DECAY OF ENTROPY SOLUTIONS OF NONLINEAR CONSERVATION LAWS

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Dedicated to Peter D. Lax and Louis Nirenberg
On the Occasion of Their 70th Birthdays

1. Introduction

We are concerned with the asymptotic behavior of entropy solutions of nonlinear conservation laws. The main objective of this paper is to present an analytical approach and to explore its applications to studying the large-time behavior of periodic entropy solutions of hyperbolic conservation laws.

The asymptotic decay of periodic solutions of nonlinear hyperbolic conservation laws is an important nonlinear phenomenon. It is observed that the genuine nonlinearity of equations forces the waves of each characteristic family to interact vigorously and to cancel each other. The insightful analysis of Glimm-Lax [GL], for scalar equations and 2×2 systems, has indicated that the resultant mutual cancellation of interacting shock and rarefaction waves of the same family induces the decay of periodic solutions. Such a result was first shown by Lax [L1] in 1957 for one-dimensional convex scalar conservation laws. Dafermos [D1], applying his uniform processes, proved the decay result for the case that the set of inflection points of the flux does not have an accumulation point. The Glimm-Lax theory [GL] indicates that, for 2×2 strictly hyperbolic and genuinely nonlinear systems, any periodic Glimm solution decays like $O(1/t)$. This result was proved by using the approximate characteristic method in the Glimm difference solutions, provided that the oscillation of the corresponding initial data is small. Recently, using the method of generalized characteristics, Dafermos [D3] showed that any periodic solution with local bounded variation and small oscillation for 2×2 systems decays asymptotically, with the same detailed structure pas found by Lax [L1] for the

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scalar case. Also see Engquist-E [E] for the decay of periodic solutions with local bounded variation for the two-dimensional scalar conservation laws.

In this paper we develop an analytical approach and study the decay problem for L^∞ periodic solutions of hyperbolic conservation laws. The approach we develop here is motivated by some essential features of the underlying conservation laws, such as scale-invariance and compactness. The goal of this effort is to prove the decay of L^∞ periodic entropy solutions in any L^p norm, $1 \le p < \infty$, in a general framework for (a) any large L^∞ periodic solutions without restrictions of either small oscillation or local bounded variation and (b) more general nonlinear hyperbolic systems, especially including hyperbolic systems with degeneracy and multidimensional scalar conservation laws. Our main observation is that the compactness of an L^∞ solution operator in L^1_{loc} coupling with the weak convergence of the periodic initial data to its mean yields the decay of the L^∞ periodic solution; and, for hyperbolic systems endowed with a strictly convex entropy, the decay is actually in any L^p norm, $1 \le p < \infty$.

In Section 3, we introduce an analytical approach to study the large-time behavior of periodic entropy solutions and discuss the relation between the decay problem and several main features of hyperbolic conservation laws such as scale-invariance of equations and compactness of solution operators.

Then we apply this approach to the decay problem for large L^∞ periodic solutions of multidimensional scalar conservation laws in Section 4 and for hyperbolic systems of conservation laws in Section 5. In §5.1, we first consider 2×2 strictly hyperbolic and genuinely nonlinear systems and the equations of elasticity, a system with reflection points of genuine nonlinearity. In particular, we prove that any periodic solution of the equations of elasticity decays in any L^p norm, provided that the stress-strain function σ is in C^2, in contrast with the counterexample of Greenberg-Rascle [GR] when σ is only allowed to be in C^1. Then we study the decay problem for the Euler equations for isentropic flow of compressible fluids, a prototype of hyperbolic systems with parabolic degeneracy, in §5.2, and hyperbolic systems with umbilic degeneracy, an example of hyperbolic systems with hyperbolic degeneracy, in §5.3. In §5.4, we study the large-time behavior of entropy solutions of the 3×3 Euler system of thermoelasticity, an example of hyperbolic systems with linear degeneracy. Our result indicates for such systems that, although periodic solutions do not decay in general, some important quantities from the solutions, such as velocity, pressure, and temperature, do asymptotically decay.

In Section 6, we consider the asymptotic decay of periodic solutions for hyperbolic conservation laws with relaxation. We show how to extend the approach and techniques developed in Sections 3-5 to this type of systems. We then apply our framework and the compactness results in [CL1,CL2] to establishing the asymptotic decay of periodic solutions for three physical systems with relaxation.

The same decay results can be obtained for the corresponding conservation laws with viscosity in a straightforward manner. The ideas and techniques developed here have been applied to the asymptotic stability of Riemann solutions for conservation laws (cf. [CF2,CF3]). Some results of this paper were announced in [CF1].

2. Entropy and Periodic Solutions

In this section, we review some basic preliminaries for subsequent developments. For more details, see [L2,Sm].

Consider the Cauchy problem for a hyperbolic system of conservation laws:

$$\partial_t u + \nabla_x \cdot f(u) = 0, \qquad u \in \mathbf{R}^m,\ x \in \mathbf{R}^n, \tag{2.1}$$

$$u\big|_{t=0} = u_0(x), \tag{2.2}$$

where $f : \mathbf{R}^m \to (\mathbf{R}^m)^n$ is a nonlinear mapping. The condition of hyperbolicity requires that, for any wave number $\xi \in S^{n-1}$, the matrix $\xi \cdot \nabla f(u)$ have m real eigenvalues and left (right) eigenvectors. For the one-dimensional case, system (2.1) is called strictly hyperbolic if the Jacobian $\nabla f(u)$ of f has m real and distinct eigenvalues, $\lambda_1(u) < \cdots < \lambda_m(u)$, and thus has m linearly independent right and left eigenvectors $r_j = r_j(u)$ and $l_j = l_j(u)$:

$$\nabla f(u) r_j(u) = \lambda_j(u) r_j(u), \qquad l_j(u) \nabla f(u) = \lambda_j(u) l_j(u). \tag{2.3}$$

The jth characteristic field is genuinely nonlinear or linearly degenerate in the sense of Lax [L2] if

$$r_j \cdot \nabla\lambda_j \neq 0 \quad \text{or} \quad r_j \cdot \nabla \lambda_j \equiv 0. \tag{2.4}$$

That is, the jth eigenvalue changes monotonically or remains constant along the jth characteristic field for the genuinely nonlinear case or the linearly degenerate case, respectively.

It is well known that the Cauchy problem (2.1)–(2.2) does not, in general, have globally defined smooth solutions because the eigenvalues are nonlinear; hence only discontinuous solutions may exist in the large. One of the main features of conservation laws is that uniqueness is lost within the class of the discontinuous solutions in the sense of distributions; many weak solutions may share the same initial data. Thus, the problem arises of identifying an appropriate class of weak solutions, entropy solutions, to single out physically relevant solutions.

Definition 2.1. *A function $\eta : \mathbf{R}^m \to \mathbf{R}$ is called an entropy of (2.1) if there exists $q : \mathbf{R}^m \to \mathbf{R}^n$ such that*

$$\nabla q_k(u) = \nabla\eta(u) \nabla f_k(u), \qquad k = 1, 2, \cdots, n. \tag{2.5}$$

The function $q(u)$ is called the entropy flux associated with the entropy $\eta(u)$, and the pair $(\eta(u), q(u))$ is called the entropy pair. The entropy $\eta(u)$ is convex on the domain $K \subset \mathbf{R}^m$ if the Hessian matrix $\nabla^2\eta(u) \geq 0$, for $u \in K$. The entropy is strictly convex on the domain K if $\nabla^2\eta(u) > 0$, for $u \in K$.

Consider a 2×2 strictly hyperbolic system with globally defined Riemann invariants $w_j, j = 1, 2$. The Riemann invariants $w_j : \mathbf{R}^2 \to \mathbf{R}$ satisfy

$$\nabla w_j(u) \nabla f(u) = \lambda_j(u) \nabla w_j(u), \qquad j = 1, 2,$$

and hence diagonalize system (2.1), for smooth solutions, into

$$\partial_t w_j + \lambda_j \partial_x w_j = 0, \qquad j = 1, 2.$$

Lax's theorem [L3] indicates for such a system that, given any bounded domain $K \Subset \mathbf{R}^2$, there exists a strictly convex entropy pair $(\eta(u), q(u))$ on the domain K. That is,

$$\nabla^2 \eta(u) \geq c_K > 0, \qquad u \in K.$$

For $m \geq 3$, system (2.5) is overdetermined, thereby generally preventing the existence of nontrivial entropies. Friedrichs-Lax [FL] observed that most of the systems of conservation laws that result from continuum mechanics are endowed with a globally defined, strictly convex entropy. Systems endowed with a rich family of entropies were described by Serre [S2].

Definition 2.2. *A function $u(x,t) \in L^\infty(\mathbf{R}^{n+1}_+), \mathbf{R}^{n+1}_+ \equiv \mathbf{R}^n \times [0,\infty)$, is called an entropy solution of (2.1)-(2.2), if, for any convex entropy pair (η, q) of (2.1) and any nonnegative function $\phi(x,t) \in C^1_0(\mathbf{R}^{n+1}_+)$,*

$$\int_0^\infty \int_{\mathbf{R}^n} \{\eta(u)\phi_t + q(u) \cdot \nabla_x \phi\}\, dxdt + \int_{\mathbf{R}^n} \eta(u_0(x))\phi(x,0)\, dx \geq 0. \tag{2.6}$$

In particular,

$$\partial_t \eta(u) + \nabla_x \cdot q(u) \leq 0$$

in the sense of distributions over $\mathbf{R} \times (0, \infty)$.

If we take $\eta(u) = \pm u$, we see that any entropy solution satisfies

$$\int_0^\infty \int_{\mathbf{R}^n} \{u\phi_t + f(u) \cdot \nabla_x \phi\}\, dxdt + \int_{\mathbf{R}^n} u_0(x)\phi(x,0)\, dx = 0, \tag{2.7}$$

for any function $\phi(x,t) \in C^1_0(\mathbf{R}^{n+1}_+)$.

Definition 2.3. *An entropy solution $u(x,t) \in L^\infty(\mathbf{R}^{n+1}_+; \mathbf{R}^m)$ is called a periodic solution of (2.1) in $x \in \mathbf{R}^n$ with period $P = \Pi_{i=1}^n [0, p_i]$ if, in addition to (2.6), for any continuous function $h : \mathbf{R}^m \to \mathbf{R}$,*

$$\int_0^\infty \int_{\mathbf{R}^n} h(u(x + p_i \mathbf{e}_i, t))\phi(x,t) dxdt = \int_0^\infty \int_{\mathbf{R}^n} h(u(x,t))\phi(x,t) dxdt, \tag{2.8}$$

for any test function $\phi(x,t) \in C^\infty_0(\mathbf{R}^{n+1}_+)$ and all $i = 1, \cdots, n$, where $\mathbf{e}_i$ is the ith element of the canonical basis of $\mathbf{R}^n$.

3. Scale-Invariance, Compactness, and Decay of Solutions

In this section we introduce an analytical approach to study the asymptotic decay of entropy solutions of hyperbolic conservation laws via scale-invariance and compactness properties of the underlying equations (2.1).

Let $u(x,t)$ be an entropy solution of (2.1)-(2.2), periodic in $x \in \mathbf{R}^n$, with period $P = \Pi_{i=1}^n [0, p_i] \subset \mathbf{R}^n$ and periodic initial data:

$$u_0(x + p_i \mathbf{e}_i) = u_0(x), \qquad i = 1, \dots, n. \tag{3.1}$$

3.1. Notion of Decay via Scale-Invariance

Consider the self-similar scaling sequence of the solution $u(x,t)$:

$$u^T(x,t) = u(Tx, Tt). \tag{3.2}$$

One of the main features of hyperbolic conservation laws (2.1) is the scale-invariance in the sense that the self-similar scaling sequence $u^T(x,t)$ also satisfies (2.1).

DEFINITION 3.1. *The periodic solution $u(x,t)$ of the Cauchy problem (2.1)-(2.2) asymptotically decays to $\bar{u}$ provided that*

$$\|u^T - \bar{u}\|_{L^p_{loc}(\mathbf{R}^{n+1}_+)} \to 0, \quad \text{when } T \to \infty, \quad \text{for some } p \in [1, \infty). \tag{3.3}$$

Because of the self-similar structure of $u^T(x,t)$, the limit in Definition 3.1 can be translated in terms of decay along the rays $x/t = \xi$, $\xi \in \mathbf{R}^n$, emanating from the origin in the (x,t)-plane. Actually, Definition 3.1 is equivalent to the following notion (also see [CF1]).

DEFINITION 3.2. *The periodic solution $u(x,t) = U(x/t, t)$ of the Cauchy problem (2.1)-(2.2) with respect to the space variable x asymptotically decays to $\bar{u}$ provided that, for some $p \in [1, \infty)$,*

$$\frac{1}{T}\int_0^T |U(\xi,t) - \bar{u}|^p dt \to 0, \qquad \text{in} \quad L^1_{loc}(\mathbf{R}^n_\xi), \quad \text{when } T \to \infty. \tag{3.4}$$

Therefore, the geometrical interpretation of Definition 3.1 is the decay of the periodic solution $u(x,t)$ along rays $x/t = \xi$, $\xi \in \mathbf{R}^n$, in the sense of (3.4). The equivalence between Definitions 3.1 and 3.2 may be seen from the following proposition.

PROPOSITION 3.1. *The limit (3.4) is equivalent to*

$$\frac{n+1}{T^{n+1}}\int_0^T |U(\xi,t) - \bar{u}|^p t^n dt \to 0, \qquad \text{in} \quad L^1_{loc}(\mathbf{R}^n_\xi), \quad \text{when} \quad T \to \infty, \tag{3.5}$$

provided that there exists $C > 0$, independent of ξ, such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |U(\xi,t) - \bar{u}|^p dt \le C. \tag{3.6}$$

Furthermore, (3.5) is equivalent to (3.3).

Proof. The limit (3.4) is clearly stronger than (3.5). It suffices to show that the limit (3.4) is a corollary of (3.5)-(3.6). For any sufficiently large N,

$$\begin{aligned}
&\frac{1}{T}\int_0^T |U(\xi,t)-\bar{u}|^p dt = \frac{1}{T}\int_0^{\frac{T}{N}} |U(\xi,t)-\bar{u}|^p dt + \frac{1}{T}\int_{\frac{T}{N}}^T |U(\xi,t)-\bar{u}|^p dt \\
&\le \frac{1}{N}\left(\frac{N}{T}\int_0^{\frac{T}{N}} |U(\xi,t)-\bar{u}|^p dt\right) + \frac{N^n}{n+1}\left(\frac{n+1}{T^{n+1}}\int_0^T |U(\xi,t)-\bar{u}|^p t^n dt\right).
\end{aligned}$$

Now, for any compact set $K \subset \mathbf{R}^n$, using (3.5)-(3.6) and taking $T \to \infty$, we obtain

$$\limsup_{T\to\infty}\int_K \left(\frac{1}{T}\int_0^T |U(\xi,t)-\bar{u}|^p dt\right) d\xi \le \frac{C|K|}{N},$$

for any sufficiently large N, where $|K|$ denotes the measure of K. Then (3.4) follows. The equivalence of (3.3) and (3.5) is obvious.

Under this notion of decay, if an L^∞ periodic solution asymptotically decays for some $p_0 \in [1,\infty)$, it asymptotically decays for all $p \in [1,\infty)$. Definition 3.1 is equivalent to $\frac{1}{T}\int_0^T \|U(\cdot,t)-\bar{u}\|^p_{L^p_{loc}(\mathbf{R}^n_\xi)} dt \to 0$, when $T \to \infty$.

As we will show in this section, for systems endowed with a strictly convex entropy, Definition 3.1 actually implies

$$\operatorname*{ess\,lim}_{t\to\infty}\int_P |u(x,t)-\bar{u}|^p dx = 0,$$

which is the classical sense of the asymptotic decay of periodic solutions for conservation laws.

Remark 3.1. Under this definition, the decay of periodic solutions is in the sense of a long-time average. Set $\mu^T(t) = \frac{1}{T}\chi_{[0,T]}(t)dt$, which is a family of probability measures. Then Definition 3.1 means that

$$< \mu^T, |U-\bar{u}|^p > (\xi) \longrightarrow 0, \qquad \text{in} \quad L^1_{loc}(\mathbf{R}^n_\xi), \quad \text{when } T \to \infty. \tag{3.7}$$

Such an average measure has been widely used to understand macroscopically the asymptotic behavior of physical quantities in statistical mechanics, kinetic theory, ergodic theory, and probability theory (see Boltzmann [Bo], Maxwell [Ma], Birkhoff [Bi], Glimm-Jaffe [GJ], and Varadhan [Va]). We can extend these notions to more general settings. A periodic solution asymptotically decays with a family of probability measures $\mu^T(t)$ if the corresponding global entropy solution $u(x,t) = U(x/t,t)$, locally integrable in $\mathbf{R}^{n+1}_+$, satisfies

$$\int_0^\infty |U(\xi,t)-\bar{u}|^p d\mu^T(t) \to 0, \qquad \text{in} \quad L^1_{loc}(\mathbf{R}^n_\xi), \quad \text{when } T \to \infty,$$

for some $p \in [1, \infty)$, where the family of probability measures $\mu^T(t)$ satisfies

$$\text{supp } \mu(t) = \{+\infty\}, \tag{3.8}$$

for $\mu(t) = w - \lim \mu^T(t)$ in the space of Radon measures over some compactification for $[0, +\infty)$. It would be natural to develop such notions to understand the asymptotic behavior of stochastic solutions of nonlinear conservation laws.

REMARK 3.2. If the periodic solution is globally bounded in L^∞, condition (3.6) holds. In this case, the average measure sequence $\mu^T(t) = \frac{n+1}{T^{n+1}}\chi_{[0,T]}(t)t^n dt$ satisfies (3.7)-(3.8).

Using the Dominated Convergence Theorem, we have

REMARK 3.3. If any periodic solution $u(x,t) = U(x/t,t)$ decays along almost all rays $x/t = \xi$, $\xi \in \mathbf{R}^n$, that is, $\frac{1}{T}\int_0^T |U(\xi,t) - \bar{u}|^p dt \to 0$, a.e. $\xi \in \mathbf{R}^n$, when $T \to \infty$, then the periodic solution asymptotically decays in the sense of Definition 3.1.

3.2. Decay via Compactness

In this section we develop a general framework for the asymptotic decay of periodic solutions of hyperbolic conservation laws. First we have the following asymptotic decay theorem.

THEOREM 3.1. *Assume that $u(x,t) \in L^\infty(\mathbf{R}^{n+1}_+)$ is a periodic solution of (2.1)-(2.2) and that its self-similar scaling sequence $u^T(x,t)$ is compact in $L^1_{loc}(\mathbf{R}^{n+1}_+)$. Then $u(x,t)$ asymptotically decays to the mean of $u_0(x)$ over the period:*

$$\bar{u} \equiv \frac{1}{|P|}\int_P u_0(x)dx, \tag{3.9}$$

in the sense of Definition 3.1.

PROOF. It suffices to show that the whole sequence $u^T(x,t)$ strongly converges to $\bar{u}$ in $L^1_{loc}(\mathbf{R}^{n+1}_+)$. Since the self-similar scaling sequence $u^T(x,t)$ is compact in $L^1_{loc}(\mathbf{R}^{n+1}_+)$, then, for any subsequence of $u^T(x,t)$, there exists a further subsequence $u^{T_k}(x,t)$ converging to some function $\bar{u}(x,t) \in L^\infty \cap L^1_{loc}(\mathbf{R}^{n+1}_+)$:

$$u^{T_k}(x,t) \to \bar{u}(x,t), \qquad \text{in} \quad L^1_{loc}(\mathbf{R}^{n+1}_+), \quad \text{as } k \to \infty.$$

Assertion 1. $\bar{u}(x,t) = \bar{u}(t)$; that is, $\bar{u}(x,t)$ does not depend on x.
It suffices to show that, for any $y \in \mathbf{R}^n$,

$$\bar{u}(x+y,t) = \bar{u}(x,t), \tag{3.10}$$

for all $(x,t) \in \mathbf{R}^{n+1}_+$ such that (x,t) and $(x+y,t)$ are both Lebesgue points of the function $\bar{u}$. Indeed, in this case, given any two points (z,t) and (z',t) at the same time level such that both are Lebesgue points of $\bar{u}$, then $\bar{u}(z,t) = \bar{u}(z',t)$. Therefore, we can find a set of measure zero $\mathcal{N} \subset (0,\infty)$ such that, for any $t \in (0,\infty)\backslash\mathcal{N}$, the function $\bar{u}(\cdot,t)$ is constant a.e. in $\mathbf{R}^n$, and this is exactly the claim.

To prove (3.10), we first note that

$$\mathcal{Q} = \{l/T_k \mid l \in \mathbf{Z}, k \in \mathbf{N}\}^n \quad \text{is dense in } \mathbf{R}^n.$$

It suffices to show that the set of the rational numbers $\{\, p/q \,|\, p,\, q \in \mathbf{Z}\,\}$ is a subset of the closure of $\{l/T_k \mid l \in \mathbf{Z}, k \in \mathbf{N}\}$. Now, given $p/q, q \neq 0$, there exists r_k, $|r_k| < q$, and integers m_k such that $T_k = m_k q + r_k$. Pick $l_k = p m_k$. Then

$$l_k/T_k \longrightarrow p/q, \quad \text{as } k \to \infty.$$

Therefore, for any $y \in \mathbf{R}^n$, there exist T_{k_j} and $y_j = l_j/T_{k_j} \in \mathcal{Q}$ such that

$$y_j \to y, \qquad u^{T_{k_j}}(x + y_j, t) = u^{T_{k_j}}(x,t), \qquad j \to \infty.$$

Then, for any test function $\phi(x,t)$, we have

$$\begin{aligned}
\int_0^\infty \int_{\mathbf{R}^n} \bar{u}(x+y,t)\phi(x,t)dxdt &= \int_0^\infty \int_{\mathbf{R}^n} \bar{u}(x,t)\phi(x-y,t)dxdt \\
&= \lim_{j\to\infty} \int_0^\infty \int_{\mathbf{R}^n} u^{T_{k_j}}(x,t)\phi(x-y_j,t)dxdt \\
&= \lim_{j\to\infty} \int_0^\infty \int_{\mathbf{R}^n} u^{T_{k_j}}(x+y_j,t)\phi(x,t)dxdt \\
&= \lim_{j\to\infty} \int_0^\infty \int_{\mathbf{R}^n} u^{T_{k_j}}(x,t)\phi(x,t)dxdt \\
&= \int_0^\infty \int_{\mathbf{R}^n} \bar{u}(x,t)\phi(x,t)dxdt,
\end{aligned}$$

where, in the fourth equality, we used the periodicity property as stated in Definition 2.3. This implies (3.10) for all (x,t) such that (x,t) and $(x+y,t)$ are both Lebesgue points of $\bar{u}$, which concludes Assertion 1.

Assertion 2. $\bar{u}(t) = \bar{u}$; that is, $\bar{u}(x,t)$ is a constant.

Notice that the self-similar scaling sequence $u^T(x,t)$ still satisfies (2.6), especially (2.7). Taking the limit $T_k \to \infty$, we see that (2.7) also holds for $\bar{u}(t)$. In (2.7), we pick a test function $\phi(x,t) = \alpha(t)\zeta(x)$ with $\alpha(t)$ satisfying $\alpha(0) = 1$ and approximating the characteristic function $\chi_{[0,\tau]}$, for any given $\tau > 0$, which is a Lebesgue point of the function $\bar{u}(t)$. We then arrive at

$$\bar{u}(\tau) \int_{\mathbf{R}^n} \zeta(x)\, dx = \bar{u}(0) \int_{\mathbf{R}^n} \zeta(x)\, dx,$$

for any such $\tau > 0$. This implies that $\bar{u}(t) = \bar{u} \equiv \bar{u}(0)$.

Since $u_0(x)$ is periodic, it is well known that

$$u_0(Tx) \rightharpoonup \frac{1}{|P|} \int_P u_0(x)dx \equiv \bar{u}, \qquad \text{weak star in } L^\infty,$$

which concludes the proof of (3.9).

Since the limit is unique, the whole sequence $u^T(x,t)$ strongly converges to $\bar{u}$:

$$u^T(x,t) \to \bar{u}, \qquad L^1_{loc}(\mathbf{R}^{n+1}_+), \quad T \to \infty.$$

This completes the proof of Theorem 3.1. $\square$

Now we show that the asymptotic decay of a periodic solution in the sense of Definition 3.1 coupled with an entropy inequality implies the asymptotic decay of the periodic solution in any L^p norm, $1 \le p < \infty$.

THEOREM 3.2. *Let system (2.1) be endowed with a strictly convex entropy η_*. Then the asymptotic decay of an L^∞ periodic entropy solution $u(x,t)$ of (2.1)-(2.2) to the mean $\bar{u}$ of $u_0(x)$ over the period P in the sense of Definition 3.1 implies its asymptotic decay in $L^p, 1 \le p < \infty$:*

$$\operatorname*{ess\,lim}_{t\to\infty} \int_P |u(x,t) - \bar{u}|^p dx = 0, \qquad \textit{for any} \quad 1 \le p < \infty. \tag{3.11}$$

PROOF. We assume that $P = [0,1]^n$ for simplicity. The periodic entropy solution $u(x,t)$ satisfies the entropy inequality (2.6) for the strictly convex entropy pair $(\eta_*(u), q_*(u))$. In (2.6), we use

$$\begin{aligned} &\eta_*(u,\bar{u}) = \eta_*(u) - \eta_*(\bar{u}) - \nabla\eta_*(\bar{u})(u - \bar{u}) \ge 0, \\ &q_*(u,\bar{u}) = q_*(u) - q_*(\bar{u}) - \nabla\eta_*(\bar{u})(f(u) - f(\bar{u})). \end{aligned}$$

For simplicity of notation, we set $\eta(u) = \eta_*(u,\bar{u})$ and $q(u) = q_*(u,\bar{u})$. By Definition 2.3 for the periodicity of $u(x,t)$ in x, given smooth functions $\rho(t) \in C_0^\infty(\mathbf{R}_+)$, $\zeta_1(x_1)$,$\cdots$, $\zeta_n(x_n) \in C_0^\infty(\mathbf{R})$, we have

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^n} q(u(x+\mathbf{e}_i,t))\rho(t)\zeta_1(x_1)\cdots\zeta_n(x_n)\, dx_1\cdots dx_i\cdots dx_n dt \\ &\qquad = \int_0^\infty \int_{\mathbf{R}^n} q(u(x,t))\rho(t)\zeta_1(x_1)\cdots\zeta_n(x_n)\, dx_1\cdots dx_i\cdots dx_n dt. \end{aligned} \tag{3.12}$$

We assume that a_i is a Lebesgue point of the functions

$$\int_0^\infty \int_{\mathbf{R}^{n-1}} q(u(x+\mathbf{e}_i,t))\rho(t)\zeta_1(x_1)\cdots\widehat{\zeta_i(x_i)}\cdots\zeta_n(x_n)\, dx_1\cdots\widehat{dx_i}\cdots dx_n dt,$$

and

$$\int_0^\infty \int_{\mathbf{R}^{n-1}} q(u(x,t))\rho(t)\zeta_1(x_1)\cdots\widehat{\zeta_i(x_i)}\cdots\zeta_n(x_n)\, dx_1\cdots\widehat{dx_i}\cdots dx_n dt,$$

when ρ, $\zeta_1, \cdots, \widehat{\zeta_i}, \cdots, \zeta_n$ run through a countable dense subset of $C_0^1(\mathbf{R})$, where the sign $\widehat{\ }$ indicates the omitted part. Equality (3.12) then implies

$$\begin{aligned}\int_0^\infty \int_{x_i=a_i} q(u(x+\mathbf{e}_i,t))\rho(t)\zeta_1(x_1)\cdots\widehat{\zeta_i(x_i)}\cdots\zeta_n(x_n)\,dx_1\cdots\widehat{dx_i}\cdots dx_n dt\\ =\int_0^\infty \int_{x_i=a_i} q(u(x,t))\rho(t)\zeta_1(x_1)\cdots\widehat{\zeta_i(x_i)}\cdots\zeta_n(x_n)\,dx_1\cdots\widehat{dx_i}\cdots dx_n dt,\end{aligned}$$

for such a_i, $i=1,\cdots,n$. Now, in (2.6), we choose $\phi(x,t)=\rho(t)\zeta_1(x_1)\cdots\zeta_n(x_n)$, with $\rho(t)$ and $\zeta_i(x_i)$ as above, and make $\zeta_i(x_i)$ approach the characteristic function $\chi_{[a_i,a_i+1]}$, $i=1,\cdots,n$. We then obtain

$$-\int_0^\infty\int_{a+P}\eta(u(x,t))\rho'(t)\,dxdt \le \int_{a+P}\eta(u_0(x))\rho(0)\,dx,$$

where $a=(a_1,\cdots,a_n)$ for a_i, $i=1,\cdots,n$, as above. Using the Dominated Convergence Theorem, we obtain that the above inequality holds for any $a\in\mathbf{R}^n$ and any ρ in the countable dense set mentioned above. Using the Dominated Convergence Theorem again, we realize that it holds for any nonnegative function $\rho\in C_0^1[0,\infty)$. Now, we assume that $0\le\tau_1<\tau_2$ are Lebesgue points of the functions

$$\int_{a+P}\eta(u(x,t))\,dx$$

when a runs through a countable dense subset of $\mathbf{R}^n$. Then we make $\rho(t)=\alpha^{\tau_2}(t)-\alpha^{\tau_1}(t)$, with $\alpha^\tau(t)=1-\int_0^t\alpha(s-\tau)\,ds$, where $\alpha(t)$ stands for a Dirac sequence, to arrive at

$$\int_{a+P}\eta(u(x,\tau_2))\,dx \le \int_{a+P}\eta(u(x,\tau_1))\,dx, \tag{3.13}$$

for all $0\le\tau_1<\tau_2, \tau_1,\tau_2\in\mathcal{T}$, where meas $((0,\infty)-\mathcal{T})=0$. Again the Dominated Convergence Theorem immediately implies that (3.13) holds for all $a\in\mathbf{R}$, for $\tau_1,\tau_2\in\mathcal{T}$.

Given $T>0, T\in\mathcal{T}$, we take all the right n-prisms given by $x\in a+P$, for $a\in\mathbf{Z}^n$, and $t\in[[rT]/(2r),T]$, in the interior of the cone $\{|x|\le rt, 0\le t\le T\}$. The number of such prisms is greater than $[rT]^n$. Using the periodicity of $u(x,t)$, inequality (3.13) with $\tau_2=T$, which holds for a.e. $t=\tau_1\in(0,T)$ over the period P, and the strict convexity of the entropy η, we obtain that there exists $c_0>0$, independent of T, such that

$$\begin{aligned}\frac{1}{T^{n+1}}\int_0^T\int_{|x|\le rt}\eta_*(u(x,t),\bar u)dxdt &\ge \frac{[rT]^n}{T^{n+1}}\int_{\frac{[rT]}{2r}}^T\int_P\eta_*(u(x,t),\bar u)dxdt\\ &\ge \frac{[rT]^n}{T^{n+1}}\int_{\frac{[rT]}{2r}}^T\int_P\eta_*(u(x,T),\bar u)dxdt\\ &\ge \frac{c_0}{T^{n+1}}T^{n+1}\int_P|u(x,T)-\bar u|^2dx,\end{aligned} \tag{3.14}$$

where we used $[a]$ as the largest integer less than or equal a.

Noting the uniform boundedness of the periodic solution, we have

(3.15)
$$\begin{aligned}\frac{1}{T^{n+1}}\int_0^T\int_{|x|\le rt}\eta_*(u(x,t),\bar{u})dxdt &\le C_1\frac{1}{T^{n+1}}\int_0^T\int_{|x|\le rt}|u(x,t)-\bar{u}|^2dxdt\\ &\le C_2\frac{n+1}{T^{n+1}}\int_0^T\int_{|\xi|\le r}|U(\xi,t)-\bar{u}|^2t^nd\xi dt\\ &\to 0, \qquad T\to+\infty.\end{aligned}$$

Combining (3.14) with (3.15), we obtain

$$\int_P|u(x,T)-\bar{u}|^2dx\to 0, \qquad \text{when } T\to\infty,\ T\in\mathcal{T}.$$

The boundedness of the periodic solution and the Hölder inequality yield (3.11) for all $p\in[1,\infty)$. This completes the proof of Theorem 3.2. □

REMARK 3.4. The first part of the above proof can be given in a simpler way with the aid of the theory of divergence-measure fields (see [CF4]).

REMARK 3.5. In Theorem 3.1, we assume that the self-similar scaling sequence $u^T(x,t)$ is compact in $L^1_{loc}(\mathbf{R}^{n+1}_+)$. Such a compactness can be achieved by the compensated-compactness method and other analytical techniques. See §3.3.

REMARK 3.6. In Theorem 3.1, we assume the existence of a periodic solution of (2.1)-(2.2), with periodic initial data. Such existence theorems of the Cauchy problem for (2.1) with L^∞ periodic data can be proved by using the compensated-compactness method and, in some cases, the Glimm scheme for genuinely nonlinear systems (see §3.3).

REMARK 3.7. In Theorems 3.1-3.2, the assumption $u(x,t)\in L^\infty(\mathbf{R}^{n+1}_+)$ can be replaced by $u(x,t)\in L^q(\mathbf{R}^{n+1}_+), q>2$. Then the asymptotic decay of an L^q periodic entropy solution $u(x,t)$ of (2.1)-(2.2), with period P, in the sense of Definition 3.1 in the L^2 norm implies (3.11) for any $p\in[2,q)$.

3.3. Compactness of the Self-Similar Scaling Sequence

Now we explore possible ways to obtain the L^1_{loc}-compactness of the self-similar scaling sequence $u^T(x,t)$. We first introduce the following embedding lemmas.

LEMMA 3.1. *Let $1<q\le p<r\le\infty$. Then*

$$\begin{aligned}&(\textit{compact set of } W^{-1,q}_{loc}(\mathbf{R}^k))\cap(\textit{bounded set of } W^{-1,r}_{loc}(\mathbf{R}^k))\\ &\qquad\subset(\textit{compact set of } W^{-1,p}_{loc}(\mathbf{R}^k)).\end{aligned}$$

LEMMA 3.2. *The embedding of the positive cone of $W^{-1,q}(\Omega)$ in $W^{-1,p}(\Omega)$ is completely continuous for $p < q$, for any $\Omega \Subset \mathbf{R}^k$.*

The proof of Lemma 3.1 can be found in [C1,DC1]. Lemma 3.2 is Murat's lemma [Mu1]. Lemma 3.1 says that the compactness in $W^{-1,q}_{loc}(\mathbf{R}^k)$ coupled with boundedness in $W^{-1,r}_{loc}(\mathbf{R}^k)$ yields compactness in $W^{-1,p}_{loc}(\mathbf{R}^k)$ for $1 < q \le p < r \le \infty$. Lemma 3.2 is used later for $p \in [1,\infty)$ and $q = \infty$.

THEOREM 3.3. *Consider a hyperbolic system of conservation laws (2.1) with a strictly convex entropy pair (η_*, q_*). Assume that the uniformly bounded sequence $u^T(x,t) \in L^\infty(\mathbf{R}^{n+1}_+)$ satisfies*

$$\partial_t \eta(u^T) + \nabla_x \cdot q(u^T) \le 0 \tag{3.16}$$

in the sense of distributions for any <u>convex</u> entropy pair $(\eta, q) \in \Lambda$, where Λ is a linear space of entropy pairs of (2.1) including (η_*, q_*). Then

$$\partial_t \eta(u^T) + \nabla_x \cdot q(u^T) \qquad \text{is compact in } W^{-1,p}_{loc}(\mathbf{R}^{n+1}_+), \quad p \in (1,\infty), \tag{3.17}$$

for <u>any</u> entropy pair $(\eta, q) \in \Lambda$ satisfying $|\nabla^2 \eta(u)| \le C_\eta \nabla^2 \eta_*(u)$ when $u \in K \Subset \mathbf{R}^m$ for some C_η depending only on η and the compact set K.

PROOF. Since the sequence $u^T(x,t)$ is uniformly bounded in $L^\infty(\mathbf{R}^{n+1}_+)$, then

$$\mu^T_* \equiv -\mathrm{div}_{\,t,x}(\eta_*(u^T), q_*(u^T)) \quad \text{is a bounded subset of } \ W^{-1,\infty}_{loc}(\mathbf{R}^{n+1}_+).$$

Then Murat's lemma (Lemma 3.2) indicates that

$$\mu^T_* \quad \text{is compact in } W^{-1,p}_{loc}(\mathbf{R}^{n+1}_+), \ p \in (1,\infty).$$

Analogously, for any <u>convex</u> entropy pair $(\eta, q) \in \Lambda$, (3.17) holds.

Next, using an idea in [C4], for any (not necessarily convex) entropy pair $(\eta(u), q(u)) \in \Lambda$, satisfying $|\nabla^2 \eta(u)| \le C_\eta \nabla^2 \eta_*(u)$, we use the fact that $(\eta + C_\eta \eta_*, q + C_\eta q_*)$ is a convex entropy pair. Setting

$$\mu^T_\eta \equiv -\mathrm{div}_{\,t,x}(\eta(u^T), q(u^T)),$$

we then get that $\mu^T_\eta + C_\eta \mu^T_*$ is compact in $W^{-1,p}_{loc}(\mathbf{R}^{n+1}_+)$. By linearity, we conclude that (3.17) holds. □

Theorem 3.3 is useful for proving the compactness of the self-similar scaling sequence $u^T(x,t)$ in Theorem 3.1 for the asymptotic decay of periodic entropy solutions in $L^\infty(\mathbf{R}^{n+1}_+)$ for (2.1).

As a corollary, we can obtain the existence of L^∞ solutions for the Cauchy problem with arbitrary L^∞ initial data (without the $L^2(\mathbf{R})$ assumption).

Theorem 3.4. *Consider the Cauchy problem (2.1)–(2.2) in one dimension with a strictly convex entropy (η_*, q_*). Let system (2.1) satisfy the following conditions.*

(i) *For any L^∞ (or BV) initial data $u_0(x)$ such that $u_0(x) - \bar{u}$ has compact support, for some $\bar{u}$, there always exists an L^∞ entropy solution $u(x,t)$ such that*

$$\|u\|_{L^\infty} \le C\|u_0\|_{L^\infty},$$

where C depends only on the flux function $f(u)$.

(ii) *There exists a certain class Λ of entropy pairs such that*

(a). For any $(\eta, q) \in \Lambda$,

$$|\nabla^2 \eta(u)| \le C_\eta \nabla^2 \eta_*(u),$$

where $u \in K \Subset \mathbf{R}^m$, for some C_η depending on η and K.

(b). For any uniformly bounded sequence $u^T(x,t)$, the condition that

$$\partial_t \eta(u^T) + \partial_x q(u^T) \qquad \text{is compact in } H^{-1}_{loc}(\mathbf{R}^2_+), \text{ for all } (\eta, q) \in \Lambda,$$

implies the compactness of $u^T(x,t)$ in $L^1_{loc}(\mathbf{R}^2_+)$.

Then there exists a global entropy solution for any L^∞ initial data $u_0(x)$.

Proof. For any L^∞ initial data $u_0(x)$, there exists an approximate sequence of L^∞ or BV functions $u_0^\varepsilon(x)$ such that

$$\begin{cases} u_0^\varepsilon(x) - \bar{u} & \text{have compact support,} \\ u_0^\varepsilon(x) \longrightarrow u_0(x) & \text{a.e.} \end{cases}$$

Assumption (i) indicates that there exist L^∞ entropy solutions $u^\varepsilon(x,t)$ with corresponding initial data $u_0^\varepsilon(x)$ satisfying

$$\begin{aligned} &|u^\varepsilon(x,t)| \le C, \qquad C \text{ independent of } \varepsilon, \\ &\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \le 0, \qquad \text{for any convex } (\eta, q) \in \Lambda. \end{aligned}$$

Following the proof of Theorem 3.3, we conclude that, for any entropy pair $(\eta, q) \in \Lambda$,

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \qquad \text{is compact in } H^{-1}_{loc}(\mathbf{R}^2_+).$$

Assumption (ii) implies the compactness of $u^\varepsilon(x,t)$ in $L^1_{loc}(\mathbf{R}^2_+)$. That is, there exists a subsequence of $u^\varepsilon(x,t)$ converging to an L^∞ function $u(x,t)$ almost everywhere. Using the standard procedures, one can verify that the limit function $u(x,t)$ is an entropy solution with L^∞ initial data $u_0(x)$ of arbitrarily large oscillation. $\square$

The compactness of the self-similar scaling sequence $u^T(x,t)$ satisfying (3.17) in Theorem 3.3 and assumption (ii) of Theorem 3.4 are corollaries of the compactness of solution operators of hyperbolic conservation laws. The compactness of solution operators can be achieved by the compensated-compactness method, the averaging lemma, and other analytical techniques. The compensated-compactness method, first introduced by Tartar [T] and Murat [Mu2] and a related observation by Ball [B], is one of the efficient methods to achieve this.

Theorem 3.5 [T]. *Suppose that* $u^T : \mathbf{R}^2_+ \to \mathbf{R}^m$ *is a sequence of bounded measurable functions*

$$u^T(x,t) \in K, \qquad a.e. \tag{3.18}$$

for a bounded set $K \subset \mathbf{R}^m$ *and that, for function pairs* (η_i, q_i), $i = 1, 2$,

$$\partial_t \eta_i(u^T) + \partial_x q_i(u^T) \qquad \text{is compact in } H^{-1}_{loc}(\mathbf{R}^2_+). \tag{3.19}$$

Then there exist a subsequence (still labeled) u^T *and Young measures* $\nu_{x,t}(\omega) \in \mathrm{Prob.}(\mathbf{R}^m)$ *such that*

(i) *for any continuous function* g, $w^* - \lim\ g(u^T) =< \nu_{x,t}(\omega), g(\omega) >$, *and*

$$< \nu_{x,t}, \eta_1 q_2 - \eta_2 q_1 >=< \nu_{x,t}, \eta_1 >< \nu_{x,t}, q_2 > - < \nu_{x,t}, \eta_2 >< \nu_{x,t}, q_1 >; \tag{3.20}$$

(ii) $u^T(x,t) \to u(x,t)$ *a.e. if and only if* $\nu_{x,t}$ *is a Dirac mass:*

$$\nu_{x,t} = \delta_{u(x,t)}, \qquad \text{for almost all } (x,t).$$

This theorem provides a framework by which one can prove strong convergence of the sequence $u^T(x,t)$ satisfying (3.18)-(3.19), by deducing $\nu_{x,t}(\omega) = \delta_{u(x,t)}(\omega)$ from the functional relation (3.20), where δ_v denotes the Dirac measure concentrated in v and $u(x,t) = w^* - \lim u^T(x,t)$.

Remark 3.8. With the aid of Theorem 3.4, some existence theorems, obtained by the compensated-compactness method with initial assumption $u_0(x) \in L^\infty \cap L^2(\mathbf{R})$ and by the Glimm method with BV initial assumption, for systems with bounded invariant regions, can be extended to the corresponding existence theorems without such initial assumptions. That is, the existence theorems also hold for any L^∞ initial data for some systems. This extension can be achieved by approximating L^∞ initial data $u_0(x) \in L^\infty(\mathbf{R})$ by $L^\infty \cap L^2(\mathbf{R})$ or $L^\infty \cap BV(\mathbf{R})$ in the sense of convergence almost everywhere.

Such existence theorems for systems include 2×2 strictly hyperbolic and genuinely nonlinear systems as well as the equations of elasticity [Di1,Di2], the Euler equations for isentropic flow (see [DC1,DC2] for $\gamma = 3/2$, [C1,C2] for $1 < \gamma \le 5/3$, [LP2] for $\gamma \ge 3$, [LP3] for $5/3 < \gamma < 3$, and [CL] for a general pressure law), hyperbolic systems with umbilic degeneracy [CK1,CK2], the chromatography system [JP], and the 3×3 Euler system of thermoelasticity [CD].

Remark 3.9. Similar frameworks can be established for the corresponding conservation laws with viscosity in a straightforward manner.

Remark 3.10. The approach and ideas we developed above have been applied to studying the large-time behavior of entropy solutions of hyperbolic conservation laws with general initial data. See [CF2,CF3] for the details.

4. Multidimensional Scalar Conservation Laws

In this section we study the asymptotic decay of entropy solutions of multidimensional scalar conservation laws with periodic initial data:

$$\begin{cases} \partial_t u + \nabla_x \cdot f(u) = 0, & x \in \mathbf{R}^n,\ t > 0, \\ u\big|_{t=0} = u_0(x), \end{cases} \tag{4.1}$$

where $u \in \mathbf{R}$ and $f(u) = (f_1(u), \cdots, f_n(u))$ is a vector function in $C^2(\mathbf{R}; \mathbf{R}^n)$, and $u_0(x) \in L^\infty(\mathbf{R}^n)$ is a periodic function with period $P = \Pi_{i=1}^n [0, p_i]$.

The existence of global entropy solutions of (4.1) is well-known when $u_0 \in L^\infty(\mathbf{R}^n)$, which was first proved by Kruzkov [Kr] by improving an earlier result of Volpert [Vo] for $u_0 \in BV(\mathbf{R}^n)$. We now apply the approach established in Section 3 to analyze the asymptotic behavior of periodic solutions.

Theorem 4.1. *Assume that the flux function $f(u)$ satisfies*

$$\begin{aligned} &\text{meas}\,\{\, v \in \mathbf{R} \mid \tau + f'(v) \cdot k = 0 \,\} = 0, \\ &\qquad \textit{for any } (\tau, k) \in \mathbf{R} \times \mathbf{R}^n, \textit{ with } \tau^2 + |k|^2 = 1. \end{aligned} \tag{4.2}$$

Then any L^∞ entropy solution operator is compact in $L^1_{\text{loc}}(\mathbf{R}^n \times (0, T))$.

This theorem is due to Lions-Perthame-Tadmor [LP1]. A direct proof can be found in [CF3].

Remark 4.1. Condition (4.2) is implied by the following generalized genuine nonlinearity condition:

$$\text{meas}\{\, v \in \mathbf{R} \mid k \cdot f''(v) = 0 \,\} = 0, \qquad \text{for all } |k| = 1. \tag{4.3}$$

This is a consequence of the fact that the derivative of $\tau + k \cdot f'(v)$, viewed as a function of v, *i.e.*, $k \cdot f''(v)$, equals 0 a.e. on the set where $\tau + k \cdot f'(v) = 0$, from a well-known result of real analysis (see, *e.g.*, [Ev]). As a simple example, when $n = 2$, one easily verifies that the function $f(v) = (|v|^{p+2}, |v|^{q+2})$, with $p \neq q$, $p, q > 0$, satisfies condition (4.3) and hence (4.2).

We then have the following immediate corollary of Theorem 4.1.

Corollary 4.1. *Let $u(x,t)$ be the entropy solution of* (4.1) *in $\mathbf{R}^{n+1}_+$. Assume that $u_0(x)$ is periodic and the flux function $f(u)$ satisfies* (4.2). *Then the self-similar scaling sequence $u^T(x,t) = u(Tx, Tt)$ is compact in $L^1_{\text{loc}}(\mathbf{R}^n \times (0, \infty))$.*

Proof. Since $u(x,t) \in L^\infty(\mathbf{R}^{n+1}_+)$, then

$$\|u^T\|_{L^\infty} \le C < \infty,$$

where C is a constant, independent of T.

Since $u(x,t)$ is a periodic entropy solution, it satisfies

$$\partial_t \eta(u) + \nabla_x \cdot q(u) \le 0$$

in the sense of distributions, for any convex entropy pair (η, q). Thus u^T also satisfies

$$\partial_t \eta(u^T) + \nabla_x \cdot q(u^T) \le 0,$$

which implies that $u^T(x,t)$ is a sequence of entropy solutions of (4.1) in $\mathbf{R}^{n+1}_+$ with oscillatory initial data $u_0(Tx)$. Theorem 4.1 implies the result we expected. □

Corollary 4.1 together with Theorems 3.1-3.2 yields the following decay result.

THEOREM 4.2. *Let $u(x,t)$ be an entropy solution of* (4.1) *in $\mathbf{R}^{n+1}_+$ with periodic data $u_0(x)$. Assume that condition* (4.2) *holds. Then*

$$\operatorname*{ess\,lim}_{t\to\infty} \|u(\cdot,t) - \frac{1}{|P|}\int_P u_0(x)dx\|_{L^p(P)} = 0, \qquad \textit{for } 1 \le p < \infty.$$

5. Hyperbolic Systems of Conservation Laws

In this section, we study the asymptotic behavior of periodic entropy solutions of hyperbolic systems (2.1) using the approach we developed in Section 3.

5.1. 2×2 Strictly Hyperbolic Systems

Consider a 2×2 strictly hyperbolic and genuinely nonlinear system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u \in \mathbf{R}^2, \tag{5.1}$$

with periodic initial data

$$u|_{t=0} = u_0(x) \in L^\infty(\mathbf{R}). \tag{5.2}$$

We will show that any L^∞ periodic entropy solution of such a system asymptotically decays in L^p, $1 \le p < \infty$. We will assume neither the small oscillation and the local bounded variation of the periodic solution nor further conditions on the system.

THEOREM 5.1. *Let $u(x,t) \in L^\infty(\mathbf{R}^2_+)$ be a periodic entropy solution of (5.1)-(5.2) with period P. Then $u(x,t)$ asymptotically decays to $\bar u = \frac{1}{|P|}\int_P u_0(x)dx$ in L^p, $1 \le p < \infty$, in the sense of (3.11).*

PROOF. From Theorems 3.1-3.2, it suffices to prove that the corresponding self-similar scaling sequence $u^T(x,t) = u(Tx, Tt)$ is compact in $L^1_{loc}(\mathbf{R}^2_+)$. As in the arguments in the proof of Corollary 4.1, the sequence $u^T(x,t)$ satisfies

$$\|u^T\|_{L^\infty} \le C < \infty, \tag{5.3}$$

for a constant C, independent of T, and

$$\partial_t \eta(u^T) + \partial_x q(u^T) \le 0 \tag{5.4}$$

in the sense of distributions for any convex entropy pair (η, q).

Lax's theorem [L3] indicates that, for any 2×2 strictly hyperbolic and genuinely nonlinear system, on any bounded domain in the u-plane, there exists a strictly convex entropy pair (η_*, q_*). Then (5.3)-(5.4) imply that

$$\partial_t \eta(u^T) + \partial_x q(u^T) \qquad \text{is compact in } H^{-1}_{loc}(\mathbf{R}^2_+),$$

for any C^2 entropy pair (η, q) from Theorem 3.4. Then DiPerna's theorem [Di1] and Theorem 3.5 imply the L^1_{loc}-compactness of the self-similar scaling sequence $u^T(x,t)$. This completes the proof with the aid of Theorems 3.1-3.2. □

The genuine nonlinearity condition of system (5.1) can be relaxed to allow some reflection points. A typical example is given by the equations of elasticity:

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v - \partial_x \sigma(\tau) = 0, \qquad \sigma(\tau) \in C^2. \end{cases} \tag{5.5}$$

System (5.5) is strictly hyperbolic provided that $\sigma'(\tau) > 0$. However, the genuine nonlinearity is typically precluded by the fact that the medium in question can sustain discontinuities in both the compressive and expansive phases of the motion. In the simplest model for common rubber, one postulates that the stress σ as a function of the strain τ switches from concave in the compressive mode $\tau < 0$ to convex in the expansive mode $\tau > 0$, that is, $\mathrm{sign}(\tau\sigma''(\tau)) > 0, \tau \neq 0$. For this case, DiPerna's existence theorem [Di1] implies the existence of an $L^\infty(\mathbf{R}^2_+)$ periodic entropy solution $(\tau(x,t), v(x,t))$ with L^∞ periodic initial data $(\tau_0(x), v_0(x))$, following the arguments in §3.3. Employing similar arguments in the proof of Theorem 5.1, we have

Theorem 5.2. *Let $(\tau(x,t), v(x,t)) \in L^\infty(\mathbf{R}^2_+)$ be a periodic entropy solution of the equations of elasticity (5.5) with period P. Then $(\tau(x,t), v(x,t))$ asymptotically decays to $(\bar{\tau}, \bar{v}) = (\frac{1}{|P|}\int_P \tau_0(x)dx, \frac{1}{|P|}\int_P v_0(x)dx)$ in L^p, $1 \le p < \infty$, in the sense of (3.11) with $u(x,t) = (\tau(x,t), v(x,t))$.*

Remark 5.1. For system (5.5) with $\sigma(\tau) \notin C^2$, Greenberg-Rascle [GR] constructed a counterexample that the periodic solution $(\tau(x,t), v(x,t))$ does not asymptotically decay in $L^p, 1 \le p < \infty$. This fact indicates that the asymptotic behavior of entropy solutions is sensitive to the regularity of the flux function of the corresponding system.

5.2. Isentropic Euler Equations

Consider the Euler equations for isentropic flow of compressible fluids:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x(\frac{m^2}{\rho} + p(\rho)) = 0, \end{cases} \tag{5.6}$$

where ρ, m, and p are the density, the momentum, and the pressure, respectively. In the non-vacuum state $(\rho \neq 0)$, $v = m/\rho$ is the velocity. The pressure $p(\rho)$ is a given function of the density ρ depending on the compressible fluids under consideration. For the polytropic case, $p(\rho) = k^2\rho^\gamma$, $\gamma > 1$.

Consider the Cauchy problem for (5.6) with the initial data

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)), \qquad 0 \le \rho_0(x) \le C_0,\ |\frac{m_0(x)}{\rho_0(x)}| \le C_0 < \infty. \tag{5.7}$$

The main difficulty of this system is that strict hyperbolicity fails, the eigenvalues coalesce, and the flux function is only Lipschitz continuous on the vacuum $\rho = 0$, which cannot be avoided in general. This degeneracy is parabolic: $\nabla f(\rho, m)$ is not diagonalizable. Nevertheless, a similar compactness theorem has been established by using only weak entropy pairs, a subspace of entropy pairs, consisting of those η vanishing on the vacuum $\rho = 0$ for any fixed $m/\rho \in (-\infty, \infty)$. For example, the mechanical energy-energy flux pair

$$\eta_* = \frac{1}{2}\frac{m^2}{\rho} + \rho \int_0^\rho \frac{p(s)}{s^2} ds, \qquad q_* = m\left(\frac{1}{2}\frac{m^2}{\rho^2} + \rho \int_0^\rho \frac{p'(s)}{s} ds\right) \tag{5.8}$$

is a convex weak entropy pair. One can prove that, for $0 \le \rho \le C$, $|m/\rho| \le C$,

$$|\nabla \eta(u)| \le C_\eta, \qquad |\nabla^2 \eta(u)| \le C_\eta \nabla^2 \eta_*(u), \tag{5.9}$$

for any weak entropy η, where the constant C_η is independent of u. For this system, we have the following theorem.

THEOREM 5.3. *(a) There exists a global solution $u(x,t) = (\rho(x,t), m(x,t))$ of the Cauchy problem (5.6)-(5.7), satisfying $0 \le \rho(x,t) \le C, |m(x,t)/\rho(x,t)| \le C$, for some C depending only on C_0 and p, and*

$$\partial_t \eta(u) + \partial_x q(u) \le 0 \tag{5.10}$$

in the sense of distributions for any convex weak entropy pairs (η, q).

(b) Suppose that the sequence $u^T = (\rho^T, m^T)$ satisfies the following.

(1) *There is a constant $C > 0$ such that*

$$0 \le \rho^T(x,t) \le C, \qquad \left|m^T(x,t)/\rho^T(x,t)\right| \le C. \tag{5.11}$$

(2) *For all weak entropy pairs (η, q), the measure sequence*

$$\partial_t \eta(u^T) + \partial_x q(u^T) \quad \textit{is compact in } H^{-1}_{\mathrm{loc}}(\mathbf{R}^2_+). \tag{5.12}$$

Then the sequence $(\rho^T(x,t), m^T(x,t))$ is compact in $L^1_{loc}(\mathbf{R}^2_+)$.

This theorem was proved by DiPerna [Di2] for the case $\gamma = 1 + \frac{1}{2m+1}, m \ge 2$ an integer, for $L^2 \cap L^\infty(\mathbf{R})$ initial data, by Ding-Chen-Luo [DC1] for the case $\gamma = 3/2$, and by Chen [C2] for the general case $1 < \gamma \le 5/3$ for usual gases with general L^∞ initial data. This theorem was also proved for the case $\gamma \ge 3$ by Lions-Perthame-Tadmor [LP2] and for the case $5/3 < \gamma < 3$ by Lions-Perthame-Souganidis [LP3]. Recently, this theorem was further proved by Chen-LeFloch [CL] for a general pressure law $p(\rho)$. With the aid of this theorem, we conclude the following.

THEOREM 5.4. *Let* $(\rho(x,t), m(x,t)), 0 \leq \rho(x,t) \leq C, |m(x,t)/\rho(x,t)| \leq C$, *be a periodic entropy solution of (5.6)-(5.7) with period* P. *Then* $(\rho(x,t), m(x,t))$ *asymptotically decays to* $\frac{1}{|P|}\int_P (\rho_0(x), m_0(x))dx$ *in the sense of (3.11).*

PROOF. It suffices to prove that the corresponding self-similar scaling sequence $(\rho^T(x,t), m^T(x,t)) = (\rho(Tx,Tt), m(Tx,Tt))$ satisfies (5.11) and (5.12).

Condition (5.11) is trivial. From (5.10), we have that (ρ^T, m^T) satisfy

$$\partial_t \eta(\rho^T, m^T) + \partial_x q(\rho^T, m^T) \leq 0$$

in the sense of distributions, for any *convex* weak entropy pair (η, q). Then, using (5.9) and Theorem 3.3, we conclude that $u^T(x,t)$ satisfies (5.12). □

REMARK 5.2. The same analysis can be applied to proving the asymptotic decay of periodic solutions for the viscous system of isentropic gas dynamics:

$$\begin{cases} \partial_t \rho + \partial_x m = \partial_{xx}\rho, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p(\rho)\right) = \partial_{xx} m. \end{cases}$$

In this system the flux function is only Lipschitz continuous near the vacuum. The standard energy estimates can not be used to obtain the results we expected for the viscous case because the density is not uniformly bounded away from the vacuum for $t \in [0, \infty)$. This case can be handled in the same fashion as the hyperbolic case with the aid of our approach without further difficulty.

REMARK 5.3. The same decay result is true for the chromatography system, considered by James-Peng-Perthame [JP] with the same type of parabolic degeneracy, by using our approach.

5.3. Hyperbolic Systems with Umbilic Degeneracy

Now we study the asymptotic behavior of entropy solutions for hyperbolic systems of conservation laws with hyperbolic degeneracy, that is, the Jacobian $\nabla f(u)$ is diagonalizable. A typical example is the class of quadratic systems with umbilic degeneracy:

$$\partial_t u + \partial_x(\nabla C(u)) = 0, \qquad u = (u_1, u_2) \in \mathbf{R}^2, \tag{5.13}$$

where

$$C(u) = \frac{1}{2}(\frac{1}{3}au_1^3 + bu_1^2 u_2 + u_1 u_2^2), \tag{5.14}$$

and a and b are real parameters with $a \neq 1 + b^2$.

This class of systems is generic in the following sense. Consider a hyperbolic system of conservation laws $\partial_t u + \partial_x f(u) = 0, u \in \mathbf{R}^2$, with an isolated umbilic point u_0. That is, $\nabla f(u_0)$ is diagonalizable, and there is a neighborhood $\mathcal{N}$ of u_0 such that $\nabla f_T(u)$ has distinct eigenvalues for all $u \in \mathcal{N} - u_0$, where

$$\nabla f_T(u) = f(u_0) + \nabla f(u_0)(u - u_0) + \frac{1}{2}(u - u_0)^\top \nabla^2 f(u_0)(u - u_0). \tag{5.15}$$

Take the Taylor expansion for $f(u)$ around $u = u_0$:

$$f(u) = \nabla f_T(u) + h.o.t.,$$

where $h.o.t.$ represents the remainder. The flux function $\nabla f_T(u)$ determines the local behavior of the hyperbolic singularity near the umbilic point u_0. Since $\nabla f(u_0)$ is diagonalizable, we can make a coordinate transformation to eliminate the linear term from (5.15) and relabel $u - u_0$ as u to obtain

$$\partial_t u + \partial_x Q(u) = 0, \tag{5.16}$$

where $Q(u) = \frac{1}{2} u^\top \nabla^2 f(u_0) u$. From the normal form theorem in [SS], there is a nonsingular linear coordinate transformation to transform system (5.16) into (5.13)-(5.14) in which a and b are real parameters with $a \neq 1 + b^2$.

It is easy to check that, as long as $a \neq 1 + b^2$, $\lambda_1 = \lambda_2$ if and only if $u = (0,0)$ so that $(0,0)$ is the unique umbilic point with hyperbolic degeneracy for (5.13)-(5.14). Even so, a compactness theorem has been established by Chen-Kan [CK1] by using C^2 entropy pairs, especially at the umbilic point. A special entropy is $\eta_*(u) = u_1^2 + u_2^2$. In general, the entropy function $\eta(u)$ may not be in C^2 (see [CK1]). However, any C^2 entropy can be controlled by $\eta_*(u)$, that is, $|\nabla^2 \eta(u)| \leq C_\eta \nabla^2 \eta_*(u)$, for any C^2 entropy η.

More precisely, we consider the Cauchy problem (5.13)-(5.14) for the case $a > 1 + b^2$:

$$u|_{t=0} = u_0(x), \qquad u_0(x) \in \mathbf{R}^2_+ \equiv \{u \mid u_2 \geq 0\}, \qquad |u_0(x)| \leq C_0 < \infty. \tag{5.17}$$

Then we have the following theorem.

Theorem 5.5. *(a) There exists a global solution $u(x,t)$ of the Cauchy problem (5.13) and (5.17) satisfying*

$$u(x,t) \in \mathbf{R}^2_+, \qquad |u(x,t)| \leq C, \tag{5.18}$$

for some C depending only on C_0, a, and b.

(b) The solution operator $u(\cdot,t) = S_t u_0(\cdot)$, defined by (a), is compact in $L^1_{loc}(\mathbf{R} \times (0,\infty))$.

This theorem is proved by Chen-Kan [CK1]. Using our approach and this compactness theorem, we immediately have the following theorem.

Theorem 5.6. *Let $u(x,t)$ be a periodic entropy solution of (5.13) and (5.17), $a > 1 + b^2$, with period P satisfying (5.18). Then $u(x,t)$ asymptotically decays to $\bar{u} = \frac{1}{|P|} \int_P u_0(x) dx$ in L^p, $1 \leq p < \infty$.*

Remark 5.4. The decay result is also true for system (5.13) with $a \leq 1 + b^2$. For more details, see Chen-Kan [CK2].

Remark 5.5. Similarly, Theorem 5.6 can include the systems of the conjugate type in Frid-Santos [FS1] and Rubino [Ru].

REMARK 5.6. The decay in $L^p, 1 \leq p < 2$, in the sense of Definition 3.1, of periodic entropy solutions of the system $z_t - (\bar{z}^\gamma)_x = 0$, with $1 \leq \gamma < 2$, which approximate the complex Burgers' equation with zero viscosity, $z_t - (\bar{z}^2)_x = 0$, can be obtained from the compactness theorem established in [FS2] (see also [Fr1]).

5.4. 3×3 **Euler Equations in Thermoelasticity**

We now consider hyperbolic systems of conservation laws with linear degeneracy, that is, systems in which at least one characteristic field of the systems is linearly degenerate. A typical example is the 3×3 system of Euler equations in thermoelasticity:

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t (e + \frac{1}{2} v^2) + \partial_x (vp) = 0, \end{cases} \tag{5.19}$$

where τ, v, p, and e denote respectively the deformation gradient (the specific volume for fluids, the strain for solids), the velocity, the pressure, and the internal energy. Other relevant fields are the entropy s and the temperature θ. System (5.19) is complemented by the Clausius inequality

$$\partial_t s \geq 0, \tag{5.20}$$

which expresses the second law of thermodynamics.

We consider the following class of constitutive relations for the new state vector (w, v, s) with the form

$$\tau = w + \alpha s, \quad p = h(w), \quad e = -\int^w h(\omega) d\omega + \beta s, \quad \theta = \alpha h(w) + \beta, \tag{5.21}$$

where h is a smooth function with $h'(w) < 0$ satisfying

$$h''(w) - 4 \frac{\alpha h'(w)^2}{\alpha h(w) + \beta} \begin{cases} > 0, & \text{if } w < \hat{w}, \\ < 0, & \text{if } w > \hat{w}, \end{cases} \tag{5.22}$$

and α and β are positive constants. Observe that equations (5.21) are compatible with the thermodynamic relation

$$\theta \, ds = de + p \, dv.$$

The model (5.21) can be regarded as a "first-order correction" to the general constitutive equations (see Chen-Dafermos [CD] for the details).

Consider the Cauchy problem for (5.19) with periodic initial data

$$(w, v, s)|_{t=0} = (w_0(x), v_0(x), s_0(x)), \tag{5.23}$$

satisfying

$$|w_0(x)| \leq C_0, \qquad |v_0(x)| \leq C_0, \qquad s_0(x) \in \mathcal{M}_{loc}(\mathbf{R}), \tag{5.24}$$

and

$$(w_0(x), v_0(x)) \in \Sigma_{C_1} \equiv \{(w, v) \,|\, -C_1 \leq v \pm \int_{\hat{w}}^w \sqrt{-h'(\omega)} d\omega \leq C_1\}, \tag{5.25}$$

which contains only physically admissible states. In particular, if $(w, v) \in \Sigma_{C_1}$, then $\theta = \alpha h(w) + \beta > 0$.

THEOREM 5.7. *(a) There exists a global periodic distributional solution* $(w(x,t), v(x,t), s(x,t))$ *for the Cauchy problem (5.19) and (5.23)-(5.25), satisfying*

$$(w,v) \in L^\infty(\mathbf{R}^2_+), \quad (s, s_t) \in \mathcal{M}_{loc}(\mathbf{R}^2_+), \quad \theta(w(x,t)) \geq 0, \tag{5.26}$$

$$|s|\{[-cT_0, cT_0] \times [0, T_0]\} \leq CT_0^2, \tag{5.27}$$

for any $c > 0, T_0 > 0$*, with* $C > 0$ *independent of* T_0*, where* $|s|$ *denotes the variation measure associated with the signed measure* s*. Moreover,* $(w(x,t), v(x,t), s(x,t))$ *satisfies the entropy condition:*

$$\partial_t \eta(w,v) + \partial_x q(w,v) \leq 0, \qquad s_t \geq 0, \tag{5.28}$$

in the sense of distributions for any C^2 *entropy pair* $(\eta(v,w), q(v,w))$ *of the prototypical system*

$$\partial_t w - \partial_x v = 0, \qquad \partial_t v + \partial_x h(w) = 0, \tag{5.29}$$

for which the strong convexity condition holds:
(5.30)
$$\theta\eta_{ww} - \alpha h'(w)\eta_w \geq 0, \, \theta\eta_{vv} + \alpha\eta_w \geq 0, \, (\theta\eta_{ww} - \alpha h'(w)\eta_w)(\theta\eta_{vv} + \alpha\eta_w) - \eta_{wv}^2 \geq 0.$$

(b) Assume that the sequence $(w^T(x,t), v^T(x,t))$ *satisfies the following:*

(1) *There exists a constant* $C > 0$ *such that*

$$\|(w^T, v^T)\|_{L^\infty} \leq C. \tag{5.31}$$

(2) *The sequence*

$$\partial_t \eta(w^T, v^T) + \partial_x q(w^T, v^T) \leq 0 \qquad \textit{in the sense of distributions}, \tag{5.32}$$

for any C^2 *convex entropy pair* $(\eta(w,v), q(w,v))$ *of system (5.29) satisfying (5.30).*

Then the sequence $(w^T(x,t), v^T(x,t))$ *is compact in* $L^1_{loc}(\mathbf{R}^2_+)$*.*

This theorem was proved by Chen-Dafermos [CD] by using the vanishing viscosity method and the arguments as in Section 3. Condition (5.30) is a strong version of the convexity of entropy functions. In particular, the pair

$$\eta_*(w,v) = \frac{1}{2}v^2 - \int^w h(\omega)d\omega, \quad q_* = vh(w)$$

is a convex entropy pair satisfying (5.30).

THEOREM 5.8. *Let* $(\tau(x,t), v(x,t), s(x,t))$, $|v(x,t)| + |\tau(x,t) - \alpha s(x,t)| \le C$, *be a periodic entropy solution of (5.19) and (5.23)-(5.25) with period* P *satisfying (5.26)-(5.27). Then the velocity* $v(x,t)$ *asymptotically decays to* $\bar{v} = \frac{1}{|P|}\int_P v_0(x)dx$ *in* L^p, $1 \le p < \infty$. *Moreover, the pressure* $p(w(x,t))$ *and the temperature* $\theta(w(x,t))$ *decay to*

$$\tilde{p} = p\left(\Theta^{-1}(\frac{1}{|P|}\int_P \Theta(w_0(x))dx)\right), \ \textit{and}\ \tilde{\theta} = \theta\left(\Theta^{-1}(\frac{1}{|P|}\int_P \Theta(w_0(x))dx)\right),$$

in $L^p, 1 \le p < \infty$, *respectively, where* $\Theta(w) = \beta w + \alpha\int_0^w h(\omega)d\omega$.

PROOF. Let $(\tau^T(x,t), v^T(x,t), s^T(x,t))$ be the scaling sequence associated with the periodic solution $(\tau(x,t), v(x,t), s(x,t))$, where now the scaling of $s(x,t)$ must be taken in the sense of distributions. By using rescaling arguments, it is not difficult to verify that (τ^T, v^T, s^T) are also periodic solutions of (5.19) satisfying (5.31)–(5.32). Using Theorem 5.7, we conclude that

$$(w^T(x,t), v^T(x,t)) \qquad \text{is compact in } L^1_{loc}(\mathbf{R}^2_+).$$

From the uniform boundedness of (w^T, v^T), we have that there exists a subsequence $\{T_k\}_{k=1}^\infty$, $T_k \to \infty$ as $k \to \infty$, such that

$$(w^{T_k}(x,t), v^{T_k}(x,t)) \to (w(x,t), v(x,t)), \quad a.e. \qquad \text{as } k \to \infty.$$

Using the same arguments as in the proof of Theorem 3.1, we conclude that the function $(w(x,t), v(x,t))$ depends only on t. Then, using the conservation of momentum

$$\partial_t v + \partial_x h(w) = 0$$

in the sense of distributions, we conclude that

$$v = \bar{v} \equiv \frac{1}{|P|}\int_P v_0(x)dx,$$

following the arguments in the proof of Theorem 3.1.

We now return to the equations in (5.19). For the limits in the sense of distributions of (τ^T, p^T, e^T), $(\bar{\tau}, \bar{p}, \bar{e})$, we get

$$\partial_t\bar{\tau} = 0, \quad \partial_x\bar{p} = 0, \quad \partial_t\bar{e} = 0.$$

This implies that

$$\partial_t(\beta w - \alpha H(w)) = \partial_t(\beta\bar{\tau} - \alpha\bar{e}) = 0$$

in the sense of distributions, where $H(w) = -\int_0^w h(w)\,dw$. Hence, the function $\Theta(w(x,t)) \equiv \beta w(x,t) - \alpha H(w(x,t))$ does not depend on t either. As in the proof of Theorem 3.1, we obtain

$$\Theta(w(x,t)) \equiv \frac{1}{|P|}\int_P \Theta(w_0(x))dx.$$

Since $\Theta'(w) = \theta(w) > 0$, $\Theta(w)$ is a monotone function. Therefore, $w(x,t)$ also does not depend on t. In fact, we have

$$w(x,t) = \tilde{w} \equiv \Theta^{-1}\left(\frac{1}{|P|}\int_P \Theta(w_0(x))dx\right).$$

Hence the same procedure as used in the proof of Theorem 3.2 yields

$$\operatorname*{ess\,lim}_{t\to\infty} \int_P |(v(x,t) - \bar{v}, w(x,t) - \tilde{w})|^p dx = 0, \quad \text{for any } 1 \le p < \infty.$$

The decay of $p(w)$ and $\theta(w)$ follows from that of w. This completes the proof.

Remark. For the hyperbolic systems with symmetry, which are linearly degenerate, considered in [C3,C4], the decay of the quantity $\sqrt{u_1^2(x,t) + \cdots + u_m^2(x,t)}$ can be achieved by using similar arguments as in the above proof.

6. Hyperbolic Conservation Laws with Relaxation

We are now concerned with the asymptotic behavior of entropy solutions of hyperbolic systems of conservation laws with relaxation mechanism, which is provided by zero-order stiff terms with respect to a small parameter ε, the relaxation time:

$$\partial_t u + \partial_x f_1(u,v) = 0, \tag{6.1}$$

$$\partial_t v + \partial_x f_2(u,v) + \frac{1}{\varepsilon} r(u,v) = 0, \tag{6.2}$$

where $u \in \mathbf{R}^m$, $v \in \mathbf{R}^k$, $f_1(u,v) \in \mathbf{R}^m$, and $f_2(u,v), r(u,v) \in \mathbf{R}^k$. Denote $U = (u,v)$, $F = (f_1, f_2)$, and $R = (0, r)$. Then system (6.1)–(6.2) may be rewritten as

$$\partial_t U + \partial_x F(U) + \frac{1}{\varepsilon} R(U) = 0.$$

Intuitively, when the relaxation time ε goes to 0, equation (6.2) reduces to $r(u,v) = 0$, which can be solved as $v = e(u)$. Then equation (6.1) becomes the following conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad f(u) = f_1(u, e(u)). \tag{6.3}$$

The main goal of this section is to develop the approach in Section 3 to study the large-time behavior of solutions of (6.1)–(6.2).

Consider the Cauchy problem for system (6.1)–(6.2):

$$(u,v)\big|_{t=0} = (u_0(x), v_0(x)). \tag{6.4}$$

System (6.1)–(6.2) has the general form of nonhomogeneous systems of conservation laws. Therefore, the concept of entropy and entropy flux for (6.1)–(6.2) is the same as that of the homogeneous case: $R \equiv 0$. The difference here is that we are interested in the existence of entropy functions that not only are convex but also satisfy a dissipation condition. Specifically, if (Φ, Ψ) is a convex entropy pair for (6.1)-(6.2), the dissipation condition is

$$\nabla\Phi(U)R(U) \ge c_0(v - e(u))^2, \tag{6.5}$$

for a certain constant $c_0 > 0$.

DEFINITION 6.1. *A bounded measurable function $U(x,t)$ is called an entropy solution of (6.1)–(6.2) and (6.4) if, for every convex entropy pair (Φ,Ψ) of (6.1)–(6.2) and for any nonnegative $\phi \in C_0^1(\mathbf{R}_+^2)$, $U(x,t)$ satisfies*

$$\int_0^\infty \int_{-\infty}^\infty \{\Phi(U)\phi_t + \Psi(U)\phi_x - \frac{1}{\varepsilon}\nabla\Phi(U)R(U)\phi\}\,dxdt + \int_{-\infty}^\infty \Phi(U_0(x))\phi(x,0)\,dx \geq 0. \tag{6.6}$$

If there exists a convex entropy satisfying the dissipation condition (6.5), on any bounded domain $\Omega \subset \mathbf{R}_+^2$, then one obtains

$$\iint_\Omega (v-e(u))^2\,dxdt \leq C\varepsilon, \tag{6.7}$$

by choosing an appropriate sequence of test functions in (6.6) converging to the characteristic function of a suitable rectangle containing Ω, where $C \equiv C(\Omega, \|U\|_\infty)$ is independent of the parameter ε. In particular, this implies that, if a sequence of entropy solutions $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ of (6.1)–(6.2) and (6.4) is such that $u^\varepsilon(x,t)$ converges to some function $u(x,t)$ in L^1_{loc} when $\varepsilon \to 0$, then $v^\varepsilon(x,t)$ converges to $e(u(x,t))$ in L^2_{loc}, which means that $u(x,t)$ is a weak solution of the Cauchy problem for (6.3) with initial data $u_0(x)$, in the sense that

$$\iint_{\Pi_T} \{u(x,t)\phi_t + f(u(x,t))\phi_x\}\,dxdt + \int_{-\infty}^\infty u_0(x)\phi(0,x)\,dx = 0, \tag{6.8}$$

for any $\phi \in C_0^1(\mathbf{R}_+^2)$. Equality (6.8) does not guarantee that u is an entropy solution, which is what one needs to prove. The latter could be easily achieved from the L^1_{loc}–compactness of $u^\varepsilon(x,t)$ and $v^\varepsilon(x,t)$, if one proves that any convex entropy $\eta(u)$ for (6.3) could be extended, in a neighborhood of the equilibrium surface, $v = e(u)$, to an entropy $\Phi(u,v)$ for (6.1)–(6.2) in the sense that $\Phi(u,e(u)) = \eta(u)$ such that

$$\nabla\Phi(U)R(U) \geq 0. \tag{6.9}$$

The problem of the local extendibility of convex entropy functions for (6.3) to convex entropy functions of (6.1)–(6.2) satisfying (6.9) has been solved in [CL2] in the case $m = k = 1$ under the assumption that (6.1) is strictly hyperbolic and its eigenvalues $\lambda_1(U)$ and $\lambda_2(U)$ satisfy the stability condition:

$$\lambda_1(u,e(u)) < f'(u) < \lambda_2(u,e(u)), \qquad \text{on } v = e(u), \tag{6.10}$$

where $f'(u) = \frac{d}{du}f_1(u,e(u)) = \partial_u f_1(u,e(u)) + \partial_v f_1(u,e(u))e'(u)$. Another observation about (6.9) is that it also depends on the form of the relaxation term $r(u,v)$. A typical example is given by $r(u,v) = v - e(u)$. For this particular example in the case $k = 1$, it is then clear that the field $-R(U)$ points toward the equilibrium surface $v = e(u)$ for all U outside this surface, as one would require for the stability of the equilibrium surface with respect to the flow of

$$\frac{dU}{dt} = -\frac{1}{\varepsilon}R(U). \tag{6.11}$$

The local stability of the equilibrium $v = e(u)$ with respect to the flow generated by (6.11) would be guaranteed for more general $r(U)$ under the assumptions that $r(u, e(u)) = 0$ and $\partial_v r(u, e(u)) > 0$.

6.1. Frameworks

We now discuss how the frameworks, established in Sections 3-4, are developed to study the large-time behavior of entropy solutions of (6.1)-(6.2). Our first observation is that, if $(u(x,t), v(x,t))$ is an L^∞ solution of (6.1)–(6.2) and (6.4), the self-similar scaling functions $(u^T(x,t), v^T(x,t))$ are L^∞ solutions of the Cauchy problems:

$$\left\{ \begin{array}{l} \partial_t u^T + \partial_x f_1(u^T, v^T) = 0, \\ \partial_t v^T + \partial_x f_2(u^T, v^T) + \frac{T}{\varepsilon} r(u^T, v^T) = 0, \end{array} \right. \tag{6.12}$$

$$(u^T, v^T)\big|_{t=0} = (u_0(Tx), v_0(Tx)). \tag{6.13}$$

The notion of asymptotic decay of periodic solutions of (6.1)–(6.2) under a perturbation evolved by the relaxation system (6.1)–(6.2) is a direct adaptation of the notion for hyperbolic conservation laws.

Definition 6.2. *We say that an L^∞ periodic solution of (6.1)–(6.2) and (6.4) with period P asymptotically decays to the equilibrium $(\bar{u}, e(\bar{u}))$ with $\bar{u} = \frac{1}{|P|}\int_P u_0(x)\,dx$, if*

$$\|(u^T - \bar{u}, v^T - e(\bar{u}))\|_{L^p_{loc}(\mathbf{R}^2_+)} \to 0, \qquad \text{as } T \to \infty. \tag{6.14}$$

The limit (6.14) is equivalent to

$$\frac{1}{T}\int_0^T \{|u(\xi t, t) - \bar{u}| + |v(\xi t, t) - e(\bar{u})|\}\,dt \to 0, \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbf{R}_\xi) \text{ as } T \to \infty. \tag{6.15}$$

Then, we have an analogous version of Theorem 3.1.

Theorem 6.1. *Assume that $(u(x,t), v(x,t)) \in L^\infty(\mathbf{R}^2_+)$ is a periodic solution of (6.1)-(6.2) assuming periodic data (6.4), and that its self-similar scaling sequence $(u^T(x,t), v^T(x,t))$ is compact in $L^1_{loc}(\mathbf{R}^2_+)$. Suppose also that there exists a convex entropy for (6.1)-(6.2) satisfying the dissipation condition (6.5). Then $(u(x,t), v(x,t))$ asymptotically decays to $(\bar{u}, e(\bar{u}))$ in the sense of Definition 6.2.*

The proof is the same as that of Theorem 3.1. The only fact to be added is that the existence of a convex entropy for (6.1)–(6.2) satisfying the dissipation condition (6.5) guarantees the convergence of $v^T(x,t)$ to $e(\bar{u})$ in L^1_{loc}. Once we prove the convergence of $u^T(x,t)$ to $\bar{u}$ in L^1_{loc}, the proof follows from that for Theorem 3.1.

In the same way, we also obtain an analogous version of Theorem 3.2.

Theorem 6.2. *Suppose that there exists a strictly convex entropy $\Phi(U)$ for (6.1)–(6.2) satisfying (6.9) and*

$$\partial_v \Phi(u, e(u)) = 0, \qquad \text{for any } u \in \mathbf{R}^m. \tag{6.16}$$

Then the decay of an L^∞ periodic entropy solution $(u(x,t), v(x,t))$ of (6.1)–(6.2) and (6.4) with period P to $(\bar{u}, e(\bar{u}))$, in the sense of Definition 6.2, implies its decay in $L^p, 1 \le p < \infty$:

$$\operatorname*{ess\,lim}_{t\to\infty} \int_P \{|u(x,t) - \bar{u}|^p + |v(x,t) - e(\bar{u})|^p\} dx = 0. \tag{6.17}$$

Proof. We notice that, given any strictly convex entropy for (6.1)–(6.2) satisfying (6.9) and (6.16), we may define the quadratic function $\Phi(U, \bar{U}) = \Phi(U) - \Phi(\bar{U}) - \nabla\Phi(\bar{U})(U - \bar{U})$, which is also an entropy for (6.1) and satisfies (6.9) because of (6.16), where $\bar{U} = (\bar{u}, e(\bar{u}))$ is a certain equilibrium state. Furthermore, we have

$$C_1|U - \bar{U}|^2 \le \Phi(U, \bar{U}) \le C_2|U - \bar{U}|^2,$$

for certain positive constants C_1 and C_2. The proof then follows identically from that of Theorem 3.2. □

To prove the compactness assumptions in Theorems 6.1-6.2 in the applications, the compensated-compactness method is again one of the efficient methods. The compactness results that we will use in these applications have been proved in [CL1,CL2]; the $L^2(\mathbf{R})$ requirement on the initial data made in those papers can be easily replaced by an $L^2_{\text{loc}}(\mathbf{R})$ requirement.

6.2. Applications

We focus on the case $m = k = 1$ so that (6.1)-(6.2) are 2×2 systems and (6.3) are one-dimensional scalar conservation laws.

6.2.1. p-System

We first consider the Cauchy problem for a model studied in [CL2]:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x p(u) + \frac{1}{\varepsilon}(v - f(u)) = 0, \end{cases} \tag{6.18}$$

and

$$(u, v)\big|_{t=0} = (u_0(x), v_0(x)) \in \mathbf{O} \subset \mathbf{R}^2. \tag{6.19}$$

We assume that $p(u)$ is a C^2 function satisfying $p'(u) > 0$, for any $u \in \mathbf{R}$; $p''(u)u > 0$, if $u \ne 0$; and $f(u)$ is a C^1 function satisfying $meas\{u|\ f''(u) = 0\} = 0$ and the stability condition:

$$p'(u) - (f'(u))^2 \ge 0.$$

We also assume that the functions p and f satisfy the following conditions.

(i) There exists an invariant region, containing $\mathbf{O}$, under the flow given by the parabolic regularization for (6.18) for any $\delta > 0$,

$$\begin{cases} \partial_t u + \partial_x v = \delta\partial^2_{xx} u, \\ \partial_t v + \partial_x p(u) + \frac{1}{\varepsilon}(v - f(u)) = \delta\partial^2_{xx} v. \end{cases} \tag{6.20}$$

(ii) There exists a strictly convex entropy of (6.18) satisfying $\Phi_v|_{v=f(u)} = 0$, which is an extension of a convex entropy $\eta(u) = \Phi(u, f(u))$ for the scalar equation (6.3).

Remark 6.1. Condition (6.20) holds, in particular, if p and f satisfy $f(-u) = f(u)$ and $p(-u) = -p(u)$. Part (ii) is proved by using the extension result in [CL2], which provides a neighborhood of the graph of f where one can define such a strictly convex entropy possessing the required properties. Part (i) is proved by taking a symmetric diamond-shape invariant region for the homogeneous parabolic system ((6.20) without the relaxation term), such that a symmetric pair of its vertices lies on the graph of f. Its center of symmetry can be made the new origin by Galilean invariance. The existence of an invariant region like that, for the homogeneous (and, by construction, for the nonhomogeneous) parabolic system, is a well-known fact. We choose such an invariant region so that its closure is contained in the neighborhood given by the cited extension result. The stability condition ensures the possibility of this construction.

Thus, as in [CL2], we first prove the existence of an L^∞ solution of (6.18)-(6.19) by showing the convergence of a subsequence of solutions of (6.18)-(6.19) in $L^1_{\text{loc}}(\mathbf{R}^2_+)$ with the aid of the compactness theorem in [Di1]. Moreover, using the existence of the above mentioned convex entropy satisfying (6.5) exactly as that in [CL2], we obtain the following compactness result.

Theorem 6.3. *Let $(u(x,t), v(x,t))$ be an L^∞ solution of (6.18)-(6.19), taking values in a small region $\mathbf{O}$ around the origin obtained as above. Then the scaling sequence $(u^T(x,t), v^T(x,t))$ is compact in $L^1_{\text{loc}}(\mathbf{R}^2_+)$.*

Now, from Theorems 6.1-6.3, we arrive at

Theorem 6.4. *Let $(u(x,t), v(x,t))$ be an L^∞ periodic solution of (6.18)–(6.19), with periodic initial data in the neighborhood of the origin $\mathbf{O}$ obtained as above. Then $(u(x,t), v(x,t))$ asymptotically decays to $(\bar{u}, e(\bar{u}))$, $\bar{u} = \frac{1}{|P|}\int_P u_0(x)\,dx$, in any L^p norm, $1 \le p < \infty$.*

6.2.2. Model for Viscoelasticity

Consider the following model for viscoelasticity studied in [CL1]:

$$\begin{cases} \partial_t u + \partial_x (f(u) - v) = 0, \\ \partial_t v + \frac{1}{\varepsilon}\big(v - (1-\mu) f(u)\big) = 0, \end{cases} \tag{6.21}$$

where μ is a constant satisfying $0 < \mu < 1$. We assume that u and $f(u)$ represent nonnegative physical quantities and the function f is in $C^2[0,\infty)$ satisfying

$$f(0) = 0, \quad f'(u) > 0, \quad \text{and} \quad (u - u_*) f''(u) > 0, \text{ for } u \neq u_*,$$

for some $u_* > 0$. For initial data $(u_0(x), v_0(x))$ in $L^\infty(\mathbf{R})$ with $u_0(x) \ge 0$, one can construct an entropy solution of (6.21) and (6.4) using the vanishing viscosity method. One can check that the regions

$$\Sigma_C = \{\,(u,v)\,|\,0 \le v \le C,\ f^{-1}(v) \le u \le f^{-1}(v + \mu C)\,\}, \qquad C > (1-\mu) f(u_*),$$

are invariant for system (6.21).

To prove the L^1_{loc}-compactness of the scaling sequence $(u^T(x,t), v^T(x,t))$, one uses the compensated-compactness method as in [CL1]. Another important ingredient in the proof of the L^1_{loc}-compactness of the vanishing relaxation sequence of solutions of (6.21) and (6.4) is the fact that, given any convex entropy for the scalar conservation law obtained by setting $v = (1-\mu)f(u)$ in the first equation of (6.21), one can always define a convex entropy for (6.21) that satisfies the dissipation condition (6.5) and coincides with that entropy when $v = (1-\mu)f(u)$, as shown in [CL1]. From the compactness result proved in [CL1], one immediately obtains the following theorem.

THEOREM 6.5. *Let $(u(x,t), v(x,t))$ be an L^∞ solution of (6.21) and (6.4) taking its values in a region Σ_C for sufficiently large C. Then the scaling sequence $(u^T(x,t), v^T(x,t))$ is compact in $L^1_{\mathrm{loc}}(\mathbf{R}^2_+)$.*

Applying Theorems 6.1-6.2 and 6.5, we obtain

THEOREM 6.6. *Let $(u(x,t), v(x,t))$ be an L^∞ periodic solution of (6.21) and (6.4) with initial data taking their values in a region Σ_C for sufficiently large C. Then $(u(x,t), v(x,t))$ asymptotically decays to $(\bar{u}, e(\bar{u}))$, $\bar{u} = \frac{1}{|P|}\int_P u_0(x)\,dx$, in $L^p, 1 \le p < \infty$, where $e(u) = (1-\mu)f(u)$.*

6.2.3. Model for Phase Transitions

When the stability condition (6.11) is not satisfied, the study of the zero relaxation limit becomes much more difficult. This is the case of the following simple model for phase transitions studied in [CL1]:

$$\begin{cases} \partial_t u + \partial_x \big(\frac{1+(\mu-1)v}{u}\big)^2 = 0, \\ \partial_t v + \frac{1}{\varepsilon}(v - e(u)) = 0. \end{cases}$$

Here, u represents a nonnegative physical quantity, v represents a physical quantity satisfying $0 \le v \le 1$, μ is a constant satisfying $\mu > 1$, and $e(u)$ is given by

$$e(u) = \begin{cases} 0, & \text{for } 0 < u < 1, \\ (u-1)/(\mu-1), & \text{for } 1 < u < \mu, \\ 1, & \text{for } \mu < u. \end{cases}$$

We can easily verify that the stability condition (6.11) is no longer valid; instead we have the weaker condition $\lambda_1(u, e(u)) \le f'(u) \le \lambda_2(u, e(u))$, where $f(u) = (1+(\mu-1)e(u))/u$. One way to compensate for the loss of stability in the vanishing relaxation process caused by this marginal failure of condition (6.11) is to introduce an artificial viscosity that is comparable to the relaxation parameter. This leads us to the system

$$\begin{cases} \partial_t u + \partial_x \big(\frac{1+(\mu-1)v}{u}\big)^2 = \varepsilon \partial_{xx} u, \\ \partial_t v + \frac{1}{\varepsilon}(v - e(u)) = \varepsilon \partial_{xx} v. \end{cases} \tag{6.22}$$

An important point concerning the parabolic system (6.22) is the existence of bounded invariant regions. Specifically, one can easily verify that the regions

$$\Sigma_{C_1,C_2} = \{\, (u,v) \,|\, 0 \le v \le 1,\, C_1(1+(\mu-1)v) \le u \le C_2(1+(\mu-1)v) \,\},$$

with $0 < C_1 < 1 < C_2 < \infty$, are invariant for (6.22), according to the principle of invariant regions (see [Sm]). In particular, given initial data in Σ_{C_1,C_2} for some C_1 and C_2 satisfying the above inequalities, one can find a solution of the Cauchy problem (6.22) and (6.4) taking its values in Σ_{C_1,C_2}. The solution is smooth as soon as $t > 0$. One then poses the question about the asymptotic decay of solutions with periodic initial data under the perturbation evolved by the viscous-relaxation system (6.22). As above, using the L^1_{loc}-compactness of the vanishing relaxation-dissipation sequence, proved in [CL1], and the results in §6.1, one obtains the following results.

THEOREM 6.7. *Let $(u(x,t), v(x,t)) \in \Sigma_{C_1,C_2}$ be a smooth solution of (6.22) and (6.4). Then the scaling sequence $(u^T(x,t), v^T(x,t))$ is compact in $L^1_{\mathrm{loc}}(\mathbf{R}^2_+)$.*

THEOREM 6.8. *Let $(u(x,t), v(x,t))$ be a smooth periodic solution of (6.22) and (6.4) with initial data in a region Σ_{C_1,C_2}. Then $(u(x,t), v(x,t))$ asymptotically decays to $(\bar{u}, e(\bar{u}))$, $\bar{u} = \frac{1}{|P|}\int_P u_0(x)\,dx$, in $L^p, 1 \le p < \infty$.*

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