Hyperbolic Conservation Laws with Stiff Relaxation Terms and Entropy

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Comm. Pure & Appl. Math.
submitted February 27, 1992
accepted November 20, 1992

Abstract

We study the limiting behavior of systems of hyperbolic conservation laws with stiff relaxation terms. Reduced systems, inviscid and viscous local conservation laws, and weakly nonlinear limits are derived through asymptotic expansions. An entropy condition is introduced for $N \times N$ systems that ensures the hyperbolicity of the reduced inviscid system. The resulting characteristic speeds are shown to be interlaced with those of the original system. Moreover, the first correction to the reduced system is shown to be dissipative. A partial converse is proved for 2×2 systems. This structure is then applied to study the convergence to the reduced dynamics for the 2×2 case.

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1. Introduction

We are concerned with the phenomena of relaxation, particularly the question of stability and singular limits of zero relaxation time. Relaxation is important in many physical situations. For example, it arises in kinetic theory [5], gases not in local thermodynamic equilibrium [17,27], elasticity with memory [11,15,22], multiphase and phase transition [14,23], and linear and nonlinear waves [28]. In general, relaxations can be modeled as having a functional dependence on the basic dependent variables. In this article we consider the case where the relaxation depends on the local values of these variables. Thus, we consider an evolutionary system of partial differential equations in the form

$$\partial_t U + \nabla\cdot F(U) + \frac{1}{\epsilon} R(U) = 0\,. \tag{1.1}$$

Here $U = U(t,x)$, which takes on values in $\mathbf{R}^N$, represents the density vector of basic physical variables over the space variable $x \in \mathbf{R}^D$. Consistent with the linear well-posedness of physical examples, we assume that the system is hyperbolic, that is, that the flux vector $F = F(U)$ is such that for every wavenumber $\xi \in \mathbf{R}^D$ the $N \times N$ matrix $\partial_U F(U)\cdot\xi$ has real eigenvalues and is diagonalizable.

The relaxation term is endowed with a $n \times N$ matrix $\mathcal{Q}$ with rank $n < N$ such that

$$\mathcal{Q}R(U) = 0 \qquad \text{for all } U\,. \tag{1.2}$$

This yields n independent conserved quantities $u = \mathcal{Q}U$. In addition, we assume that each such u uniquely determines a local equilibrium value $U = \mathcal{E}(u)$ satisfying $R(\mathcal{E}(u)) = 0$ and such that

$$\mathcal{Q}\mathcal{E}(u) = u, \qquad \text{for all } u\,. \tag{1.3}$$

The image of $\mathcal{E}$ then constitutes the manifold of local equilibria of R.

Associated with $\mathcal{Q}$ are n local conservation laws that are satisfied by every solution of (1.1) and that take the form

$$\partial_t(\mathcal{Q}U) + \nabla\cdot\big(\mathcal{Q}F(U)\big) = 0\,. \tag{1.4}$$

These can be closed as a reduced system for $u = \mathcal{Q}U$ if we make the local equilibrium approximation for U, namely

$$U = \mathcal{E}(u)\,, \tag{1.5}$$

$$\partial_t u + \nabla\cdot f(u) = 0\,, \tag{1.6}$$

where the reduced flux f is defined by

$$f(u) \equiv \mathcal{Q}F\big(\mathcal{E}(u)\big)\,. \tag{1.7}$$

The quantity ϵ is the relaxation time, which is small in many physical situations. In the kinetic theory it is the mean free path, in elasticity the duration of memory. One would expect solutions of the full system (1.1) to tend to those of the local system (1.6) as ϵ goes to zero. This local equilibrium limit turns out to be highly singular because of shock and initial layers. We are also interested in the weakly nonlinear limits when the characteristic values tend to infinity, so as to obtain incompressible flow equations.

Basic to our understanding of system (1.1) is the question of stability. In the classical study of homogeneous systems of hyperbolic conservation laws ($R \equiv 0$), this question is addressed through the notion of entropy, which is a function $\Phi : \mathbf{R}^N \to \mathbf{R}$ such that for every U and ξ the matrix

$$\partial_{UU}\Phi(U)\,\partial_U F(U)\cdot\xi \qquad \text{is symmetric}\,. \tag{1.8}$$

This ensures the existence of an entropy flux $\Psi : \mathbf{R}^N \to \mathbf{R}^D$ such that

$$\partial_U\Phi(U)\,\partial_U F(U) = \partial_U\Psi(U) \qquad \text{for all } U\,. \tag{1.9}$$

If Φ is convex, zero viscosity solutions of the homogeneous conservation laws should satisfy

$$\partial_t\Phi(U) + \nabla\cdot\Psi(U) \leq 0\,, \tag{1.10}$$

with equality for classical solutions. Now, given such a Φ, every classical solution of (1.1) satisfies

$$\partial_t\Phi(U) + \nabla\cdot\Psi(U) + \frac{1}{\epsilon}\partial_U\Phi(U)\,R(U) = 0\,. \tag{1.11}$$

If this equation is to be consistent with (1.10), the relaxation term should be dissipative in the sense that

$$\partial_U \Phi(U)\, R(U) \geq 0\,, \qquad \text{for all } U\,. \tag{1.12}$$

This is reminiscent of the notion of entropy introduced by Boltzmann into his kinetic theory to describe kinetic relaxation to fluid dynamics. His key observation was that his entropy characterizes the local equilibria of his kinetic equation, the celebrated H theorem. In this work we adopt a notion of entropy that shares all of the above properties. Among our main results is the conclusion that if system (1.1) is endowed with such an entropy, the characteristic speeds for the local equilibrium equation (1.6) are real and interlaced with those of the full system. More precise statements are given in the next section. For 2×2 systems the converse also holds, and the aforementioned singular limits can also be justified.

Many physical systems of the form (1.1) exist; we list a few below. The compressible Euler equations are

$$\begin{aligned} &\partial_t \rho + \nabla\cdot(\rho v) = 0\,, \\ &\partial_t(\rho v) + \nabla\cdot(\rho v \otimes v) + \nabla p = 0\,, \\ &\partial_t(\rho E) + \nabla\cdot(\rho E v + p v) = 0\,. \end{aligned} \tag{1.13}$$

In local thermodynamic equilibrium, the system is closed by the constitutive relation

$$p = p(\rho, e)\,, \qquad E = \tfrac{1}{2}|v|^2 + e\,.$$

When the temperature varies over a wide range, the gas may not be in local thermodynamic equilibrium, and the pressure p should then be regarded as a function of only a part e of the specific internal energy, while another part q is governed by a rate equation

$$\partial_t q + \nabla\cdot(q v) = \frac{Q(\rho, e) - q}{\tau(\rho, e)}\,, \tag{1.14}$$

$$p = p(\rho, e)\,, \qquad E = \tfrac{1}{2}|v|^2 + e + q\,. \tag{1.15}$$

Using the notation of the general system (1.1), one identifies

$$U = (\rho, \rho v, \rho E, q)^\top\,, \qquad u = (\rho, \rho v, \rho E)^\top\,.$$

The relaxation time τ and the equilibrium value Q are given functions of the density ρ and the part e of the specific internal energy.

A large class of systems from kinetic theory, the discrete velocity kinetic equations, is represented by the Broadwell model, which has the form

$$\begin{aligned} \partial_t f_- - \partial_x f_- &= \frac{f_+ f_- - f_0^2}{\epsilon}\,, \\ \partial_t f_0 &= \frac{f_0^2 - f_+ f_-}{\epsilon}\,, \\ \partial_t f_+ + \partial_x f_+ &= \frac{f_+ f_- - f_0^2}{\epsilon}\,. \end{aligned} \tag{1.16}$$

Here the relaxation time is the mean free path ϵ, and the conserved quantities u are the density and the momentum as given by

$$u = \begin{pmatrix} \rho \\ \rho v \end{pmatrix} = \mathcal{Q} \begin{pmatrix} f_- \\ f_0 \\ f_+ \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Considerable interest exists in the question of the local equilibrium approximation in kinetic theory, where it is better known as a fluid dynamical approximation. See [3,4,24,29] for such studies in the context of the Broadwell model (1.16), and [2] or [19] for a discussion of more general models.

The simplest physical models are 2×2 systems. Here we mention the river equations [25,28], a shallow-water wave model describing the vertical depth h and mean velocity v by

$$\begin{aligned} \partial_t h + \partial_x (hv) &= 0\,, \\ \partial_t (hv) + \partial_x (hv^2 + \tfrac{1}{2} g h^2) &= ghS - C_f |v| v\,, \end{aligned} \tag{1.17}$$

where g is the gravitational constant, S is the slope of the river bottom, and C_f is the friction coefficient. The local equilibrium limit is identified with the long space-time behavior of solutions of the system (1.17) that is associated with the description of flooding. Care must be taken when devising numerical simulations of the river equations in such regimes [16,25].

As an illustrative 2×2 model, consider the p-system of conservation laws

$$\begin{aligned} &\partial_t u + \partial_x v = 0\,, \\ &\partial_t v + \partial_x p(u) + \frac{1}{\epsilon}\big(v - f(u)\big) = 0\,. \end{aligned} \tag{1.18}$$

This system is hyperbolic provided $p'(u) > 0$; its characteristic speeds then are $\pm\sqrt{p'(u)}$. The positive parameter ϵ is the relaxation time for the system; in the absence of spatial gradients, the value of u remains fixed while that of v evolves to the equilibrium $v = f(u)$. The relaxation term is stiff when $\epsilon << 1$; that is, the relaxation time is much shorter than the time it takes for a hyperbolic wave (sound wave) to propagate over a gradient length. In such a case, one might naively expect that the solution would achieve local equilibrium and the dynamics would be governed by the so-called local equilibrium approximation

$$\begin{aligned} v &= f(u)\,, \\ \partial_t u + \partial_x f(u) &= 0\,. \end{aligned} \tag{1.19}$$

This approximation has two potential problems. First, its solution will develop an infinite spatial derivative at some location in a finite time. This not only violates the key smoothness assumption that led to this approximation, but also complicates the nature of the solutions involved in the analysis. Second, equation (1.2b) has the characteristic speed $f'(u)$; this will exceed the characteristic speeds of the original system unless

$$-\sqrt{p'(u)} \le f'(u) \le \sqrt{p'(u)}\,. \tag{1.20}$$

The consistency of the approximation would seem to require that this bound be satisfied, if only to preserve the proper causality. Indeed, further analysis shows that the local equilibrium $(\bar{u}, f(\bar{u}))$ is linearly stable if and only if $u = \bar{u}$ satisfies the bound (1.20); hence (1.20) will be referred to as a stability criterion.

This issue can be better understood by improving upon the local equilibrium approximation. Let (u^ϵ, v^ϵ) be a family parameterized by ϵ that solves

$$\begin{aligned} &\partial_t u^\epsilon + \partial_x v^\epsilon = 0\,, \\ &\partial_t v^\epsilon + \partial_x p(u^\epsilon) + \frac{1}{\epsilon}\big(v^\epsilon - f(u^\epsilon)\big) = 0\,. \end{aligned}$$

Consider a formal expansion of just v^ϵ in the form

$$v^\epsilon = f(u^\epsilon) + \epsilon\, v_1^\epsilon + \epsilon^2 v_2^\epsilon + \cdots\,,$$

where the ϵ dependence in each v_k^ϵ arises only through u^ϵ and its derivatives. One finds from the above two identities that

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \mathcal{O}(\epsilon)\,,$$
$$\partial_t f(u^\epsilon) + \partial_x p(u^\epsilon) + v_1^\epsilon = \mathcal{O}(\epsilon)\,.$$

Eliminating the time derivative of u^ϵ up to $\mathcal{O}(\epsilon)$ above leads to

$$\Big(p'(u^\epsilon) - f'(u^\epsilon)^2\Big)\partial_x u^\epsilon + v_1^\epsilon = \mathcal{O}(\epsilon)\,.$$

The simplest way to satisfy this relation is to set

$$v_1^\epsilon = -\Big(p'(u^\epsilon) - f'(u^\epsilon)^2\Big)\partial_x u^\epsilon\,.$$

Dropping all the higher-order terms in expansion (1.6) leads to a first-order correction to the local equilibrium approximation in the form

$$\begin{aligned} v &= f(u) - \epsilon\,\big(p'(u) - f'(u)^2\big)\partial_x u\,,\\ \partial_t u + \partial_x f(u) &= \epsilon\,\partial_x\Big(\big(p'(u) - f'(u)^2\big)\partial_x u\Big)\,. \end{aligned} \tag{1.21}$$

This evolution equation will be dissipative provided that the stability criterion (1.20) holds. The criterion is now intrinsically manifest in the approximation (1.21), whereas before it was inferred through reference to the original system (1.18).

The approximation is in the spirit of that of Chapman-Enskog for the kinetic theory by which the Navier-Stokes equations are derived. The approximation (1.21) cannot be viewed as the local equilibrium limit of (1.1) in any rigorous sense. Instead, it holds when the solutions are dissipative so that higher differentiations of the solutions are smaller. Thus it holds in the time-asymptotic sense.

Such 2×2 systems have been studied in [8] and [20]. In [20] the viscous approximation and time-asymptotic justification are shown for general systems. Particular solutions of traveling waves and rarefaction waves are constructed, and their stability is studied. In the local equilibrium limit, solutions of (1.1) tend to those of the local equilibrium approximation (1.6). The limit is highly singular because of shock and initial layers. In [8] this limit is studied for the physical models in elasticity and phase transition. Specifically, entropy pairs are constructed to derive the energy estimates, and the compensated compactness method is then applied to control the oscillations.

In the present article we study the limiting behavior of general systems with stiff relaxation terms. In Section 2 we construct the local equilibrium approximation and its first correction for a general $N \times N$ system of hyperbolic conservation laws with appropriate relaxation terms. A general notion of entropy is introduced for such systems, the existence of which ensures the hyperbolicity of the local equilibrium approximation and the dissipativity of its first correction. In Section 3 we show that for general 2×2 strictly hyperbolic systems the existence of dissipative entropies is implied by a strict stability criterion that the equilibrium characteristic lies between the frozen characteristics (cf. (1.20)). The validity of the local equilibrium limit for such 2×2 systems is then established in Section 4. Finally, in Section 5 we derive the weakly nonlinear limit for 2×2 systems and justify it through energy estimates. This limit is the third approximation; it shares the feature of the local equilibrium approximation (1.6) that it does not contain the relaxation time. It is based on the observation that the linearization of the local equilibrium approximation (1.2) about an equilibrium $u = \bar{u}$ gives a simple advection dynamics with speed $f'(\bar{u})$. This suggests that, for solutions of the original system (1.1), a small perturbation about an equilibrium $(u, v) = (\bar{u}, f(\bar{u}))$ will be slowly varying in the corresponding moving frame.

2. The Structure for General Systems

Consider the $N \times N$ system of conservation laws over $\Omega \subset \mathbf{R}^D$ in the form

$$\partial_t U + \nabla \cdot F(U) + \frac{1}{\epsilon} R(U) = 0 \,. \tag{2.1}$$

Here the density vector $U = U(t, x)$ takes on values in $\bar{\mathbf{O}}$, the closure of an open convex set $\mathbf{O} \subset \mathbf{R}^N$. The flux vector $F : \mathbf{O} \to \mathbf{R}^{N \times D}$ is a twice-differentiable function such that the system is hyperbolic. More precisely, this means that for every $U \in \mathbf{O}$ and every wavenumber $\xi \in \mathbf{R}^D$, the $N \times N$ matrix $\partial_U F(U) \cdot \xi$ has real eigenvalues and is diagonalizable.

The relaxation term $R : \mathbf{O} \to \mathbf{R}^N$ is a vector field that leaves $\mathbf{O}$ invariant under the flow

$$\frac{d}{dt}U + \frac{1}{\epsilon}R(U) = 0\,, \tag{2.2}$$

such that it has $n < N$ independent linear conserved quantities

$$\mathcal{Q} : \mathbf{R}^N \to \mathbf{R}^n \qquad \text{such that } \mathcal{Q} \text{ has rank } n\,, \tag{2.3a}$$

$$\mathcal{Q}R(U) = 0 \qquad \text{for every } U \in \mathbf{O}\,. \tag{2.3b}$$

Moreover, each orbit has an equilibrium that is uniquely determined by the constants $u = \mathcal{Q}U$ denoted $\mathcal{E}(u)$. The map

$$\mathcal{E} : \mathbf{o} \to \mathbf{O}\,, \qquad \text{where } \mathbf{o} \equiv \mathcal{Q}\mathbf{O} \subset \mathbf{R}^n \text{ is an open convex set}\,, \tag{2.4}$$

and satisfies the identities

$$\mathcal{Q}\mathcal{E}(u) = u \qquad \text{for every } u \in \mathbf{o}\,, \tag{2.5a}$$

$$\mathcal{Q}\,\partial_u\mathcal{E}(u) = I \qquad \text{for every } u \in \mathbf{o}\,. \tag{2.5b}$$

This last implies that for every $u \in \mathbf{o}$ the $N \times N$ matrix $P(u) \equiv \partial_u\mathcal{E}(u)\mathcal{Q}$ is a projection ($P^2 = P$) onto the tangent space of the image of $\mathcal{E}$, which is therefore the n-dimensional manifold of local equilibria.

Throughout this section we shall adopt the following notational conventions. Variables uniquely associated with either $\mathbf{R}^n$ or $\mathbf{R}^N$ will be denoted by lower or upper case Roman fonts respectively. The calligraphical font is used for quantities that go between $\mathbf{R}^n$ and $\mathbf{R}^N$ like $\mathcal{E}$ and $\mathcal{Q}$. Products are matrix multiplication with the understanding that vectors in $\mathbf{R}^n$ or $\mathbf{R}^N$ are column vectors, while those from $\mathbf{R}^{n*}$ or $\mathbf{R}^{N*}$ are row vectors. Gradients of scalars with respect to column vectors are row vectors, and vice versa.

Associated with $\mathcal{Q}$ are n local conservation laws that are satisfied by every solution of (2.1) and take the form

$$\partial_t(\mathcal{Q}U) + \nabla\cdot\big(\mathcal{Q}F(U)\big) = 0\,. \tag{2.6}$$

These can be closed as a reduced system for $u = \mathcal{Q}U$ if we make the local equilibrium approximation

$$U = \mathcal{E}(u)\,, \tag{2.7a}$$

$$\partial_t u + \nabla\cdot f(u) = 0\,, \tag{2.7b}$$

where the reduced flux f is defined by

$$f(u) \equiv \mathcal{Q}F\big(\mathcal{E}(u)\big)\,. \tag{2.8}$$

This approximation may not even be hyperbolic, much less have excessive characteristic speeds. For example, consider the symmetric 3×3 linear system in one spatial dimension:

$$\partial_t \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \frac{1}{\epsilon} \begin{pmatrix} 0 \\ 0 \\ w - 2u \end{pmatrix} = 0\,. \tag{2.9}$$

Here the local equilibrium approximation is obtained by setting $w = 2u$ in the first two equations. This yields

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = 0\,,$$

which are the Cauchy-Riemann equations, and hence elliptic. These do not even have a well-posed initial-value problem. Although the initial-value problem for system (2.9) is well posed for each positive ϵ, the bounds are not uniform as ϵ tends to zero. This behavior can also be connected with the fact that all of the equilibria $(u, v, w) = (\bar{u}, \bar{v}, 2\bar{u})$ are unstable. Even if the reduced system (2.7b) is hyperbolic, the approximation is subject to the same kind of questions as were raised regarding the p-system (1.18), such as whether it has excessive characteristic speeds.

Whenever the local equilibrium approximation is hyperbolic, it makes sense to seek a first-order correction. The validity of the local equilibrium approximation would suggest that the conserved density functions u could be used as coordinates for a subset of the density functions U that is invariant, or at least approximately invariant, under the evolution (2.1). Such a set would be the image of the conserved densities u under a coordinate map $u \mapsto U = \mathcal{M}[u]$ such that $\mathcal{Q}\mathcal{M}[u] = u$. Here the notation indicates that, unlike $\mathcal{E}(u)$, the value of $\mathcal{M}[u]$ at x generally depends on more than the value of u at x. For example, the dependence may be nonlocal, or possibly local through the values of spatial derivatives of u. Assuming differentiability, denote the Frechét derivative of $\mathcal{M}$ with respect to u by $D_u\mathcal{M}[u]$.

If the image of $\mathcal{M}$ were invariant under the evolution (2.1) the conserved densities u would satisfy the closed system

$$\partial_t u + \nabla\cdot\Big(\mathcal{Q}F\big(\mathcal{M}[u]\big)\Big) = 0\,. \tag{2.10}$$

Assume that the dynamics of u is now governed by (2.10). The evolution of $U = \mathcal{M}[u]$ is then given by

$$\partial_t U = D_u\mathcal{M}[u]\,\partial_t u = -D_u\mathcal{M}[u]\,\mathcal{Q}\,\nabla\cdot F\big(\mathcal{M}[u]\big)\,,$$

and hence

$$\begin{aligned}\partial_t U + \nabla\cdot F(U) + \frac{1}{\epsilon}R(U)& \\ = \big(I - D_u\mathcal{M}[u]\mathcal{Q}\big)\,\nabla\cdot F\big(\mathcal{M}[u]\big) + \frac{1}{\epsilon}R\big(\mathcal{M}[u]\big)&\,. \end{aligned}\tag{2.11}$$

If $\mathcal{M} = \mathcal{M}[u]$ could be found such that

$$\big(I - D_u\mathcal{M}[u]\mathcal{Q}\big)\,\nabla\cdot F\big(\mathcal{M}[u]\big) + \frac{1}{\epsilon}R\big(\mathcal{M}[u]\big) = 0\,, \tag{2.12}$$

the image of $\mathcal{M}$ would be an invariant set under the evolution (2.1). While this is too much to ask, we can seek an approximately invariant set by finding an $\mathcal{M}$ for which this term is small.

Since the local equilibrium approximation suggests the possibility of such an approximately invariant set for small ϵ, it is natural to seek a formal expansion for $\mathcal{M}^\epsilon[u]$ in powers of ϵ as

$$\mathcal{M}^\epsilon[u] = \mathcal{E}(u) + \epsilon\,\mathcal{M}^{(1)}[u] + \epsilon^2\mathcal{M}^{(2)}[u] + \cdots\,. \tag{2.13}$$

The equations to be satisfied are

$$\big(I - D_u\mathcal{M}^\epsilon[u]\mathcal{Q}\big)\,\nabla\cdot F\big(\mathcal{M}^\epsilon[u]\big) + \frac{1}{\epsilon}R\big(\mathcal{M}^\epsilon[u]\big) = 0\,, \tag{2.14a}$$

$$\mathcal{Q}\mathcal{M}^\epsilon[u] = u\,. \tag{2.14b}$$

Setting expansion (2.13) into (2.14) and matching terms order by order in ϵ, at first order one obtains

$$\begin{aligned} \big(I - P(u)\big)\,\nabla\cdot F\big(\mathcal{E}(u)\big) + \partial_U R\big(\mathcal{E}(u)\big)\,\mathcal{M}^{(1)}[u] = 0\,, \\ \mathcal{Q}\mathcal{M}^{(1)}[u] = 0\,, \end{aligned} \tag{2.15}$$

while at higher orders one finds the general form

$$\partial_U R\big(\mathcal{E}(u)\big)\,\mathcal{M}^{(k)}[u] = \mathcal{J}^{(k)}[u]\,, \qquad \mathcal{Q}\mathcal{M}^{(k)}[u] = 0\,, \tag{2.16}$$

for $k > 1$, where the term $\mathcal{J}^{(k)}[u]$ depends explicitly on those $\mathcal{M}^{(j)}[u]$ with order j strictly less than k.

At this point we will assume that the null space of the $N \times N$ matrix $\partial_U R\big(\mathcal{E}(u)\big)$ has dimension exactly n. The linear system

$$\partial_U R\big(\mathcal{E}(u)\big)V = J\,, \qquad \mathcal{Q}V = 0\,, \tag{2.17}$$

will then have a unique solution if and only if

$$\mathcal{Q}J = 0\,; \tag{2.18}$$

we denote this solution by

$$V = \Big(\partial_U R\big(\mathcal{E}(u)\big)\Big)^{-1} J\,. \tag{2.19}$$

Since $I - P(u)$ is annihilated by left multiplication by $\mathcal{Q}$, the solution of (2.15) is then simply

$$\mathcal{M}^{(1)}[u] = -\Big(\partial_U R\big(\mathcal{E}(u)\big)\Big)^{-1}\big(I - P(u)\big)\,\nabla\cdot F\big(\mathcal{E}(u)\big)\,. \tag{2.20}$$

Furthermore, since (2.14a) is annihilated upon left multiplication by $\mathcal{Q}$, that will also be the case order by order. Since left multiplication by $\mathcal{Q}$ also annihilates the left side of (2.15b), it follows that

$$\mathcal{Q}\,\mathcal{J}^{(k)}[u] = 0\,, \qquad \text{for every } k > 1\,. \tag{2.21}$$

Thus, equation (2.16) can be solved recursively to systematically obtain each term in expansion (2.13) as

$$\mathcal{M}^{(k)}[u] = \Big(\partial_U R\big(\mathcal{E}(u)\big)\Big)^{-1} \mathcal{J}^{(k)}[u]\,. \tag{2.22}$$

This formal expansion is just the quasi-linear analogue of the classical Chapman-Enskog expansion of kinetic theory [5], the local equilibrium approximation being analogous to the compressible Euler approximation.

The first order correction to the local equilibrium approximation is obtained by truncating the above expansion after order ϵ; it is the analogue of the compressible Navier-Stokes approximation in the kinetic theory context. Thus, the first correction is

$$\begin{aligned} &U = \mathcal{E}(u) - \epsilon \Big(\partial_U R\big(\mathcal{E}(u)\big)\Big)^{-1} \big(I - P(u)\big)\, \nabla\!\cdot\! F\big(\mathcal{E}(u)\big)\,, \\ &\partial_t u + \nabla\!\cdot\! f(u) = \epsilon\, \nabla\!\cdot\! \Big[\mathcal{Q}\partial_U F\big(\mathcal{E}(u)\big) \Big(\partial_U R\big(\mathcal{E}(u)\big)\Big)^{-1} \big(I - P(u)\big)\, \nabla\!\cdot\! F\big(\mathcal{E}(u)\big)\Big]\,. \end{aligned} \tag{2.23}$$

It is not generally clear, even when the zero-order approximation is hyperbolic and its characteristic speeds are not excessive, that this first-order correction will be dissipative. It can be shown that this will be the case whenever the linear constant coefficient problem obtained by linearizing the original problem about any absolute equilibrium $\mathcal{E}(\bar{u})$ is stable as $\epsilon \to 0$. However, this is a cumbersome criterion to check. Here we present a simple alternative criterion, namely, the existence of a strictly convex entropy.

Definition 2.1. *A twice-differentiable function $\Phi : \mathbf{O} \to \mathbf{R}$ is said to be an entropy for the system (2.1) provided*

(i) $\partial_{UU}\Phi(U)\, \partial_U F(U)\!\cdot\!\xi$ *is symmetric for all* $U \in \mathbf{O}$ *and* $\xi \in \mathbf{R}^D$*;*

(ii) $\partial_U \Phi(U)\, R(U) \geq 0 \qquad$ *for all* $U \in \mathbf{O}$ *;*

(iii) for any $U \in \mathbf{O}$ *the following are equivalent*

(a) $R(U) = 0$ *,*

(b) $\partial_U \Phi(U)\, R(U) = 0$ *,*

(c) $\partial_U \Phi(U) = vQ$, *for some* $v \in \mathbf{R}^{n*}$.

An entropy Φ *is called convex if*

(iv) $\partial_{UU}\Phi(U) \geq 0$ *as a quadratic form for all* $U \in \mathbf{O}$*;*

if the inequality in (iv) is strict, the entropy is called strictly convex.

Notice that (i) is the classical Lax entropy condition for hyperbolic conservation laws [18]. It ensures the existence of an entropy flux $\Psi : \mathbf{O} \to \mathbf{R}^D$ such that

$$\partial_U \Phi(U)\, \partial_U F(U) = \partial_U \Psi(U)\,, \qquad \text{for all } U \in \mathbf{O}. \tag{2.24}$$

Formally, every classical solution of (2.1) then satisfies

$$\partial_t \Phi(U) + \nabla\cdot\Psi(U) + \frac{1}{\epsilon}\partial_U \Phi(U)\, R(U) = 0\,, \tag{2.25}$$

so that by (ii) the entropy Φ is locally dissipated. Moreover, if Φ is strictly convex, the characteristic speeds associated with any wave vector $\xi \in \mathbf{R}^D$ are determined by the critical values of the Rayleigh quotient

$$W \mapsto \frac{W^\top \partial_{UU}\Phi(U)\, \partial_U F(U)\cdot\xi\, W}{W^\top \partial_{UU}\Phi(U)\, W}\,, \qquad \text{for } W \in \mathbf{R}^N. \tag{2.26}$$

Note that (ii) and (iii) are an abstraction of Boltzmann's H theorem [5]. In particular, (iii) completely characterizes the local equilibria $\mathcal{E} = \mathcal{E}(u)$ in terms of Φ and $\mathcal{Q}$ as follows.

Let $\Phi^* : \mathbf{O}^* \to \mathbf{R}$ be the Legendre dual function of the strictly convex entropy function Φ. Its domain $\mathbf{O}^*$ is given by

$$\mathbf{O}^* \equiv \left\{ V \in \mathbf{R}^{N*} \,\middle|\, V = \partial_U \Phi(U) \text{ for some } U \in \mathbf{O} \right\},$$

and for every Φ it satisfies

$$\Phi(U) + \Phi^*(V) = VU\,, \tag{2.27}$$

where $U \in \mathbf{O}$ and $V \in \mathbf{O}^*$ are related by

$$V = \partial_U \Phi(U)\,, \qquad U = \partial_V \Phi^*(V)\,. \tag{2.28}$$

Defining

$$\mathbf{o}^* \equiv \left\{ v \in \mathbf{R}^{n*} \,\middle|\, v\mathcal{Q} \in \mathbf{O}^* \right\},$$

we can recast (c) of (iii) as

$$\text{(d) } U = \partial_V \Phi^*(v\mathcal{Q}) \qquad \text{for some } v \in \mathbf{o}^*.$$

The conserved densities corresponding to these equilibria are then

$$u = \mathcal{Q}U = \mathcal{Q}\,\partial_V \Phi^*(v\mathcal{Q}) = \partial_v \phi^*(v)\,, \tag{2.29}$$

where $\phi^*(v) \equiv \Phi^*(v\mathcal{Q})$. Note that since Φ^* is strictly convex, so is $\phi^* : \mathbf{o}^* \to \mathbf{R}$. Let $\phi = \phi(u)$ be its Legendre dual; its domain $\mathbf{o}^{**}$ is given by

$$\begin{aligned} \mathbf{o}^{**} &\equiv \left\{ u \in \mathbf{R}^n \,\middle|\, u = \partial_v \phi^*(v) \text{ for some } v \in \mathbf{o}^* \right\} \\ &= \left\{ u \in \mathbf{R}^n \,\middle|\, u = \mathcal{Q}\partial_V \Phi^*(v\mathcal{Q}) \text{ for some } v \in \mathbf{o}^* \right\}, \end{aligned}$$

and it satisfies

$$\phi(u) + \phi^*(v) = vu\,, \tag{2.30}$$

where $u \in \mathbf{o}^{**}$ and $v \in \mathbf{o}^*$ are related by

$$v = \partial_u \phi(u)\,, \qquad u = \partial_v \phi^*(v)\,. \tag{2.31}$$

Now recall the set $\mathbf{o}$ defined in (2.4), which by (2.28) can be expressed as

$$\begin{aligned} \mathbf{o} &\equiv \left\{ u \in \mathbf{R}^n \,\middle|\, u = \mathcal{Q}U \text{ for some } U \in \mathbf{O} \right\} \\ &= \left\{ u \in \mathbf{R}^n \,\middle|\, u = \mathcal{Q}\partial_V \Phi^*(V) \text{ for some } V \in \mathbf{O}^* \right\}. \end{aligned}$$

It is clear upon comparing this with the definition of $\mathbf{o}^{**}$ that $\mathbf{o}^{**} \subset \mathbf{o}$. On the other hand, given any $u \in \mathbf{o}$, the orbit of any solution of (2.2) with conserved values $u = \mathcal{Q}U$ has a unique equilibrium $\mathcal{E}(u)$. By (d) of (iii) this must have the form

$$\mathcal{E}(u) = \partial_V \Phi^*(v\mathcal{Q}) \qquad \text{for some } v \in \mathbf{o}^*.$$

Applying $\mathcal{Q}$ to both sides above and using (2.5a), we see that $u \in \mathbf{o}^{**}$; hence $\mathbf{o} \subset \mathbf{o}^{**}$. Therefore $\phi = \phi(u)$ is defined over the entire set $\mathbf{o}$, and the equilibria in (d) written as a function of $u \in \mathbf{o}$ must be

$$\mathcal{E}(u) = \partial_V \Phi^*\big(\partial_u \phi(u)\mathcal{Q}\big)\,. \tag{2.32}$$

One sees, from the duality relations (2.30) and (2.31) for ϕ^*, the definition of ϕ^*, and the duality relations (2.27) and (2.28) for Φ^*, that

$$\begin{aligned}\phi(u) &= v\partial_v\phi^*(v) - \phi^*(v) \\ &= v\mathcal{Q}\partial_V\Phi^*(v\mathcal{Q}) - \Phi^*(v\mathcal{Q}) \\ &= \Phi\big(\partial_V\Phi^*(v\mathcal{Q})\big) = \Phi\big(\mathcal{E}(u)\big)\,.\end{aligned} \tag{2.33}$$

Simply stated, ϕ is just the restriction of Φ to the manifold of local equilibria.

The functions Φ, ϕ, and $\mathcal{E}$ are implicitly related by the relation

$$\partial_U\Phi\big(\mathcal{E}(u)\big) = \partial_u\phi(u)\,\mathcal{Q}\,. \tag{2.34}$$

Differentiating the above relation yields

$$\big(\partial_u\mathcal{E}(u)\big)^{\top}\partial_{UU}\Phi\big(\mathcal{E}(u)\big) = \partial_{uu}\phi(u)\,\mathcal{Q}\,. \tag{2.35}$$

Multiplying the last identity by $\mathcal{Q}^{\top}$ and envoking the symmetry of the resulting right side, we obtain the relation

$$P^{\top}(u)\,\partial_{UU}\Phi\big(\mathcal{E}(u)\big) = \partial_{UU}\Phi\big(\mathcal{E}(u)\big)\,P(u)\,. \tag{2.36}$$

This in turn implies that $P(u)$ is an orthogonal projection with respect to the inner product defined by $\partial_{UU}\Phi\big(\mathcal{E}(u)\big)$.

Assuming the existence of a strictly convex entropy for the system (2.1) as defined by (i)-(iv) above, we have the following stability theorems.

Theorem 2.1. *The local equilibrium approximation*

$$\partial_t u + \nabla\cdot f(u) = 0\,, \tag{2.37}$$

is hyperbolic with the strictly convex entropy pair

$$\phi(u) = \Phi\big(\mathcal{E}(u)\big)\,, \qquad \psi(u) = \Psi\big(\mathcal{E}(u)\big)\,; \tag{2.38}$$

Moreover, its characteristic speeds associated with any wave number $\xi \in \mathbf{R}^D$ are determined as the critical values of the restricted Rayleigh quotient

$$w \mapsto \frac{W^{\top}\partial_{UU}\Phi(\mathcal{E})\,\partial_U F(\mathcal{E})\cdot\xi\, W}{W^{\top}\partial_{UU}\Phi(\mathcal{E})\,W}\,, \qquad W = \partial_u\mathcal{E}(u)w \text{ for } w \in \mathbf{R}^n\,. \tag{2.39}$$

Proof. First check that (ϕ, ψ) is an entropy pair for the reduced system (2.37) by the following direct calculation. Use the definition (2.8) of f, the relation (2.34), and the defining relation (2.24) of Ψ to show

$$\begin{aligned} \partial_u\phi(u)\,\partial_u f(u) &= \partial_u\phi(u)\,\mathcal{Q}\,\partial_U F\big(\mathcal{E}(u)\big)\,\partial_u\mathcal{E}(u) \\ &= \partial_U\Phi\big(\mathcal{E}(u)\big)\,\partial_U F\big(\mathcal{E}(u)\big)\,\partial_u\mathcal{E}(u) \\ &= \partial_U\Psi\big(\mathcal{E}(u)\big)\,\partial_u\mathcal{E}(u) = \partial_u\psi(u)\,. \end{aligned} \tag{2.40}$$

The strict convexity of ϕ follows from its construction through Legendre duals.

Next, since the existence of this strictly convex entropy pair ensures that the reduced system is symmetrizable, its characteristic speeds associated with any wave number $\xi \in \mathbf{R}^D$ are determined as the critical values of the Rayleigh quotient

$$w \mapsto \frac{w^\top \partial_{uu}\phi(u)\,\partial_u f(u)\cdot\xi\,w}{w^\top \partial_{uu}\phi(u)\,w}\,. \tag{2.41}$$

By the definition (2.8) of f and the identity (2.35), the matrix appearing in the numerator of (2.41) can be written as

$$\partial_{uu}\phi\,\partial_u f = \partial_{uu}\phi\,\mathcal{Q}\,\partial_U F(\mathcal{E})\,\partial_u\mathcal{E} = (\partial_u\mathcal{E})^\top\partial_{UU}\Phi(\mathcal{E})\,\partial_U F(\mathcal{E})\,\partial_u\mathcal{E}\,, \tag{2.42}$$

while, by the identities (2.5b) and (2.35), the matrix in the denominator becomes

$$\partial_{uu}\phi = \partial_{uu}\phi\,\mathcal{Q}\,\partial_u\mathcal{E} = (\partial_u\mathcal{E})^\top\,\partial_{UU}\Phi(\mathcal{E})\,\partial_u\mathcal{E}\,. \tag{2.43}$$

Substituting these into the above quotient yields the result.

Remark. The reduced system's characteristic speeds associated with any wave number $\xi \in \mathbf{R}^D$ are determined as the critical values of a restriction (2.39) of the Rayleigh quotient for the full system (2.26) to the tangent space of the manifold of local equilibria. Hence, they are interlaced with the characteristic apeeds for the full system. More precisely, given a wave number $\xi \in \mathbf{R}^D$, for each $u \in \mathbf{o}$ let

$$\Lambda_1 \leq \cdots \leq \Lambda_k \leq \Lambda_{k+1} \leq \cdots \leq \Lambda_N\,,$$

where $\Lambda_k = \Lambda_k(\mathcal{E}(u))$ are the characteristic speeds for the full system (2.1), while

$$\lambda_1 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots \leq \lambda_n,$$

where $\lambda_j = \lambda_j(u)$ are those for the reduced system. Then the λ_j are interlaced with the Λ_k in the sense that each of the λ_j lies in the closed interval $[\Lambda_j, \Lambda_{j+N-n}]$, cf. (1.20). This follows from the classical min-max characterizations of the Λ_k and λ_j given by

$$\begin{aligned}\Lambda_k &= \min_{\mathbf{W}\subset\mathbf{R}^N}\left\{\max_{W\in\mathbf{W}}\left\{\frac{W^\top\partial_{UU}\Phi(\mathcal{E})\,\partial_U F(\mathcal{E})\cdot\xi\, W}{W^\top\partial_{UU}\Phi(\mathcal{E})\,W}\right\}\;\middle|\;\dim\mathbf{W}=k\right\}\\ &= \max_{\mathbf{W}\subset\mathbf{R}^N}\left\{\min_{W\in\mathbf{W}}\left\{\frac{W^\top\partial_{UU}\Phi(\mathcal{E})\,\partial_U F(\mathcal{E})\cdot\xi\, W}{W^\top\partial_{UU}\Phi(\mathcal{E})\,W}\right\}\;\middle|\;\mathrm{codim}\mathbf{W}=k-1\right\},\end{aligned}$$

$$\begin{aligned}\lambda_j &= \min_{\mathbf{w}\subset\mathbf{R}^n}\left\{\max_{w\in\mathbf{w}}\left\{\frac{w^\top\partial_{uu}\phi\,\partial_u f\cdot\xi\, w}{w^\top\partial_{uu}\phi\, w}\right\}\;\middle|\;\dim\mathbf{w}=j\right\}\\ &= \max_{\mathbf{w}\subset\mathbf{R}^n}\left\{\min_{w\in\mathbf{w}}\left\{\frac{w^\top\partial_{uu}\phi\,\partial_u f\cdot\xi\, w}{w^\top\partial_{uu}\phi\, w}\right\}\;\middle|\;\mathrm{codim}\mathbf{w}=j-1\right\},\end{aligned}$$

and the identities established in the proof of the proceeding theorem.

Theorem 2.2. *The first-order correction is locally dissipative with respect to the entropy $\phi(u)$. It has the form*

$$\partial_t u + \nabla\cdot f(u) = \epsilon\,\nabla\cdot\big[g(u)\,\nabla\partial_u\phi^\top(u)\big]\,, \tag{2.44}$$

with the diffusion tensor $g(u)$ defined by

$$g(u) = \mathcal{S}(u)\,L(u)^{-1}\mathcal{S}(u)^\top\,, \tag{2.45}$$

where

$$L(u)^{-1} \equiv \Big(\partial_{UU}\Phi\big(\mathcal{E}(u)\big)\,\partial_U R\big(\mathcal{E}(u)\big)\Big)^{-1}, \tag{2.46a}$$

$$\mathcal{S}(u) \equiv \mathcal{Q}\,\partial_U F\big(\mathcal{E}(u)\big)\,\big(I-P(u)\big)\,, \tag{2.46b}$$

is a nonnegative 4-tensor in $\mathbf{R}^{N\times N}\otimes\mathbf{R}^{D\times D}$.

Proof. Multiplying the equation (2.45) by $\partial_u\phi(u)$ gives

$$\partial_t\phi + \nabla\cdot\psi = \epsilon\,\nabla\cdot\big[\partial_u\phi\, g\,\nabla(\partial_u\phi^\top)\big] - \epsilon\,(\nabla\partial_u\phi)\, g\,(\nabla\partial_u\phi)^\top\,. \tag{2.47}$$

Integrating (neglecting boundary contributions), we have that

$$\frac{d}{dt}\int \phi(u)\,dx = -\epsilon \int (\nabla\partial_u\phi)\,g\,(\nabla\partial_u\phi)^\top\,dx \le 0\,. \tag{2.48}$$

The key step in this proof is checking that

$$\partial_{UU}\Phi(\mathcal{E})\,\partial_U R(\mathcal{E}) + \big(\partial_{UU}\Phi(\mathcal{E})\,\partial_U R(\mathcal{E})\big)^\top \ge 0\,, \tag{2.49}$$

in the sense of forms. But this follows directly from (iv) which states

$$\partial_U\Phi(U)\,R(U) \ge 0\,, \qquad \text{for every } U \in \mathbf{O}, \tag{2.50}$$

and the fact that equality is attained when $U = \mathcal{E}$,

$$\partial_U\Phi(\mathcal{E})\,R(\mathcal{E}) = 0\,. \tag{2.51}$$

Together, these imply

$$\partial_{UU}\big(\partial_U\Phi(U)\,R(U)\big)\Big|_{U=\mathcal{E}} \ge 0\,. \tag{2.52}$$

A direct calculation then shows

$$\partial_{UU}\big(\partial_U\Phi(U)\,R(U)\big)\Big|_{U=\mathcal{E}} = \partial_{UU}\Phi(\mathcal{E})\,\partial_U R(\mathcal{E}) + \big(\partial_{UU}\Phi(\mathcal{E})\,\partial_U R(\mathcal{E})\big)^\top\,, \tag{2.53}$$

and (2.49) then follows from (2.52). What remains is to compute the formula for $g(u)$. Note that by the definition (2.44a) of L^{-1}, the orthogonality (2.36) of P, the symmetry (i), the relation (2.35), and the definition (2.44b) of $\mathcal{S}$, one has

$$\begin{aligned}
&\big(\partial_U R(\mathcal{E})\big)^{-1}(I-P)\,\nabla\cdot F(\mathcal{E})\\
&\qquad = \big(\partial_{UU}\Phi(\mathcal{E})\,\partial_U R(\mathcal{E})\big)^{-1}\partial_{UU}\Phi(\mathcal{E})\,(I-P)\,\partial_U F(\mathcal{E})\,\nabla\cdot\mathcal{E}\\
&\qquad = L^{-1}(I-P)^\top\partial_{UU}\Phi(\mathcal{E})\,\partial_U F(\mathcal{E})\,\partial_u\mathcal{E}\,\nabla u\\
&\qquad = L^{-1}(I-P)^\top\big(\partial_U F(\mathcal{E})\big)^\top\partial_{UU}\Phi(\mathcal{E})\,\partial_u\mathcal{E}\,\nabla u\\
&\qquad = L^{-1}(I-P)^\top\big(\partial_U F(\mathcal{E})\big)^\top\mathcal{Q}^\top\partial_{uu}\phi\,\nabla u\\
&\qquad = L^{-1}\mathcal{S}^\top(\nabla\partial_u\phi)^\top\,. \qquad (2.54)
\end{aligned}$$

Substituting this identity into our formula (2.23b) then yields the result.

3. The Structure for 2×2 Systems

The remainder of this paper considers the general 2×2 system of conservation laws over a one-dimensional spatial domain in the form

$$\begin{aligned} &\partial_t u + \partial_x f_1(u, v) = 0\,, \\ &\partial_t v + \partial_x f_2(u, v) + \frac{1}{\epsilon} r(u, v) = 0\,. \end{aligned} \tag{3.1}$$

This system is assumed to be strictly hyperbolic with (real and distinct) characteristic speeds given by

$$\Lambda_\pm(u, v) \equiv \tfrac{1}{2}\Big(\partial_u f_1 + \partial_v f_2 \pm \sqrt{(\partial_u f_1 - \partial_v f_2)^2 + 4\partial_v f_1\, \partial_u f_2}\Big)\,. \tag{3.2}$$

Here, as before, the values of $(u, v)^\top$ lie in $\bar{\mathbf{O}}$, the closure of an open convex set $\mathbf{O} \subset \mathbf{R}^2$. For many physical models the set $\mathbf{O}$ can be one generated from an invariant region, for example, as for the p-system (1.18) (see Section 4) and the elastic model (see [8]).

The first equation of (3.1) represents a conservation law for u and the second equation a rate equation for v. A typical form for the relaxation term $r(u, v)$ is

$$r(u, v) = v - e(u)\,; \tag{3.3}$$

however, in general we assume only that for each $u \in \mathbf{o}$ the vector field $v \mapsto r(u, v)$ has a unique stable equilibrium $v = e(u)$ satisfying

$$r(u, e(u)) = 0\,, \qquad \partial_v r(u, e(u)) > 0\,. \tag{3.4}$$

We also assume that the two equations in (3.1) are coupled in a nontrivial way. More precisely, for Theorem 3.2 we assume the coupling condition

$$\partial_v f_1(u, e(u)) \neq 0\,, \qquad \text{for all } u \in \mathbf{o}\,. \tag{3.5}$$

In addition, we assume smoothness as needed.

When cast in the notation of the previous section, system (3.1) takes the form

$$\partial_t U + \partial_x F(U) + \frac{1}{\epsilon} R(U) = 0\,, \tag{3.6}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad F(U) = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}, \qquad R(U) = \begin{pmatrix} 0 \\ r(u,v) \end{pmatrix}, \tag{3.7}$$

while

$$\mathcal{Q} = (\, 1 \quad 0\,)\,, \qquad \mathcal{E}(u) = \begin{pmatrix} u \\ e(u) \end{pmatrix}, \qquad P(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.8}$$

The general formulas for the local equilibrium approximation (2.7) and its first-order correction (2.23) can now be specialized to system (3.1).

The local equilibrium equation is now just the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0\,, \tag{3.9}$$

where the flux $f(u)$ is simply

$$f(u) \equiv f_1(u, e(u))\,, \tag{3.10}$$

and its characteristic speed is

$$\lambda(u) \equiv f'(u) = \partial_u f_1(u, e(u)) + \partial_v f_1(u, e(u))\, e'(u)\,. \tag{3.11}$$

While by its scalar nature the reduced equation (3.9) is hyperbolic, the stability criterion analogous to that of (1.20) for the p-system (1.18) is

$$\Lambda_- \le \lambda \le \Lambda_+ \qquad \text{on } v = e(u)\,. \tag{3.12}$$

As with the p-system, this can be inferred by requiring the linear stability of the local equilibrium $(\bar{u}, e(\bar{u}))$. By Theorem 2.1, the existence of an entropy for system (3.1) would also yield (3.12); later we will establish a partial converse.

The first-order relaxation correction for system (3.1) is simply the scalar convection-diffusion equation

$$\partial_t u + \partial_x f(u) = \epsilon\, \partial_x \big(g(u) \partial_x u \big)\,, \tag{3.13}$$

where

$$g(u) = \frac{\partial_v f_1 \big(\partial_u f_2 + (\partial_v f_2 - \partial_u f_1)\, e' - \partial_v f_1\, {e'}^2 \big)}{\partial_v r} \qquad \text{on } v = e(u)\,. \tag{3.14}$$

We have the following theorem.

Theorem 3.1. *The first-order relaxation correction (3.13) is dissipative if and only if the stability criterion (3.12) holds. Moreover,*

$$g(u) = \frac{(\Lambda_+ - \lambda)(\lambda - \Lambda_-)}{\partial_v r} \quad \text{on } v = e(u)\,. \tag{3.15}$$

Proof. Since $\partial_v r(u, e(u)) > 0$ by (3.4), the nonnegativity of $g(u)$ is equivalent to that of the numerator on the right side of (3.15), which in turn is equivalent to the stability criterion (3.12). Thus, the theorem follows immediately once (3.15) is established. But a direct calculation starting from the definitions (3.2) of $\Lambda_\pm$ and (3.11) of λ leads to the important identity

$$\begin{aligned}(\Lambda_+ - \lambda)&(\lambda - \Lambda_-) \\ &= \partial_v f_1\big(\partial_u f_2 + (\partial_v f_2 - \partial_u f_1)\, e' - \partial_v f_1\, {e'}^2\big) \quad \text{on } v = e(u)\,.\end{aligned} \tag{3.16}$$

Setting this into the numerator on the right side of (3.14) gives (3.15), and thus proves the theorem.

Theorem 2.1 and its subsequent remark state that if (Φ, Ψ) is a strictly convex entropy pair for the system (3.1), then (ϕ, ψ) defined by

$$\phi(u) \equiv \Phi(u, e(u))\,, \qquad \psi(u) \equiv \Psi(u, e(u))\,, \tag{3.17}$$

is a strictly convex entropy pair for the local equilibrium equation (3.9) and, moreover, that the stability criterion (3.12) holds. Since (3.9) is a scalar conservation law, any strictly convex $\phi(u)$ gives an entropy pair. Therefore the first assertion is no deeper than the fact that $\phi(u)$ defined in (3.17) is strictly convex by construction. The main content is therefore the second assertion of the validity of the stability criterion. For 2×2 systems (3.1) satisfying the coupling condition (3.5) there is the following partial converse.

Theorem 3.2. *Let (ϕ, ψ) be a strictly convex entropy pair for the local equilibrium equation (3.9). Assume that the strict stability criterion*

$$\Lambda_- < \lambda < \Lambda_+ \qquad \text{on } v = e(u)\,, \tag{3.18}$$

holds. Then there exists a strictly convex entropy pair (Φ, Ψ) for the system (3.1) over an open set $\mathbf{O}_\phi \subset \mathbf{O}$ containing the local equilibria curve $v = e(u)$, along which it satisfies (3.17).

Proof. If Φ is to be a strictly convex entropy for the system (3.1), it must satisfy (i)-(iv) of Definition 2.1 as they are now manifest:

(i') Φ solves the following entropy equation over $(u,v)^\top \in \mathbf{O}_\phi$,

$$\partial_u f_2 \, \partial_{vv}\Phi - (\partial_v f_2 - \partial_u f_1)\, \partial_{uv}\Phi - \partial_v f_1 \, \partial_{uu}\Phi = 0\,; \tag{3.19}$$

(ii') $\partial_v \Phi(u,v)\, r(u,v) \geq 0 \quad$ for all $(u,v)^\top \in \mathbf{O}_\phi$;

(iii') for any $(u,v)^\top \in \mathbf{O}_\phi$ the following are equivalent

(a) $r(u,v) = 0$,

(b) $\partial_v \Phi(u,v)\, r(u,v) = 0$,

(c) $v = e(u) \quad$ for some $u \in \mathbf{o}$;

(iv') Φ satisfies the following strict convexity conditions over $(u,v)^\top \in \mathbf{O}_\phi$

$$\partial_{vv}\Phi > 0\,, \qquad \partial_{uu}\Phi\, \partial_{vv}\Phi - (\partial_{uv}\Phi)^2 > 0\,. \tag{3.20}$$

Given (iv') and assumptions (3.4) regarding the equilibria of $r(u,v)$, it is clear that (ii') and (iii') will both follow from the simple requirement that

$$\partial_v \Phi(u, e(u)) = 0\,. \tag{3.21}$$

Indeed, for any fixed $u \in \mathbf{o}$ conditions (3.21) and (iv) ensure that $v \mapsto \partial_v \Phi(u,v)$ has one simple zero at the point $v = e(u)$, and no more. Condition (3.4) does the same for $v \mapsto r(u,v)$. As both functions are locally increasing at this zero, their product must be nonnegative, with $v = e(u)$ being the only zero of both r and the product, in accord with (ii') and (iii'). Thus, given $\phi = \phi(u)$ we must construct a $\Phi = \Phi(u,v)$ that satisfies (3.17), (3.19), (3.20), and (3.21).

First, by the strict stability criterion (3.18) and the coupling condition (3.6), the identity (3.16) implies that

$$\partial_u f_2 + (\partial_v f_2 - \partial_u f_1)\, e' - \partial_v f_1 \, {e'}^2 \neq 0\,. \tag{3.22}$$

But this is exactly the condition that the local equilibria curve $v = e(u)$ is a noncharacteristic curve for the entropy equation (3.19). This is a second-order hyperbolic equation for which we consider Cauchy data consistent with (3.19) and (3.21) in the form

$$\Phi(u, e(u)) = \phi(u)\,, \qquad \partial_v \Phi(u, e(u)) = 0\,. \tag{3.23}$$

The classical local existence theory ensures that there is a solution $\Phi = \Phi(u, v)$ of this Cauchy problem over an open domain containing the local equilibria curve.

If the strict convexity conditions (3.20) are satisfied along the local equilibria curve, then by continuity they will also be satisfied in some (possibly smaller) open domain containing the local equilibria curve; this smaller domain is the $\mathbf{O}_\phi$ asserted in the theorem.

Differentiating the Cauchy data (3.23) with respect to u will lead to the identities

$$\partial_{uu}\Phi(u, e(u)) + \partial_{uv}\Phi(u, e(u))\, e'(u) = \partial_{uu}\phi(u)\,, \tag{3.24a}$$

$$\partial_{uv}\Phi(u, e(u)) + \partial_{vv}\Phi(u, e(u))\, e'(u) = 0\,, \tag{3.24b}$$

Starting with the entropy equation (3.19), first use (3.24a) to eliminate $\partial_{uu}\Phi$, and then use (3.24b) to eliminate $\partial_{uv}\Phi$ to obtain

$$\begin{aligned} 0 &= \partial_v f_1\, \partial_{uu}\Phi + (\partial_v f_2 - \partial_u f_1)\, \partial_{uv}\Phi - \partial_u f_2\, \partial_{vv}\Phi \\ &= \partial_v f_1\, \partial_{uu}\phi + (\partial_v f_2 - \partial_u f_1 - \partial_v f_1\, e')\, \partial_{uv}\Phi - \partial_u f_2\, \partial_{vv}\Phi \\ &= \partial_v f_1\, \partial_{uu}\phi - \big(\partial_u f_2 + (\partial_v f_2 - \partial_u f_1)\, e' - \partial_v f_1\, {e'}^2\big)\, \partial_{vv}\Phi\,, \end{aligned} \tag{3.25}$$

on $v = e(u)$. Multiplying this by $\partial_v f_1$ and employing the identity (3.16) give

$$(\Lambda_+ - \lambda)(\lambda - \Lambda_-)\, \partial_{vv}\Phi = (\partial_v f_1)^2\, \partial_{uu}\phi\,, \qquad \text{on } v = e(u)\,. \tag{3.26}$$

The strict stability criterion (3.18), the coupling condition (3.6), and the strict convexity of ϕ then imply that

$$\partial_{vv}\Phi > 0 \qquad \text{on } v = e(u)\,, \tag{3.27}$$

which is the first of the strict convexity conditions (3.20).

To obtain the second, start with the identity (3.16), use (3.24b) to eliminate e', and then use the entropy equation (3.19) to find

$$
\begin{aligned}
&(\Lambda_+ - \lambda)(\lambda - \Lambda_-) \\
&= \partial_v f_1 \big(\partial_u f_2 + (\partial_v f_2 - \partial_u f_1)\, e' - \partial_v f_1\, {e'}^2\big) \\
&= \frac{\partial_v f_1}{(\partial_{vv}\Phi)^2} \big(\partial_u f_2 (\partial_{vv}\Phi)^2 - (\partial_v f_2 - \partial_u f_1)\, \partial_{vv}\Phi\, \partial_{uv}\Phi - \partial_v f_1 (\partial_{uv}\Phi)^2\big) \\
&= \frac{(\partial_v f_1)^2}{(\partial_{vv}\Phi)^2} \big(\partial_{uu}\Phi\, \partial_{vv}\Phi - (\partial_{uv}\Phi)^2\big) \qquad \text{on } v = e(u)\,.
\end{aligned}
\tag{3.28}
$$

Now the strict stability criterion (3.18), the coupling condition (3.6), and the first strict convexity condition (3.27) imply

$$
\partial_{uu}\Phi\, \partial_{vv}\Phi - (\partial_{uv}\Phi)^2 > 0 \qquad \text{on } v = e(u)\,,
\tag{3.29}
$$

which is the second of the strict convexity conditions (3.20). By the aforementioned continuity argument, Theorem 3.2 follows.

Remark. For any bounded set $\mathbf{B} \subset \mathbf{R}^2$ there is a constant $\gamma > 0$ such that

$$
\mathbf{B}_\gamma \equiv \mathbf{B} \cap \big\{(u,v) \,\big|\, |v - e(u)| \leq \gamma\big\} \subset \mathbf{O}_\phi\,.
\tag{3.30}
$$

On the set $\mathbf{B}_\gamma$ one has

$$
\begin{aligned}
&\partial_{vv}\Phi > 0\,, \\
&\det(\partial_{UU}\Phi) > 0\,, \\
&\partial_v\Phi(u,v)\, r(u,v) > 0\,, \qquad \text{for } (u,v) \in \mathbf{B}_\gamma - \{v = e(u)\}\,.
\end{aligned}
\tag{3.31}
$$

The constant γ depends on the flux functions (f_1, f_2) as well as the entropy function ϕ.

4. The Local Equilibrium Limit for 2×2 Systems

Suppose that a sequence $U^\epsilon = (u^\epsilon, v^\epsilon) \in \mathbf{B}_\gamma$, bounded open convex set, are solutions of the systems of conservation laws with a stiff relaxation term:

$$
\begin{aligned}
&\partial_t U^\epsilon + \partial_x F(U^\epsilon) + \frac{1}{\epsilon}\begin{pmatrix} 0 \\ r(U^\epsilon) \end{pmatrix} = 0\,, \\
&U^\epsilon\big|_{t=0} = U_0^\epsilon\,,
\end{aligned}
\tag{4.1}
$$

and satisfy the entropy condition: for any convex entropy pair (Φ, Ψ) on $\mathbf{B}_\gamma$,

$$\partial_t \Phi(U^\epsilon) + \partial_x \Psi(U^\epsilon) + \frac{1}{\epsilon}\Phi_v(U^\epsilon)\, r(U^\epsilon) \leq 0\,. \tag{4.2}$$

For simplicity, we can assume from Theorem 3.2 that there are two convex and dissipative entropy pairs (Φ_i, Ψ_i), $i = 1, 2$, on $\mathbf{B}_\gamma$ such that

$$\phi_2(u) - \phi_1(u) = Cf(u)\,, \tag{4.3}$$

where $\phi_i(u) = \Phi_i|_{v=e(u)}$, $f(u) = f_1(u, e(u))$, and

$$C < \frac{\sup\limits_{(u,v)\in\mathbf{B}_\gamma} f''(u)}{\inf\limits_{(u,v)\in\mathbf{B}_\gamma} \phi_1''(u)}\,.$$

In fact, we first choose any strictly convex function $\phi_1(u)$ as the Cauchy data in the Cauchy problem for (3.19) to get a convex and dissipative entropy pair (Φ_1, Ψ_1) on $\mathbf{B}_{\gamma_1}$; then we take convex function $\phi_2 = Cf(u) + \phi_1$ as the Cauchy data in the Cauchy problem for (3.19) to get another convex and dissipative entropy pair (Φ_2, Ψ_2) on $\mathbf{B}_{\gamma_2}$; and finally we choose $\gamma = \min(\gamma_1, \gamma_2)$.

Theorem 4.1. *Suppose that*

$$meas\{u \,|\, \lambda'(u) = 0\} = 0\,,$$

and

$$\|(u_0^\epsilon - \bar{u}, v_0^\epsilon - \bar{v})\|_{L^2} \leq C\,,$$

with $\bar{v} = e(\bar{u})$. Then there exists a subsequence (still denoted) (u^ϵ, v^ϵ) strongly converging almost everywhere:

$$(u^\epsilon, v^\epsilon) \longrightarrow (u, v) \quad a.e.\,,$$

and the limit functions (u, v) satisfy

(i) $v(t,x) = e(u(t,x))$ for a.e. $t > 0$;

(ii) $u(t,x)$ is unique entropy solution of the Cauchy problem

$$\partial_t u + \partial_x f(u) = 0\,,$$
$$u\big|_{t=0} = w^*\text{-}\lim u_0^\epsilon(x)\,.$$

where w^*-lim *denotes the weak-star limit in the space* L^∞.

Remark 1. Notice that $v_0^\epsilon(x)$ generally is not equal to $e(u_0^\epsilon(x))$; indeed, the initial data may even be far from equilibrium. Theorem 4.1 indicates that, in the local equilibrium limit, the functions (u, v) indeed come into local equilibrium as soon as $t > 0$. This shows that the limit is highly singular. In fact, this limit consists of two processes simultaneously: one is the initial layer limit, and the other is the shock layer limit.

Remark 2. Theorem 4.1 indicates that the sequence (u^ϵ, v^ϵ) is compact no matter how oscillatory the initial data $(u_0^\epsilon(x), v_0^\epsilon(x))$ are. Note that systems with the stiff relaxation term are allowed to be linearly degenerate; in particular, the second characteristic field may be linearly degenerate, and the initial oscillations can propagate along the linearly degenerate fields for the systems without the stiff relaxation term (see [7]). This fact shows that the relaxation mechanism and the nonlinearity of the equilibrium equations can kill the initial oscillations, just as the nonlinearity for the full system can kill the initial oscillations (see [6,13]).

Proof of Theorem 4.1. Notice that if (Φ, Ψ) is a convex entropy pair, then so is

$$\hat{\Phi}(U) = \Phi(U) - \Phi(\bar{U}) - \partial_U \Phi(\bar{U})(U - \bar{U}),$$
$$\hat{\Psi}(U) = \Psi(U) - \Psi(\bar{U}) - \partial_U \Phi(\bar{U})(F(U) - F(\bar{U})),$$

with $\bar{U} = (\bar{u}, \bar{v}) = (\bar{u}, e(\bar{u}))$. Without loss of generality, we assume that $\bar{U} = (0, 0)$. Therefore, we have

$$\partial_t \Phi(U^\epsilon) + \partial_x \Psi(U^\epsilon) + \frac{1}{\epsilon}\Phi_v(U^\epsilon)\, r(U^\epsilon) \le 0\,. \tag{4.4}$$

Integrate (4.4) over $[0, t] \times (-\infty, \infty)$ to obtain

$$\int_{-\infty}^{\infty} \Phi(U^\epsilon(t,x))\, dx + \frac{1}{\epsilon}\int_0^t \int_{-\infty}^{\infty} \Phi_v(U^\epsilon)\, r(U^\epsilon)\, dx d\tau \le \int_{-\infty}^{\infty} \Phi(U_0^\epsilon(x))\, dx\,.$$

Therefore, there exists a constant $C > 0$ such that

$$\frac{1}{\epsilon}\int_0^t \int_{-\infty}^{\infty} (v^\epsilon - e(u^\epsilon))^2\, dx d\tau \le C \int_{-\infty}^{\infty} \left\{(u_0^\epsilon(x) - \bar{u})^2 + (v_0^\epsilon(x) - \bar{v})^2\right\} dx\,. \tag{4.5}$$

Define $(\phi_i(u), \psi_i(u)) = (\Phi_i, \Psi_i)|_{v=e(u)}$, $i = 1, 2$. Then $(\phi_i(u), \psi_i(u))$, $i = 1, 2$, are convex entropy from Theorem 3.2. Notice that

$$
\begin{aligned}
&\partial_t \phi_i(u^\epsilon) + \partial_x \psi_i(u^\epsilon) \\
&\qquad\qquad \le \partial_t \big(\Phi_i(u^\epsilon, e(u^\epsilon)) - \Phi_i(u^\epsilon, v^\epsilon)\big) \\
&\qquad\qquad\quad + \partial_x \big(\Psi_i(u^\epsilon, e(u^\epsilon)) - \Psi_i(u^\epsilon, v^\epsilon)\big) \\
&\qquad\qquad\quad + \frac{1}{\epsilon} \Phi_{iv}(u^\epsilon, v^\epsilon) \big(r(u^\epsilon, e(u^\epsilon)) - r(u^\epsilon, v^\epsilon)\big) \\
&\qquad\qquad \equiv I^\epsilon_{i1} + I^\epsilon_{i2} + I^\epsilon_{i3}\,, \qquad i = 1, 2\,.
\end{aligned}
\tag{4.6}
$$

Using the estimate (4.5), we have

$$
\begin{aligned}
\|I^\epsilon_{i1}\|_{H^{-1}} &= \sup_{\varphi \in H^1_0} \Big| \int\!\!\int \partial_t \big(\Phi_i(u^\epsilon, e(u^\epsilon)) - \Phi_i(u^\epsilon, v^\epsilon)\big)\, \varphi \, dx d\tau \Big| \\
&\le C\, \|v^\epsilon - e(u^\epsilon)\|_{L^2} \|\varphi_t\|_{L^2} \\
&\le \sqrt{\epsilon}\, C\, \|\varphi\|_{H^1} \longrightarrow 0\,, \qquad \text{as } \epsilon \longrightarrow 0\,.
\end{aligned}
$$

Similarly, we have

$$
\|I^\epsilon_{i2}\|_{H^{-1}} \le C\, \sqrt{\epsilon}\, \|\varphi\| \longrightarrow 0\,.
$$

Notice that $\Phi_{iv} = 0$ as $v = e(u)$. Therefore, we have from (4.4) and (4.6)

$$
\|I^\epsilon_3\|_{L^1} \le C\,;
$$

that is,

$$
I^\epsilon_{i3} \qquad \text{is compact in } W^{-1,p_1}\,,
$$

from the Sobolev embedding theorem. Therefore, we have

$$
I^\epsilon_i = \sum_{j=1}^{3} I^\epsilon_{ij} \qquad \text{is compact in } W^{-1,p_1}\,. \tag{4.7}
$$

Using the fact

$$
\begin{aligned}
&\partial_t \phi_i(u^\epsilon) + \partial_x \psi_i(u^\epsilon) - I^\epsilon \le 0\,, \\
&\partial_t \phi_i(u^\epsilon) + \partial_x \psi_i(u^\epsilon) - I^\epsilon \quad \text{is bounded in } W^{-1,p_1}\,,
\end{aligned}
$$

from (4.6) and the boundedness of u^ϵ, and the Murat lemma [21], we conclude

$$
\partial_t \phi_i(u^\epsilon) + \partial_x \psi_i(u^\epsilon) - I^\epsilon \quad \text{is compact in } W^{-1,p_2}\,, \quad 1 < p_2 < p_1\,. \tag{4.8}
$$

Combining (4.7) and (4.8) with the boundedness of u^ϵ, we obtain

$$\partial_t\phi_i(u^\epsilon)+\partial_x\psi_i(u^\epsilon)\quad\begin{cases}\text{is compact in } W^{-1,p}\,, & 1<p<p_2\,,\\ \text{is bounded in } W^{-1,\infty}\,.\end{cases}$$

Thus

$$\partial_t\phi_i(u^\epsilon)+\partial_x\psi_i(u^\epsilon)\quad\text{is compact in } H^{-1}\,,\qquad i=1,2\,, \tag{4.9}$$

from a compactness embedding lemma [12]: Let $1<p\le q<r<\infty$. Then

$$\big(\text{compact set of } W^{-1,p}_{loc}\big)\cap\big(\text{bounded set of } W^{-1,r}_{loc}\big)\subset\big(\text{compact set of } W^{-1,q}_{loc}\big)\,.$$

Therefore, we have from (4.3) and (4.9)

$$\partial_t f(u^\epsilon)+\partial_x\Big(\int^{u^\epsilon}\big(f'(y)\big)^2\,dy\Big)\qquad\text{is compact in } H^{-1}\,.$$

Similarly, we can show that

$$\partial_t u^\epsilon+\partial_x f(u^\epsilon)\qquad\text{is compact in } H^{-1}\,.$$

The compactness theorem that was established in [9] by using the Div-Curl Lemma of compensated compactness (see [26]) then ensures that there exists a subsequence (still denoted) $u^\epsilon(t,x)$ converging almost everywhere to a function $u(t,x)$:

$$u^\epsilon(t,x)\longrightarrow u(t,x)\quad\text{a.e.}\,,$$

and

$$v^\epsilon(t,x)\longrightarrow e(u(t,x))\quad\text{a.e.}\,,$$

from the estimate (4.5). The proof is complete.

The above theorem is based on the L^∞ a priori estimate. In many physical systems, the estimate can be achieved. As an illustration, we now apply this theorem to the p-system (1.18), a model in which a stiff relaxation term is appended to the equations of elasticity. Consider the solutions (u^ϵ,v^ϵ) of the Cauchy problem

$$\begin{aligned}&\partial_t u+\partial_x v=0\,,\\ &\partial_t v+\partial_x p(u)+\frac{1}{\epsilon}(v-f(u))=0\,,\qquad p'(u)>0\,,\end{aligned} \tag{4.10}$$

with the Cauchy data

$$(u,v)\big|_{t=0}=(u_0^\epsilon(x),v_0^\epsilon(x))\,. \tag{4.11}$$

We first construct the solutions of the Cauchy problem (4.9)-(4.10).

Lemma 4.1. *Suppose that $p(u)$ and $f(u)$ satisfy*

$$\begin{cases} p \in C^2\,; \quad p''(u)(u-\bar{u}) > 0\,, \quad \text{for all } u \neq 0\,, \\ f \in C^1\,; \quad f(u) = \bar{f} = const.\,, \quad \text{as } |u-\bar{u}| \geq M_0\,, \end{cases} \tag{4.12}$$

and the stability condition

$$p'(u) - (f'(u))^2 \geq 0\,. \tag{4.13}$$

Assume that the initial data $(u_0^\epsilon(x), v_0^\epsilon(x))$ are bounded in L^∞, uniformly in ϵ, and $(u_0^\epsilon(x)-\bar{u}, v_0^\epsilon(x)-\bar{v}) \in L^2$. Then there exists a weak solution $(u^\epsilon(t,x), v^\epsilon(t,x))$ for the Cauchy problem (4.10) and (4.11) for any fixed ϵ. The solution sequence $(u^\epsilon(t,x), v^\epsilon(t,x))$ is bounded in L^∞, uniformly in ϵ. Moreover, there exists a constant $N_0 > 0$ such that $(u^\epsilon(t,x), v^\epsilon(t,x)) \in \mathbf{B}_\gamma$ provided $\|(u_0^\epsilon-\bar{u}, v_0^\epsilon-\bar{v})\|_{L^\infty} \leq M_0 \leq N_0$.

Proof. Consider the following parabolic systems:

$$\begin{aligned} &\partial_t u + \partial_x v = \delta\, \partial_{xx} u\,, \\ &\partial_t v + \partial_x p(u) + \frac{1}{\epsilon}(v - f(u)) = \delta\, \partial_{xx} v\,, \end{aligned} \tag{4.14}$$

with the Cauchy data

$$(u,v)|_{t=0} = (u_0^\epsilon(x), v_0^\epsilon(x))\,. \tag{4.15}$$

The Riemann invariants associated with system (4.14) are

$$w_\pm(u,v) = v - \bar{f} \pm \int_{\bar{u}}^{u} \sqrt{p'(\xi)}\, d\xi\,.$$

The invariant region principle [10] indicates that the domains

$$\Sigma_K = \left\{(u,v)\,\middle|\, M_1 \leq |w_\pm(u,v)| \leq M_2\right\},$$

with $K \geq M_0$ and

$$\begin{aligned} M_1 &= \min\left\{ \int_{\bar{u}}^{K} \sqrt{p'(\xi)}\, d\xi\,, \int_{\bar{u}}^{-K} \sqrt{p'(\xi)}\, d\xi \right\}, \\ M_2 &= \max\left\{ \int_{\bar{u}}^{K} \sqrt{p'(\xi)}\, d\xi\,, \int_{\bar{u}}^{-K} \sqrt{p'(\xi)}\, d\xi \right\}, \end{aligned}$$

are positively invariant for the homogeneous system associated with (4.14). This is guaranteed by the condition (4.12). Therefore, the domains Σ will remain invariant for the inhomogeneous system provided that

$$-\vec{n}\cdot(0, v-f(u))^{\top} \leq 0\,, \qquad \text{on } \partial\Sigma\,, \tag{4.16}$$

where $\vec{n}$ denotes the outer unit normal to $\partial\Sigma$. The condition (4.16) is an immediate corollary to the stability criteria (4.13).

For the uniformly bounded initial data $(u_0^\epsilon(x), v_0^\epsilon(x))$, there exists $K_0 > M_0$ such that

$$(u_0^\epsilon, v_0^\epsilon) \in \Sigma_{K_0}\,,$$

and hence

$$(u_\delta^\epsilon(t,x), v_\delta^\epsilon(t,x)) \in \Sigma_{K_0}\,,$$

which means that $(u_\delta^\epsilon(t,x), v_\delta^\epsilon(t,x))$ are uniformly bounded. Then the classical local solution can be extended to the global solution for the Cauchy problem (4.14)–(4.15). DiPerna's compactness theorem [13] for such p-system ensures that the viscosity solutions $(u_\delta^\epsilon(t,x), v_\delta^\epsilon(t,x))$ converge strongly and pointwise almost everywhere to $(u^\epsilon(t,x), v^\epsilon(t,x))$:

$$\big(u_\delta^\epsilon(t,x), v_\delta^\epsilon(t,x)\big) \longrightarrow \big(u^\epsilon(t,x), v^\epsilon(t,x)\big) \quad \text{a.e.}\,,$$

and that for every $\epsilon > 0$ the limit $(u^\epsilon(t,x), v^\epsilon(t,x))$ is an entropy weak solution, uniformly bounded in ϵ. This completes the proof.

Theorem 4.2. *Suppose that $p(u)$ and $f(u)$ satisfy (4.12) and (4.13). Then there exists a constant $N_0 > 0$ such that, when*

$$\|(u_0^\epsilon, v_0^\epsilon)\|_{L^\infty} \leq M_0 \leq N_0\,, \tag{4.17}$$

there exist global weak solutions

$$\big(u^\epsilon(t,x), v^\epsilon(t,x)\big) \in \mathbf{B}_\gamma \tag{4.18}$$

for the Cauchy problem (4.10) and (4.11) subsequently converging pointwise almost everywhere:

$$\big(u^\epsilon(t,x), v^\epsilon(t,x)\big) \longrightarrow \big(u(t,x), v(t,x)\big) \quad \textit{a.e.}\,,$$

Moreover, the limit function (u, v) satisfies the following conditions:

(i) $v(t,x) = f(u(t,x))$ *a.e.* , *for* $t > 0$;

(ii) $u(t,x)$ *is the unique entropy solution of the Cauchy problem*

$$
\begin{aligned}
&\partial_t u + \partial_x f(u) = 0\,, \\
&u\big|_{t=0} = w^*\text{-}\lim u_0^\epsilon(x)\,.
\end{aligned} \tag{4.19}
$$

Proof. Theorem 4.2 is a direct corollary of Theorem 4.1 by using Lemma 4.1.

Remark. The constant N_0 depends only on γ. If there exists a global strictly convex entropy for the p-system, the condition (4.17) can be removed by only assuming the uniform boundedness of the initial data $(u_0^\epsilon(x), v_0^\epsilon(x))$, and the assumption $f(u) = \bar{f} = const.$, as $|u - \bar{u}| \geq M_0$, does not affect the limit equation if one chooses a sufficiently large $M_0 \geq \sup |u_0^\epsilon(x)|$.

5. The Weakly Nonlinear Limit for 2×2 Systems

Now we are concerned with the weakly nonlinear limit for the Cauchy problem of 2×2 systems

$$
\begin{aligned}
&\partial_t u^\epsilon + \partial_x f_1(u^\epsilon, v^\epsilon) = 0\,, \\
&\partial_t v^\epsilon + \partial_x f_2(u^\epsilon, v^\epsilon) + \frac{1}{\epsilon} r(u^\epsilon, v^\epsilon) = 0\,,
\end{aligned} \tag{5.1}
$$

and

$$
(u^\epsilon, v^\epsilon)\big|_{t=0} = (u_0^\epsilon(x), v_0^\epsilon(x))\,. \tag{5.2}
$$

about an equilibrium $(\bar{u}, \bar{v}) = (\bar{u}, e(\bar{u}))$:

$$
(u^\epsilon, v^\epsilon) = (\bar{u}, \bar{v}) + \epsilon\,(w^\epsilon, z^\epsilon)\,. \tag{5.3}
$$

Upon rescaling the time variable t and translating the space variable x as the slow time variable ϵt and the moving space variable $x - \lambda(\bar{u})t$, respectively:

$$
(t, x) \longmapsto (\epsilon\, t, x - \lambda(\bar{u})\, t)\,,
$$

the flux functions in (5.1) with the stability condition (3.9) satisfy

$$
\begin{aligned}
\bar\lambda &\equiv \lambda(\bar u) = 0\,,\\
\bar\lambda_-\bar\lambda_+ &\equiv \Lambda_-(\bar u)\Lambda_+(\bar u) < 0\,,
\end{aligned}
\tag{5.4}
$$

The limit process as $\epsilon \to 0$ is a weakly nonlinear limit just as the limit from the Boltzmann equations to the incompressible Navier-Stokes equations [1].

Let us first consider a formal expansion. Suppose that (u^ϵ, v^ϵ) is a family of solutions of (5.1) parameterized by ϵ. Define the new dependent variables $(\hat w^\epsilon, \hat z^\epsilon)$ and $(\hat u^\epsilon, \hat v^\epsilon)$ by the relations

$$
u^\epsilon(t,x) = \bar u + \epsilon\,\hat w^\epsilon + \epsilon^2 \hat u^\epsilon(\hat t, \hat x)\,,
\tag{5.5}
$$

$$
v^\epsilon(t,x) = e(\bar u) + \epsilon\,\hat z^\epsilon + \epsilon^2 \hat v^\epsilon(\hat t, \hat x)\,.
\tag{5.6}
$$

Rendering system (5.1) in terms of these new variables yields

$$
\begin{aligned}
&\epsilon\,\partial_t \hat w^\epsilon + \epsilon^2 \partial_t \hat u^\epsilon + \bar f_{1u}(\hat w^\epsilon_x + \epsilon\,\hat u^\epsilon) + \bar f_{1v}(\hat z^\epsilon_x + \epsilon\,\hat v^\epsilon_x)\\
&\quad + \frac{\epsilon}{2}\partial_x\{\bar f_{1uu}(\hat w^\epsilon + \epsilon\,\hat u)^2 + 2\bar f_{1uv}(\hat w^\epsilon + \epsilon\,\hat u^\epsilon)(\hat z^\epsilon + \epsilon\,\hat v^\epsilon) + \bar f_{1vv}(\hat z^\epsilon_x + \epsilon\,\hat v^\epsilon)^2\}\\
&\quad + \epsilon^2 \partial_x Q_1(f_1) = 0\,,
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
&\epsilon\,\partial_t \hat z^\epsilon + \epsilon^2 \partial_t \hat v^\epsilon + \bar f_{2u}(\hat w^\epsilon_x + \epsilon\,\hat u^\epsilon) + \bar f_{2v}(\partial_x \hat z^\epsilon + \epsilon\,\partial_x \hat v^\epsilon) + \epsilon\,\partial_x Q_0(f_2)\\
&\quad + \frac{1}{\epsilon}\{\bar r_u(\hat w^\epsilon + \epsilon\,\hat u^\epsilon) + \bar r_v(\hat z^\epsilon + \epsilon\,\hat v^\epsilon)\}\\
&\quad + \epsilon\{\bar r_{uu}(\hat w^\epsilon + \epsilon\,\hat u^\epsilon)^2 + 2\bar r_{uv}(\hat w^\epsilon + \epsilon\,\hat u^\epsilon)(\hat z^\epsilon + \epsilon\,\hat v^\epsilon) + \bar r_{vv}(\hat z^\epsilon + \epsilon\,\hat v^\epsilon)^2\}\\
&\quad + \epsilon^2 Q_1(r) = 0\,.
\end{aligned}
\tag{5.8}
$$

where

$$
\begin{aligned}
\hat Q_0(\beta) &= \frac{1}{\epsilon^2}\Big(\beta(u^\epsilon, v^\epsilon) - \beta(\bar u, e(\bar u)) - \epsilon\bar\beta_u(\hat w^\epsilon + \epsilon\hat u^\epsilon) - \epsilon\bar\beta_v(\hat z^\epsilon + \epsilon\hat v^\epsilon)\Big)\\
&= \mathcal{O}(1)\Big((\hat w^\epsilon + \epsilon\hat u^\epsilon)^2 + (\hat z^\epsilon + \epsilon\hat v^\epsilon)^2\Big)\,,\\
\hat Q_1(\beta) &= \frac{1}{\epsilon^3}\Big(Q_0(\beta) - \tfrac{1}{2}\epsilon^2\big(\bar\beta_{uu}(\hat w^\epsilon + \epsilon\hat u^\epsilon)^2\\
&\qquad + 2\bar\beta_{uv}(\hat w^\epsilon + \epsilon\hat u^\epsilon)(\hat z^\epsilon + \epsilon\hat v^\epsilon) + \bar\beta_{vv}(\hat z^\epsilon + \epsilon\hat v^\epsilon)^2\big)\Big)\\
&= \mathcal{O}(1)\big(|\hat w^\epsilon + \epsilon\hat u^\epsilon|^3 + |\hat z^\epsilon + \epsilon\hat v^\epsilon|^3\big)\,,\\
\bar\beta &= \beta(\bar u, \bar v)\,, \qquad \bar\beta_u = \partial_u\beta(\bar u, \bar v)\,, \qquad \bar\beta_v = \partial_v\beta(\bar u, \bar v)\,, \text{ etc.}
\end{aligned}
\tag{5.9}
$$

We have from (5.7)–(5.8)

$$\bar{r}_u \hat{w}^\epsilon + \bar{r}_v \hat{z}^\epsilon = 0\,, \tag{5.10}$$

$$\bar{r}_v \hat{v}^\epsilon + \bar{r}_u \hat{u}^\epsilon + \tfrac{1}{2}\Big(\bar{r}_{uu}(\hat{w}^\epsilon)^2 + 2\bar{r}_{uv}\hat{w}^\epsilon\hat{z}^\epsilon + \bar{r}_{vv}(\hat{z}^\epsilon)^2\Big) + \big(\bar{f}_{2u} + \bar{f}_{2v}e'(\bar{u})\big)\partial_x\hat{w}^\epsilon = 0\,, \tag{5.11}$$

$$\partial_t\hat{w}^\epsilon + \bar{f}_{1u}\partial_x\hat{u}^\epsilon + \bar{f}_{1v}\partial_x\hat{v}^\epsilon + \tfrac{1}{2}\Big(\bar{f}_{1uu}\partial_x(\hat{w}^\epsilon)^2 + 2\bar{f}_{1uv}\partial_x(\hat{w}^\epsilon\hat{z}^\epsilon) + \bar{f}_{1vv}\partial_x(\hat{z}^\epsilon)^2\Big) = 0\,. \tag{5.12}$$

If $(\hat{w}^\epsilon, \hat{z}^\epsilon) \to (w, z)$ and $(\hat{u}^\epsilon, \hat{v}^\epsilon) \to (\hat{u}, \hat{v})$ as ϵ tends to zero so that, consistent with their leading-order formal expansions, one also has the limits

$$z = e'(\bar{u})w\,, \tag{5.13}$$

$$\hat{v} = e'(\bar{u})\hat{u} - \frac{1}{2\bar{r}_v}\Big(\bar{r}_{uu} + 2\bar{r}_{uv}\,e'(\bar{u}) + \bar{r}_{vv}\,e'(\bar{u})^2\Big)w^2 - \frac{1}{\bar{r}_v}\big(\bar{f}_{2u} + \bar{f}_{2v}\,e'(\bar{u})\big)\partial_x w\,, \tag{5.14}$$

$$\partial_t w + \bar{f}_{1v}\big(\partial_x\hat{v} - e'(\bar{u})\partial_x\hat{u}\big) + \Big(\bar{f}_{1uu} + 2\bar{f}_{1uv}\,e'(\bar{u}) + \bar{f}_{1vv}\,e'(\bar{u})^2\Big)\partial_x\big(\tfrac{1}{2}w^2\big) = 0\,, \tag{5.15}$$

by using

$$\begin{aligned}\bar{r}_u + \bar{r}_v\,e'(\bar{u}) &= 0\,,\\ \bar{f}_{1u} + \bar{f}_{1v}\,e'(\bar{u}) &= 0\,.\end{aligned}$$

Plug (5.14) into (5.15) and use

$$\begin{aligned}&\bar{r}_{uu} + 2\bar{r}_{uv}\,e'(\bar{u}) + \bar{r}_{vv}\,e'(\bar{u})^2 + \bar{r}_v\,e''(\bar{u}) = 0\,,\\ &\bar{\Lambda}_-\bar{\Lambda}_+ = -\bar{f}_{1v}\big(\bar{f}_{2u} + \bar{f}_{2v}\,e'(\bar{u})\big)\,.\end{aligned}$$

We have

$$\bar{r}_v\Big(\partial_t w + \bar{\lambda}'\partial_x\big(\tfrac{1}{2}w^2\big)\Big) + \bar{\lambda}_-\bar{\lambda}_+\partial_{xx}w = 0\,. \tag{5.16}$$

This is the dominant balance for this asymptotic scaling provided

$$\bar{\lambda}' \neq 0\,, \quad \text{and} \quad \bar{\lambda}_-\bar{\lambda}_+ < 0\,. \tag{5.17}$$

Not unexpectedly, (5.16) is just the classical Burgers equation.

If one applies the same asymptotic scaling as above to the first correction to the local equilibrium approximation (1.21), one again arrives at the weakly nonlinear approximation (5.16). This shows the latter to be a distinguished limit of the former and makes clear why it inherits the good features of the former. Its big advantage is that the solutions of the Burgers equation (5.16) are so nice. If the initial data has spatial derivatives that are order one, then so does the solution for all time. Moreover, the solutions are smooth for any positive time. Thus, the solutions of (5.16) remain globally consistent with all the assumptions that were used to derive the weakly nonlinear approximation.

We justify this approximation by using the energy estimate technique. From (5.1) and (5.3), we obtain that (w^ϵ, z^ϵ) satisfies

$$\begin{aligned} &\epsilon^2 \partial_t w^\epsilon + \partial_x f_1(\bar{u} + \epsilon\, w^\epsilon, \bar{v} + \epsilon\, z^\epsilon) = 0\,, \\ &\epsilon^2 \partial_t z^\epsilon + \partial_x f_2(\bar{u} + \epsilon\, w^\epsilon, \bar{v} + \epsilon\, z^\epsilon) + \frac{1}{\epsilon} r(\bar{u} + \epsilon\, w^\epsilon, \bar{v} + \epsilon\, z^\epsilon) = 0\,, \end{aligned} \tag{5.18}$$

and

$$(w^\epsilon, z^\epsilon)\big|_{t=0} = (w_0^\epsilon(x), z_0^\epsilon(x))\,. \tag{5.19}$$

Theorem 5.1. *There exist constants $\epsilon_0 > 0$ and $C_0 > 0$ independent of ϵ such that, if*

$$\begin{aligned} &0 < \epsilon \le \epsilon_0\,, \\ &\|(w_0^\epsilon, z_0^\epsilon)\|_{H^3} \le C_0\,, \end{aligned} \tag{5.20}$$

and

$$\|z_0^\epsilon - \frac{e(\bar{u} + \epsilon\, w_0^\epsilon) - e(\bar{u})}{\epsilon}\|_{L^2(\mathbf{R}^1)} \le C_0 \epsilon\,, \tag{5.21}$$

such that there exists a unique global solution $(w^\epsilon, z^\epsilon) \in H^3$ for the Cauchy problem (5.18)–(5.19) (also (5.1)–(5.2)) such that

$$\begin{aligned} &\sum_{i,j=1, i+j\le 3} \epsilon^{(i-1)} \|\partial_t^i \partial_x^j (w^\epsilon, \epsilon^i z^\epsilon)\|_{L^2(\mathbf{R}^2_+)} \le C\,, \\ &\|z^\epsilon - \frac{e(\bar{u} + \epsilon\, w^\epsilon) - e(\bar{u})}{\epsilon}\|_{L^2(\mathbf{R}^2_+)} \le C\epsilon\,, \end{aligned} \tag{5.22}$$

where C is the constant independent of ϵ.

Proof. For simplicity, we drop the index ϵ in the functions (w^ϵ, z^ϵ). Define

$$
\begin{aligned}
B(t) &= \int_{-\infty}^{\infty} \left\{ \sum_{i,j=0, i+j\le 3}^{3} \epsilon^{2i} |\partial_t^i \partial_x^j (w, \epsilon^i z)|^2 (t,x) \right\} dx \,, \\
C(t) &= \int_0^t \int_{-\infty}^{\infty} \left\{ \sum_{i,j=1, i+j\le 3} \epsilon^{2(i-1)} |\partial_t^i \partial_x^j (w, \epsilon^i z)|^2 (\tau,x) \right\} dx d\tau \,,
\end{aligned}
\tag{5.23}
$$

We first obtain the a-priori estimates.

Suppose there exist solutions $(w^\epsilon, z^\epsilon) \in H^3$ for the Cauchy problem (5.18) and (5.19). We wish to prove that there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ independent of ϵ such that, when

$$
\begin{gathered}
0 < \epsilon \le \epsilon_0 \,, \\
\|(w_0^\epsilon, z_0^\epsilon)\|_{H^3} \le C_0 \,, \\
\int_{-\infty}^{\infty} \left(v_0^\epsilon(x) - e(u_0^\epsilon(x)) \right)^2 dx \le \epsilon^4 C_0 \,,
\end{gathered}
\tag{5.24}
$$

the estimates

$$
B(t) + C(t) \le B_0 \,,
$$

hold. These are proved under the a-priori assumption

$$
B_0 \equiv \max_{0\le\tau\le t} B(\tau) \ll 1 \,.
$$

We have from the first equation of (5.18) that

$$
z_x = -\frac{1}{f_{1v}} \{ \epsilon\, w_t + f_{1u} w_x \} \,.
\tag{5.25}
$$

Differentiating both sides of the second equation of (5.18) with respect to x, we have

$$
\begin{aligned}
&\epsilon^2 z_{xt} + f_2(\bar{u} + \epsilon\, w, \bar{v} + \epsilon\, z)_x \\
&= \bar{r}_u w + \bar{r}_v z + \frac{\epsilon}{2} \left(\bar{r}_{uu} w^2 + 2\bar{r}_{uv} wz + \bar{r}_{vv} z^2 \right) + \frac{\epsilon^2}{6} Q_1(r) \,.
\end{aligned}
\tag{5.26}
$$

Henceforth the functions $r_u \equiv \partial_u r$, etc. are evaluated at the point $(\bar u, \bar v) \equiv (\bar u, e(\bar u))$ and for any function β,

$$
\begin{aligned}
[\beta] &= \frac{1}{\epsilon}\big(\beta(\bar u + \epsilon w, \bar v + \epsilon z) - \beta(\bar u, \bar v)\big) = \mathcal{O}(1)(|w| + |z|)\,,\\
Q_0(\beta) &= \frac{1}{\epsilon^2}\big([\beta] - \epsilon\bar\beta_u w - \epsilon\bar\beta_v z\big) = \mathcal{O}(1)(w^2 + z^2)\,,\\
Q_1(\beta) &= \frac{1}{\epsilon^3}\Big([\beta] - \epsilon\bar\beta_u w - \epsilon\bar\beta_v z - \tfrac{1}{2}\epsilon^2\big(\bar\beta_{uu}w^2 + 2\bar\beta_{uv}wz + \bar\beta_{vv}z^2\big)\Big)\\
&= \mathcal{O}(1)(|w|^3 + |z|^3)\,,\\
\bar\beta &= \beta(\bar u, \bar v)\,, \qquad \bar\beta_u = \partial_u\beta(\bar u, \bar v)\,, \qquad \bar\beta_v = \partial_v\beta(\bar u, \bar v)\,, \quad \text{etc.}
\end{aligned}
$$

Plug (5.25) into (5.26) to eliminate z and obtain

(5.27)

$$
\begin{aligned}
&\bar r_v\big(w_t + \bar\lambda'\big(\tfrac{1}{2}w^2\big)_x\big) + \bar\lambda_-\bar\lambda_+ w_{xx} + \epsilon^2 w_{tt}\\
&= -\epsilon(f_{1u} + f_{2v})w_{xt}\\
&\quad - \epsilon\{\bar f_{1u}[f_{1v}] + \bar f_{2v}[f_{1u}] - \bar f_{1v}[f_{2u}] - \bar f_{2u}[f_{1v}] + [f_{1u}][f_{2v}] - [f_{1v}][f_{2u}]\}w_{xx}\\
&\quad + \frac{\epsilon}{2}\bar r_v\{(\bar f_{1uv} + \bar f_{1vv}\bar v')(\bar v'' w^2 + \frac{\epsilon}{3}Q_1(r) - \frac{2}{\epsilon}f_{2x} - 2\epsilon z_t) - 2\frac{\bar r_{uu}}{\bar r_v}[f_{1v}]w\\
&\qquad + \frac{2[f_{1v}]}{\bar r_v f_{1v}}((\bar r_{uv}w + 2\bar r_{vv}z)f_{1u} - \bar r_{uv}f_{1v}z)Q_0(f_{1u}) + Q_0(f_{1v})\}w_x\\
&\quad - \{(f_{1u}f_{2v})_x - f_{1v}f_{2ux} - \frac{f_{2v}f_{1u}}{f_{1v}}f_{1vx} + \epsilon^2(f_{1uu} - \frac{f_{1u}}{f_{1v}}f_{1uv})w_t\\
&\qquad\qquad + \epsilon^3(f_{1uv} - \frac{f_{1u}}{f_{1v}}f_{1vv})z_t\}w_x\\
&\quad - \epsilon\{f_{2vx} - \frac{f_{2v}}{f_{1v}}f_{1vx} - \frac{\epsilon}{f_{1v}}f_{1vt} + \frac{\epsilon^2}{f_{1v}}[f_v](\bar r_{uv}w + 2\bar r_{uv}z)\}w_t\\
&\quad + \epsilon\frac{\bar r_v\bar f_{1v}}{4\bar r_v^3}\{\epsilon\bar r_{vv}(\bar r_{uu}w^2 + 2\bar r_{uv}wz + \bar r_{vv}z^2 + \frac{\epsilon}{3}Q_1(r) + \frac{1}{\epsilon}f_{2x} + \epsilon z_t)^2\\
&\qquad - (\bar r_{vv}(\bar v')^2 w^2 - 2\bar r_{uv}\bar r_v w)\\
&\qquad\qquad (\bar r_{uu}w^2 + 2\bar r_{uv}wz + \bar r_{vv}z^2 + \frac{\epsilon}{3}Q_1(r) + \frac{1}{\epsilon}f_{2x} + \epsilon z_t)\}_x\\
&\quad + \frac{\epsilon}{6}f_{1v}Q_1(r)_x\\
&\equiv E^\epsilon(t,x)\,.
\end{aligned}
$$

On the other hand, we know from Theorem 3.2 that there exists a convex entropy pair $(\Phi(u,v), \Psi(u,v))$ near the equilibrium with $\Phi_v = 0$ and

(5.28)
$$\Phi(u,v) \le \epsilon^2 C(w^2 + z^2)\,.$$

Otherwise we can replace (Φ,Ψ) by $(\hat\Phi,\hat\Psi)$ where

$$\begin{aligned}\hat\Phi(U) &= \Phi(U)-\Phi(\bar U)-\partial_U\Phi(\bar U)(U-\bar U)\,,\\ \hat\Psi(U) &= \Psi(U)-\Psi(\bar U)-\partial_U\Phi(\bar U)(F(U)-F(\bar U))\,.\end{aligned}$$

Multiplying (5.1) by $\partial_U\Phi$, we have

$$\epsilon\,\Phi_t+\Psi_x+\Phi_v\frac{r(u,v)}{\epsilon}\le 0\,.$$

Integrate the above inequality, and notice that

$$\Phi_v r(u,v)\ge \alpha_0(v-e(u))^2\,,$$

for some $\alpha_0>0$ from $\Phi_v(u,e(u))=0, \Phi_{vv}>0$, and $r_v\neq 0$. We obtain

$$\begin{aligned}&\epsilon\int_0^t \Phi(u,v)\,dx+\alpha_0\int_0^t\int_{-\infty}^{\infty}\frac{(v-e(u))^2}{\epsilon}\,dxdt\\ &\le \epsilon\int_{-\infty}^{\infty}\Phi(u_0(x),v_0(x))\,dx\\ &\le \epsilon^3 C\int_{-\infty}^{\infty}\big(w_0^2(x)+z_0^2(x)\big)\,dx\,,\end{aligned}$$

and, therefore,

$$\|z-\frac{e(\bar u+\epsilon\,w)-e(\bar u)}{\epsilon}\|_{L^2(\mathbf{R}^2_+)}\le C\epsilon\,. \tag{5.29}$$

Now we estimate the relationship between w and z. From (5.18) we can obtain

$$|z_x|\le C\{|w_x|+\epsilon|w_t|\}\,, \tag{5.30}$$

$$|z_t|\le\frac{C}{\epsilon}\{|w_x|+\epsilon|w_t|+\frac{|v-e(u)|}{\epsilon^2}\}\,, \tag{5.31}$$

$$|z_{xx}|\le C\big\{|w_{xx}|+\epsilon|w_{xt}|+\epsilon\big(w_x^2+\epsilon^2w_t^2\big)\big\}\,, \tag{5.32}$$

$$|z_{xt}|\le C\Big\{|w_{xt}|+\epsilon|w_{tt}|+w_x^2+\epsilon^2w_t^2\big(|w_x|+\epsilon|w_t|\big)\frac{|v-e(u)|}{\epsilon^2}\Big\}\,, \tag{5.33}$$

$$(5.34)\quad |z_{tt}| \le \frac{C}{\epsilon^3}\Big\{\epsilon^2 w_{xt} + \epsilon^3 w_{tt} + \epsilon^2(w_x^2 + \epsilon^2 w_t^2) + |w_x| + \epsilon|w_t| + \frac{|v-e(u)|}{\epsilon}\Big\}\,,$$

(5.35)

$$|z_{xxx}| \le C\big\{|w_{xxx}|+\epsilon|w_{xxt}|+\epsilon(|w_x|+\epsilon|w_t|)(|w_{xx}|+\epsilon|w_{xt}|)+\epsilon^3(|w_x|^3+\epsilon^3|w_t|^3)\big\}\,,$$

$$(5.36)\quad \begin{aligned} |z_{xxt}| \le C\Big\{&|w_{xxt}| + \epsilon|w_{xtt}| \\ &+ (|w_x| + \epsilon|w_t|)(|w_{xx}| + \epsilon|w_{xt}| + \epsilon^2|w_{tt}| + \epsilon w_x^2 + \epsilon^3 w_t^2) \\ &+ (w_{xx} + \epsilon w_{xt} + \epsilon w_x^2 + \epsilon^3 w_t^2)\frac{|v-e(u)|}{\epsilon^2}\Big\}\,, \end{aligned}$$

$$(5.37)\quad \begin{aligned} |z_{xtt}| \le \epsilon^{-2} C\Big\{&\epsilon^2|w_{xtt}| + \epsilon^3|w_{ttt}| + w_x^2 + \epsilon^2 w_t^2 + \epsilon^2|w_x|^3 + \epsilon^5|w_t|^3 \\ &+ \epsilon^2(|w_x| + \epsilon|w_t|)(|w_{xt}| + \epsilon|w_{tt}|) \\ &+ (|w_x| + \epsilon|w_t| + \epsilon^2|w_{xt}| + \epsilon^3 w_{tt})\frac{|v-e(u)|}{\epsilon^2}\Big\}\,, \end{aligned}$$

(5.38)

$$\begin{aligned} |z_{ttt}| \le &\frac{C}{\epsilon^5}\Big\{\epsilon^4|w_{xtt}| + \epsilon^5|w_{ttt}| + \epsilon^3|w_{tt}| + \epsilon^2|w_{xt}| + |w_x| + \epsilon|w_t| + \epsilon^2 w_x^2 + \epsilon^4 w_t^2 \\ &+ \epsilon^4|w_x|^3 + \epsilon^7|w_t|^3 + \epsilon^4(|w_x| + \epsilon|w_t|)(|w_{xt}| + \epsilon|w_{tt}|) + \frac{1}{\epsilon^2}|v-e(u)|\Big\}\,, \end{aligned}$$

and

$$(5.39)\quad \begin{aligned} &\epsilon^2\int_{-\infty}^{\infty} z^2(t,x)\,dx + \int_0^t\int_{-\infty}^{\infty} z^2(\tau,x)\,dxd\tau \\ &\le \epsilon^2\int_{-\infty}^{\infty} z^2(0,x)\,dx + \int_0^t\int_{-\infty}^{\infty}\{w^2 + \epsilon^2(w_x^2 + \epsilon^2 w_t^2)\}\,dxd\tau\,, \end{aligned}$$

$$(5.40)\quad \int_0^t\int_{-\infty}^{\infty} z_x^2\,dxd\tau \le C\int_0^t\int_{-\infty}^{\infty}(w_x^2 + \epsilon^2 w_t^2)\,dxd\tau\,,$$

$$(5.41)\quad \begin{aligned} &\epsilon^2\int_{-\infty}^{\infty} z_t^2(t,x)dx + \int_0^t\int_{-\infty}^{\infty}\big\{1 - \epsilon^2 C(|w_x| + \epsilon|w_t|)\big\}z_t^2\,dxd\tau \\ \le &C\int_{-\infty}^{\infty}|\nabla_x(w_0(x), z_0(x))|^2\,dx + \frac{1}{\epsilon^4}\int_{-\infty}^{\infty}(v_0(x) - e(u_0(x)))^2\,dx \\ &+ C\int_0^t\int_{-\infty}^{\infty}\big\{w_t^2 + \epsilon^2(w_x^4 + \epsilon^4 w_t^4) + \epsilon^2 w_{xt}^2 + \epsilon^4 w_{tt}^2\big\}\,dxd\tau\,, \end{aligned}$$

$$\int_0^t\int_{-\infty}^{\infty} z_{xx}^2\,dxd\tau \le C\int_0^t\int_{-\infty}^{\infty}\left\{w_{xx}^2+\epsilon^2 w_{xt}^2+\epsilon^2(w_x^4+\epsilon^4 w_t^4)\right\}dxd\tau\,, \tag{5.42}$$

$$\begin{aligned}&\int_0^t\int_{-\infty}^{\infty} z_{xt}^2\,dxd\tau\\&\le C\int_0^t\int_{-\infty}^{\infty}\left\{w_{xt}^2+\epsilon^2 w_{tt}^2+w_x^4+\epsilon^4 w_t^4+(w_x^2+\epsilon^2 w_t^2)\frac{(v-e(u))^2}{\epsilon^4}\right\}dxd\tau\,,\end{aligned} \tag{5.43}$$

$$\begin{aligned}&\epsilon^8\int_{-\infty}^{\infty} z_{tt}^2(t,x)\,dx+\epsilon^6\int_0^t\int_{-\infty}^{\infty}\{1-\epsilon^2 C(|w_x|+\epsilon|w_t|)\}z_{tt}^2\,dxd\tau\\&\le C\epsilon^2\int_{-\infty}^{\infty}|\nabla_x(w_0(x),z_0(x))|^2\,dx+\frac{2}{\epsilon^4}\int_{-\infty}^{\infty}(v_0(x)-e(u_0(x)))^2\,dx\\&\quad+\epsilon^6 C\int_0^t\int_{-\infty}^{\infty}\{w_t^2+w_x^2+w_{tt}^2+\epsilon^2 w_{xt}^2+\epsilon^2 w_{xtt}^2+\epsilon^4 w_{ttt}^2\}\,dxd\tau\,,\end{aligned} \tag{5.44}$$

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty} z_{xxx}^2\,dxd\tau \le C\int_0^t\int_{-\infty}^{\infty}&\Big\{w_{xxx}^2+\epsilon^2 w_{xxt}^2+\epsilon^6(w_x^6+\epsilon^6 w_t^6)\\&+\epsilon^2(w_x^2+\epsilon^2 w_t^2)(w_{xx}^2+\epsilon^2 w_{xt}^2)\Big\}\,dxd\tau\,,\end{aligned} \tag{5.45}$$

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty} z_{xxt}^2\,dxd\tau \le C\int_0^t\int_{-\infty}^{\infty}&\Big\{w_{xxt}^2+\epsilon^2 w_{xtt}^2\\&+(w_x^2+\epsilon^2 w_t^2)(w_{xx}^2+\epsilon^2 w_{xt}^2+\epsilon^4 w_{tt}^2+\epsilon^2 w_x^4+\epsilon^6 w_t^4)\\&+(w_{xx}^2+\epsilon^2 w_{xt}^2+\epsilon^2 w_x^4+\epsilon^6 w_t^4)\frac{(v-e(u))^2}{\epsilon^4}\Big\}\,dxd\tau\,,\end{aligned} \tag{5.46}$$

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty} z_{xtt}^2\,dxd\tau \le \frac{C}{\epsilon^4}\int_0^t\int_{-\infty}^{\infty}&\Big\{\epsilon^4 w_{xtt}^2+\epsilon^6 w_{ttt}^2+w_x^4+\epsilon^4 w_t^4\\&+\epsilon^4 w_x^6+\epsilon^{10}w_t^6+\epsilon^4(w_x^2+\epsilon^2 w_t^2)(w_{xt}^2+\epsilon^2 w_{tt}^2)\\&+(w_x^2+\epsilon^2 w_t^2+\epsilon^2 w_{xt}^2+\epsilon^3 w_{tt}^2)\Big\}\,dxd\tau\,,\end{aligned} \tag{5.47}$$

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty} z_{ttt}^2\,dxd\tau \le \frac{C}{\epsilon^{10}}\int_0^t\int_{-\infty}^{\infty}&\Big\{\epsilon^8 w_{xtt}^2+\epsilon^{10}w_{ttt}^2+\epsilon^4 w_{xt}^2+\epsilon^6 w_{tt}^2\\&+w_x^2+\epsilon^2 w_t^2+\epsilon^4 w_x^4+\epsilon^8 w_t^4+\epsilon^8 w_x^6+\epsilon^{14}w_t^6\\&+\epsilon^8(w_x^2+\epsilon^2 w_t^2)(w_{xt}^2+\epsilon^2 w_{tt}^2)\Big\}\,dxd\tau\,.\end{aligned} \tag{5.48}$$

Multiply (5.25) by w, and integrate over $[0,t]\times(-\infty,\infty)$. We obtain from (5.4) and (3.4)

$$
\begin{aligned}
&\frac{\bar r_v}{2}\int_{-\infty}^{\infty} w^2(t,x)\,dx + |\bar\lambda_-\bar\lambda_+|\int_0^t\int_{-\infty}^{\infty} w_x^2\,dxd\tau \\
&= \frac{\bar r_v}{2}\int_{-\infty}^{\infty} w^2(0,x)\,dx + \epsilon^2\int_{-\infty}^{\infty}\{(ww_t)(t,x)-(ww_t)(0,x)\}\,dx \\
&\quad + \epsilon^2\int_0^t\int_{-\infty}^{\infty} w_t^2\,dxd\tau + \int_0^t\int_{-\infty}^{\infty} E(\tau,x)w(\tau,x)\,dxd\tau\,.
\end{aligned}
\tag{5.49}
$$

Now we estimate the integral of the E term. From the Schwartz inequality

$$
w^2(t,x)=\int_{-\infty}^{x} 2ww_x\,dx \le \left(2\int_{-\infty}^{\infty} w^2\,dx\int_{-\infty}^{\infty} w_x^2\,dx\right)^{1/2},
$$

and so

$$
\begin{aligned}
\int_0^t\int_{-\infty}^{\infty} w^6(t,x)\,dxdt &\le \max_{0\le\tau\le t}\left\{\left(\int_{-\infty}^{\infty} w^2(\tau,x)\,d\tau\right)^2\right\}\int_0^t\int_{-\infty}^{\infty} w_x^2\,dxd\tau \\
&\le B_0\int_0^t\int_{-\infty}^{\infty} w_x^2\,dxd\tau\,.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\int_0^t\int_{-\infty}^{\infty}(w^6+z^6)\,dxd\tau &\le B_0\int_0^t\int_{-\infty}^{\infty}(w_x^2+z_x^2)\,dxd\tau \\
&\le B_0\int_0^t\int_{-\infty}^{\infty}(w_x^2+\epsilon^2 w_t^2)\,dxd\tau\,.
\end{aligned}
\tag{5.50}
$$

Moreover,

$$
\epsilon\int_{-\infty}^{\infty} ww_t\,dx \le \int_{-\infty}^{\infty} w^2(t,x)\,dx + \epsilon^2\int_{-\infty}^{\infty} w_t^2(t,x)\,dx\,.
\tag{5.51}
$$

and

$$
\begin{aligned}
&\left|\epsilon\int_0^t\int_{-\infty}^{\infty}(f_{1u}+f_{1v})w_{xt}w\,dxd\tau\right| \\
&= \left|\epsilon\int_0^t\int_{-\infty}^{\infty}\left\{(f_{1u}+f_{1v})w_xw_t+(f_{1u}+f_{1v})_x ww_t\right\}dxd\tau\right| \\
&\le C\int_0^t\int_{-\infty}^{\infty}\left\{\delta w_x^2+\epsilon^2 w_t^2\right\}dxd\tau\,, \quad \text{for } \delta \text{ sufficiently small.}
\end{aligned}
\tag{5.52}
$$

Using (5.50), (5.51) and the argument of estimating (5.52), we obtain from (5.49)

$$
\begin{aligned}
&\int_{-\infty}^{\infty} w^2(t,x)\,dx + \int_0^t\int_{-\infty}^{\infty} w_x^2\,dxd\tau \\
&\quad\le C\Big\{\int_{-\infty}^{\infty}\big(w^2(0,x)+\epsilon^4 w_t^2(0,x)\big)\,dx + \int_{-\infty}^{\infty}\big(w^2(t,x)+\epsilon^4 w_t^2(t,x)\big)\,dx \\
&\qquad\qquad + \int_0^t\int_{-\infty}^{\infty}\big(\delta w_x^2+\epsilon^2 w_t^2+\epsilon^4 z_t^2\big)\,dxd\tau\Big\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} w^2(t,x)\,dx + \int_0^t\int_{-\infty}^{\infty} w_x^2\,dxd\tau \\
&\le C\Big\{\int_{-\infty}^{\infty}\big(w^2(0,x)+\epsilon^2|\nabla_x(w,z)|^2(0,x)+(v-e(u))^2(0,x)\big)\,dx \\
&\qquad + \epsilon^4\int_{-\infty}^{\infty} w_t^2(t,x)\,dx + \epsilon^2\int_0^t\int_{-\infty}^{\infty}\big(w^2(t,x)+w_t^2(t,x)\big)\,dxd\tau\Big\};
\end{aligned}
\tag{5.53}
$$

henceforth we assume that C is the universal constant depending only on B_0.

Similar arguments yield the estimates for higher derivatives of w by integrating $(5.27)\times w$, $(5.27)\times w_t$, $(5.27)\times w_x$, $(5.27)_x\times w_{xx}$, $(5.27)_x\times w_{xt}$, $(5.27)_t\times w_{tt}$, $(5.27)_{xx}\times w_{xxt}$, $(5.27)\times w_{xxt}$, $(5.27)_{xt}\times w_{xtt}$, and $(5.27)_{tt}\times w_{ttt}$ over $[0,t]\times(-\infty,\infty)$, respectively,

$$
\begin{aligned}
&\epsilon^2\int_{-\infty}^{\infty} w_t^2(t,x)\,dx + \int_0^t\int_{-\infty}^{\infty} w_t^2\,dxd\tau \\
&\le \epsilon^2\int_{-\infty}^{\infty} w_t(0,x)\,dx + C\int_0^t\int_{-\infty}^{\infty}\{w_x^2+w_{xx}^2+\epsilon^4 w_{xt}^2\}\,dxd\tau\,,
\end{aligned}
\tag{5.54}
$$

(5.55)
$$
\begin{aligned}
&\int_{-\infty}^{\infty} w_x^2(t,x)\,dx + \int_0^t\int_{-\infty}^{\infty} w_{xx}^2\,dxd\tau \\
&\le C\Big\{\int_{-\infty}^{\infty}\Big(w_x^2(0,x)+\epsilon\big(w_{xx}(0,x)+z_{xx}(0,x)\big)+\epsilon\big(w_x^4(0,x)+z_x^4(0,x)\big)\Big)\,dx \\
&\qquad + \epsilon^3\int_{-\infty}^{\infty} w_{xt}^2(t,x)\,dx + \int_0^t\int_{-\infty}^{\infty}\big(w_x^2+\epsilon^2 w_t^2+\epsilon^2 w_{xt}^2\big)\,dxd\tau\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\epsilon^4 \int_{-\infty}^{\infty} w_{tt}^2(t,x)\,dx + \epsilon^2 \int_0^t \int_{-\infty}^{\infty} w_{tt}^2\,dxd\tau \\
&\le \epsilon^4 \int_{-\infty}^{\infty} w_{tt}(0,x)\,dx \\
&\quad + C \int_0^t \int_{-\infty}^{\infty} \{w_x^2 + w_t^2 + w_{xx}^2 + \epsilon^2 w_{xt}^2 + \epsilon^2 w_{xxt}^2 + \epsilon^6 w_{xtt}^2\}\,dxd\tau\,,
\end{aligned} \tag{5.56}
$$

$$
\begin{aligned}
&\epsilon^2 \int_{-\infty}^{\infty} w_{xt}^2(t,x)\,dx + \int_0^t \int_{-\infty}^{\infty} w_{xt}^2\,dxd\tau \\
&\le \epsilon^4 \int_{-\infty}^{\infty} w_{xt}(0,x)\,dx + C \int_0^t \int_{-\infty}^{\infty} \{w_x^2 + \epsilon^2 w_t^2 + w_{xx}^2 + \delta w_{xxt}^2\}\,dxd\tau\,,
\end{aligned} \tag{5.57}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} w_{xx}^2(t,x)\,dx + \int_0^t \int_{-\infty}^{\infty} w_{xxx}^2\,dxd\tau \\
&\quad \le \int_{-\infty}^{\infty} w_{xx}(0,x)\,dx \\
&\quad + C \int_0^t \int_{-\infty}^{\infty} \{w_x^2 + \epsilon^4 w_t^2 + w_{xx}^2 + \epsilon^6 w_{tt}^2 + \epsilon^2 w_{xxt}^2 + \epsilon^4 w_{xtt}^2\}\,dxd\tau\,,
\end{aligned} \tag{5.58}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} (\epsilon^2 w_{xxt}^2 + w_{xxx}^2(t,x))\,dx + \int_0^t \int_{-\infty}^{\infty} w_{xxt}^2\,dxd\tau \\
&\quad \le \int_{-\infty}^{\infty} \big(\epsilon^2 w_{xxt}^2(0,x) + w_{xxx}^2(0,x)\big)\,dx \\
&\quad + C \int_0^t \int_{-\infty}^{\infty} \{w_x^2 + \epsilon^4 w_t^2 + \epsilon^2 w_{xx}^2 + \epsilon^4 w_{xt}^2 + w_{xxx}^2 + \epsilon^6 w_{xtt}^2\}\,dxd\tau\,,
\end{aligned} \tag{5.59}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \big(\epsilon^4 w_{xtt}^2 + \epsilon^2 w_{xxt}^2(t,x)\big)\,dx + \epsilon^2 \int_0^t \int_{-\infty}^{\infty} w_{xtt}^2\,dxd\tau \\
&\le \int_{-\infty}^{\infty} \big(\epsilon^4 w_{xtt}^2(0,x) + \epsilon^2 w_{xxt}^2(0,x)\big)\,dx \\
&+ C \int_0^t \int_{-\infty}^{\infty} \{w_x^2 + \delta w_t^2 + w_{xx}^2 + \delta w_{xt}^2 + \epsilon^2 w_{tt} + w_{xxx}^2 + \delta w_{xxt}^2 + \epsilon^4 w_{ttt}^2\}\,dxd\tau\,,
\end{aligned} \tag{5.60}
$$

(5.61)

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left(\epsilon^6 w_{ttt}^2+\epsilon^4 w_{xtt}^2(t,x)\right)dx+\epsilon^4\int_0^t\int_{-\infty}^{\infty} w_{ttt}^2\,dxd\tau\\
&\quad\le \int_{-\infty}^{\infty}\left(\epsilon^6 w_{xtt}^2(0,x)+\epsilon^4 w_{xxt}^2(0,x)\right)dx\\
&\qquad + C\int_0^t\int_{-\infty}^{\infty}\Big\{w_x^2+w_t^2+\delta w_{xx}^2+\epsilon^2 w_{xt}^2+\epsilon^4 w_{tt}+\epsilon^2 w_{xxt}^2+\epsilon^4 w_{xtt}^2+\epsilon^5 w_{ttt}^2\\
&\qquad\qquad +\epsilon^8\Big(\frac{\bar f_{1vv}}{\bar f_{1v}}\Big)^2 z_{tt}^2+\epsilon^8\Big(\frac{f_{1vv}}{f_{1v}}-\frac{\bar f_{1vv}}{\bar f_{1v}}\Big)^2 z_{tt}^2\Big\}\,dxd\tau\,,
\end{aligned}
$$

We obtain, from (5.53)–(5.61) and (5.39)–(5.48), that there exists $\epsilon_0>0$ such that, when $0<\epsilon\le\epsilon_0$,

$$B(t)+C(t)\le C(B_0)D_0\,,$$

where

$$D_0=\|(w_0^\epsilon,z_0^\epsilon)\|_{H^3}^2+\frac{1}{\epsilon^2}\left(\frac{\bar f_{1vv}}{\bar f_{1v}}\right)^2\|z_0^\epsilon-\frac{e(\bar u+\epsilon w_0^\epsilon)-e(\bar u)}{\epsilon}\|_{L^2(\mathbf{R})}^2\,, \tag{5.62}$$

and $C(B_0)$ is the constant depending only on B_0. Choose $C_0>0$ sufficiently small that, when $D_0\le C_0$,

$$C(B_0)D_0\le B_0\,.$$

Then we do have

$$B(t)\le B_0\,,$$

as we assumed a priori.

Using above estimates, we have from (5.23)

$$\|\sum_{i,j=0,i+j\le 3}\epsilon^{2i}\partial_t^i\partial_x^j(w,\epsilon^i z)\|_{L^2}^2\le B_0\,. \tag{5.63}$$

The classical theory ensures the local solutions for the Cauchy problem (5.18)–(5.19). The energy estimates (5.63) enable us to extend the local solutions to the global solutions and

$$(w^\epsilon,z^\epsilon)\in H^3(\mathbf{R}_+^2)\,.$$

This completes the proof of Theorem 5.1.

The condition (5.20) in Theorem 5.1 can be removed provided that the flux function $f_1(u,v)$ satisfies the following condition (5.64).

Theorem 5.2. *Suppose that $f_1(u,v)$ satisfies*

$$f_{1vv}(\bar{u},\bar{v}) = 0\,, \tag{5.64}$$

on the equilibrium state $(\bar{u},\bar{v}) = (\bar{u}, e(\bar{u}))$. Then there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ independent of ϵ such that if (5.20) holds, there exists a unique global solution $(w^\epsilon, z^\epsilon) \in H^3$ for the Cauchy problem (5.18)–(5.19) (also (5.1)–(5.2)) such that (5.22) still holds.

In fact, the condition (5.64) ensures

$$D_0 = \|(w_0^\epsilon, z_0^\epsilon)\|_{H^3}$$

in (5.62). The results follows.

Remark. The condition (5.64) is satisfied by many physical examples such as the elasticity model (see [8]) and the p-system (1.18).

Theorem 5.3. *Suppose that the functions (w^ϵ, z^ϵ) are solutions of the Cauchy problem (5.18)–(5.19) uniquely determined by Theorem 5.1 and Theorem 5.2. Then there exists a subsequence (still denoted) $(w^\epsilon(t,x), z^\epsilon(t,x))$ strongly converging to $(w(t,x), z(t,x))$ in L^2:*

$$(w^\epsilon(t,x), z^\epsilon(t,x)) \longrightarrow (w(t,x), z(t,x)) \in L^2\,,$$

and the limit function $(w(t,x), z(t,x))$ satisfies the Burgers equation:

$$\begin{aligned} z(t,x) &= e'(\bar{u})\, w(t,x)\,, \\ \bar{r}_v\Big(w_t + \bar{\lambda}'\big(\tfrac{1}{2}w^2\big)_x\Big) &+ \bar{\lambda}_-\bar{\lambda}_+ w_{xx} = 0\,. \end{aligned}$$

Proof. Notice that the sequence $(w^\epsilon(t,x), z^\epsilon(t,x))$ satisfied the estimates (5.22):

$$\int_0^t\int_{-\infty}^{\infty}\Big\{\sum_{i,j=1,i+j\le 3}\epsilon^{2(i-1)}|\partial_t^i\partial_x^j(w^\epsilon, \epsilon^i z^\epsilon)|^2(\tau,x)\Big\}\,dxd\tau \le C\,.$$

Using the estimates (5.22), we obtain from (5.27)

$$\begin{aligned} \bar{r}_u w + \bar{r}_v z &= o_\epsilon(1) \longrightarrow 0\,, \\ w_t + \bar{\lambda}\big(\tfrac{1}{2}w^2\big)_x + \bar{\lambda}_-\bar{\lambda}_+ w_{xx} &= o_\epsilon(1) \longrightarrow 0, \quad \text{as } \epsilon \to 0\,. \end{aligned} \tag{5.65}$$

Notice that

$$\|w^\epsilon\|_{H^1(\mathbf{R}^2_+)}$$

are uniformly bounded with respect to ϵ. Using the Sobolev embedding theorem, we obtain that there exists a subsequence (still denoted) w^ϵ converging strongly in L^2:

$$w^\epsilon(t,x) \longrightarrow w(t,x) \in H^1 \,.$$

Using the estimate (5.29), we conclude that z^ϵ strongly converges in L^2:

$$z^\epsilon(t,x) \longrightarrow e'(\bar{u})\, w(t,x) \,.$$

Taking ϵ goes to zero in the equality (5.65), we finally have that the limit function $(w(t,x), z(t,x))$ satisfies

$$\begin{gathered} z(t,x) = e'(\bar{u})\, w(t,x), \\ \bar{r}_v \Big(\partial_t w + \bar{\lambda}' \partial_x \big(\tfrac{1}{2} w^2 \big) \Big) + \bar{\lambda}_- \bar{\lambda}_+ \partial_{xx} w = 0 \,. \end{gathered}$$

This completes the proof of Theorem 5.3.

Acknowledgments. Some of this work was done while the authors were visiting the Mathematical Sciences Research Institute (MSRI) in Berkeley, which was supported by the NSF under grant DMS-8505550. Gui-Qiang Chen's research was supported in part by an Alfred P. Sloan Foundation Fellowship, by ONR grant N00014-91-J-1384, and by DOE grant W-31-109-Eng-38. David Levermore's research was supported in part by NSF grant DMS-8914420, and by the AFOSR under contract AFOSR-90-0021 which is associated with the Arizona Center for Mathematical Sciences (ACMS) at the University of Arizona. Tai-Ping Liu's research was supported in part by NSF grant DMS-900226, and by Army grant DAAL03-91-G-0017.

References

[1] Bardos, C., Golse, F., and Levermore, C.D.: *Fluid Dynamic Limits of Kinetic Equations II: Convergence Proofs for the Boltzmann Equation*, Commun. Pure & Applied Math. **46**, 667–753.

[2] Bardos, C., Golse, F., and Levermore, D.: *Fluid dynamics limits of discrete velocity kinetic equations*, in "Advances in Kinetic Theory and Continuum Mechanics", R. Gatignol and Soubbaramayer eds., Springer-Verlag (1991), 57–71.

[3] Caflisch, R., and Liu, T.-P.: *Nonlinear stability of shock waves for the Broadwell model*, Commun. Math. Phys. **114** (1988), 103–130.

[4] Caflisch, R., and Papanicolaou, G.C.: *The fluid dynamic limit of a nonlinear model Boltzmann equations*, Comm. Pure Appl. Appl. **32** (1979), 521–554.

[5] Cerecignani, C.: *The Boltzmann Equation and Its Applications*, Springer-Verlag, New York (1988).

[6] Chen, G.-Q.: *The compensated compactness method and the system of isentropic gas dynamics*, Preprint MSRI-00527-91 (1990), Mathematical Science Research Institute, Berkeley.

[7] Chen, G.-Q.: *Quasidecoupling method for discontinuous solutions to conservation laws*, Preprint MSRI-02227-91 (1991), Mathematical Science Research Institute, Berkeley.

[8] Chen, G.-Q., and Liu, T.-P.: *Zero relaxation and dissipation limits for conservation laws*, IMA Preprint Series **866** (1991), Institute for Mathematics and Its Applications, Minneapolis.

[9] Chen, G.-Q, and Lu, Y.-G.: *A study on application approaches of the theory of compensated compactness*, Chinese Science Bulletin 34, 15–19 (1989); 9, 641–644 (1988) (Chinese).

[10] Chueh, K.N., Conley, C.C., and Smoller, J.: *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J. **26** (1977), 372–411.

[11] Dafermos, C.M.: *Contemporary Issues in the Dynamic Behavior of Continuous Media*, LCDS Lecture Notes #85-1, January 1985.

[12] Ding, X., Chen, G.-Q., and Luo, P.: *Convergence of the Lax-Friedrichs schemes for the isentropic gas dynamics (I)*, Acta Mathematica Scientia 5, 415–432; 7, 467–480 (1987) (Chinese).

[13] DiPerna, R.: *Convergence of approximate solutions to conservation laws*, Arch. Rat. Mech. Anal. **82** (1983), 27–70.

[14] Glimm, J.: *The continuous structure of discontinuities*, Lecture Notes in Physics **344** (1986), 177–186.

[15] Greenberg, J., and Hsiao, L.: *The Riemann problem for the system $u_t+\sigma_x = 0$ and $(\sigma - f(u))_t + (\sigma - \mu f(u)) = 0$*, Arch. Rat. Mech. Anal. **82** (1983), 87–108.

[16] Jin, S., and Levermore, C.D.: *Numerical Schemes for Hyperbolic Conservation Laws with Stiff Relaxation Terms*, submitted to J. Comp. Phys. (1993).

[17] Keizer, J.: *Statistical Thermodynamics of Nonequilibrium Processes*, Springer-Verlag, New York (1987).

[18] Lax, P.D.: *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, Philadelphia (1973).

[19] Levermore, C.D.: *Fluid Dynamical Limits of Discrete Kinetic Theories*, in "Macroscopic Simulations of Complex Hydrodynamic Phenomena", M. Mareschal and B.L. Holian eds.; NATO ASI Series B **292**, Plenum Press, New York (1992), 173–185.

[20] Liu, T.-P.: *Hyperbolic Conservation Laws with Relaxation*, Commun. Math. Phys. **108** (1987), 153–175.

[21] Murat, F.: *L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$*, J. Math. Pures Appl. **60** (1981), 309–322.

[22] Renardy, M., Hrusa, W., and Nohel, J.: *Mathematical Problems in Viscoelasticiy*, Pitman Monographs and Surveys in Pure and Applied Mathematics **35**, Longman Scientific & Technical, Essex, England (1987).

[23] Rabie, R. L., Fowles, G. R., and Fickett, W.: *The Polymorphic Detonation*, Physics of Fluids **22** (1979), 422–435.

[24] Slemrod, M.: *Fluid Dynamic Limit for a Modified Broadwell System*, Preprint, 1991.

[25] Stoker, J. J.: *Water Waves*, Interscience, New York (1957).

[26] Tartar, L.: *Compensated compactness and applications to partial differential equations*, in "Research Notes in Mathematics, Nonlinear Analysis and Mechanics: Heriot-Watt Symposium **4**", R. J. Knops ed., Pitman Press, New York (1979), 136–212.

[27] Vincenti, W. G., and Kruger, Jr., C. H.: *Introduction to Physical Gasdynamics*, R. E. Krieger Publication Co. (1982).

[28] Whitham, G. B.: *Linear and Nonlinear Waves*, Wiley, New York (1974).

[29] Xin, Z.: *The fluid-dynamic limit of the Broadwell model of the nonlinear Boltzmann Equation in the presence of shocks*, Commun. Pure & Applied Math **44** (1991), 679–713.