } \ \Omega, && (3.2)\\
&\hat{p} = 0 \qquad\qquad \text{on } \ \Omega^1, && (3.3)\\
&\Delta'(\hat{p}|_{\Omega^0}) + \frac{\mu_9}{r^1 - r_b}\partial_0\hat{p}|_{\Omega^0} + \mu_7\, \hat{p}|_{\Omega^0} = \bar{g}_8(U, U^-, \psi, DU, D\psi, D^2U, D^2\psi, D^3\psi) \quad \text{on } \Omega^0. && (3.4)
\end{aligned}$$

Thanks to Theorem 1.5 in [16] for the Venttsel problem (note that $\frac{\mu_9}{r^1 - r_b} < 0$) and Theorem 6.6 in [13] for the Dirichlet problem, with the aid of a standard higher regularity argument as in Theorem

6.19 of [13], by considering $(r^1-r_b)^2 e_4\,\hat{p}|_{\Omega^0}$ as a nonhomogeneous term and using interpolation inequalities, for the solution $\hat{p} \in C^{3,\alpha}(\Omega)$, we have the apriori Schauder estimate

$$\|\hat{p}\|_{C^{3,\alpha}(\Omega)} \le C_2\Big(\|\hat{p}\|_{C^0(\Omega)} + \big\|\bar{f}_9\big\|_{C^{1,\alpha}(\Omega)} + \|\bar{g}_8\|_{C^{1,\alpha}(\Omega^0)} \Big) \tag{3.5}$$

with constant C_2 depending only on U_b.

Next, by Lemma A.4, let $u_{n,m}(y)$ be the eigenfunctions of Δ' on $\mathbf{S}^2$ with respect to the eigenvalues $\lambda_n = n(n+1) \ge 0, n = 0, 1, 2, \cdots$. Then

$$\hat{p} = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} v_{n,m}(y^0) u_{n,m}(y).$$

For $\bar{f}_9 = 0$ and $\bar{g}_8 = 0$, each $v_{n,m}$ satisfies the nonlocal differential equation:

$$e_1 v''_{n,m} + (r^1-r_b)e_2 v'_{n,m} + (r^1-r_b)^2(e_3-\lambda_n)v_{n,m} + (r^1-r_b)^2 e_4 v_{n,m}(0) = 0, \tag{3.6}$$

$$v_{n,m}(1) = 0, \tag{3.7}$$

$$\mu_9 v'_{n,m}(0) + (r^1-r_b)(\lambda_n+\mu_7)v_{n,m}(0) = 0. \tag{3.8}$$

First, if $v_{n,m}(0) = 0$, (3.8) says that $v'_{n,m}(0) = 0$. Hence, by uniqueness of solutions of the Cauchy problem of differential equations, we infer $v_{n,m} \equiv 0$.

Now, if $v_{n,m}(0) \ne 0$, then, by considering $\frac{v_{n,m}(y^0)}{v_{n,m}(0)}$ as the unknown, we see that it solves

$$e_1 v''_{n,m} + (r^1-r_b)e_2 v'_{n,m} + (r^1-r_b)^2(e_3-\lambda_n)v_{n,m} = -(r^1-r_b)^2 e_4, \tag{3.9}$$

$$v_{n,m}(0) = 1, \qquad v_{n,m}(1) = 0, \tag{3.10}$$

$$v'_{n,m}(0) = -\frac{\lambda_n+\mu_7}{\mu_9}(r^1-r_b). \tag{3.11}$$

The S-condition in §2.6 guarantees the nonexistence of a solution to this problem. Thus, $v_{n,m}(0) = 0$, which implies $\hat{p} \equiv 0$.

Therefore, the S-Condition implies the uniqueness of solutions of the Venttsel problem (3.2)–(3.4). Then, by (3.5) and a standard argument of contradiction based on the compactness (cf. Lemma 9.17 in [13]), we have the apriori estimate:

$$\|\hat{p}\|_{C^{3,\alpha}(\Omega)} \le C_2\Big(\big\|\bar{f}_9\big\|_{C^{1,\alpha}(\Omega)} + \|\bar{g}_8\|_{C^{1,\alpha}(\Omega^0)} \Big) \tag{3.12}$$

for any $C^{3,\alpha}$ solution of the above nonlocal Venttsel problem. Then, by the method of continuity as carried out in [16], we see that this problem has a unique solution $\hat{p} \in C^{3,\alpha}(\Omega)$ satisfying estimate (3.12).

3.3. **Update of the Candidate Free Boundary.** Once we get $\hat{p}$, according to (2.52)–(2.53), we may obtain the new profile of the free boundary $\hat{\psi}^p$ and the position $\hat{r}^p$ by

$$\hat{\psi}^p = \frac{1}{\mu_2}\Big(\hat{p}|_{\Omega^0} - \frac{\mu_8}{r^1-r_b}\int_{\mathbf{S}^2} \partial_0\hat{p}|_{\Omega^0}\,\mathrm{vol} + \bar{g}_7(U, U^-, \psi, D\psi)\Big), \tag{3.13}$$

$$\hat{r}^p - r_b = -\frac{1}{4\pi\mu_6}\int_{\mathbf{S}^2}\Big(\frac{\mu_5}{r^1-r_b}\,\partial_0\hat{p}|_{\Omega^0} + \bar{g}_5(U, U^-, \psi, DU, D\psi)\Big)\,\mathrm{vol}. \tag{3.14}$$

In fact, one may show that $\int_{\mathbf{S}^2}\hat\psi^p\,\mathrm{vol}=0$ by (3.13) and (3.4); However, we do not need this in the process. We need to improve the regularity of $\hat\psi$. By (3.4) and (3.13), $\hat\psi^p$ satisfies this elliptic equation on $\mathbf{S}^2$:

$$\Delta'\hat\psi^p+\mu_7\hat\psi^p=\mu_0\mu_6(\hat r^p-r_b)+\frac{\mu_0\mu_5}{r^1-r_b}\,\partial_0\hat p|_{\Omega^0}+\bar g_6(U,U^-,\psi,DU,D\psi,D^2\psi). \tag{3.15}$$

The right-hand side belongs to $C^{2,\alpha}(\mathbf{S}^2)$. Thus, by Theorem 6.19 in [13], $\hat\psi=\hat\psi^p+\hat r^p$ obviously obeys the estimate:

$$\|\hat\psi-r_b\|_{C^{4,\alpha}(\mathbf{S}^2)}\quad\le\quad C_2\Big(\,\|(\bar g_5,\bar g_7)\|_{C^0(\Omega^0)}+\big\|\bar f_9\big\|_{C^{1,\alpha}(\Omega)}+\|\bar g_8\|_{C^{1,\alpha}(\Omega^0)}+\|\bar g_6\|_{C^{2,\alpha}(\Omega^0)}\Big) \tag{3.16}$$

with the aid of (3.12).

3.4. **Solving the Candidate Velocity on Ω^0.** Now we need to solve $\hat u$ on Ω^0. To this end, by (2.45) and (2.50), we reformulate this problem as

$$d(\hat{\bar u}^1|_{\Omega^0})=\chi(U,U^-,\psi):=-\frac{d\bar g_0(U,U^-,\psi,D\psi)}{\mu_0}\in C_2^{2,\alpha}(\mathbf{S}^2), \tag{3.17}$$

$$d^*(\hat{\bar u}^1|_{\Omega^0})=\frac{\mu_5}{r^1-r_b}\partial_0\hat p|_{\Omega^0}+\mu_6(\hat\psi-r_b)+\bar g_5(U,U^-,\psi,DU,D\psi)\in C^{2,\alpha}(\mathbf{S}^2), \tag{3.18}$$

where d and d^* are respectively the exterior differential and codifferential operator of forms on $\mathbf{S}^2$. By integrating (3.4) on $\mathbf{S}^2$, we see that the integration of the right-hand side of (3.18) on $\mathbf{S}^2$ also vanishes. Then, by Lemma A.1, there exists a unique solution $\hat{\bar u}^1|_{\Omega^0}\in C_1^{3,\alpha}(\mathbf{S}^2)$ with the estimate:

$$\big\|\hat{\bar u}^1|_{\Omega^0}\big\|_{C_1^{3,\alpha}(\mathbf{S}^2)}\le C_2\Big(\,\|\bar g_0\|_{C_1^{3,\alpha}(\Omega^0)}+\|(\bar g_5,\bar g_6,\bar g_7)\|_{C^{2,\alpha}(\Omega^0)}+\big\|\bar f_9\big\|_{C^{1,\alpha}(\Omega)}+\|\bar g_8\|_{C^{1,\alpha}(\Omega^0)}\Big). \tag{3.19}$$

Therefore, combining this with (2.46), the velocity on the candidate free boundary is obtained.

3.5. **Solving the Candidate Entropy, Density, and Vorticity in Ω.** We note that the entropy can be solved according to (2.4) and (2.49) by the following Cauchy problem of the linear transport equations:

$$D_u\widehat{A(S)}=0\qquad\text{in}\ \ \Omega, \tag{3.20}$$

$$\widehat{A(S)}=\mu_4(\hat\psi-r_b)+\bar g_4(U,U^-,\psi,D\psi)\quad\text{on}\ \ \Omega^0. \tag{3.21}$$

Note that $A(S)_b^+(x^0)=A(S)_b^+(r_b)$ is a constant. By the theory of ordinary differential equations, since u is close to $(u^0)_b^+\partial_0$ in $C^{3,\alpha}(\Omega)$, the trajectories of u still fill Ω. We also have the estimate:

$$\|\widehat{A(S)}\|_{C^{3,\alpha}(\Omega)}\le C_2\Big(\|\hat\psi-r_b\|_{C^{3,\alpha}(\mathbf{S}^2)}+\|\bar g_4\|_{C^{3,\alpha}(\Omega^0)}\Big).$$

Hence, we may solve the candidate density $\hat\rho+\rho_b^+$ by the state function $p=A(S)\rho^\gamma$. Then

$$\|\hat\rho\|_{C^{3,\alpha}(\Omega)}\le C_2\Big(\,\|\hat p\|_{C^{3,\alpha}(\Omega)}+\Big\|\widehat{A(S)}\Big\|_{C^{3,\alpha}(\Omega)}\Big).$$

For $\widehat{d\bar u}=d\bar u=d\bar u-d\bar u_b^+$, by subtracting the background solution from (2.6), we formulate a linear transport equation:

$$\mathcal{L}_u\widehat{d\bar u}=-d(\frac{1}{\rho_b^+})\wedge d\hat p+\frac{d\hat\rho\wedge dp_b^+}{(\rho_b^+)^2}+\bar f_{10}(U,DU). \tag{3.22}$$

According to (2.43), the initial value on Ω^0 can be taken as

$$\begin{aligned}\widehat{d\bar{u}}|_{\Omega^0} &= d(\hat{\bar{u}}|_{\Omega^0}) + E_1(U,\psi,DU,D\psi,D^2\psi)|_{\Omega^0}\wedge d\hat{\psi}\\ &+(r^1-r_b)\Big((r^1-r_b)r_b g_{\alpha\beta}\hat{u}^\beta|_{\Omega^0} - \frac{\partial_\alpha\hat{p}}{\rho_b^+(u^0)_b^+}\Big|_{\Omega^0}\Big)dy^0\wedge dy^\alpha\\ &+\bar{g}_9(U,\psi,DU,D\psi). \end{aligned} \tag{3.23}$$

Note that $\hat{\bar{u}}|_{\Omega^0}$ has been obtained in §3.4:

$$\hat{\bar{u}}|_{\Omega^0} = \hat{\bar{u}}^1|_{\Omega^0} + (r^1-r_b)\mu_1(\hat{\psi}+\hat{r}^p-r_b)dy^0 + \bar{g}_{10}(U,U^-,\psi,D\psi). \tag{3.24}$$

Therefore, we may solve $\widehat{d\bar{u}}$ from (3.22)–(3.23) and obtain the estimate (see Lemma A.5):

$$\begin{aligned}\|\widehat{d\bar{u}}\|_{C_2^{2,\alpha}(\Omega)} &\le C_2\Big(\|\bar{f}_{10}\|_{C_2^{2,\alpha}(\Omega)} + \|\bar{g}_9\|_{C_2^{2,\alpha}(\Omega^0)} + \|(\hat{\rho},\hat{p})\|_{C^{3,\alpha}(\Omega)}\\ &\qquad + \|\hat{\psi}-r_b\|_{C^{3,\alpha}(\mathbf{S}^2)} + \big\|\hat{\bar{u}}^1|_{\Omega^0}\big\|_{C_1^{3,\alpha}(\mathbf{S}^2)} + \|\bar{g}_{10}\|_{C^{3,\alpha}(\Omega^0)}\Big).\end{aligned}$$

3.6. **Solving the Lower Regular Candidate Velocity in Ω and on Ω^1.** From (2.5), by subtracting the background solution, we formulate a transport equation of the velocity $\hat{\bar{u}}$ in Ω (to distinguish the lower regular velocity obtained here from the candidate velocity in the next subsection, we write u as u_l):

$$\mathcal{L}_{u_b^+}\hat{\bar{u}}_l = -\frac{d\hat{p}}{\rho_b^+} + \frac{\hat{\rho}dp_b^+}{(\rho_b^+)^2} + \bar{f}_{11}(U,\psi,DU,D\psi). \tag{3.25}$$

Here we used that $\mathcal{L}_{\hat{u}_l}\bar{u}_b^+ - d\langle\bar{u}_b^+,\hat{u}_l\rangle = 0$. With the Cauchy data $\hat{\bar{u}}_l|_{\Omega^0}$ as the right-hand side of (3.24), we may uniquely solve $\hat{\bar{u}}_l$, particularly its restriction on Ω^1, i.e., $\hat{\bar{u}}_l|_{\Omega^1}$. We also have an estimate:

$$\begin{aligned}\big\|\hat{\bar{u}}_l\big\|_{C_1^{2,\alpha}(\Omega)} &\le C_2\Big(\big\|\bar{f}_{11}\big\|_{C_1^{2,\alpha}(\Omega)} + \|(\hat{p},\hat{\rho})\|_{C^{3,\alpha}(\Omega)} + \big\|\hat{\bar{u}}^1|_{\Omega^0}\big\|_{C_1^{3,\alpha}(\mathbf{S}^2)}\\ &\qquad + \|\hat{\psi}-r_b\|_{C^{3,\alpha}(\mathbf{S}^2)} + \|\bar{g}_{10}\|_{C_1^{3,\alpha}(\Omega^0)}\Big).\end{aligned} \tag{3.26}$$

3.7. **Solving the Candidate Velocity in Ω.** Note that $\hat{\bar{u}}_l$ obtained in the above step is only in $C_1^{2,\alpha}$; it is not our desired candidate velocity. In fact, we will solve the velocity $\hat{\bar{u}}$ by the following elliptic equation motivated by (2.12):

$$\Delta\hat{\bar{u}} = d\langle\hat{\bar{u}}_l, \frac{d(\ln p_b^+)}{\gamma}\rangle + d^*(\widehat{d\bar{u}}) + d\Big(\frac{D_{u_b^+}\hat{p}}{\gamma p_b^+} - \frac{D_{u_b^+}p_b^+}{\gamma(p_b^+)^2}\hat{p}\Big) + \bar{f}_{12}(U,U^-,\psi,DU,D^2U). \tag{3.27}$$

We impose the Dirichlet condition (3.24) on Ω^0 and the Neumann condition on Ω^1 according to (3.25) by

$$D_{u_b^+}\hat{\bar{u}} = D_{u_b^+}\hat{\bar{u}}_l - \mathcal{L}_{u_b^+}\hat{\bar{u}}_l + \Big(-\frac{d\hat{p}}{\rho_b^+} + \frac{dp_b^+}{(\rho_b^+)^2}\hat{\rho} + \bar{f}_{11}(U,\psi,DU,D\psi)\Big)\Big|_{\Omega^1}. \tag{3.28}$$

Note that, in local spherical coordinates,

$$D_{u_b^+}\hat{\bar{u}}_l - \mathcal{L}_{u_b^+}\hat{\bar{u}}_l = -(\hat{\bar{u}}_l)_0\partial_0((u^0)_b^+)dx^0 - \frac{1}{x^0}((u^0)_b^+)((\hat{\bar{u}}_l)_\alpha dx^\alpha),$$

so it does not contain the derivatives of $\hat{\bar{u}}_l$. Therefore, we may solve $\hat{u}$ (i.e., $\hat{\bar{u}}$) in Ω by Lemma A.2 to obtain

$$\|\hat{\bar{u}}\|_{C^{3,\alpha}_1(\Omega)} \le C_2\Big(\|\hat{\bar{u}}_l\|_{C^{2,\alpha}_1(\Omega)} + \|d\hat{\bar{u}}\|_{C^{2,\alpha}_2(\Omega)} + \|(\hat{p},\hat{\rho})\|_{C^{3,\alpha}(\Omega)} + \|\bar{f}_{12}\|_{C^{1,\alpha}_1(\Omega)} + \|\bar{f}_{11}\|_{C^{2,\alpha}(\Omega^1)} \Big).$$

3.8. **Well-Definedness of the Mapping $\mathcal{T}:\mathcal{K}_\sigma\times O_\delta\to\mathcal{K}_\sigma\times O_\delta$.** First we notice that, for any $\psi\in\mathcal{K}_\sigma$, $U\in U_b^+ + O_\delta$, and U^- satisfying (1.11), it is clear from the definition that a higher order term f satisfies

$$\|f\| \le C_2(\varepsilon+\delta^2+\sigma^2). \tag{3.29}$$

Then, combining this with the estimates in §3.2–§3.7 yields

$$\|\hat{U}\|_{C^{3,\alpha}(\Omega)} + \|\hat{\psi}-r_b\|_{C^{4,\alpha}(\mathbf{S}^2)} \le C_2(\delta^2+\sigma^2+\varepsilon). \tag{3.30}$$

Now we choose $C_0 = 2C_2$ and $\varepsilon_0 \le \frac{1}{8C_2^2}$. Then, for $\delta=\sigma=C_0\varepsilon$, the above estimate shows that $\hat{U}\in O_\delta$ and $\hat{\psi}\in\mathcal{K}_\sigma$.

3.9. **Contraction of the Mapping.** Now, for $i=1,2$, choose arbitrarily $\psi^{(i)}\in\mathcal{K}_\sigma$ and $\check{U}^{(i)}\in O_\delta$, and let $\hat{U}^{(i)}$ and $\hat{\psi}^{(i)}$ be obtained by the above process correspondingly. Then

$$\|\hat{U}^{(1)}-\hat{U}^{(2)}\|_{C^{2,\alpha}(\Omega)} \le C_3\varepsilon\Big(\|\check{U}^{(1)}-\check{U}^{(2)}\|_{C^{2,\alpha}(\Omega)} + \|\psi^{(1)}-\psi^{(2)}\|_{C^{3,\alpha}(\mathbf{S}^2)}\Big), \tag{3.31}$$

$$\|\hat{\psi}^{(1)}-\hat{\psi}^{(2)}\|_{C^{3,\alpha}(\mathbf{S}^2)} \le C_3\varepsilon\Big(\|\check{U}^{(1)}-\check{U}^{(2)}\|_{C^{2,\alpha}(\Omega)} + \|\psi^{(1)}-\psi^{(2)}\|_{C^{3,\alpha}(\mathbf{S}^2)}\Big). \tag{3.32}$$

This can be achieved by employing the equations of $\hat{U}^{(1)}-\hat{U}^{(2)}$ and the estimates of higher order terms as sketched below.

First, for any higher order term f, when $\delta=\sigma=C_0\varepsilon$, $\psi^{(i)}\in\mathcal{K}_\sigma$, and $\check{U}^{(i)}\in O_\delta$, we have

$$\begin{aligned}&\|f(\psi^{(1)},U^{(1)},U^{-(1)},\cdots)-f(\psi^{(2)},U^{(2)},U^{-(2)},\cdots)\|_*\\ &\le C_2\varepsilon\Big(\|\psi^{(1)}-\psi^{(2)}\|_{C^{3,\alpha}(\mathbf{S}^2)} + \|\check{U}^{(1)}-\check{U}^{(2)}\|_{C^{2,\alpha}(\Omega)}\Big),\end{aligned}$$

where $U^{(i)} := \check{U}^{(i)}+U_b^+$, $U^{-(i)} = {\psi^{(i)}}^*(U^-)$, and $\|\cdot\|_*$ is the corresponding norm for f when $\psi\in C^{3,\alpha}$ and $U\in C^{2,\alpha}$. As two examples, we have

(a). By the mean value theorem and (1.11),

$$\begin{aligned}&\|(\psi^{(1)})^*(U^- - U_b^-) - (\psi^{(2)})^*(U^- - U_b^-)\|_{C^{2,\alpha}(\mathbf{S}^2)}\\ &\le C\|D(U^- - U_b^-)\|_{C^{2,\alpha}(\mathcal{M})}\|\psi^{(1)}-\psi^{(2)}\|_{C^{2,\alpha}(\mathbf{S}^2)} \le C\varepsilon\|\psi^{(1)}-\psi^{(2)}\|_{C^{3,\alpha}(\mathbf{S}^2)};\end{aligned} \tag{3.33}$$

(b). For $\bar{f}_9^{(i)} = \bar{f}_9(U^{(i)},U^{-(i)},\psi^{(i)},DU^{(i)},D^2U^{(i)})$,

$$\Big\|\bar{f}_9^{(1)}-\bar{f}_9^{(2)}\Big\|_{C^\alpha(\Omega)} \le C\varepsilon\Big(\|\psi^{(1)}-\psi^{(2)}\|_{C^{3,\alpha}(\mathbf{S}^2)} + \|\check{U}^{(1)}-\check{U}^{(2)}\|_{C^{2,\alpha}(\Omega)}\Big) \tag{3.34}$$

by the definition of higher order terms.

Next, consider the equations of $\hat{U}^{(1)}-\hat{U}^{(2)}$. The right-hand sides of the elliptic equations of $\hat{p}^{(1)}-\hat{p}^{(2)}$, such as $\bar{f}_9^{(1)}-\bar{f}_9^{(2)}$ (cf. (3.2)) and $\bar{g}_8^{(1)}-\bar{g}_8^{(2)}$ (cf.(3.4)), are in C^α. Therefore, we can obtain a $C^{2,\alpha}$–estimate of $\hat{U}^{(1)}-\hat{U}^{(2)}$, rather than a $C^{3,\alpha}$–estimate. The reason is that the loss of derivative occurs in solving the transport equations. For example, from (3.20) and (3.21), we have

$$D_{u^{(1)}}(\widehat{A(S)}^{(1)}-\widehat{A(S)}^{(2)})=-D_{u^{(1)}-u^{(2)}}\widehat{A(S)}^{(2)} \qquad \text{in } \Omega, \tag{3.35}$$

$$\begin{aligned}&\widehat{A(S)}^{(1)}-\widehat{A(S)}^{(2)}\\&=\mu_4(\hat{\psi}^{(1)}-\hat{\psi}^{(2)})+\big(\bar{g}_4(U^{(1)},U^{-(1)},\psi^{(1)},D\psi^{(1)})-\bar{g}_4(U^{(2)},U^{-(2)},\psi^{(2)},D\psi^{(2)})\big) \quad \text{on } \Omega^0.\end{aligned} \tag{3.36}$$

Note that the right-hand side of (3.35) is only in $C^{2,\alpha}(\Omega)$.

We omit the details of deriving estimates (3.31)–(3.32), since the process is similar to those in §3.2–§3.7 except the two points explained above.

Then the mapping $\mathcal{T}$ has a unique fixed point if ε_0 is small enough by a simple generalized Banach fixed point theorem. In particular, the uniqueness of the fixed point implies the transonic shock solution of (1.1)–(1.3) satisfying the requirements in Theorem 1.1 is unique, as claimed there. We also note that, although $\mathcal{T}$ always has a fixed point as we proved, it is not clear yet whether this fixed point is a solution to (1.1)–(1.3); therefore, in order to obtain the existence result as assumed in Theorem 1.1, it requires further work, which is out of scope of this paper.

Appendix A. Some Notations and Facts of Differential Geometry

In this appendix, we present some notations in differential geometry and some basic facts used above for self-containedness.

A.1. **The Metric of $\mathcal{M}$ in Local Spherical Coordinates.** In local spherical coordinates, for $r^0\le x^0\le r^1, 0\le x^1<\pi, -\pi\le x^2<\pi$, the standard Euclidean metric of $\mathcal{M}$ can be written as

$$G=G_{ij}dx^i\otimes dx^j=dx^0\otimes dx^0+(x^0)^2dx^1\otimes dx^1+(x^0\,\sin x^1)^2dx^2\otimes dx^2.$$

Hence, $\sqrt{G}:=\sqrt{\det(G_{ij})}=(x^0)^2\sin x^1$. For the Christoffel symbols, since $\Gamma^i_{jk}=\Gamma^i_{kj}$, only the following are nonzero:

$$\begin{aligned}&\Gamma^0_{11}=-x^0,\quad \Gamma^0_{22}=-x^0(\sin x^1)^2,\quad \Gamma^1_{01}=\Gamma^1_{10}=\frac{1}{x^0},\\&\Gamma^1_{22}=-\sin x^1\cos x^1,\quad \Gamma^2_{02}=\Gamma^2_{20}=\frac{1}{x^0},\quad \Gamma^2_{12}=\Gamma^2_{21}=\cot x^1.\end{aligned}$$

We also use (G^{ij}) to denote the inverse of the matrix (G_{ij}), and $|u|^2=G(u,u)=G_{ij}u^iu^j$.

In local spherical coordinates, we write the standard metric of $\mathbf{S}^2$ as

$$g=g_{\alpha\beta}dx^\alpha\otimes dx^\beta:=dx^1\otimes dx^1+(\sin x^1)^2dx^2\otimes dx^2.$$

Therefore, we have

$$\sqrt{g}:=\sqrt{\det(g_{ij})}=\sin x^1=\frac{\sqrt{G}}{(x^0)^2}.$$

The nonzero Christoffel symbols are

$$\gamma^1_{22}=-\sin x^1\cos x^1,\qquad \gamma^2_{12}=\gamma^2_{21}=\cot x^1. \tag{A.1}$$

A.2. **Some Lemmas.** The following results are used in the text.

Lemma A.1. *There exists a unique* $\omega \in A^1(\mathbf{S}^2)$ *that solves*

$$d\omega = \chi, \qquad d^*\omega = \psi, \tag{A.2}$$

if $\chi \in A^2(\mathbf{S}^2)$ *and* $\psi \in A^0(\mathbf{S}^2)$ *satisfy* $\int_{\mathbf{S}^2} \chi = 0$ *and* $\int_{\mathbf{S}^2} \psi \,\mathrm{vol} = 0$.

Proof. Since the first Betti number b_1 of $\mathbf{S}^2$ is 0 (i.e., $b_1 = 0$), by the Hodge theorem, we can uniquely solve ω via

$$\Delta\omega = d^*\chi + d\psi \in A^1(\mathbf{S}^2). \tag{A.3}$$

It suffices to show that (A.2) holds.

First, since $\Delta d = d\Delta$, we have

$$\Delta(d\omega - \chi) = d\Delta\omega - (dd^*\chi + d^*d\chi) = 0. \tag{A.4}$$

In addition, $\int_{\mathbf{S}^2}(d\omega - \chi) = 0$ by the Stokes theorem and the assumption. Since the second Betti number b_2 of $\mathbf{S}^2$ is 1, by the Hodge theorem, the space $\mathcal{H}^2$ of harmonic 2-forms on $\mathbf{S}^2$ is one-dimensional. Note that $\mathrm{vol} \in \mathcal{H}^2$ because $d^* = - * d*$ and $*\mathrm{vol} = 1$. Therefore, $d\omega - \chi = 0$.

Similarly, we have

$$\Delta(d^*\omega - \psi) \quad = \quad d^*\Delta\omega - (dd^*\psi + d^*d\psi) = 0 \tag{A.5}$$

due to $\Delta d^* = d^*\Delta$, and

$$\int_{\mathbf{S}^2} (d^*\omega - \psi)\,\mathrm{vol} = 0,$$

by the divergence theorem and the assumption. Note that the zero-th Betti number b_0 of $\mathbf{S}^2$ is 1; according to the Hodge theorem, the space $\mathcal{H}^0$ of harmonic functions on $\mathbf{S}^2$ is one-dimensional. One easily sees that $1 \in \mathcal{H}^0$. Therefore, $d^*\omega - \psi = 0$ as desired. $\square$

Lemma A.2. *Let the two disjoint components of the boundary* $\partial\mathcal{M}$ *be* $\mathcal{M}^0$ *and* $\mathcal{M}^1$, $k = 2, 3$, *and* $\alpha \in (0, 1)$. *Assume that* $\beta \in C_1^{k-2,\alpha}(\mathcal{M})$, ω_0 *is a* $C^{k,\alpha}$ *1-form, and* ω_1 *is a* $C^{k-1,\alpha}$ *1-form in* $\mathcal{M}$. *Then there exists a unique* $C^{k,\alpha}$ *1-form* ω *that solves the following problem:*

$$\Delta\omega = \beta \qquad in \ \ \mathcal{M}, \tag{A.6}$$

$$\omega|_{\mathcal{M}^0} = \omega_0|_{\mathcal{M}^0}, \tag{A.7}$$

$$D_{\partial_0}\omega|_{\mathcal{M}^1} = \omega_1|_{\mathcal{M}^1}. \tag{A.8}$$

Moreover,

$$\|\omega\|_{C_1^{k,\alpha}(\mathcal{M})} \le C\Big(\|\beta\|_{C_1^{k-2,\alpha}(\mathcal{M})} + \|\omega_0\|_{C_1^{k,\alpha}(\mathcal{M})} + \|\omega_1\|_{C_1^{k-1,\alpha}(\mathcal{M})} \Big). \tag{A.9}$$

Proof. For $\mathcal{M} \subset \mathbb{R}^3$, we choose the spherical coordinates:

$$\tilde{y}^0 = (r^1 - r_b)y^0 + r_b, \quad \tilde{y}^\alpha = y^\alpha, \qquad \alpha = 1, 2.$$

Then we use the standard global Descartes coordinates:

$$(z^0, z^1, z^2) \qquad \text{with } \tilde{y}^0 = \sqrt{(z^0)^2 + (z^1)^2 + (z^2)^2}.$$

Let $\omega = \omega_i dz^i$. By the Weizenböck formula,

$$\Delta\omega = -\sum_{i=0}^{2}(\partial^2_{z^i}\omega_j)dz^j \tag{A.10}$$

holds globally (cf. [12]). Therefore, (A.6)–(A.8) represent three decoupled boundary value problems of the Poisson equations. The uniqueness, existence, and estimates of the solution are then clear. □

The following result follows from Lemma 4.6 in [5]. It particularly implies that the norm of a smooth function in a manifold Ω_2 is equivalent to the norm of its pull back in another manifold Ω_1 which is homeomorphic to Ω_2.

Lemma A.3. *Let Ω_1 and Ω_2 be two open sets in $\mathbb{R}^n$ and $u \in C^{k,\alpha}(\Omega_2)$. Let k be a positive integer and $\alpha \in (0,1)$. Let $\Phi : \Omega_1 \to \Omega_2$ satisfy $\Phi \in C^{k,\alpha}(\Omega_1;\Omega_2)$. Then $u \circ \Phi \in C^{k,\alpha}(\Omega_1)$ and satisfies*

$$\|u \circ \Phi\|_{C^{k,\alpha}(\Omega_1)} \le C\|u\|_{C^{k,\alpha}(\Omega_2)}, \tag{A.11}$$

where $C = C(n, \|\Phi\|_{C^{k,\alpha}(\Omega_1;\Omega_2)})$.

Lemma A.4 ([3])**.** *The eigenvalues of the Hodge Laplacian on $\mathbf{S}^2$ are $\lambda_n = n(n+1), n = 0, 1, 2, \cdots$, and there are $2n+1$ linear independent eigenfunctions $\{u_{n,m}\}_{m=1,\cdots,2n+1}$ corresponding to λ_n, so that the eigenfunctions $\{u_{n,m}\}_{n\in\mathbb{N}\cup\{0\},m=1,\cdots,2n+1}$ are smooth and form a complete unit orthogonal basis of $L^2(\mathbf{S}^2)$.*

A.3. **On the Transport Equations Involving Lie Derivatives in Manifolds.** In Section 3.5, we need solve the differential forms from the Cauchy problems of the transport equations involving Lie derivatives. Here we present the basic theorem with a proof.

Let M be an n-dimensional closed C^∞ differentiable manifold, $\mathcal{M} = [0,T] \times M$, X a C^{k+1}-vector field in $\mathcal{M}$ which is transverse to $\Gamma^t = \{t\} \times M$ for $t \in [0,T]$, and f a C^k–function in $\mathcal{M}$ (k is a nonnegative integer). Without loss of generality, we assume that X points to the interior of $\mathcal{M}$ when restricted on Γ^0. We wish to solve a r-form ω ($r \ge 1$) in $\mathcal{M}$ which satisfies the following problem:

$$\mathcal{L}_X\omega + f\omega = \theta \quad \text{in} \quad \mathcal{M}, \tag{A.12}$$

$$\omega = \omega^0 \qquad \text{on} \quad \Gamma^0. \tag{A.13}$$

Here $\mathcal{L}$ is the Lie derivative in $\mathcal{M}$, θ is a given C^k r-form in $\mathcal{M}$, and ω^0 is a given point-wise defined r-form of class C^k on Γ^0.

We have the following existence and uniqueness results:

Lemma A.5. *Under the above assumptions, there is a unique r-form ω in $\mathcal{M}$ that solves* (A.12)–(A.13)*. In addition, there holds*

$$\|\omega\|_{C^k(\mathcal{M})} \le C\big(\|\theta\|_{C^k(\mathcal{M})} + \|\omega^0\|_{C^k(\Gamma^0)}\big), \tag{A.14}$$

with a positive constant C depending only on $\|f\|_{C^k(\mathcal{M})}$ and $\|X\|_{C^{k+1}(\mathcal{M})}$.

For the proof, we first get familiar what (A.12) stands for in a local coordinate chart.

Let E be a local coordinate chart of M. Then $\tilde{E} = [0,T] \times E$ is a coordinate chart of $\mathcal{M}$. In $\tilde{E}$, problem (A.12)–(A.13) is an initial value problem of the transport equations. To see this, for simplicity, suppose that $X = X^0\partial_0 + X^\alpha\partial_\alpha$, with $X^\alpha = 0$ for $\alpha = 1, \cdots, n$ in $\mathcal{M}$. Since x^0 is a global coordinate, $X^0 = dx^0(X)$ is a C^k–function defined in $\mathcal{M}$. By our assumption, X^0 is positive and bounded away from zero since $\mathcal{M}$ is compact.

Suppose that

$$\omega = \omega_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r} = r! \sum_{0\le i_1<\cdots<i_r\le n} \omega_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \tag{A.15}$$

$$\theta = \theta_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r} = r! \sum_{0\le i_1<\cdots<i_r\le n} \theta_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}. \tag{A.16}$$

Since, for the differential forms α and β, there hold

$$\mathcal{L}_X(\alpha\wedge\beta) = (\mathcal{L}_X\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_X\beta),$$

$d\mathcal{L}_X\alpha = \mathcal{L}_X(d\alpha)$, and $\mathcal{L}_X f = Xf$ (cf. [12]), we obtain

$$\begin{aligned}
\mathcal{L}_X\omega \;=\;& r! \sum_{0<i_2<\cdots<i_r\le n} \Big(X^0\partial_0\omega_{0i_2\cdots i_r} + \omega_{0i_2\cdots i_r}\partial_0 X^0\Big) dx^0\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}\\
&+ r! \sum_{0<i_1<\cdots<i_r\le n} X^0\partial_0\omega_{i_1\cdots i_r} dx^{i_1}\wedge\cdots\wedge dx^{i_r}\\
&+ r! \sum_{0<i_2<\cdots<i_r\le n} \omega_{0i_2\cdots i_r}\partial_\alpha X^0 dx^\alpha\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}\\
=\;& r! \sum_{0<i_2<\cdots<i_r\le n} \Big(X^0\partial_0\omega_{0i_2\cdots i_r} + \omega_{0i_2\cdots i_r}\partial_0 X^0\Big) dx^0\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}\\
&+ r! \sum_{0<i_1<\cdots<i_r\le n} \Big(X^0\partial_0\omega_{i_1\cdots i_r} + \frac{1}{(r-1)!}\\
&\sum_{\sigma\in\mathcal{P}(r)} \big(\mathrm{sign}(\sigma)\ \omega_{0i_{\sigma(2)}\cdots i_{\sigma(r)}}\partial_{i_{\sigma(1)}} X^0\big)\Big) dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}.
\end{aligned} \tag{A.17}$$

Here $\mathcal{P}(r)$ is the permutation group of $\{1, \cdots, r\}$, and $\mathrm{sign}(\sigma)$ is the sign of a permutation σ.

Hence, by dividing X^0 from both sides of equation (A.12), we have

$$\partial_0\omega_{0i_2\cdots i_r} + \big(\frac{1}{X^0}\partial_0 X^0 + \frac{f}{X^0}\big)\omega_{0i_2\cdots i_r} = \frac{1}{X^0}\theta_{0i_2\cdots i_r}, \quad 0 < i_2 < \cdots < i_r \le n; \tag{A.18}$$

and

$$\begin{aligned}
&\partial_0\omega_{i_1\cdots i_r} + \frac{f}{X^0}\omega_{i_1\cdots i_r}\\
&= \frac{1}{X^0}\theta_{i_1\cdots i_r} - \frac{1}{(r-1)!X^0}\sum_{\sigma\in\mathcal{P}(r)} \big(\mathrm{sign}(\sigma)\partial_{i_{\sigma(1)}} X^0\omega_{0i_{\sigma(2)}\cdots i_{\sigma(r)}}\big), \quad 0 < i_1 < \cdots < i_r \le n.
\end{aligned} \tag{A.19}$$

These are linear transport equations in the x^0-direction.

We can first solve (A.18) in $\tilde{E}$ by using the initial data $\omega_{0i_2\cdots i_r}|_{\Gamma^0} = (\omega^0)_{0i_2\cdots i_r}$ and then substitute $\omega_{0i_2\cdots i_r}$ in the right-hand side of (A.19) to solve $\omega_{i_1\cdots i_r}$ in $\tilde{E}$ with initial data $\omega_{i_1i_2\cdots i_r}|_{\Gamma^0} = (\omega^0)_{i_1i_2\cdots i_r}$. By antisymmetry of the lower indices, we obtain all the coefficients $\omega_{i_1\cdots i_r}$. Since the quantities in (A.12)–(A.13) are defined globally, $\omega = \omega_{i_1\cdots i_r} dx^{i_1}\wedge\cdots dx^{i_r}$ we solved is also well defined in $\mathcal{M}$. Estimate (A.14) is obvious from these initial value problems of ordinary differential equations.

For the general case that $X \neq X^0\partial_0$, equation (A.12) would be a first-order hyperbolic system with $\binom{n+1}{r}$ unknowns. In addition, we can not use the rather simple coordinate charts like $\tilde{E}$, since the characteristic curves (i.e., integral curves of X) may escape $\tilde{E}$ at some $t < T$ by solving the initial value problem.

We note that the Lie derivative behaves well under the homeomorphisms of differentiable manifolds.

Proposition A.1. *Let Φ be a homeomorphism of $\mathcal{M}$. Then*

$$(\Phi^{-1})^*\mathcal{L}_X\omega = \mathcal{L}_{\Phi_* X}((\Phi^{-1})^*\omega). \tag{A.20}$$

This can be shown by using the Cartan formula $\mathcal{L}_X\omega = di_X\omega + i_X d\omega$ and the formulae

$$\Phi^* d\omega = d\Phi^*\omega,\ (\Phi^{-1})^*(i_X\omega) = i_{\Phi_* X}(\Phi^{-1})^*\omega.$$

(cf. [12]), where $i_X\omega$ is the interior product of ω and X.

In addition, by Lemma A.3, the C^k–norm of a differential form in $\mathcal{M}$ is equivalent under C^k–homeomorphism of the manifold. Thus, to prove Lemma A.5 for the general case, it suffices to straighten the vector field X to the form $X^0\partial_0$ globally by a suitable homeomorphism of $\mathcal{M}$.

Proposition A.2. *For given C^{k+1}–vector field X in $\mathcal{M}$ which is transverse to $\{x^0\}\times M$ for every $x^0 \in [0,T]$, there is a C^{k+1}–homeomorphism Φ: $\mathcal{M} \to \mathcal{M}$ such that, for any fixed $x^0 \in [0,T]$, it is also a homeomorphism of $\{x^0\}\times M$, and $\Phi_* X = (\Phi^{-1})^*(dx^0(X))\partial_0$.*

Proof. Step 1. Let $\tilde{E}$ be a coordinate chart of $\mathcal{M}$ as introduced above and $X = X^i\partial_i$ with $X^0 > 0$. Since X^0 is a nonzero function in $\mathcal{M}$, we may define a C^{k+1}–vector field $\tilde{X}$ in $\mathcal{M}$ by

$$\tilde{X} = \tilde{X}^i\partial_i = \partial_0 + \frac{X^\alpha}{X^0}\partial_\alpha. \tag{A.21}$$

Step 2. Now $\tilde{X}$ generates a flow ϕ_t in $\mathcal{M}$ by theory of ordinary differential equations since $\mathcal{M}$ is compact: For any $P \in M$, $\gamma(t) = \phi_t(P)$, $t \in [0,T]$, is a curve in $\mathcal{M}$ with initial value $\gamma(0) = (0,P) \in \Gamma^0$. We then define $\Phi : \mathcal{M} \to \mathcal{M}$ by

$$\Phi(\phi_t(P)) = (t,P), \qquad P \in M. \tag{A.22}$$

Note that $\phi_t(P) \in \Gamma^t$ since $\tilde{X}^0 = 1$. We now show that Φ is a homeomorphism.

Step 3. For any $Q \in \mathcal{M}$, it is easy to see that there is uniquely a pair (t,P) with $P \in M$ and $t \in [0,T]$ such that $\phi_t(P) = Q$ by solving backward the integral curve of $\tilde{X}$ through Q. So Φ is defined for all the points in $\mathcal{M}$. In addition, Φ is obviously surjective and injective by the uniqueness and existence results of the initial value problem of ordinary differential equations. By continuous dependence on the initial data $(0,P)$ and t, we see that Φ and Φ^{-1} are also continuous. If $\tilde{X} \in C^{k+1}$, then Φ and Φ^{-1} are C^{k+1}–mappings by C^{k+1}–dependence of solutions of ordinary differential equations on t and initial data. This proves that Φ is a C^{k+1}–homeomorphism.

Step 4. Now, by definition of push-forward mapping (tangent mapping) of vectors, we have

$$\begin{aligned}\Phi_*(X) &= \Phi_*(X^0\tilde{X}) = (\Phi^{-1})^*(X^0)\Phi_*(\tilde{X})\\ &= (\Phi^{-1})^*(X^0)\frac{d}{dt}\Phi(\gamma(t))\\ &= (\Phi^{-1})^*(X^0)\partial_0.\end{aligned} \tag{A.23}$$

This completes the proof. □

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