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Concentration in Solutions to Hyperbolic Conservation Laws

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Abstract. The phenomenon of concentration in solutions to hyperbolic systems of conservation laws and related recent analytical techniques for studying such solutions are discussed. In particular, for the Euler equations for both isentropic and nonisentropic fluids, the phenomenon of concentration in solutions is fundamental as the pressure vanishes, which occurs not only in the multidimensional situations, but also even naturally in the one-dimensional case. ¿From the point of view of hyperbolic conservation laws, since the limit system loses hyperbolicity, the phenomenon of concentration in the process of vanishing pressure limit can be regarded as a phenomenon of resonance among the characteristic fields. This phenomenon indicates that the flux-function limit can be very singular: the limit functions of solutions are no longer in the spaces of functions, BV or L^∞, yet the space of Radon measures, for which the divergences of certain entropy and entropy flux fields are also Radon measures, is a natural space to deal with such a limit in general. In this regard, a theory of divergence-measure fields developed recently is also presented. This theory especially includes normal traces, a generalized Gauss-Green theorem, and product rules, among others. Some applications of this theory to solving various nonlinear problems in conservation laws and related areas are also discussed.

1. Introduction

We are concerned with the phenomenon of concentration in solutions to hyperbolic systems of conservation laws:

$$\partial_t u + \nabla \cdot f(u) = 0, \qquad x \in \mathbf{R}^d, u \in \mathbf{R}^m, \tag{1.1}$$

and related recent analytical techniques for studying such solutions for (1.1).

One of the archetypes of hyperbolic systems of conservation laws is the Euler equations of isentropic gas dynamics:

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.2}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + P) = 0, \tag{1.3}$$

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where ρ, P, and $m = \rho v$ represent the density, the scalar pressure, and the momentum, respectively; and ρ and m are in the physical region $\{(\rho, m) : \rho \geq 0, |m| \leq V_0\rho\}$ for some $V_0 > 0$. For $\rho > 0$, $v = m/\rho$ is the velocity with $|v| \leq V_0$. The scalar pressure P is a function of the density ρ and a small parameter $\epsilon > 0$ satisfying

$$\lim_{\epsilon \to 0} P(\rho, \epsilon) = 0;$$

its prototypical example is the pressure function for polytropic gases:

$$P(\rho, \epsilon) = \epsilon p(\rho), \qquad p(\rho) = \rho^\gamma/\gamma, \quad \gamma > 1. \tag{1.4}$$

Another archetype of hyperbolic systems of conservation laws is the Euler equations of nonisentropic gas dynamics:

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.5}$$

$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + P\right) = 0, \tag{1.6}$$

$$\partial_t(\rho E) + \partial_x\left((\rho E + P)\, v\right) = 0, \tag{1.7}$$

where $\rho E = \rho e + \frac{1}{2}\rho v^2$ is the total energy with e the internal energy, ρ and m are in a physical region $\{(\rho, m) : \rho \geq 0, |m| \leq V_0\rho\}$ for some $V_0 > 0$ and, for $\rho > 0$, $v = m/\rho$ is the velocity with $|v| \leq V_0$. Now the scalar pressure P is a function of the density ρ, the internal energy e, and a small parameter $\epsilon > 0$ satisfying

$$\lim_{\epsilon \to 0} P(\rho, e, \epsilon) = 0;$$

its prototypical example is now the pressure function for polytropic gases:

$$P(\rho, e, \epsilon) = \epsilon\, p(\rho, e), \qquad p(\rho, e) = (\gamma - 1)\rho e. \tag{1.8}$$

The phenomenon of concentration in solutions of the two-dimensional Riemann problem was first observed numerically in Chang-Chen-Yang [**10, 11**], which leads to the occurrence of so called smoothed δ-shocks, for the Euler equations of gas dynamics, when the Riemann data produces four initial contact discontinuities with different signs and the initial pressure data is close to zero.

In this paper, we first discuss the results in Chen-Liu [**17, 18**] to show that the phenomenon of concentration in solutions, observed numerically in [**10, 11**], for inviscid compressible fluid flow is fundamental, which occurs not only in the multidimensional situations, but also even naturally in the one-dimensional case.

As $\epsilon \to 0$, the limit system formally becomes the following transport equations:

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.9}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2) = 0, \tag{1.10}$$

which are also called the one-dimensional system of pressureless Euler equations, modeling the motion of free particles which stick under collision (see [**7, 22, 49, 64**]). System (1.9)–(1.10) has been analyzed extensively since 1994; for example, see [**4, 5, 7, 22, 28, 31, 37, 38, 39, 44, 56, 61**] and the references cited therein. In particular, the existence of measure solutions of the Riemann problem was first presented in Bouchut [**4**], and a connection of (1.9)–(1.10) with adhesion particle dynamics and the behavior of global weak solutions with random initial data were discussed in E-Rykov-Sinai [**22**]. Also see [**33, 34, 35, 48, 54, 55**] for related equations and results. It has been shown that, for the transport equations (1.9)–(1.10), δ-shocks do occur in the Riemann solutions.

In Sections 2–4, we discuss the phenomenon of concentration and the formation of δ-shocks in solutions to the Euler equations for both isentropic and nonisentropic fluids as the pressure vanishes. We show that the phenomenon of concentration occurs naturally in the one-dimensional case as the pressure vanishes: any Riemann solution containing two shocks to the Euler equations for isentropic or nonisentropic fluids tends to a δ-shock solution to the Euler equations for pressureless fluids, and the intermediate density between the two shocks tends to a weighted δ-measure that forms a δ-shock. These results show that the δ-shocks for the transport equations result from the phenomenon of concentration in the process of vanishing pressure limit.

From the point of view of hyperbolic conservation laws, since the limit system loses hyperbolicity, the phenomenon of concentration in the process of vanishing pressure limit can be regarded as a phenomenon of resonance between the characteristic fields. This phenomenon shows that the flux-function limit can be very singular: the limit functions of solutions are no longer in the spaces of functions, BV or L^∞, yet the space of Radon measures, for which the divergences of certain entropy and entropy flux fields are also Radon measures, is a natural space to deal with such a limit in general. In this regard, in Section 5, we discuss a theory of divergence-measure fields, developed recently in Chen-Frid [**13, 14, 15**], for dealing with entropy solutions with concentration, that is, measure solutions satisfying the Lax entropy inequality. Some applications of this theory to solving several nonlinear problems in conservation laws and related areas are also discussed.

2. δ-Shocks for the Euler Equations for Pressureless Fluids

Consider the Riemann problem of the transport equations (1.9)–(1.10) with Riemann initial data

$$(\rho, v)(0, x) = (\rho_\pm, v_\pm), \qquad \pm\, x > 0, \tag{2.1}$$

with $\rho_\pm > 0$. Since the equations and the Riemann data are invariant under uniform stretching of coordinates:

$$(t, x) \to (\beta t, \beta x), \qquad \beta \text{ constant},$$

we consider the self-similar solutions of (1.9), (1.10), and (2.1):

$$(\rho, v)(t, x) = (\rho, v)(\xi), \quad \xi = x/t.$$

Then the Riemann problem is reduced to a boundary value problem for ordinary differential equations:

$$\begin{aligned} &-\xi\rho_\xi + (\rho v)_\xi = 0,\\ &-\xi(\rho v)_\xi + (\rho v^2)_\xi = 0,\\ &(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \end{aligned}$$

In the case $v_- > v_+$, a key observation in [**56**] is that the singularity cannot be a jump with finite amplitude, that is, there is no solution which is piecewise smooth and bounded; hence a solution containing a weighted δ-measure (i.e. δ-shock) supported on a line was constructed in order to establish the existence in a space of measures (also see [**54, 55**]).

To define the measure solutions, the weighted δ-measure $w(t)\delta_S$ supported on a smooth curve $S=\{(t(s),x(s))\,:\,a<s<b\}$ can be defined by

$$\langle w(\cdot)\delta_S,\psi(\cdot,\cdot)\rangle=\int_a^b w(t(s))\psi(x(s),t(s))\sqrt{t'(s)^2+x'(s)^2}ds$$

for any $\psi\in C_0^\infty(\mathbf{R}_+\times\mathbf{R})$.

With this definition, a family of δ-measure solutions with parameter σ in the case $v_->v_+$ can be obtained as

$$\rho(t,x)=\rho_0(t,x)+w_1(t)\delta_S,\qquad v(t,x)=v_0(t,x),$$

where $S=\{(t,\sigma t)\,:\,0\le t<\infty\}$,

$$\rho_0(t,x)=\rho_-+[\rho]\,\chi(x-\sigma t),\;\; v_0(t,x)=v_-+[v]\,\chi(x-\sigma t),\;\; w_1(t)=\frac{t}{1+\sigma^2}(\sigma[\rho]-[\rho v]),$$

in which $[h]:=h_+-h_-$ denotes the jump of function h across the discontinuity, and $\chi(x)$ is the characteristic (or indicator) function that is 0 when $x<0$ and 1 when $x>0$.

As shown in [**56**], the δ-measure solutions (ρ,v) constructed above satisfy

$$\langle\rho,\phi_t\rangle+\langle\rho v,\phi_x\rangle=0,$$
$$\langle\rho v,\phi_t\rangle+\langle\rho v^2,\phi_x\rangle=0,$$

for any $\phi\in C_0^\infty(\mathbf{R}_+\times\mathbf{R})$, where

$$\langle\rho,\phi\rangle=\int_0^\infty\int_{-\infty}^\infty\rho_0\phi\,dxdt+\langle w_1\delta_S,\phi\rangle,$$

and

$$\langle\rho v,\phi\rangle=\int_0^\infty\int_{-\infty}^\infty\rho_0 v_0\phi\,dxdt+\langle w_2\delta_S,\phi\rangle,$$

where $w_2(t)=\frac{t}{1+\sigma^2}(\sigma[\rho v]-[\rho v^2])$.

A unique solution can be singled out by the so called δ-Rankine-Hugoniot condition:

$$\sigma=\frac{\sqrt{\rho_+}v_++\sqrt{\rho_-}v_-}{\sqrt{\rho_+}+\sqrt{\rho_-}}\tag{2.2}$$

that satisfies the $\delta-$entropy condition:

$$v_+<\sigma<v_-.\tag{2.3}$$

The entropy condition (2.3) means that, in the (t,x)-plane, all the characteristic lines on either side of a δ-shock run into the line of δ-shock, which implies that a δ-shock is an overcompressive shock. The δ-entropy condition can be derived from the Lax entropy inequality:

$$\partial_t(\rho v^2)+\partial_x(\rho v^3)\le 0\tag{2.4}$$

in the sense of distributions, that is, for any $\phi\in C_0^\infty(\mathbf{R}_+\times\mathbf{R})$, $\phi\ge 0$, we have

$$\langle\rho v^2,\phi_t\rangle+\langle\rho v^3,\phi_x\rangle\ge 0.$$

3. Concentration in Solutions to the Euler Equations for Isentropic Fluids

In this section, we discuss the phenomenon of concentration in solutions to system (1.2)–(1.4). We start with Riemann solutions.

3.1. Riemann Solutions. The eigenvalues of system (1.2)–(1.4) are

$$\lambda_1 = v - \sqrt{\epsilon\, p'(\rho)}, \quad \lambda_2 = v + \sqrt{\epsilon\, p'(\rho)}, \qquad \text{for } \rho > 0.$$

The Riemann invariants are

$$w = v + \int_0^\rho \frac{\sqrt{\epsilon p'(s)}}{s} ds, \qquad z = v - \int_0^\rho \frac{\sqrt{\epsilon p'(s)}}{s} ds.$$

Then the Riemann solutions, which are functions of $\xi = x/t$, are solutions of

$$-\xi \rho_\xi + (\rho v)_\xi = 0, \tag{3.1}$$

$$-\xi (\rho v)_\xi + (\rho v^2 + \epsilon\, p(\rho))_\xi = 0, \tag{3.2}$$

$$(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \tag{3.3}$$

Shock Curves. The Rankine-Hugoniot conditions for discontinuous solutions of (1.2)–(1.4) are

$$-\sigma[\rho] + [\rho v] = 0, \qquad -\sigma[\rho v] + [\rho v^2 + \epsilon\, p(\rho)] = 0.$$

The Lax entropy conditions imply

$$\rho_+ > \rho_- \ \text{(1-shock)}, \qquad \rho_+ < \rho_- \ \text{(2-shock)}. \tag{3.4}$$

Then, given a state $u_- = (\rho_-, \rho_- v_-)$, the shock curves in the phase plane, which are the sets of states that can be connected on the right by a 1-shock or 2-shock, are the following.

1-shock curve $S_1(u_-)$**:**

$$v - v_- = -\sqrt{\frac{\epsilon\,(p(\rho) - p(\rho_-))}{\rho_- \rho(\rho - \rho_-)}}(\rho - \rho_-), \qquad \rho > \rho_-;$$

2-shock curve $S_2(u_-)$**:**

$$v - v_- = -\sqrt{\frac{\epsilon\,(p(\rho) - p(\rho_-))}{\rho_- \rho(\rho - \rho_-)}}(\rho - \rho_-), \qquad \rho < \rho_-.$$

Then the shock curves are concave or convex, respectively, with respect to the point $u_- = (\rho_-, \rho_- v_-)$ in the $\rho - m$ plane with $m = \rho v$, that is, the quotient $\frac{m - m_-}{\rho - \rho_-}$ as a function of ρ is monotone.

Similarly, we can define the 1-rarefaction wave curve $R_1(u_-)$ and the 2-rarefaction wave curve $R_2(u_-)$.

Given a left state $u_- = (\rho_-, \rho_- v_-)$, the set of states that can be connected on the right by a shock or a rarefaction wave in the phase plane consists of the 1-shock curve $S_1(u_-)$, the 1-rarefaction curve $R_1(u_-)$, the 2-shock curve $S_2(u_-)$, and the 2-rarefaction curve $R_2(u_-)$. These curves divide the phase plane into four regions $S_2S_1(u_-), S_2R_1(u_-), R_2S_1(u_-)$, and $R_2R_1(u_-)$; any right state of the Riemann data staying in one of them yields a unique global Riemann solution $R(x/t)$, which contains a 1-shock (or 1-rarefaction wave) and/or a 2-shock (or 2-rarefaction wave) satisfying

$$w(R(x/t)) \leq w(u_+), \ \ z(R(x/t)) \geq z(u_-), \ \ w(R(x/t)) - z(R(x/t)) \geq 0.$$

In particular, when $u_+ \in S_2S_1(u_-)$, $R(x/t)$ contains a 1-shock, a 2-shock, and a nonvacuum intermediate constant state. For more details about Riemann solutions, see [**20, 51**].

3.2. Concentration in the Vanishing Pressure Limit. We now discuss the phenomenon of concentration and the formation of δ-shocks in the Riemann solutions to (1.2)–(1.4) in the case $u_+ \in S_2S_1(u_-)$ with $v_- > v_+$ and $\rho_\pm > 0$ as the pressure vanishes.

For fixed $\epsilon > 0$, let $u_*^\epsilon := (\rho_*^\epsilon, \rho_*^\epsilon v_*^\epsilon)$ be the intermediate state in the sense that u_- and u_*^ϵ are connected by 1-shock S_1 with speed σ_1^ϵ, and that u_*^ϵ and u_+ are connected by 2-shock S_2 with speed σ_2^ϵ. Then $(\rho_*^\epsilon, v_*^\epsilon)$ is determined by

$$v_*^\epsilon - v_- = -\sqrt{\frac{\epsilon\,(p(\rho_*^\epsilon) - p(\rho_-))}{\rho_-\rho_*^\epsilon(\rho_*^\epsilon - \rho_-)}}(\rho_*^\epsilon - \rho_-), \qquad \rho_*^\epsilon > \rho_-, \tag{3.5}$$

and

$$v_+ - v_*^\epsilon = -\sqrt{\frac{\epsilon\,(p(\rho_+) - p(\rho_*^\epsilon))}{\rho_+\rho_*^\epsilon(\rho_+ - \rho_*^\epsilon)}}(\rho_+ - \rho_*^\epsilon), \qquad \rho_*^\epsilon > \rho_+. \tag{3.6}$$

A combination of the jump conditions (3.5) and (3.6) with the Lax entropy conditions (3.4) yields

LEMMA 3.1. (i) $\lim_{\epsilon\to 0} \epsilon^{1/\gamma}\rho_*^\epsilon = \Big(\frac{\sqrt{\rho_-\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(v_- - v_+)\Big)^{1/\gamma}$.
(ii) *Set* $\sigma = \frac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} \in (v_+, v_-)$. *Then*

$$\lim_{\epsilon\to 0} v_*^\epsilon = \lim_{\epsilon\to 0}\sigma_1^\epsilon = \lim_{\epsilon\to 0}\sigma_2^\epsilon = \sigma, \quad \lim_{\epsilon\to 0} \rho_*^\epsilon(\sigma_2^\epsilon - \sigma_1^\epsilon) = \sigma[\rho] - [\rho v].$$

The quantity σ, which is the limit of $v_*^\varepsilon, \sigma_1^\varepsilon$, and σ_2^ε, uniquely determines the δ-shock solution of (1.9)–(1.10) as the limit of the Riemann solutions when $\varepsilon \to 0$ and is consistent with both the δ-Rankine-Hugoniot condition (2.2) and the δ-entropy condition (2.3), as proposed for the Riemann solutions of the pressureless Euler equations in Section 2.

The following theorem characterizes the vanishing pressure limit in the case $v_- > v_+$, which can be proved from the weak formulations and Lemma 3.1 and by tracking the time-dependence of the weights of δ-measures as $\epsilon \to 0$.

THEOREM 3.1. *Let* $v_- > v_+$. *For each fixed* $\epsilon > 0$, *assume that* $(\rho^\epsilon, m^\epsilon) = (\rho^\epsilon, \rho^\epsilon v^\epsilon)$ *is a two-shock solution of* (1.2)–(1.4) *with Riemann data* $u_\pm = (\rho_\pm, \rho_\pm v_\pm)$. *Then, as* $\epsilon \to 0$, ρ^ϵ *and* m^ϵ *converge in the sense of distributions, and the limit functions* ρ *and* m *are the sums of a step function and a* δ*-measure with weights*

$$\frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) \quad \textit{and} \quad \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho v] - [\rho v^2]),$$

respectively, which form a δ*-shock solution of* (1.9)–(1.10) *with the same Riemann data* $u_\pm$.

The formation process of δ-shocks in the Riemann solutions can be described as the pressure decreases as follows: As ε decreases, the locations of the two shocks become closer, and the density of the intermediate state increases dramatically, while the velocity is closer to a step function. In the vanishing pressure limit, the two shocks coincide to form, along with the intermediate state, a δ-shock of the transport equations (1.9)–(1.10), while the velocity becomes a step function.

4. Concentration in Solutions to the Euler Equations for Nonisentropic Fluids

In this section, we discuss the phenomenon of concentration in solutions to the Euler equations for nonisentropic fluids.

4.1. Riemann Solutions. The Euler equations (1.5)–(1.8) for smooth solutions with $\rho > 0$ can be written as

$$\partial_t u + A(u)\partial_x u = 0, \quad u = (\rho, \rho v, \rho E)^\top,$$

where

$$A(u) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{(\gamma-1)\epsilon-2}{2}v^2 & (2-\epsilon(\gamma-1))v & \epsilon(\gamma-1) \\ -\frac{1}{\epsilon(\gamma-1)}(c^\varepsilon)^2 v + \frac{\epsilon(\gamma-1)-1}{2}v^3 & \frac{1}{\epsilon(\gamma-1)}(c^\varepsilon)^2 + \frac{1-2\epsilon(\gamma-1)}{2}v^2 & (1+\epsilon(\gamma-1))v \end{pmatrix}.$$

Then the eigenvalues of system (1.5)–(1.8) are

$$\lambda_1 = v - c^\varepsilon, \quad \lambda_2 = v, \quad \lambda_3 = v + c^\varepsilon, \qquad \text{for } \rho > 0,$$

with associated right eigenvectors

$$r_j = \begin{pmatrix} 1 \\ v + \operatorname{sgn}(j-2)c^\varepsilon \\ \frac{1}{2}v^2 + \operatorname{sgn}(j-2)c^\varepsilon v + \frac{\operatorname{sgn}|j-2|}{\epsilon(\gamma-1)}(c^\varepsilon)^2 \end{pmatrix},$$

and left eigenvectors

$$l_j = \left(\frac{1}{2}v^2 - \frac{\operatorname{sgn}(j-2)}{\epsilon(\gamma-1)}c^\varepsilon v + \frac{(\operatorname{sgn}|j-2|-1)}{\epsilon(\gamma-1)}(c^\varepsilon)^2, -v + \frac{\operatorname{sgn}(j-2)}{\epsilon(\gamma-1)}c^\varepsilon, 1\right),$$

where $c^\varepsilon = c(e,\epsilon) := \sqrt{\varepsilon\gamma_\epsilon(\gamma-1)e}$ and $\gamma_\epsilon = 1 + \epsilon(\gamma-1)$.

The Riemann invariants along the characteristic fields are

$$\left\{S^\varepsilon, v + \frac{2}{\epsilon(\gamma-1)}c^\epsilon\right\}, \quad \{v, p\}, \quad \left\{S^\epsilon, v - \frac{2}{\epsilon(\gamma-1)}c^\epsilon\right\}, \tag{4.1}$$

respectively, where S^ϵ is determined by

$$p = \kappa e^{S^\epsilon}\rho^{\gamma_\varepsilon}, \quad \kappa > 0. \tag{4.2}$$

The Riemann solutions, which are the functions of $\xi = x/t$, are solutions of

$$\begin{aligned} &-\xi\rho_\xi + (\rho v)_\xi = 0, \\ &-\xi(\rho v)_\xi + (\rho v^2 + \epsilon p)_\xi = 0, \\ &-\xi(\rho E)_\xi + ((\rho E + \epsilon p)v)_\xi = 0, \\ &(\rho, v, E)(\pm\infty) = (\rho_\pm, v_\pm, E_\pm). \end{aligned}$$

Then the Rankine-Hugoniot conditions for discontinuous solutions to (1.5)–(1.8) on a discontinuity are

$$\sigma[\rho] = [\rho v], \quad \sigma[\rho v] = [\rho v^2 + \epsilon p], \quad \sigma[\rho E] = [(\rho E + \epsilon p)v],$$

which can be rewritten in terms of $w := v - \sigma$ and $J := \rho w = \rho v - \sigma\rho$ as

$$[J] = 0, \quad [Jw + \epsilon p] = 0, \quad J\left[\frac{2}{\epsilon(\gamma-1)}(c^\epsilon)^2 + w^2\right] = 0.$$

Contact Discontinuities. If $\sigma = v_l = v_r$, the discontinuity is a contact discontinuity S_c, which corresponds to the second family of characteristic fields. Then

$$[v] = 0, \qquad [p] = 0.$$

Shock Curves. If $\sigma \neq v_l, v_r$, then the discontinuity is a shock wave, which corresponds to either the first or the third family of characteristic fields. Then the Lax entropy conditions and the Rankine-Hugoniot conditions imply that, on a 1-shock S_1,

$$[p] > 0, \quad [\rho] > 0, \quad [v] < 0, \tag{4.3}$$

and, on a 3-shock S_3,

$$[p] < 0, \quad [\rho] < 0, \quad [v] < 0. \tag{4.4}$$

We now examine the dependence of the two one-parameter families of shock curves on ϵ.

Set

$$\alpha_\epsilon = \frac{\gamma - 1}{2\gamma_\epsilon}, \quad \beta_\epsilon = \frac{2}{\epsilon(\gamma - 1)} + 1.$$

Then we obtain the following formulas for the j-shock curves, $j = 1, 3$, in terms of the parametrization $s \leq 0$, respectively.

1-shock curve: For $s \leq 0$,

$$\frac{p_r}{p_l} = e^{-s}, \quad \frac{\rho_r}{\rho_l} = f_1^\varepsilon(s) := \frac{\beta_\epsilon + e^s}{1 + \beta_\epsilon e^s}, \quad \frac{v_r - v_l}{c_l^\varepsilon} = h_1^\varepsilon(s) := \frac{2\sqrt{\alpha_\epsilon/\varepsilon}}{\gamma - 1} \frac{1 - e^{-s}}{\sqrt{1 + \beta_\epsilon e^{-s}}}.$$

3-shock curve: For $s \leq 0$,

$$\frac{p_r}{p_l} = e^{s}, \quad \frac{\rho_r}{\rho_l} = f_3^\varepsilon(s) := \frac{\beta_\epsilon + e^{-s}}{1 + \beta_\epsilon e^{-s}}, \quad \frac{v_r - v_l}{c_l^\varepsilon} = h_3^\varepsilon(s) := \frac{2\sqrt{\alpha_\epsilon/\varepsilon}}{\gamma - 1} \frac{e^{s} - 1}{\sqrt{1 + \beta_\epsilon e^{s}}}.$$

Riemann Solutions. Consider the Riemann problem (1.5)–(1.8) with Riemann data:

$$u(0, x) = u_\pm, \quad \pm x > 0, \tag{4.5}$$

and $u_+ \in S_2 S_c S_1(u_-)$. Then the Riemann solution $R(x/t)$ may contain a 1-shock, a 2-contact discontinuity, a 3-shock, and two nonvacuum intermediate constant states between the shocks and the contact discontinuity. Hence, there exist $s_1^\varepsilon, s_2^\varepsilon$, and s_3^ε determined by

$$\rho_+ = f_1^\varepsilon(s_1^\varepsilon) e^{s_2^\varepsilon} f_3^\varepsilon(s_3^\varepsilon) \rho_-, \tag{4.6}$$

$$p_+ = e^{s_3^\varepsilon - s_1^\varepsilon} p_-, \tag{4.7}$$

$$v_+ = v_- + c_-^\varepsilon \left(h_1^\varepsilon(s_1^\varepsilon) + \frac{e^{-(s_1^\varepsilon + s_2^\varepsilon)/2}}{\sqrt{f_1^\varepsilon(s_1^\varepsilon)}} h_3^\varepsilon(s_3^\varepsilon) \right). \tag{4.8}$$

Define

$$A = \rho_+/\rho_-, \quad B = p_+/p_-, \quad C^\epsilon = (v_+ - v_-)/c_-^\epsilon.$$

Then one can find $s_i^\varepsilon, i=1,2,3$, successively as follows:

$$h_1^\varepsilon(s_1^\varepsilon)+\sqrt{\frac{B}{A}}h_1^\varepsilon(s_1^\varepsilon+logB)=C^\epsilon, \tag{4.9}$$

$$s_3^\varepsilon=s_1^\varepsilon+logB, \tag{4.10}$$

$$s_2^\varepsilon=log\left(\frac{A}{f_1^\varepsilon(s_1^\varepsilon)f_3^\varepsilon(s_3^\varepsilon)}\right), \tag{4.11}$$

where we have used the fact that $f_1^\varepsilon(s)f_3^\varepsilon(s)=1$ and $h_3^\varepsilon(s)=e^{s/2}h_1^\varepsilon(s)\sqrt{f_1^\varepsilon(s)}$.

In general, we have

THEOREM 4.1. *For any fixed $\varepsilon>0$, there exists a unique Riemann solution $R(x/t)$, staying away from vacuum, of the Riemann problem* (1.5)–(1.8) *with Riemann data* (4.5) *if and only if $v_+-v_-<\frac{2(c_-^\epsilon+c_+^\epsilon)}{\varepsilon(\gamma-1)}$. Otherwise, the vacuum is present in the Riemann solution.*

For more details about Riemann solutions, see [**51, 52**].

4.2. Concentration in the Vanishing Pressure Limit. We now discuss the phenomenon of concentration and the formation of δ-shocks in the Riemann solutions to the Euler equations for nonisentropic fluids in the case $u_+\in S_2S_cS_1(u_-)$ with $v_->v_+$ and $\rho_\pm>0$ as the pressure vanishes.

Let $u_i^\varepsilon:=(\rho_i^\varepsilon,\rho_i^\varepsilon v_*^\varepsilon,\rho_i^\varepsilon E_i^\varepsilon), i=1,2$, with $p_1^\varepsilon=p_2^\varepsilon=p_*^\varepsilon$, be the intermediate states in the sense that (p_-,ρ_-,v_-) and $(p_*^\varepsilon,\rho_1^\varepsilon,v_*^\varepsilon)$ determine a 1-shock S_1 with speed σ_1^ϵ and that $(p_*^\varepsilon,\rho_2^\varepsilon,v_*^\varepsilon)$ and (p_+,ρ_+,v_+) determine a 3-shock S_3 with speed σ_3^ε. The difference $\rho_2^\varepsilon-\rho_1^\varepsilon$ is the jump of density which determines a 2-contact discontinuity S_c. Based on these facts, we can obtain the limits of $s_i^\varepsilon, i=1,2,3$, as $\epsilon\to 0$, respectively.

LEMMA 4.1. *If $v_->v_+$, then*

(i) $\lim_{\epsilon\to 0}\epsilon\, e^{-s_1^\varepsilon}=q_1:=\frac{\rho_+\rho_-(v_+-v_-)^2}{p_-(\sqrt{\rho_+}+\sqrt{\rho_-})^2}$, $\quad\lim_{\epsilon\to 0}\epsilon\, e^{-s_3^\varepsilon}=q_3:=\frac{\rho_+\rho_-(v_+-v_-)^2}{p_+(\sqrt{\rho_+}+\sqrt{\rho_-})^2}$,
$\lim_{\epsilon\to 0}e^{s_2^\varepsilon}=q_2:=\frac{2+(\gamma-1)q_3}{2+(\gamma-1)q_1}$;

(ii) *As $\epsilon\to 0$, the intermediate densities ρ_1^ε and ρ_2^ε become unbounded, i.e.,*

$$\lim_{\epsilon\to 0}\epsilon\rho_1^\varepsilon=\frac{2\rho_-q_1}{2+(\gamma-1)q_1},\qquad \lim_{\epsilon\to 0}\epsilon\rho_2^\varepsilon=\frac{2\rho_+q_3}{2+(\gamma-1)q_3};$$

(iii) *The intermediate velocity v_*^ε and pressure p_*^ε satisfy*

$$\lim_{\epsilon\to 0}v_*^\varepsilon=\sigma:=\frac{\sqrt{\rho_-}v_-+\sqrt{\rho_+}v_+}{\sqrt{\rho_+}+\sqrt{\rho_-}},\qquad \lim_{\epsilon\to 0}\epsilon p_*^\varepsilon=\frac{\rho_+\rho_-(v_+-v_-)^2}{(\sqrt{\rho_+}+\sqrt{\rho_-})^2};$$

(iv) $\lim_{\epsilon\to 0}\sigma_1^\varepsilon=\lim_{\epsilon\to 0}\sigma_2^\varepsilon=\lim_{\epsilon\to 0}\sigma_3^\varepsilon=\frac{\sqrt{\rho_-}v_-+\sqrt{\rho_+}v_+}{\sqrt{\rho_+}+\sqrt{\rho_-}}=\sigma\in(v_+,v_-)$;

(v) $\lim_{\epsilon\to 0}\rho_1^\varepsilon(\sigma_1^\varepsilon-\sigma_2^\varepsilon)=\frac{\rho_-\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(v_+-v_-)$,
$\lim_{\epsilon\to 0}\rho_2^\varepsilon(\sigma_2^\varepsilon-\sigma_3^\varepsilon)=\frac{\rho_+\sqrt{\rho_-}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(v_+-v_-)$.

Again, the quantity σ, which is the limit of v_*^ε and $\sigma_j^\varepsilon, j=1,2,3$, in Lemma 4.1, uniquely determines the δ-shock solution as the limit of the Riemann solutions when $\varepsilon\to 0$ and is consistent with both the δ-Rankine-Hugoniot condition (2.2) and the δ-entropy condition (2.3).

The following theorem characterizes the vanishing pressure limit for the case $v_- > v_+$.

THEOREM 4.2. *Let* $(\rho^\epsilon, \rho^\epsilon v^\epsilon, \rho^\varepsilon E^\epsilon)$ *be the Riemann solutions to* (1.5)–(1.8) *and* (4.5), *which contain two shocks and possibly one contact discontinuity. Then* ρ^ϵ, $\rho^\epsilon v^\epsilon$, *and* $\rho^\varepsilon E^\epsilon$ *converge in the sense of distributions, respectively, and the limit functions are all sums of a step function and a* δ*-measure with weights*

$$\frac{t}{\sqrt{1+\sigma^2}}([\rho v]-\sigma[v]), \quad \frac{t}{\sqrt{1+\sigma^2}}([\rho v^2]-\sigma[\rho v]), \quad \frac{t}{\sqrt{1+\sigma^2}}([\rho vE]-\sigma[\rho E]),$$

respectively, where $\sigma = (\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+)/(\sqrt{\rho_+}+\sqrt{\rho_-})$.

Furthermore, we can conclude the entropy consistency by proving that the conservation law of energy (1.7) actually yields the correct entropy inequality (2.4) for the transport equations (1.9) and (1.10).

THEOREM 4.3. *The limit functions* (ρ, v) *are a measure solution of the transport equation* (1.9)–(1.10) *satisfying*

$$\partial_t(\rho v^2) + \partial_x(\rho v^3) \le 0$$

in the sense of distributions.

This can be seen as follows. Since $\partial_t(\rho^\varepsilon E^\varepsilon) + \partial_x(\rho^\varepsilon v^\varepsilon E^\varepsilon) = 0$ in the sense of distributions and $\rho E = \frac{p}{\gamma-1} + \frac{1}{2}\rho v^2$, then, for any nonnegative test function $\phi \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R})$,

$$-\left(\langle \rho^\varepsilon (v^\varepsilon)^2, \phi_t\rangle + \langle \rho^\varepsilon (v^\varepsilon)^3, \phi_x\rangle\right) = \frac{2}{\gamma-1}\left(\langle p^\varepsilon, \phi_t\rangle + \langle p^\varepsilon v^\varepsilon, \phi_x\rangle\right).$$

Note that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^\infty \int_{-\infty}^\infty (p^\varepsilon \phi_t + p^\varepsilon v^\varepsilon \phi_x)dxdt \\
&= \lim_{\epsilon\to 0}\int_0^\infty \left(\sigma_1^\varepsilon(p_*^\varepsilon - p_-) - (p_*^\varepsilon v_*^\varepsilon - p_- v_-)\right)\phi(\sigma_1^\varepsilon t, t)dt \\
&\quad + \lim_{\epsilon\to 0}\int_0^\infty \sigma_2^\varepsilon\left((p_+ - p_*^\varepsilon) - (p_+ v_+ - p_*^\varepsilon v_*^\varepsilon)\right)\phi(\sigma_2^\varepsilon t, t)dt \\
&= \int_0^\infty (v_- - v_+)(\alpha p_+ + (1-\alpha)p_-)\phi(\sigma t, t)dt < 0,
\end{aligned}$$

since $\alpha = \frac{\sqrt{\rho_-}}{\sqrt{\rho_+}+\sqrt{\rho_-}} \in (0,1)$. This verifies the entropy consistency as claimed.

The process of concentration in the Riemann solutions to the Euler equations (1.5)–(1.8) can be seen when the pressure vanishes as follows: As ε decreases, the locations of the two shocks become closer to the contact discontinuity, and the densities of the intermediate states increase dramatically, while the velocity is closer to a step function; in the vanishing pressure limit, the two shocks and the contact discontinuity coincide to form, along with the intermediate states, a δ-shock of the transport equations (1.9) and (1.10), while the velocity becomes a step function.

5. Divergence-Measure Fields for Conservation Laws

In this section, we discuss a theory of divergence-measure fields ($\mathcal{DM}$-fields, for short), developed recently in Chen-Frid [**13, 14, 15**], in order to deal with entropy solutions with concentration.

5.1. Conservation Laws and Divergence-Measure Fields. We first discuss some connection between hyperbolic conservation laws and $\mathcal{DM}$-fields.

The $\mathcal{DM}$-fields arise naturally in the study of entropy solutions of nonlinear hyperbolic systems of conservation laws, which take the form (1.1). The main feature of nonlinear hyperbolic conservation laws, especially systems (1.2)–(1.4) and (1.5)–(1.8), is that solutions may develop singularities and form shock waves in finite time, no matter how smooth the initial data is. One may expect solutions in the space of functions of bounded variation. This is indeed the case by the Glimm theorem [**26**] which indicates that, when the initial data has sufficiently small total variation and the one -dimensional system is strictly hyperbolic, there exists a global entropy solution in BV. However, when the initial data becomes large, and/or strict hyperbolicity of the system fails (especially for the multidimensional case), the solutions are no longer in the space of functions of bounded variation and may become measures in finite time, even instantaneously as $t > 0$ (see [**14, 45**]). Furthermore, as discussed in Sections 2–4, the concentration occurs and the δ-shocks form in the vanishing pressure limit of solutions to the Euler systems (1.2)–(1.4) and (1.5)–(1.8). These indicate that solutions of nonlinear hyperbolic conservation laws are generally in $\mathcal{M}(\mathbf{R}_+ \times \mathbf{R}^d)$, the space of signed Radon measures.

Definition 5.1. A function $\eta : \mathbf{R}^m \to \mathbf{R}$ is called an entropy of (1.1) if there exists $q : \mathbf{R}^m \to \mathbf{R}^d$ such that

$$\nabla q_k(u) = \nabla \eta(u) \nabla f_k(u), \qquad k = 1, 2, \cdots, d. \tag{5.1}$$

The function $q(u)$ is called the entropy flux associated with the entropy $\eta(u)$, and the pair $(\eta(u), q(u))$ is called an entropy pair. The entropy pair $(\eta(u), q(u))$ is called a convex entropy pair on the domain $K \subset \mathbf{R}^m$ if the Hessian matrix $\nabla^2 \eta(u) \geq 0$, for $u \in K$. The entropy pair $(\eta(u), q(u))$ is called a strictly convex entropy pair on the domain K if $\nabla^2 \eta(u) > 0$ for $u \in K$.

Friedrichs-Lax [**25**] observed that most of the systems of conservation laws that result from continuum mechanics are endowed with a globally defined, strictly convex entropy.

Available existence theories show that the solutions $u(t,x)$ of (1.1) are generally in the following class of entropy solutions:

i) $u(t,x) \in \mathcal{M}(\mathbf{R}_+ \times \mathbf{R}^d)$, or $L^p(\mathbf{R}_+ \times \mathbf{R}^d), 1 \leq p \leq \infty$;

ii) $u(t,x)$ satisfies the Lax entropy inequality:

$$\partial_t \eta(u(t,x)) + \nabla_x \cdot q(u(t,x)) \leq 0 \tag{5.2}$$

in the sense of distributions, for any entropy pair $(\eta, q) : \mathbf{R}^m \to \mathbf{R} \times \mathbf{R}^d$ with convex $\eta, \nabla^2 \eta(u) \geq 0$, so that $(\eta(u(t,x))$ and $q(u(t,x))$ are distributional functions. If we take $\eta = \pm u$, we see that any entropy solution is a weak solution.

One of the main issues in conservation laws is to study the behavior of solutions in this class to explore all possible information of solutions, including large-time behavior, uniqueness, stability, and traces of solutions, among others. The Schwartz lemma indicates from (5.2) that the distribution

$$\partial_t \eta(u(t,x)) + \nabla_x \cdot q(u(t,x))$$

is in fact a Radon measure, that is,

$$\operatorname{div}_{(t,x)}(\eta(u(t,x)), q(u(t,x))) \in \mathcal{M}(\mathbf{R}_+ \times \mathbf{R}^d). \tag{5.3}$$

Divergence-measure fields also arise in various nonlinear problems involving some extended vector fields whose divergences are Radon measures. For example, see Bouchitté-Buttazzo [**3**] for such fields in the characterization of optimal shapes and masses through the Monge-Kantorovich equation. It is clear that understanding more properties of $\mathcal{DM}$-fields can advance our understanding of the behavior of entropy solutions for hyperbolic conservation laws and other related nonlinear equations.

5.2. Divergence-Measure Fields. Divergence-measure fields are extended vector fields, including vector fields in L^p and vector-valued Radon measures, whose divergences are Radon measures. More precisely, we have

DEFINITION 5.2. Let $\Omega \subset \mathbf{R}^N$ be open. For $F \in L^p(\Omega; \mathbf{R}^N)$, $1 \le p \le \infty$, or $F \in \mathcal{M}(\Omega; \mathbf{R}^N)$, set

$$|\mathrm{div}\, F|(\Omega) := \sup\{\, \langle F, \nabla\varphi\rangle \ : \ \varphi \in C_0^1(\Omega),\ |\varphi(x)| \le 1,\ x \in \Omega\,\}.$$

For $1 \le p \le \infty$, we say that F is an L^p divergence-measure field over Ω, i.e., $F \in \mathcal{DM}^p(\Omega)$, if

$$\|F\|_{\mathcal{DM}^p(\Omega)} := \|F\|_{L^p(\Omega;\mathbf{R}^N)} + |\mathrm{div}\, F|(\Omega) < \infty. \tag{5.4}$$

We say that F is an extended divergence-measure field over Ω, i.e., $F \in \mathcal{DM}^{ext}(\Omega)$, if

$$\|F\|_{\mathcal{DM}^{ext}(\Omega)} := |F|(D) + |\mathrm{div}\, F|(D) < \infty. \tag{5.5}$$

If $F \in \mathcal{DM}^p(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbf{R}^N$, then we say $F \in \mathcal{DM}^p_{loc}(D)$; and, if $F \in \mathcal{DM}^{ext}(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbf{R}^N$, we say $F \in \mathcal{DM}^{ext}_{loc}(D)$. We denote $F \in \mathcal{DM}(\Omega)$ either $F \in \mathcal{DM}^p(\Omega)$ or $F \in \mathcal{DM}^{ext}(\Omega)$. Here, for open sets $A, B \subset \mathbf{R}^N$, the relation $A \Subset B$ means that the closure of A, $\bar{A}$, is a compact subset of B.

These spaces under the norms (5.4) and (5.5) are Banach spaces, respectively. These spaces are larger than the space of vector fields of bounded variation. The establishment of the Gauss-Green theorem, traces, and other properties of BV functions in the middle of last century (see Federer [**24**]) has advanced significantly our understanding of solutions of nonlinear partial differential equations and nonlinear problems in calculus of variations, differential geometry, and other areas. A natural question is whether the $\mathcal{DM}$-fields have similar properties, especially the traces and the Gauss-Green formula as for the BV functions. At a first glance, it seems unclear.

First, observe that one cannot define the traces for each component of a $\mathcal{DM}$ field over any Lipschitz boundary in general, as opposed to the case of BV fields. This fact can be easily seen through the following example.

EXAMPLE 5.1: The field $F(x,y) = (\cos(\frac{1}{x-y}), \cos(\frac{1}{x-y}))$ belongs to $\mathcal{DM}^\infty(\mathbf{R}^2)$. It is impossible to define any reasonable notion of traces over the line $x = y$ for the component $\cos(\frac{1}{x-y})$.

The following example indicates that the classical Gauss-Green theorem may fail.

EXAMPLE 5.2: The field $F(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ belongs to $\mathcal{DM}^1_{loc}(\mathbf{R}^2)$. As remarked in Whitney [**62**], for $\Omega = (0,1)\times(0,1)$,

$$\int_\Omega \operatorname{div} F dxdy = 0 \neq \int_{\partial\Omega} F\cdot\nu\, d\mathcal{H}^1 = \frac{\pi}{2},$$

if one understands $F\cdot\nu|_{\partial\Omega}$ in the classical sense.

EXAMPLE 5.3: Let $\mu \in \mathcal{M}$. Then, for any bounded open interval $I\subset\mathbf{R}$,

$$F(x,y) = (\mu(y), 0) \in \mathcal{DM}^{ext}(I\times\mathbf{R}).$$

Some efforts have been made in generalizing the Gauss-Green theorem. Some results for certain situations can be found in Anzellotti [**1**] for an abstract formulation for $F\in L^\infty$, Rodrigues [**46**] for $F\in L^2$, and Ziemer [**65**] for a related problem for $\operatorname{div} F \in L^1$ (see also Baiocchi-Capelo [**2**], Brezzi-Fortin [**8**], and Ziemer [**66**]). Also see Harrison [**30**], Harrison-Norton [**29**], Jurkat-Nonnenmacher [**32**], Nonnenmacher [**42**], Pfeffer [**43**], and Shapiro [**50**] for related problems and references.

5.3. Basic Properties of Divergence-Measure Fields. Now we discuss some basic properties of divergence-measure fields in the spaces $\mathcal{DM}^p(\Omega), 1\le p\le\infty$, and $\mathcal{DM}^{ext}(\Omega)$.

PROPOSITION 5.1. (i) *Let $\{F_j\}$ be a sequence in $\mathcal{DM}^p(\Omega)$ such that*

$$F_j \rightharpoonup F \qquad L^p_{loc}(\Omega;\mathbf{R}^N), \quad \textit{for } 1\le p<\infty, \tag{5.6}$$

$$F_j \overset{*}{\rightharpoonup} F \qquad L^\infty_{loc}(\Omega;\mathbf{R}^N), \quad \textit{for } p=\infty. \tag{5.7}$$

Then

$$\|F\|_{L^p(\Omega)} \le \liminf_{j\to\infty} \|F_j\|_{L^p(\Omega)}, \qquad |\operatorname{div} F|(\Omega) \le \liminf_{j\to\infty} |\operatorname{div} F_j|(\Omega).$$

(ii) *Let $\{F_j\}$ be a sequence in $\mathcal{DM}^{ext}(\Omega)$ such that*

$$F_j \rightharpoonup F \qquad \mathcal{M}_{loc}(\Omega;\mathbf{R}^N).$$

Then

$$|F|(\Omega) \le \liminf_{j\to\infty} |F_j|(\Omega), \qquad |\operatorname{div} F|(\Omega) \le \liminf_{j\to\infty} |\operatorname{div} F_j|(\Omega).$$

This proposition immediately implies that the spaces $\mathcal{DM}^p, 1\le p\le\infty$, and $\mathcal{DM}^{ext}(\Omega)$ are Banach spaces under the norms (5.4) and (5.5), respectively.

PROPOSITION 5.2. *Let $\{F_j\}$ be a sequence in $\mathcal{DM}(\Omega)$ satisfying*

$$\lim_{j\to\infty} |\operatorname{div} F_j|(\Omega) = |\operatorname{div} F|(\Omega)$$

and one of the following three conditions:

$$F_j \rightharpoonup F \qquad L^p_{loc}(\Omega;\mathbf{R}^N), \ \textit{for } 1\le p<\infty,$$
$$F_j \overset{*}{\rightharpoonup} F \qquad L^\infty_{loc}(\Omega;\mathbf{R}^N), \ \textit{for } p=\infty,$$
$$F_j \rightharpoonup F \qquad \mathcal{M}_{loc}(\Omega;\mathbf{R}^N).$$

Then, for every open set $A\subset\Omega$,

$$|\operatorname{div} F|(\bar A\cap\Omega) \ge \limsup_{j\to\infty} |\operatorname{div} F_j|(\bar A\cap\Omega). \tag{5.8}$$

In particular, if $|\mathrm{div}\, F|(\partial A \cap \Omega) = 0$*, then*

$$|\mathrm{div}\, F|(A) = \lim_{j\to\infty} |\mathrm{div}\, F_j|(A). \tag{5.9}$$

We use the so-called positive symmetric mollifiers $\omega : \mathbf{R}^N \to \mathbf{R}$ satisfying $\omega(x) \in C_0^\infty(\mathbf{R}^N)$, $\omega(x) \geq 0$, $\omega(x) = \omega(|x|)$, $\int_{\mathbf{R}^N} \omega(x)\, dx = 1$, $\mathrm{supp}\, \omega(x) \subset B_1 \equiv \{x \in \mathbf{R}^N : |x| < 1\}$. We denote $\omega_\varepsilon(x) = \varepsilon^{-N}\omega(\frac{x}{\varepsilon})$ and $F_\varepsilon = F * \omega_\varepsilon$, that is,

$$F^\varepsilon(x) = \varepsilon^{-N} \int_{\mathbf{R}^N} F(y)\omega(\frac{x-y}{\varepsilon})\, dy = \int_{\mathbf{R}^N} F(x+\varepsilon y)\omega(y)\, dy. \tag{5.10}$$

Then $F^\varepsilon \in C^\infty(A; \mathbf{R}^N)$ for any $A \Subset \Omega$ when ε is sufficiently small. We recall that, for any $f, g \in L^1(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} f_\varepsilon g\, dx = \int_{\mathbf{R}^N} f g_\varepsilon\, dx. \tag{5.11}$$

The following fact for $\mathcal{DM}$ fields is analogous to a well-known property of BV functions.

PROPOSITION 5.3. *Let* $F \in \mathcal{DM}(\Omega)$*. Let* $A \Subset \Omega$ *be open and* $|\mathrm{div}\, F|(\partial A) = 0$*. Then, for any* $\varphi \in C(\Omega; \mathbf{R})$*,*

$$\lim_{\varepsilon\to 0} \langle \mathrm{div}\, F^\varepsilon, \varphi\chi_A\rangle =< \mathrm{div}\, F, \varphi\chi_A > .$$

Furthermore, if $F \in \mathcal{DM}^{ext}(\Omega)$ *and* $|F|(\partial A) = 0$*, then, for any* $\varphi \in C(\Omega; \mathbf{R}^N)$*,*

$$\lim_{\varepsilon\to 0} < F^\varepsilon, \varphi\chi_A >=< F, \varphi\chi_A > .$$

Now we discuss some product rules for divergence-measure fields.

PROPOSITION 5.4. *Let* $F = (F_1, \cdots, F_N) \in \mathcal{DM}(\Omega)$*. Let* $g \in BV \cap L^\infty(\Omega)$ *be such that* $\partial_{x_j} g(x)$ *is* $|F_j|$*-integrable, for each* $j = 1, \cdots, N$*, and the set of non-Lebesgue points of* $\partial_{x_j} g(x)$ *has* $|F_j|$*-measure zero; and* $g(x)$ *is* $(|F| + |\mathrm{div}\, F|)$*-integrable and the set of non-Lebesgue points of* $g(x)$ *has* $(|F| + |\mathrm{div}\, F|)$*-measure zero. Then* $gF \in \mathcal{DM}(\Omega)$ *and*

$$\mathrm{div}\,(gF) = g\,\mathrm{div}\, F + \nabla g \cdot F. \tag{5.12}$$

In particular, if $F \in \mathcal{DM}^\infty(\Omega)$*,* $gF \in \mathcal{DM}^\infty(\Omega)$ *for any* $g \in BV \cap L^\infty(\Omega)$*; moreover, if* g *is also Lipschitz over any compact set in* Ω*,*

$$\mathrm{div}\,(gF) = g\,\mathrm{div}\, F + F \cdot \nabla g. \tag{5.13}$$

In fact, for $F \in \mathcal{DM}^\infty(\Omega)$, one may refine the above result to yield that (5.13) holds a.e. in a more general case, not only for local Lipschitz functions. In this case, we must take the absolutely continuous part of ∇g. For $g \in BV$, let $(\nabla g)_{\mathrm{ac}}$ and $(\nabla g)_{\mathrm{sing}}$ denote the absolutely continuous part and the singular part of the Radon measure ∇g, respectively. Then

PROPOSITION 5.5. *Given* $F \in \mathcal{DM}^\infty(\Omega)$ *and* $g \in BV(\Omega) \cap L^\infty(\Omega)$*, the identity*

$$\mathrm{div}\,(gF) = \bar{g}\mathrm{div}\, F + \overline{F \cdot \nabla g}$$

holds in the sense of Radon measures in Ω*, where* $\bar{g}$ *is the limit of a mollified sequence for* g *through a positive symmetric mollifier, and* $\overline{F \cdot \nabla g}$ *is a Radon measure absolutely continuous with respect to* $|\nabla g|$*, whose absolutely continuous part with respect to the Lebesgue measure in* Ω *coincides with* $F \cdot (\nabla g)_{ac}$ *almost everywhere in* Ω*.*

5.4. Normal Traces and the Gauss-Green Formula. We now discuss the generalized Gauss-Green theorem for $\mathcal{DM}$-fields over $\Omega \subset \mathbf{R}^N$ by introducing a suitable definition of normal traces over the boundary $\partial\Omega$ of a bounded open set with Lipschitz deformable boundary, established in Chen-Frid [**13, 14, 15**].

DEFINITION 5.3. Let $\Omega \subset \mathbf{R}^N$ be an open bounded subset. We say that $\partial\Omega$ is a deformable Lipschitz boundary, provided that

(i) $\forall x \in \partial\Omega$, $\exists r > 0$ and a Lipschitz map $\gamma : \mathbf{R}^{N-1} \to \mathbf{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap Q(x,r) = \{y \in \mathbf{R}^N \ : \ \gamma(y_1, \cdots, y_{N-1}) < y_N\} \cap Q(x,r),$$

where $Q(x,r) = \{y \in \mathbf{R}^N \ : \ |x_i - y_i| \le r, \ i = 1, \cdots, N\}$;

(ii) $\exists\, \Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(\omega, 0) = \omega$ for all $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Denote $\partial\Omega_s \equiv \Psi(\partial\Omega \times \{s\})$, $s \in [0,1]$, and Ω_s the open subset of Ω whose boundary is $\partial\Omega_s$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

DEFINITION 5.4. We say that the Lipschitz deformation is *regular* if

$$\lim_{s \to 0+} D\Psi_s \circ \tilde{\gamma} = D\tilde{\gamma}, \qquad \text{in } L^1_{\rm loc}(B), \tag{5.14}$$

where $\tilde{\gamma}$ is a map as in Condition (i) of Definition 5.3, and Ψ_s denotes the map of $\partial\Omega$ into Ω, given by $\Psi_s(x) = \Psi(x, s)$. Here B denotes the greatest open set such that $\tilde{\gamma}(B) \subset \partial\Omega$.

REMARK 5.1. The domains with regular deformable Lipschitz boundaries clearly include bounded domains with Lipschitz boundaries, the star-shaped domains, and the domains whose boundaries satisfy the cone property. It is also clear that, if Ω is the image through a bi-Lipschitz map of a domain $\bar{\Omega}$ with a (regular) Lipschitz deformable boundary, then Ω itself possesses a (regular) Lipschitz deformable boundary.

We first discuss the normal traces and the Gauss-Green formula for $\mathcal{DM}^p$ fields with $1 < p \le \infty$.

THEOREM 5.1. *Let $F \in \mathcal{DM}^p(\Omega)$, $1 < p \le \infty$. Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional $F \cdot \nu|_{\partial\Omega}$ over* $\mathrm{Lip}(\partial\Omega)$ *such that, for any $\phi \in \mathrm{Lip}(\mathbf{R}^N)$,*

$$\langle F \cdot \nu|_{\partial\Omega}, \phi\rangle = \langle \operatorname{div} F, \, \phi\rangle + \int_\Omega \nabla\phi \cdot F\, dx. \tag{5.15}$$

Moreover, let $\nu : \Psi(\partial\Omega \times [0,1]) \to \mathbf{R}^N$ be such that $\nu(x)$ is the unit outer normal to $\partial\Omega_s$ at $x \in \partial\Omega_s$, defined for a.e. $x \in \Psi(\partial\Omega \times [0,1])$. Let $h : \mathbf{R}^N \to \mathbf{R}$ be the level set function of $\partial\Omega_s$, that is,

$$h(x) := \begin{cases} 0, & \textit{for} \ \ x \in \mathbf{R}^N - \overline{\Omega}, \\ 1, & \textit{for} \ \ x \in \Omega - \Psi(\partial\Omega \times [0,1]), \\ s, & \textit{for} \ \ x \in \partial\Omega_s, 0 \le s \le 1. \end{cases}$$

Then, for any $\psi \in \mathrm{L}ip(\partial\Omega)$,

$$\langle F\cdot\nu|_{\partial\Omega},\psi\rangle = -\lim_{s\to 0}\frac{1}{s}\int_{\Psi(\partial\Omega\times(0,s))}\mathcal{E}(\psi)\,\nabla h\cdot F\,dx, \tag{5.16}$$

where $\mathcal{E}(\psi)$ *is any Lipschitz extension of* ψ *to all* $\mathbf{R}^N$.

In the case $p=\infty$, *the normal trace* $F\cdot\nu|_{\partial\Omega}$ *is a function in* $L^\infty(\partial\Omega)$ *satisfying* $\|F\cdot\nu\|_{L^\infty(\partial\Omega)}\le C\|F\|_{L^\infty(\Omega)}$, *for some constant* C *independent of* F; *if* $\partial\Omega$ *admits a regular Lipschitz deformation, then* $C=1$. *Furthermore, for any field* $F\in\mathcal{DM}^\infty(\Omega)$,

$$\langle F\cdot\nu|_{\partial\Omega},\psi\rangle = \operatorname*{esslim}_{s\to 0}\int_{\partial\Omega_s}\psi\circ\Psi_s^{-1}\,F\cdot\nu\,d\mathcal{H}^{N-1},\qquad \textit{for any } \psi\in L^1(\Omega). \tag{5.17}$$

Finally, for $F\in\mathcal{DM}^p(\Omega)$ with $1<p<\infty$, $F\cdot\nu|_{\partial\Omega}$ can be extended to a continuous linear functional over $W^{1-1/p,p}(\partial\Omega)\cap L^\infty(\partial\Omega)$.

Remark 5.2. As an example, consider the normal trace of $F(x,y)$ in Example 5.1 over the line $x=y$ where there is no reasonable notion of traces for the component $\cos(\frac{1}{x-y})$. Nevertheless, the unit normal ν_s to the line $x-y=s$ is the vector $(-1/\sqrt{2},1/\sqrt{2})$ so that the scalar product $F(x,x-s)\cdot\nu_s$ is identically zero over this line. Hence, we find that $F\cdot\nu\equiv 0$ over the line $x=y$ and the Gauss-Green formula in this case implies that, for any $\phi\in C_0^1(\mathbf{R}^2)$,

$$0=\langle \operatorname{div} F|_{x>y},\phi\rangle = -\int_{x>y}F\cdot\nabla\phi\,dxdy.$$

This identity could be also directly obtained by applying the dominated convergence theorem to the analogous identity, which can be derived from the classical Gauss-Green formula, for the domain $\{\,(x,y)\,|\,x>y+s\,\}$ when $s\to 0$.

As indicated by Example 5.2, it is more delicate for fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$. Then we have to define the normal traces as functionals over the spaces $\mathrm{L}ip(\gamma,\partial\Omega)$ with $\gamma>1$ (see Stein [**53**]). For $1<\gamma\le 2$, the elements of $\mathrm{L}ip(\gamma,\partial\Omega)$ are $(N+1)$-components vectors, where the first component is the function itself, and the other N components are its "first-order partial derivatives". In particular, as a functional over $\mathrm{L}ip(\gamma,\partial\Omega)$, the values of the normal trace of a field in $\mathcal{DM}^1$ or $\mathcal{DM}^{ext}$ on $\partial\Omega$ depend not only on the values of the respective functions over $\partial\Omega$, but also on the values of their first-order derivatives over $\partial\Omega$. To define the normal traces for $F\in\mathcal{DM}^1$ or $\mathcal{DM}^{ext}$, we resort to the properties of the Whitney extensions of functions in $\mathrm{L}ip(\gamma,\partial\Omega)$ to $\mathrm{L}ip(\gamma,\mathbf{R}^N)$. We have the following analogue of Theorem 5.1 which covers fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$ (see [**14**] for the proof).

Theorem 5.2. *Let* $F\in\mathcal{DM}^1(\Omega)$ *or* $F\in\mathcal{DM}^{ext}(\Omega)$. *Let* $\Omega\subset\mathbf{R}^N$ *be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional* $F\cdot\nu|_{\partial\Omega}$ *over* $\mathrm{L}ip(\gamma,\partial\Omega)$ *for any* $\gamma>1$ *such that, for any* $\phi\in\mathrm{L}ip(\gamma,\mathbf{R}^N)$,

$$\langle F\cdot\nu|_{\partial\Omega},\phi\rangle = \langle \operatorname{div} F,\phi\rangle_\Omega + \langle F,\nabla\phi\rangle_\Omega. \tag{5.18}$$

Moreover, let $h:\mathbf{R}^N\to\mathbf{R}$ *be the level set function as in Theorem 5.1; and, in the case that* $F\in\mathcal{DM}^{ext}(\Omega)$, *we also assume that* $\partial_{x_i}h$ *is* $|F_i|$*-measurable and its* *set of non-Lebesgue points has $|F_i|$-measure zero, $i = 1, \cdots, N$. Then, for any $\psi \in \mathrm{Lip}(\gamma, \partial\Omega)$, $\gamma > 1$,*

$$\langle F \cdot \nu|_{\partial\Omega}, \psi \rangle = -\lim_{s \to 0} \frac{1}{s} \langle F, \mathcal{E}(\psi)\, \nabla h \rangle_{\Psi(\partial\Omega \times (0,s))}, \tag{5.19}$$

where $\mathcal{E}(\psi) \in \mathrm{Lip}(\gamma, \mathbf{R}^N)$ is the Whitney extension of ψ to all $\mathbf{R}^N$.

Remark 5.3. In general, for $F \in \mathcal{DM}(D)$, the normal traces $F \cdot \nu|_{\partial\Omega}$ may be no longer functions. This can be seen in Example 5.2 for $F \in \mathcal{DM}^1_{loc}(\mathbf{R}^2)$ with $\Omega = (0,1) \times (0,1)$, for which $F \cdot \nu|_{\partial\Omega}$ is a measure.

Finally, as a corollary of the normal traces and the generalized Gauss-Green formula in $\mathcal{DM}^\infty$, we have

Proposition 5.6. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary and $F_1 \in \mathcal{DM}^\infty(\Omega)$, $F_2 \in \mathcal{DM}^\infty(\mathbf{R}^N - \bar{\Omega})$. Then*

$$F(y) = \begin{cases} F_1(y), & y \in \Omega, \\ F_2(y), & y \in \mathbf{R}^N - \bar{\Omega} \end{cases} \tag{5.20}$$

belongs to $\mathcal{DM}^\infty(\mathbf{R}^N)$, and

$$\|F\|_{\mathcal{DM}^\infty(\mathbf{R}^N)} \le \|F_1\|_{\mathcal{DM}^\infty(\Omega)} + \|F_2\|_{\mathcal{DM}^\infty(\mathbf{R}^N - \bar{\Omega})} + \|F_1 \cdot \nu - F_2 \cdot \nu\|_{L^\infty(\partial\Omega)} \mathcal{H}^{N-1}(\partial\Omega).$$

The theory of divergence-measure fields presented above has been applied to solving various nonlinear problems for conservation laws and related nonlinear equations. These especially include (1) the stability of Riemann solutions, which may contain rarefaction waves, contact discontinuities, and/or vacuum states, in the class of entropy solutions of the Euler equations for gas dynamics in [**14, 15**]; (2) the decay of periodic entropy solutions for hyperbolic conservation laws in [**13**]; (3) Initial and boundary layer problems for hyperbolic conservation laws in [**19**] and [**57**]; (4) initial-boundary value problems for hyperbolic conservation laws in [**13**]; and (5) nonlinear degenerate parabolic-hyperbolic equations in [**9**] and [**41**]. It would be interesting to apply the theory to solving further various problems in partial differential equations and related areas whose solutions are only measures or L^p functions.

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References

[1] Anzellotti, G., Pairings between measures and functions and compensated compactness, *Ann. Mat. Pura Appl.* **135** (1983), 293–318.

[2] Baiocchi, C. and A. Capelo, *Variational and Quasi-Variational Inequalities, Applications to Free-Boundary Problems*, Vols. **1,2**, John Wiley: Chichester-New York, 1984.

[3] Bouchitté, G. and G. Buttazzo, Characterization of optimal shapes and masses through Monge-Kantorovich equation, *J. J. Eur. Math. Soc. (JEMS)* **3** (2001), 139–168.

[4] Bouchut, F., On zero pressure gas dynamics, In: *Advances in Kinetic Theory and Computing*, pp. 171–190, *Ser. Adv. Math. Appl. Sci.* **22**, World Sci. Publishing: River Edge, NJ, 1994.

[5] Bouchut F. and F. James, Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness, *Commun. Partial Diff. Eqs.* **24** (1999), 2173–2189.

[6] Bouchut, F., Jin, S., and X. Li, Numerical approximation of pressureless and isothermal gas dynamics, Preprint, November 2001.

[7] Brenier, Y. and E. Grenier, Sticky particles and scalar conservation laws, *SIAM J. Numer. Anal.* **35** (1998), 2317–2328.

[8] Brezzi, F. and M. Fortin, *Mixed and Hydrid Finite Element Methods*, Springer-Verlag: New York, 1991.

[9] Bürger, R., Karlsen, K. H., and H. Frid, On a free boundary problem for a strongly degenerate quasilinear parabolic equation arising in a model for pressure filtration, submitted for publication in December 2001.

[10] Chang, T., Chen, G.-Q., and S. Yang, 2-D Riemann problem in gas dynamics and formation of spiral, In: *Nonlinear Problems in Engineering and Science–Numerical and Analytical Approach (Beijing, 1991)*, pp. 167–179, Science Press: Beijing, 1992.

[11] Chang, T., Chen, G.-Q., and S. Yang, On the Riemann problem for two-dimensional Euler equations, I: Interaction of shocks and rarefaction waves, *Discrete Contin. Dynam. Systems*, **1** (1995), 555–584; II: Interaction of contact discontinuities, *Discrete Contin. Dynam. Systems*, **6** (2000), 419–430.

[12] Chang T. and L. Hsiao, *The Riemann Problem and Interaction of Waves in Gas Dynamics*, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc.: New York, 1989.

[13] Chen, G.-Q. and H. Frid, Divergence-measure fields and hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **147** (1999), 89–118.

[14] Chen, G.-Q. and H. Frid, Extended divergence-measure fields and the Euler equations of gas dynamics, *Commun. Math. Phys.* 2003 (to appear).

[15] Chen, G.-Q. and H. Frid, On the theory of divergence-measure fields and its applications, *Bol. Soc. Brasil. Mat. (N.S.)* **32** (2001), 401–433.

[16] Chen, G.-Q., Frid, H., and Y. Li, Uniqueness and stability of Riemann solutions with large oscillation in gas dynamics, *Commun. Math. Phys.* **288** (2002), 201-217.

[17] Chen, G.-Q. and H. Liu, Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids, *SIAM J. Math. Anal.* 2002 (to appear).

[18] Chen, G.-Q. and H. Liu, Concentration and cavitation in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids, Preprint, Northwestern University, 2002.

[19] Chen, G.-Q. and M. Rascle, Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **153** (2000), 205-220.

[20] Chen, G.-Q. and D. Wang, The Cauchy problem for the Euler equations for compressible fluids, In: *Handbook of Mathematical Fluid Dynamics,* Vol. **1**, pp. 421-543, Elsevier: Amsterdam, The Netherlands, 2002.

[21] Dafermos, C. M., *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag: Berlin, 1999.

[22] E, W., Rykov, Yu. G., and Ya. G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics, *Commun. Math. Phys.* **177** (1996), 349–380.

[23] Evans, L. C. and R. F. Gariepy, *Lecture Notes on Measure Theory and Fine Properties of Functions*, CRC Press: Boca Raton, Florida, 1992.

[24] Federer, H., *Geometric Measure Theory*, Springer-Verlag: Berlin-Heidelberg-New York, 1969.

[25] Friedrichs, K. O. and P. D. Lax, Systems of conservation equations with a convex extension, *Proc. Nat. Acad. Sci. U.S.A.* **68** (1971), 1686–1688.

[26] Glimm, J., Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 95-105.

[27] Glimm, J. and P. D. Lax, *Decay of Solutions of Systems of Nonlinear Hyperbolic Conservation Laws*, Mem. Amer. Math. Soc. **101**, AMS: Providence, R.I. 1970.

[28] Grenier, E., Existence globale pour le système des gaz sans pression, *C. R. Acad. Sci. Paris Sér. I. Math.* **321** (1995), 171–174.

[29] Harrison, J. and A. Norton, The Gauss-Green theorem for fractal boundaries, *Duke Math. J.* **67** (1992), 575–588.

[30] Harrison, J., Stokes' theorem for nonsmooth chains, *Bull. Amer. Math. Soc. (N.S.)* **29** (1993), 235–242.

[31] Huang, F. and Z. Wang, Well posedness for pressureless flow, *Commun. Math. Phys.* **222** (2001), 117-146.

[32] Jurkat, W. B. and D. J. F. Nonnenmacher, A generalized n-dimensional Riemann integral and the divergence theorem with singularities, *Acta Sci. Math. (Szeged)* **59** (1994), 241–256.

[33] Joseph, K. T., A Riemann problem whose viscosity solutions contain delta measure, *Asymptotic Analysis*, **7** (1993), 105-120.

[34] Keyfitz, B. L. and H. Kranzer, Spaces of weighted measures for conservation laws with singular shock solutions, *J. Diff. Eqs.* **118** (1995), 420–451.

[35] Korchinski, D. J., Solutions of a Riemann problem for a 2×2 system of conservation laws possessing no classical solution, Adelphi University Thesis, 1977.

[36] Lax, P. D., *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMS. **11**, SIAM, Philadelphia, 1973.

[37] Li, J. and H. Yang, Delta-shocks as limits of vanishing viscosity for multidimensional zero-pressure gas dynamics, *Quart. Appl. Math.* **59** (2001), 315–342.

[38] Li, J. and T. Zhang, Generalized Rankine-Hugoniot relations of delta-shocks in solutions of transportation equations, In: *Advances in Nonlinear Partial Differential Equations and Related Areas* (Beijing, 1997), 219–232, eds. G.-Q. Chen et al, World Sci. Publishing: River Edge, NJ, 1998.

[39] Li, J. and T. Zhang, On the initial-value problem for zero-pressure gas dynamics, In:*Hyperbolic Problems: Theory, Numerics, Applications*, Vol. **2** (Zürich, 1998), 629–640, Birkhäuser: Basel, 1999.

[40] Liu, T.-P., The deterministic version of the Glimm scheme. *Comm. Math. Phys.* **57** (1977), 135–148.

[41] Mascia, C., Porretta, A., and A. Terracina, Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations, *Arch. Rational Mech. Anal.* **163** (2002), 87–124.

[42] Nonnenmacher, D. J. F., Sets of finite perimeter and the Gauss-Green theorem with singularities, *J. London Math. Soc. (2)* **52** (1995), 335–344.

[43] Pfeffer, W. F., *Derivation and Integration*, Cambridge Tracts in Math. **140**, Cambridge Univ. Press: Cambridge, 2001.

[44] Poupaud, F. and M. Rascle, Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients, *Commun. Partial Diff. Eqs.* **22** (1997), 337–358.

[45] Rauch, J., BV estimates fail for most quasilinear hyperbolic systems in dimensions greater than one, *Comm. Math. Phys.* **106** (1986), 481–484.

[46] Rodrigues, J.-F., *Obstacle Problems in Mathematical Physics*, North-Holland Mathematics Studies **134**, Elsevier Science Publishers B.V. 1987.

[47] Schwartz, L., *Théorie des Distributions*, Actualites Scientifiques et Industrielles **1091**, **1122**, Herman: Paris, 1950-51.

[48] Sever, M., A class of nonlinear, nonhyperbolic systems of conservation laws with well-posed initial value problem, Preprint, Hebrew University, 2000.

[49] Shandarin, S. F. and Ya. B. Zeldovich, The large-scale structure of the universe: turbulence, intermittency, structures in a self-gravitating medium, *Rev. Modern Phys.* **61** (1989), 185–220.

[50] Shapiro, V., The divergence theorem for discontinuous vector fields, *Ann. Math. (2)* **68** 1958, 604-624.

[51] Smith, R., The Riemann problem in gas dynamics, *Trans. Amer. Math. Soc.* **249** (1979), 1-50.

[52] Smoller, J., *Shock Waves and Reaction-Diffusion Equations*, 2nd edition, Springer-Verlag: New York, 1994.

[53] Stein, E., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press: Princeton, NJ, 1970.

[54] Tan, D. and T. Zhang, Two dimensional Riemann problem for a hyperbolic system of non-linear conservation laws (I): four-J cases, *J. Diff. Eqs.* **111** (1994), 203-254; (II): Initial data consists of some rarefaction, *J. Diff. Eqs.* **111** (1994), 255-283.

[55] Tan, D., Zhang, T., and Yuxi Zheng, Delta-shock waves as limits of vanishing viscosity for hyperbolic system of conservation laws, *J. Diff. Eqs.* **112** (1994), 1-32.

[56] Sheng, W. and T. Zhang, *The Riemann Problem for the Transport Equations in Gas Dynamics*, Mem. Amer. Soc. **654** (1999), AMS: Providence.

[57] Vasseur, A., Strong traces for solutions to multidimensional scalar conservation laws, *Arch. Rational Mech. Anal.* **160** (2001), 181-193.

[58] Volpert, A. I., The space BV and quasilinear equations, *Mat. Sb. (N.S.)* **73** (1967), 255–302, *Math. USSR Sbornik*, **2** (1967), 225–267 (in English).

[59] Wagner, D., Equivalence of the Euler and Lagrange equations of gas dynamics for weak solutions, *J. Diff. Eqs.* **68** (1987), 118–136.

[60] Wang, Z. and X. Ding, Uniqueness of generalized solutions for the Cauchy problem of transportation equations, *Acta Math. Sci.* (English Ed.), **17** (1997), 341-352.

[61] Wang, Z., Huang, F., and X. Ding, On the Cauchy problem of transportation equations, *Acta Math. Appl. Sinica (English Ser.)*, **13** (1997), 113–122.

[62] Whitney, H., *Geometric Integration Theory*, Princeton Univ. Press: Princeton, NJ, 1957.

[63] Whitney, H., Analytic extensions of differentiable functions defined in closed sets, *Trans. Amer. Math. Soc.* **36** (1934), 63-89.

[64] Zeldovich, Ya. B., Gravitational instability: an approximate theory for large density perturbations, *Astron. Astrophys.* **5** (1970), 84-89.

[65] Ziemer, W. P., Cauchy flux and sets of finite perimeter, *Arch. Rational Mech. Anal.* **84** (1983), 189-201.

[66] Ziemer, W. P., *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*, Springer-Verlag: New York, 1989.

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