The Method of Quasidecoupling for Discontinuous Solutions to Conservation Laws

Gui-Qiang Chen*[†]

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*Department of Mathematics, The University of Chicago

[†]Email: CHENG@ZAPHOD.UCHICAGO.EDU

1 Introduction

We are concerned with the existence, uniqueness, and qualitative behavior of discontinuous solutions to nonlinear conservation laws. The main objectives of this paper are to describe an analytical method — the quasidecoupling method — and to study its applications to these problems for discontinuous solutions to hyperbolic systems of conservation laws with linearly degenerate characteristic fields. This method is developed from the ideas in [4, 5] about propagation and cancellation of initial oscillations for conservation laws and is motivated by the transformation between Euler coordinates and Lagrangian coordinates in fluid dynamics (cf. [7, 56, 49]). The novel feature of the method described here is the quasidecoupling of the entropy conservative inequalities and, to a lesser extent, the particular forms of entropy conservative inequalities for conservation laws.

We consider a hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \qquad -\infty < x < \infty, \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^n$ and f is a smooth nonlinear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. The condition of hyperbolicity requires that the Jacobian ∇f of f has n real eigenvalues $\lambda_i(u)$, $1 \le i \le n$, and n distinct right eigenvectors $r_i(u)$, $1 \le i \le n$,

$$\nabla f(u) r_i(u) = \lambda_i(u) r_i(u); \tag{1.2}$$

that is, the Jacobian ∇f is diagonalizable for any value of u. An eigenvalue λ_i is either genuinely nonlinear or linearly degenerate in the sense of Lax [33] if its derivative in the corresponding eigendirection is either

$$r_i \cdot \nabla \lambda_i \ne 0 \qquad \text{or} \qquad r_i \cdot \nabla \lambda_i \equiv 0. \tag{1.3}$$

In the classical theory of hyperbolic equations, an efficient method is the diagonalization method for studying the qualitative behavior of smooth solutions to quasilinear hyperbolic systems. The systems are reduced to corresponding systems of diagonal forms that consist of the transport equations

$$\partial_t w_i + \lambda_i(w) \partial_x w_i = 0, \qquad 1 \le i \le n, \tag{1.4}$$

where the functions $w_i(u)$, $1 \le i \le n$, are Riemann invariants satisfying

$$\nabla w_i \cdot \nabla f = \lambda_i \nabla w_i. \tag{1.5}$$

A necessary and sufficient condition for the existence of the Riemann invariants w_i for strictly hyperbolic systems is the well-known Frobenius condition:

$$l_i\{r_j, r_k\} = 0, \qquad \text{for any } j, k \ne i, \tag{1.6}$$

where l_i denotes the left eigenvector corresponding to λ_i and $\{\cdot,\cdot\}$ is the Poisson bracket of vector fields in u-space. For $n = 2$, the Frobenius condition always holds. It is well known that the Cauchy problem for (1.1) does not, in general, have globally defined smooth solutions because of the nonlinear structure of the eigenvalues; hence only discontinuous solutions may exist in the large. For discontinuous solutions, the system (1.1) is not equivalent to the system (1.4) in general, and this method no longer works. In Section 3, we shall introduce a related method — the quasidecoupling method — and shall establish several frameworks for studying the uniqueness and qualitative behavior of discontinuous solutions to hyperbolic systems of conservation laws.

The uniqueness problem for conservation laws has been the topic of many papers (see the references and those cited therein). A complete theory of the uniqueness and L^1 stability of solutions to scalar conservation laws has been established in the space L^∞ or $L^\infty \cap BV$. We refer the reader to the work of OLEINIK [41], VOL'PERT [55], KEYFITZ [29], and KRUZKOV [31] on this problem for a scalar conservation law. Under certain smoothness (cf. piecewise Lipschitz) and ordering assumptions on generalized solutions, some uniqueness theorems for strictly hyperbolic systems of conservation laws were established by OLEINIK [42], ROZHDESTVENSKII [45], GODUNOV [24], HURD [25, 26], LIU [35], and DIPERNA [14] in the space $L^\infty \cap BV$. These results impose assumptions on generalized solutions that have not been translated into conditions on the initial data. For an isotachophoresis model, DAFERMOS AND GENG [12] established a uniqueness theorem of a BV solution with small oscillation BV initial data in which generation points of centered rarefaction waves of the two families are strictly separated. We also refer the reader to DOUGLIS [24], LYAPIDEVSKII [38] and TVEITO AND WINTHER [54] for stability results within certain special classes of solutions. In Subsection 3b, we shall first establish a framework (Framework (A)) for the uniqueness and stability of L^∞ solutions of the model Cauchy problem

$$\begin{cases} (a_0(w-k)P(x,t))_t + (a_0(w-k)\lambda(x,t)P(x,t))_x \le 0, & \text{for any } k \in (-\infty,\infty), \\ w|_{t=0} = w_0(x), \end{cases} \tag{1.7}$$

where $a_0(w)$ satisfies

$$\begin{cases} a_0(0) = 0, \quad a_0(-w) = a_0(w), \\ a_0(w) > 0, \quad \text{for} \quad w \neq 0, \end{cases} \tag{1.8}$$

and the L^∞ vector field $(P(x,t), \lambda(x,t)P(x,t))$, $P(x,t) \ge 0$, $P(x,0) \ge P_0 > 0$, is divergence free:

$$\operatorname{div}(P(x,t), \lambda(x,t)P(x,t)) = 0. \tag{1.9}$$

The major finding is that the solution of the Cauchy problem (1.7) is independent of the choice of the L^∞ function $P(x,t) > 0$ with the same initial value $P(x,0)$. This is essential for using the framework to solve the uniqueness problem for conservation laws (see Sections 4–6). We remark that Framework (A) does not require any regularity of L^∞ solutions in contrast to, for example, BV regularity that is the basic assumption of all uniqueness theorems established for systems of conservation laws, as in [12, 14, 26, 35, 42, 55].

The study of the limit behavior of dissipative approximate solutions to conservation laws is another objective in this paper. Essentially, this problem is equivalent to the oscillation problem, that is, to studying the dynamic behavior of initial oscillations of $O(1)$ amplitude with high frequency, as time evolves. More specifically, we study the limit behavior of the solutions corresponding to the oscillatory sequence of large initial data for nonlinear hyperbolic systems of conservation laws. This study can provide insight into how the behavior of the system at the microscopic level affects the behavior of the system at the macroscopic level. Such results are important for understanding both the structure of solutions for the Cauchy problem and the limit behavior of numerical and analytical approximations for the Cauchy problem with oscillatory initial data.

For genuinely nonlinear systems of conservation laws, many compactness theorems have been established in [15, 3, 13, 44] (see also the references cited therein). These theorems indicate that, for 2×2 genuinely nonlinear systems, the initial oscillations will instantaneously cancel as time evolves. The basic strategy is to transfer the compactness problem of the solution operator into proving that the corresponding Young measures are the Dirac masses by solving the Tartar-Murat functional equation for the Young measures. However, for the systems with at least a linearly degenerate field, the Tartar-Murat equation cannot describe a complete structure for the corresponding Young measures. This causes an analytical difficulty (see DiPerna [16], Rascle [43] and Serre [48]). A suitable framework must be introduced to pick up the lost information for the Young measures. For a 2×2 system with one linearly degenerate field, a rigorous framework is successfully established in [3, 4], which indicates that the initial oscillations propagate along the linearly degenerate field and cancel along the genuinely nonlinear field. We also refer the reader to [48] and [2] for some evidence of large oscillations for strictly hyperbolic and linearly degenerate systems. In Subsection 3c, we shall develop a more general framework (Framework (B)), which will be shown to be useful for conservation laws in Sections 4–6.

Section 4 is concerned with $n \times n$ systems with fully contact fields, i.e., linearly degenerate characteristic fields, which are described in [50]. In [50], Serre established a global existence and uniqueness theorem of smooth solutions for the Cauchy problem for such systems by finding a hierarchy of transport equations involving higher derivatives of Riemann invariants w_i's and by using the local existence and uniqueness theorem of Kato [28]. For $n = 2$, existence theorems of global generalized solutions were established in [47] for BV solutions and in [5] for L^∞ solutions. In Section 4 we shall study well-posedness problems in the space L^∞ and the space $L^\infty \cap BV$: the existence, uniqueness, and stability of generalized solutions, and the propagation problem of initial oscillations for such $n \times n$ systems by using the quasidecoupling method established in Section 3.

We remark that the basic assumption of all existing uniqueness theorems for systems of conservation laws as in [14, 12, 35, 42] is that the solutions lie in the space $BV \cap L^\infty$. In Section 4, our uniqueness theorems are established only in the space L^∞, which is much broader than $BV \cap L^\infty$.

In Section 5 we shall establish uniqueness and stability theorems for generalized solutions to systems with one contact field and one line field, identified by TEMPLE [52], with the aid of the quasidecoupling method. The physical background of such systems is several important models arising in the fields of elasticity theory (cf. [30, 1]), enhanced oil recovery (cf. [27, 53]), and magnetohydrodynamics (cf. [20, 1]). In this connection we note that some existence theorems of generalized solutions have been established as in [53, 4], the propagation and cancellation of initial oscillations have been identified in [4, 5] and a well-posedness result is obtained for a polymer flooding model within a special class of solutions in [54] for such systems.

Section 6 describes an example of how to reduce the uniqueness and qualitative behavior problems of solutions for large systems with linearly degenerate fields into the corresponding problems of solutions for much smaller systems by using the quasidecoupling method. This example is the system of electromagnetic plane waves, derived from the Maxwell equations by SERRE [49], where he succeeded in solving global existence of L^∞ solutions for this system by using a related technique. We believe from this example that the quasidecoupling method can be useful for dealing with large systems of conservation laws with linearly degenerate fields arising in many areas such as magnetohydrodynamics, electromagnetic theory, and elasticity theory.

In Section 7 we shall give several remarks for further applications of the quasidecoupling method to some important problems and methods for conservation laws.

2 Entropy, Solutions, and the Young Measure

We shall begin with several remarks concerning entropy, solutions for conservation laws, and the Young measure, which motivate part of the subsequent development. We recall that u is called a weak solution of (1.1) with initial data $u_0(x)$ if the integral relation

$$\int\int_{\Pi_T} (u\phi_t + f(u)\phi_x)dxdt + \int_{-\infty}^{\infty} \phi(x,0)u_0(x)dx = 0 \tag{2.1}$$

holds for any function $\phi \in C_0^1(\Pi_T)$, where $\Pi_T = (-\infty, \infty) \times [0, T]$.

One of the main features of conservation laws is that uniqueness is lost within the broader class of weak solutions; many weak solutions may share the same initial data. Thus the problem arises of identifying appropriate class of weak solutions (i.e., admissibility criteria) to single out physically relevant solutions.

In this connection we recall that a function $\eta : \mathbb{R}^n \to \mathbb{R}$ is an entropy for (1.1) with entropy flux $q : \mathbb{R}^n \to \mathbb{R}$ if all smooth solutions satisfy an additional conservation law of the form

$$\eta(u)_t + q(u)_x = 0;$$

that is,

$$\nabla q = \nabla\eta\nabla f. \tag{2.2}$$

Taking the inner product of (2.2) with the right eigenvectors r_i of ∇f produces the characteristic form

$$(\lambda_i\nabla\eta - \nabla q) \cdot r_i = 0, \tag{2.3}$$

and

$$\lambda_i\eta_{w_i} = q_{w_i}, \tag{2.4}$$

provided that the system (1.1) is endowed with globally defined Riemann invariants w_j. For $n = 2$, the linear hyperbolic system (2.2) is equivalent to (2.4), while the system (2.4) can be reduced to a linear second-order hyperbolic equation for which the Goursat problem is well posed in the coordinates of Riemann invariants. The entropy space is of infinite dimension and is represented by two family functions of one variable $\alpha(w_1)$ and $\beta(w_2)$, provided $\lambda_1 < \lambda_2$. For $n \geq 3$, the system (2.2) is overdetermined and generally prevents the existence of nontrivial entropy. Friedrichs and Lax [21] observed that most of the conservative systems that result from continuum mechanics (in one and several dimensions) are endowed with a globally defined, strictly convex entropy. A description of the systems endowed with rich entropy has been addressed by Serre [50]. The algebraic characterization for such systems is

$$\partial_{w_k}((\lambda_j - \lambda_i)^{-1}\partial_{w_j}\lambda_i) = \partial_{w_j}((\lambda_k - \lambda_i)^{-1}\partial_{w_k}\lambda_i), \tag{2.5}$$

for distinct i, j, and k.

For the class of systems (1.1) endowed with a strictly convex entropy η, Lax [32] postulated the following entropy criterion: a weak solution u is admissible if

$$\eta(u)_t + q(u)_x \leq 0, \tag{2.6}$$

in the sense of distribution. The entropy inequality (2.6) is a proper statement of the second law of thermodynamics for the equations of fluid dynamics: the entropy increases along thermodynamic processes. It is easy to check that, on the discontinuity of u with speed σ, left state u_- and right state u_+, (2.6) reduces to the local condition

$$\sigma(\eta(u_+) - \eta(u_-)) - (q(u_+) - q(u_-)) \geq 0. \tag{2.7}$$

Note that the inequality (2.7) imposes a restriction on each point of discontinuity, a fact that suggests that admissible solutions of (1.1) may be determined by means of shock criteria.

LAX [32] proposed the Lax shock criterion for systems where each of the eigenvalues is either genuinely nonlinear or linearly degenerate:

$$\lambda_i(u_+) \leq s \leq \lambda_i(u_-). \tag{2.8}$$

The criterion governs the number and type of characteristics impinging on a shock wave. For more general systems, a comprehensive shock criterion has been proposed by LIU [36]:

$$s(\tau^+; u_-) \leq s(\tau; u_-), \qquad 0 \leq \tau \leq \tau^+, \tag{2.9}$$

where $s(\tau; u_-)$ denotes the speed of the shock that joins $u(\tau; u_-)$ with u_- determined by the Rankine-Hugoniot condition. On the other hand, it is known that the second law of thermodynamics is not powerful enough to rule out all undesirable solutions in the general thermodynamics processes. DAFERMOS [8] proposed the entropy rate criterion, which recognizes as admissible those solutions that maximize the rate of entropy production.

DAFERMOS [9, 10] analyzed these criteria and showed that all these criteria are equivalent when applied to weak solutions with nondegenerate shock speed; for shocks of moderate strength, the shock criterion (2.9) and the entropy rate criterion are equivalent and generally more discriminating than either the Lax shock criterion (2.8) and the entropy criterion (2.6) or (2.7). For shocks of great strength, the relationship among these criteria is generally complicated; some peculiar phenomena for them have been discussed in [4, 37].

In view of the existence theory of conservation laws (cf. GLIMM [22]), it is natural to pose the class of solutions in the space L^∞ or $L^\infty \cap BV$. Here $BV(\Omega)$ denotes the space of functions that have locally bounded total variation in the sense of Cesari; that is, a real-valued function $u = u(y) \in \mathbb{R}^n$ defined on a region $\Omega \subset \mathbb{R}^m$ is an element of $BV(\Omega)$ if it is locally integrable and if its gradient is a locally finite Borel measure, namely,

$$\int_\Omega u \nabla \phi dy = -\int_\Omega \phi d\mu,$$

for all $\phi \in C_0^\infty(\Omega)$, where μ is locally finite Borel measure. On the other hand, $u \in BV(\Omega)$ if and only if the local total variation in each x_i is locally integrable in $x_i' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$:

$$\int_{|x_i'| \leq M} TVu(x_i', \cdot) dx_i' < \infty,$$

for all $M < \infty$. It is known (cf. see VOL'PERT [55]) that u induces a partition of $\mathbb{R}^m$ into the union of three pairwise-disjoint subsets: the set of points of Lebesgue approximate continuity of u, the set $\Gamma(u)$ of points of Lebesgue approximate jump discontinuity of u contained in the countable union of Lipschitz arcs, and the set of irregular points whose $(m-1)$-dimensional Hausdorff measure $H_{(m-1)}$ is zero.

In the context of generalized solutions $u \in BV$, the set $\Gamma(u)$ is referred to as the set of shock waves and contact discontinuities of solution u. For simplicity, we assume that, for every fixed $\bar{t} \in [0, \infty)$, $u(\cdot, \bar{t})$ is a function of bounded variation on $(-\infty, \infty)$ with one-sided limits $u(\bar{x}\pm, \bar{t})$ for every $\bar{x} \in (-\infty, \infty)$, and u is equal to the symmetric mean of the solution: $u = \bar{u}$. Here

$$\bar{u} \equiv \lim_{m \to \infty} \psi_m * u$$

is defined H_1 almost everywhere and coincides with u H_2 almost everywhere, where $\{\psi_m\}$ is a sequence of radially symmetric functions approximating the δ-function. Consider the B-measure defined by

$$B_u = \eta(u)_t + q(u)_x, \tag{2.10}$$

where (η, q) is a pair of C^2 entropy-entropy flux of (1.1). Using the generalized Green's theorem for measure [19], the B-measure can be expressed by

$$B_u(E) = B_u(E \cap \Gamma(u)) = \int_{E \cap \Gamma(u)} (n_t[\eta] + n_x[q]) dH_1. \tag{2.11}$$

Here $\vec{n} = (n_x, n_t)$ denotes the unit normal to $E \cap \Gamma(u)$ in the sense of Federer [19], and the bracket denotes the jump in the enclosed quantity across $E \cap \Gamma(u)$ in the direction of $\vec{n}$:

$$[\eta] = [\eta](P) = \eta\{l_{\vec{n}} u(P)\} - \eta\{l_{-\vec{n}} u(P)\},$$

where $l_{\pm\vec{n}} u(P)$ denote the approximate limits of u at $P \in E \cap \Gamma(u)$ with respect to the half-planes $\{Q :< Q - P, \pm\vec{n} >\geq 0\}$.

Normalize the direction of $\vec{n}$ by requiring that $n_x < 0$, and refer to

$$u_- \equiv l_{\vec{n}} u \qquad \text{and} \qquad u_+ \equiv l_{-\vec{n}} u$$

as the left and right approximate limits of u at P, respectively. Then the bracket denotes the jump from left to right across a shock or a contact discontinuity. We note that the set of irregular points and points on which $n_x = 0$ for $u \in BV(\mathbb{R}^2_+)$ are a one-dimensional Hausdorff zero measure set. Therefore, we can write (2.11) as

$$B_u(E) = \int_{E \cap \Gamma(u)} (\sigma[\eta] - [q]) dt, \tag{2.12}$$

where $\sigma \equiv -n_t(P)/n_x(P)$ denotes the speed of propagation of $\Gamma(u) \cap E$ at P and $dt \equiv -n_t dH_1$.

The Young measure provides an appropriate tool for studying oscillations in solutions to (1.1), that is, the behavior of oscillations of perturbed solutions u^ϵ as the perturbation parameter ϵ goes to zero. We recall from Tartar [51] and Murat [39, 40] that, for an arbitrary sequence of measurable maps

$$u^\epsilon : \mathbb{R}^m \to \mathbb{R}^n, \qquad \|u^\epsilon\|_{L^\infty} \leq M < \infty,$$

that converges in the weak-star topology of L^∞ to a function u,

$$w^* - \lim_{\epsilon\to 0} u^\epsilon = u,$$

there exists a subsequence (still labeled) u^ϵ and a family of Young measures

$$\nu_y \in \mathrm{Prob}(\mathbb{R}^n), \quad \mathrm{supp}\ \nu_y \subset \{\lambda : |\lambda| \le M\}, \qquad y \in \mathbb{R}^m,$$

such that for all continuous functions g,

$$w^* - \lim_{\epsilon\to 0} g(u^\epsilon(y)) = \int_{\mathbb{R}^n} g(\lambda) d\nu_y \equiv < \nu_y, g(\lambda) >,$$

for almost all points $y \in \mathbb{R}^m$. In particular, u^ϵ converges strongly to u if and only if the Young measure ν_y at almost all points y is equal to a Dirac mass concentrated at $u(y)$. Then, we have the following theorem.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^2_+$ be a bounded open set and $u^\epsilon : \Omega \to \mathbb{R}^n$ be measurable functions satisfying*

(i) $\|u^\epsilon\|_{L^\infty} \le M$.

(ii) For continuous function pairs $(\eta_i(u), q_i(u))(i = 1, 2)$,

$$\frac{\partial \eta_i(u^\epsilon)}{\partial t} + \frac{\partial q_i(u^\epsilon)}{\partial x}, \qquad \textit{compact in } H^{-1}_{\mathrm{loc}}(\Omega). \tag{2.13}$$

Then there exists a subsequence (still labeled) u^ϵ and a family of Young measures $\{\nu_{x,t} : \mathbb{R}^2_+ \to Prob(\mathbb{R}^n)\}$, $supp\nu_{x,t} \subset \{\lambda : |\lambda| \le M\}$, such that for all continuous functions g,

$$w^* - \lim_{\epsilon\to 0} g(u^\epsilon(x,t)) = \int_{\mathbb{R}^2_+} g(\lambda) d\nu_{x,t} \equiv < \nu_{x,t}, g(\lambda) >, \tag{2.14}$$

and

$$\left\langle \nu_{x,t}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda) & q_2(\lambda) \end{vmatrix} \right\rangle = \begin{vmatrix} < \nu_{x,t}, \eta_1(\lambda) > & < \nu_{x,t}, q_1(\lambda) > \\ < \nu_{x,t}, \eta_2(\lambda) > & < \nu_{x,t}, q_2(\lambda) > \end{vmatrix}, \qquad a.e.\ \mathbb{R}^2_+. \tag{2.15}$$

Next, we consider the system (1.1) endowed with globally defined Riemann invariant coordinates and satisfying the formula (2.5). Suppose that one of its eigenvalues λ_i is linearly degenerate in the sense of LAX [33]:

$$\partial_{w_i} \lambda_i \equiv 0, \qquad \text{i.e.,} \qquad \lambda_i = \lambda_i(w_i'), \tag{2.16}$$

where $w_i' = (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \in \mathbb{R}^{n-1}$. Then we deduce from (2.4) that there exists a family of entropy-entropy pairs (η_i, q_i) corresponding to the linearly degenerate field which are of the form

$$\begin{cases} \eta_i = N_i a(w_i), \\ q_i = N_i \lambda_i(w_i') a(w_i), \end{cases} \tag{2.17}$$

where a is any continuous function of the one-dimensional variable w_i and

$$N_i = \exp\left\{\int^{w_j} \frac{\partial_{w_j}\lambda_i}{\lambda_j - \lambda_i} dw_j\right\}, \qquad j \neq i, \tag{2.18}$$

is independent of $j(\neq i)$ because of (2.5). If one can choose $a(w_i)$ such that the associated η_i is convex, then we have

$$\partial_t(N_i a(w_i)) + \partial_x(\lambda_i N_i a(w_i)) \leq 0, \tag{2.19}$$

in the sense of distribution. Generally, there always exists an $a_0(w_i) \geq 0$ such that

$$\partial_t(N_i a_0(w_i)) + \partial_x(\lambda_i N_i a_0(w_i)) = 0, \tag{2.20}$$

derived from one equation of the system (1.1).

Using the notation of the Young measures $\nu_{x,t} \in \mathrm{Prob}(\mathbb{R}^n)$ for the dissipative approximate solutions, we can obtain the associated average inequality of (2.19) and equality of (2.20). In particular, if the Young measures $\nu_{x,t}$ are of the following property: for any continuous function $a(w_i)$, there exists an L^∞ function $\bar{\lambda}_i(x,t)$ such that

$$< \nu_{x,t}, \lambda_i(w_i')a(w_i)N_i >= \bar{\lambda}_i(x,t) < \nu_{x,t}, a(w_i)N_i > \qquad \text{a.e. }, \tag{2.21}$$

then we have

$$\partial_t < \nu_{x,t}, N_i a(w_i) > +\partial_x(\bar{\lambda}_i(x,t) < \nu_{x,t}, N_i a(w_i) >) \leq 0, \tag{2.22}$$

$$\partial_t < \nu_{x,t}, N_i a_0(w_i) > +\partial_x(\bar{\lambda}_i(x,t) < \nu_{x,t}, N_i a_0(w_i) >) = 0, \tag{2.23}$$

in the sense of distribution.

Finally, we remark that, if $S_0 = \{(x,t) : N_i(x,t) = 0\} \neq \phi$ for nonstrictly hyperbolic cases, we can generalize the sense of (2.19) to a stronger sense that the inequality (2.19) holds in the generalized sense of distribution; that is, for any $\phi \in C_0^1(\Pi_T - S_0) \cap BV(\Pi_T), \phi \geq 0$,

$$\int\int_{\Pi_T} N_i a(w_i)(\phi_t + \lambda_i \phi_x) dx dt + \int_{-\infty}^{\infty} (N_i a(w_i))(x,0)\phi(x,0) dx \geq 0. \tag{2.24}$$

Similarly we can also assume that the inequality (2.22), and the equalities (2.20) and (2.23) hold in the same sense.

3 Quasidecoupling Method

In this section we describe the quasidecoupling method for studying the existence, uniqueness, and qualitative behavior of solutions to systems of conservation laws with linearly degenerate fields.

a. Quasidecoupling Transformation

Let $w : \mathbb{R}^2_+ \to \mathbb{R}$ be a bounded and measurable function satisfying the following inequality:

$$\partial_t(a(w;x,t)P(x,t)) + \partial_x(a(w;x,t)\lambda(x,t)P(x,t)) \le 0, \tag{3.1}$$

with the initial condition

$$w(x,0) = w_0(x), \tag{3.2}$$

in the generalized sense of distribution; that is, the inequality

$$\int\int_{\Pi_T} a(w;x,t)P(x,t)(\phi_t + \lambda(x,t)\phi_x)dxdt + \int_{-\infty}^{\infty} a(w_0(x);x,0)P(x,0)\phi(x,0)dx \ge 0 \tag{3.3}$$

holds for any nonnegative function $\phi \in C_0^1(\Pi_T - \{P(x,t) = 0\}) \cap BV(\Pi_T)$. Here the function $a(w;x,t)$ is continuous with respect to w and uniformly bounded with respect to (x,t), and $(P(x,t), \lambda(x,t)P(x,t))$, $P(x,t) \ge 0$, is a bounded and measurable vector field whose divergence vanishes in the sense of distribution

$$\operatorname{div}(P(x,t), \lambda(x,t)P(x,t)) = 0, \tag{3.4}$$

and

$$\int_0^{\infty} P(x,0)dx = \int_{-\infty}^{0} P(x,0)dx = +\infty. \tag{3.5}$$

Then we have the following theorem.

Theorem 3.1 *There exists a globally defined coordinate transformation*

$$T : (x,t) \to (p,t) \equiv (T_t(x), t)$$

such that, in the new coordinate system $(p,t) = T(x,t)$,

$$\begin{cases} \partial_t a(w(T^{-1}(p,t)); T^{-1}(p,t)) \le 0, \\ w|_{t=0} = w_0(T_0^{-1}(p)), \end{cases} \tag{3.6}$$

in the sense of distribution; that is,

$$\int\int_{\Pi_T} a(w(T^{-1}(p,t)); T^{-1}(p,t))\psi(p,t)_t dpdt + \int_{-\infty}^{\infty} a(w_0(T_0^{-1}(p)); T^{-1}(p,0))\psi(p,0)dp \ge 0, \tag{3.7}$$

for all nonnegative functions $\psi \in C_0^1(\Pi_T)$, *where* $T_0^{-1}(p)$ *is the inverse map of* $T_t(x)|_{t=0} = p(x,0) = \int_0^x P(\xi,0)d\xi$, *uniquely determined by* $P(x,0)$.

Remark. Under the transformation T, the divergence form of the vector field

$$(a(w;x,t)P(x,t), a(w;x,t)\lambda(x,t)P(x,t))$$

is transformed to a complete single differential form $\partial_t a(w(T^{-1}(p,t)); T^{-1}(p,t))$, and the inequality in (3.6) is formally independent of the vector fields $(P, \lambda P)$ in the new coordinate system (p,t). For these reasons, we call the transformation T a "quasidecoupling transformation" corresponding to the vector field $(P, \lambda P)$.

Lemma 3.1 *Let the L^∞ vector field $(P(x,t), Q(x,t))$, $P(x,t) \geq 0$, be divergence free in the sense of distribution in Π_T and*

$$\int_\alpha^\beta P(x,0)dx = +\infty, \qquad -\infty \leq \alpha < \beta \leq +\infty.$$

Then, for almost all $t \in (0,T)$,

$$\int_\alpha^\beta P(x,t)dx = +\infty.$$

Proof. The proof is simple. Similar to [56], we choose two sequences of nonnegative smooth functions $\phi_\epsilon(\tau)$ and $\psi_\delta(x)$ with compact support for any $t \in (0,T)$,

$$\phi_\epsilon(\tau) = \begin{cases} 1, & \tau \in [0,t], \\ 0, & \tau \in [t+\epsilon, T], \end{cases}$$
$$\psi_\delta(x) = \begin{cases} 1, & x \in [\alpha, \beta], \\ 0, & x \in (-\infty, \alpha-\delta] \cup [\beta+\delta, +\infty), \end{cases}$$

such that

$$\int_0^T |\phi'_\epsilon(\tau)| d\tau = \int_{-\infty}^\infty |\psi'_\delta(x)| dx = 2.$$

Since $\operatorname{div}(P(x,t), Q(x,t)) = 0$, we have

$$\int\int_{\Pi_T} (\phi'_\epsilon(\tau)\psi_\delta(x)P(x,t) + \phi_\epsilon(\tau)\psi'_\delta(x)Q(x,t))dxdt + \int_{-\infty}^\infty \psi_\delta(x)P(x,0)dx = 0.$$

Set $\epsilon \to 0$. We obtain

$$\left| \int_{-\infty}^\infty \psi_\delta(x)(P(x,t) - P(x,0))dx \right| \leq 2\|Q\|_{L^\infty} t,$$

for any Lebesgue point t of all locally integrable functions $t \to \int \psi_\delta(x)P(x,t)dx$. Set $\delta \to 0$. We have that, for almost all $t \in (0,T)$,

$$\left| \int_\alpha^\beta (P(x,t) - P(x,0))dx \right| \leq 2\|Q\|_{L^\infty} t.$$

Therefore

$$\int_\alpha^\beta P(x,t)dx = \int_\alpha^\beta P(x,0)dx = +\infty.$$

Proof of Theorem 3.1. Notice that the vector field $(P(x,t), \lambda(x,t)P(x,t))$ is divergence free. We introduce a coordinate transformation

$$T : (x,t) \to (p,t),$$

determined by

$$\begin{cases} \frac{\partial p}{\partial x} = P(x,t), \\ \frac{\partial p}{\partial t} = -\lambda(x,t)P(x,t), \\ p(x,0) = \int_0^x P(\xi,0)d\xi. \end{cases} \tag{3.8}$$

Then the transformation T is well defined, and there exists a globally defined Lipschitz solution $(p,t) = T(x,t)$ in the space of distribution. From (3.5) and Lemma 3.1, we have

$$\int_0^\infty P(x,t)dx = \int_{-\infty}^0 P(x,t)dx = +\infty.$$

Then T is proper and onto in $\mathbb{R}^2_+$.

If $P(x,t) > 0$, then T is a bi-Lipschitz homeomorphism with the Jacobian $P(x,t) > 0$. Then we have from (3.3) that, for any nonnegative function $\psi \in C_0^\infty(\Pi_T)$,

$$\int\int_{\Pi_T} a(w;x,t)P(x,t)(\psi_t + \lambda(x,t)\psi_x)dxdt + \int_{-\infty}^\infty a(w_0(x);x,0)P(x,0)\psi(x,0)dx \geq 0. \tag{3.9}$$

Make the coordinate transformation (3.8) for the integrals in (3.9). We immediately obtain

$$\int\int_{\Pi_T} a(w(T^{-1}(p,t));T^{-1}(p,t))\psi(p,t)_t dpdt + \int_{-\infty}^\infty a(w_0(T_0^{-1}(p));T^{-1}(p,0))\psi(p,0)dp \geq 0.$$

This arrives at (3.7).

If the set $\{(x,t) : P(x,t) = 0\} \neq \phi$, we can verify the above process of proof under the assumption that (3.1) holds in the generalized sense of distribution and prove that the inequalities (3.1) and (3.6) are equivalent by arguments similar to WAGNER [56]. We omit the details.

A useful fact is the following corollary.

Corollary 3.1 *Let T be the quasidecoupling transformation (3.8). Then for any $g \in L^1(\Pi_T)$,*

$$\int_{-\infty}^\infty g(p,t)dp = \int_{-\infty}^\infty g(T(x,t))P(x,t)dx$$

holds for almost all $t \in (0,T)$.

b. Framework for the Uniqueness and Stability of Solutions

Suppose that L^∞ functions $w(x,t)$ and $z(x,t)$ satisfy the following Framework (A).

A1: $w(x,t)$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t(a_0(w-k)P_1(x,t)) + \partial_x(a_0(w-k)\lambda_1(x,t)P_1(x,t)) \le 0, \text{ for any } k \in (-\infty,\infty), \\ w(x,0) = w_0(x), \end{cases} \tag{3.10}$$

in the generalized sense of distribution in Π_T.
A2: $z(x,t)$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t(a_0(z-l)P_2(x,t)) + \partial_x(a_0(z-l)\lambda_2(x,t)P_2(x,t)) \le 0, \text{ for any } l \in (-\infty,\infty), \\ z(x,0) = z_0(x), \end{cases} \tag{3.11}$$

in the generalized sense of distribution in Π_T.
Here a_0 is a nonnegative and even Lipschitz function satisfying $a_0(0) = 0, a_0(w) > 0$ for $w \neq 0$, and the vector fields $(P_i(x,t), \lambda_i(x,t)P_i(x,t))$, $P_i(x,t) \ge 0$, $i = 1,2$, are two divergence-free fields in (x,t)-space and satisfy

$$P_1(x,0) \ge P_0 > 0, \qquad P_2(x,0) \ge P_0 > 0.$$

Then we have the following theorem.

Theorem 3.2 *Suppose that L^∞ functions $w(x,t)$ and $z(x,t)$ satisfy Framework (A) in Π_T. Then*

$$w(x,t) = w_0(T_0^{-1}(p(x,t))), \qquad z(x,t) = z_0(\bar{T}_0^{-1}(q(x,t))), \qquad a.e.\ t \in [0,T), \tag{3.12}$$

and, for any positive constant N,

$$\begin{cases} \int_{|x|\le N} a_0(w(x,t) - z(\bar{T}_t^{-1}T_t(x),t))P_1(x,t)P_2(\bar{T}_t^{-1}T_t(x),t)dx \\ \quad \le \|P_1\|_{L^\infty}\|P_2\|_{L^\infty} \int_{|x|\le K_0(N+Lt)} a_0(w_0(x) - z_0(\bar{T}_0^{-1}T_0(x)))dx, \\ \int_{|x|\le N} a_0(w(T_t^{-1}\bar{T}_t(x),t) - z(x,t))P_1(T_t^{-1}\bar{T}_t(x),t)P_2(x,t)dx \\ \quad \le \|P_1\|_{L^\infty}\|P_2\|_{L^\infty} \int_{|x|\le K_0(N+Lt)} a_0(w_0(T_0^{-1}\bar{T}_0(x)) - z_0(x))dx, \end{cases} \tag{3.13}$$

or,

$$\begin{cases} \int_{|x|\le N} a_0(w(x,t) - z(\bar{T}_t^{-1}\bar{T}_0T_0^{-1}T_t(x),t))P_1(x,t)P_2(\bar{T}_t^{-1}\bar{T}_0T_0^{-1}T_t(x),t)dx, \\ \quad \le \|P_1\|_{L^\infty}\|P_2\|_{L^\infty} \int_{|x|\le K_0(N+Lt)} a_0(w_0(x) - z_0(x))dx \\ \int_{|x|\le N} a_0(w(T_t^{-1}T_0\bar{T}_0^{-1}\bar{T}_t(x),t) - z(x,t))P_1(T_t^{-1}T_0\bar{T}_0^{-1}\bar{T}_t(x),t)P_2(x,t)dx \\ \quad \le \|P_1\|_{L^\infty}\|P_2\|_{L^\infty} \int_{|x|\le K_0(N+Lt)} a_0(w_0(x) - z_0(x))dx, \end{cases} \tag{3.14}$$

for almost all $t \in [0,T)$, where $(p,t) = T(x,t) \equiv (T_t(x),t)$ and $(q,t) = \bar{T}(x,t) \equiv (\bar{T}_t(x),t)$ are the quasidecoupling transformations corresponding to the vector fields $(P_i, \lambda_i P_i), i = 1,2$, respectively, $L = \max_{i=1,2}(\sup_{\Pi_T} |\lambda_i(x,t)|)$, and $K_0 = max_{i=1,2}\|P_i\|_{L^\infty}/P_0$.

We first introduce a lemma from [5].

Lemma 3.2 *Suppose that functions* $u(x,t) \in L^\infty([0,T]; L^1(-\infty,\infty))$ *and* $u_0(x) \in L^\infty$ *satisfy*

$$\int\int_{\Pi_T} u(x,t)\psi'(t)dxdt + \psi(0)\int_{-\infty}^{\infty} u_0(x)dx \geq 0,$$

for any nonnegative function $\psi(t) \in C_0^1[0,T)$. *Then* $\lim_{t\to 0}\int_{-\infty}^{\infty} u(x,t)dx$ *exists and satisfies*

$$\int_{-\infty}^{\infty} u(x,t)dx \leq \lim_{t\to 0}\int_{-\infty}^{\infty} u(x,t)dx \leq \int_{-\infty}^{\infty} u_0(x)dx, \qquad a.e.\ t \in [0,T).$$

Proof of Theorem 3.2. Using the quasidecoupling transformation (3.8) for (3.10) and (3.11), respectively, we obtain from Theorem 3.1 that the function $w(T^{-1}(p,t))$ satisfies

$$\begin{cases} \partial_t a(w(T^{-1}(p,t)) - k) \leq 0, \\ w|_{t=0} = w_0(T_0^{-1}(p)), \end{cases} \tag{3.15}$$

in the sense of distribution under the $(p,t) = T(x,t)$ coordinates for any $k \in \mathbb{R}^1$, and the function $z(\bar{T}^{-1}(q,t))$ satisfies

$$\begin{cases} \partial_t a(z(\bar{T}^{-1}(q,t)) - l) \leq 0, \\ z|_{t=0} = z_0(\bar{T}_0^{-1}(q)), \end{cases} \tag{3.16}$$

in the sense of distribution under the $(q,t) = \bar{T}(x,t)$ coordinates for any $l \in \mathbb{R}^1$.

First we show (3.12). Suppose that $\psi(p,t;p') \geq 0$ is a $C^\infty(\Pi_T \times \mathbb{R}^1)$ function and, for any fixed $p' \in \mathbb{R}^1 (p \in \mathbb{R}^1)$, $\psi(p,t;p') \in C_0^\infty(\Pi_T)$. Now we choose $k = w_0(T_0^{-1}(p'))$ and the test function $\psi(\cdot,\cdot;p')$ for (3.15). We obtain

$$\begin{array}{l} \int\int_{\Pi_T} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p')))\psi_t(p,t;p')dpdt \\ + \int_{-\infty}^{\infty} a_0(w_0(T_0^{-1}(p)) - w_0(T_0^{-1}(p')))\psi(p,0;p')dp \geq 0. \end{array} \tag{3.17}$$

Integrate (3.17) over $p' \in \mathbb{R}^1$. We have

$$\begin{aligned} &\int_{-\infty}^{\infty}\int\int_{\Pi_T} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p')))\psi_t(p,t;p')dpdtdp' \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_0(w_0(T_0^{-1}(p)) - w_0(T_0^{-1}(p')))\psi(p,0;p')dpdp' \geq 0. \end{aligned} \tag{3.18}$$

Motivated by Kruzkov [31], we choose

$$\psi(p,t;p') = \phi(p,t)\omega_h(p-p'). \tag{3.19}$$

Here

$$\begin{array}{l} \phi(p,t) \geq 0, \qquad \phi(p,t) \in C_0^\infty(\Pi_T), \\ \omega_h(s) = \frac{1}{h}\omega(\frac{s}{h}), \end{array}$$

where $\omega(s)$ is the function satisfying

$$\omega(s) \in C_0^\infty(-\infty,\infty), \quad \omega(s) \geq 0, \quad \omega(s) = \omega(-s), \quad \text{supp } \omega(s) \subset \{s : |s| \leq 1\},$$

and

$$\int_{-\infty}^{\infty} \omega(s)ds = 1.$$

Then we obtain from (3.18) that

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p')))\omega_h(p-p')\phi_t(p,t)dtdpdp' \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_0(w_0(T_0^{-1}(p)) - w_0(T_0^{-1}(p')))\omega_h(p-p')\phi(p,0)dpdp' \geq 0. \qquad (3.20)
\end{aligned}$$

On the other hand, for any $\phi \in C_0^\infty(\Pi_T)$, supp $\phi(\cdot,t) \subset \{p : |p| \leq N < \infty\}$, we have

$$\begin{aligned}
A_h \equiv & \Big| \textstyle\int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p')))\omega_h(p-p')\phi_t(p,t)dtdpdp' \\
& - \textstyle\int_0^T \int_{-\infty}^{\infty} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p)))\phi_t(p,t)dpdt| \\
\leq & \textstyle\frac{C}{h} \int_0^T \int\int_{|p-p'|\leq h, |p|\leq N} |a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p'))) - a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p)))|dtdpdp' \\
\leq & C \textstyle\int_{|\alpha|\leq h} H_h(\alpha)d\alpha,
\end{aligned} \qquad (3.21)$$

by using the Lipschitz condition of a_0, where

$$H_h(\alpha) = \frac{1}{h} \int_{|\beta|\leq h} |w_0(T_0^{-1}(\alpha)) - w_0(T_0^{-1}(\alpha - \beta))|d\beta.$$

Since almost all points α of $[-N, N]$ are the Lebesgue points of $H_h(\alpha)$, we deduce

$$\lim_{h\to 0} H_h(\alpha) = 0, \qquad \text{a.e. } \alpha \in [-N, N].$$

Note that $|H_h(\alpha)| \leq 4 \sup|w_0|$. Using the Lebesgue dominated theorem, we conclude from (3.21)

$$\lim_{h\to 0} A_h = 0. \qquad (3.22)$$

Moreover, we have

$$\lim_{h\to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_0(w_0(T_0^{-1}(p)) - w_0(T_0^{-1}(p')))\omega_h(p-p')\psi(p,0)dpdp' = 0. \qquad (3.23)$$

Combining (3.20) with (3.21)-(3.23), we finally obtain

$$\int\int_{\Pi_T} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p)))\phi_t(p,t)dpdt \geq 0. \qquad (3.24)$$

Choose $\phi(p,t) = \phi(p)\psi(t)$, $\phi(p) \in C_0^\infty(-\infty, \infty)$, $\psi(t) \in C_0^1[0,T)$. Then

$$\int_0^T \int_{-\infty}^{\infty} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p)))\phi(p)\psi'(t)dpdt \geq 0, \qquad (3.25)$$

for almost all $t \in [0,T)$. Using Lemma 3.2, we have

$$\int_{-\infty}^{\infty} a_0(w(T^{-1}(p,t)) - w_0(T_0^{-1}(p)))\phi(p)dp \leq 0, \qquad (3.26)$$

for almost all $t \in [0,T)$. Therefore,

$$w(T^{-1}(p,t)) = w_0(T_0^{-1}(p));$$

that is,

$$w(x,t) = w_0(T_0^{-1}(p(x,t))), \qquad \text{a.e. } \{(x,t) \in \Pi_T : P_1(x,t) > 0\}.$$

Similarly, we also have

$$z(x,t) = z_0(\bar{T}_0^{-1}(q(x,t))), \qquad \text{a.e. } \{(x,t) \in \Pi_T : P_2(x,t) > 0\}.$$

This proves (3.12).

Notice that, for fixed t, both $p(x,t)$ and $q(x,t)$ are monotonically increasing functions of x. If x lies in $[-N,N]$, then both intervals $[p(-N,t)/P_0, p(N,t)/P_0]$ and $[q(-N,t)/P_0, q(N,t)/P_0]$ lie in $[-K_0(N+Lt), K_0(N+Lt)]$, where $L = \max_{i=1,2} \sup_{\Pi_T} |\lambda_i(x,t)|$ and $K_0 = \max_{i=1,2} \|P_i\|_{L^\infty}/P_0$. Therefore, for any nonnegative function $\psi(t) \in C_0^\infty[0,T]$ and any positive constant N,

$$\begin{aligned}
&\textstyle\int_0^T \int_{|x|\le N} a_0(w(x,t) - z(\bar{T}_t^{-1}T_t(x),t))P_1(x,t)P_2(\bar{T}_t^{-1}T_t(x),t)\psi(t)dxdt \\
&= \textstyle\int_0^T \int_{|x|\le N} a_0(w_0(T_0^{-1}(T_t(x))) - z_0(\bar{T}_0^{-1}(T_t(x))))P_1(x,t)P_2(\bar{T}_t^{-1}T_t(x),t)\psi(t)dxdt \\
&\le \|P_2\|_{L^\infty} \textstyle\int_0^T \psi(t)dt \int_{|p|\le N\|P_1\|_{L^\infty}+Lt} a_0(w_0(T_0^{-1}(p)) - z_0(\bar{T}_0^{-1}(p)))dp.
\end{aligned}$$

Then, for almost all $t \in [0,T)$,

$$\begin{aligned}
&\int_{|x|\le N} a_0(w(x,t) - z(\bar{T}_t^{-1}T_t(x),t))P_1(x,t)P_2(\bar{T}_t^{-1}T_t(x),t)dx \\
&\le \|P_2\|_{L^\infty} \int_{|p|\le N\|P_1\|_{L^\infty}+Lt} a_0(w_0(T_0^{-1}(p)) - z_0(\bar{T}_0^{-1}(p)))dp \\
&\le \|P_1\|_{L^\infty}\|P_2\|_{L^\infty} \int_{|x|\le K_0(N+Lt)} a_0(w_0(x) - z_0(\bar{T}_0^{-1}T_0(x)))dx.
\end{aligned}$$

Similarly, we can also prove other inequalities in (3.13) and (3.14). This completes the proof of Theorem 3.2.

Now we consider the deviation between $(p(x,t),t) \equiv (p(x,t;w),t) = T(x,t)$ and $(q(x,t),t) \equiv (q(x,t;z),t) = \bar{T}(x,t)$.

Theorem 3.3 *Suppose that an L^∞ function $\lambda(y,t;w_0(y))$ satisfies*

$$\int_{|y|\le N_0} |\lambda(y,t;w_0(y)) - \lambda(y,t;\bar{w}_0(y))|dy \le \int_{|y|\le N_0} |w_0(y) - \bar{w}_0(y)|dy, \tag{3.27}$$

and the Lipschitz functions $p(x,t)$ and $q(x,t)$ are solutions of the Cauchy problems

$$\begin{cases} p_t + \lambda(p,t;w_0(p))p_x = 0, \\ p|_{t=0} = p_0(x), \end{cases} \tag{3.28}$$

and

$$\begin{cases} q_t + \lambda(q, t; \bar{w}_0(q)) q_x = 0, \\ q|_{t=0} = q_0(x), \end{cases} \tag{3.29}$$

respectively. Then, for almost all $t \in [0, T]$,

$$\int_{|x| \le N} |p(x,t) - q(x,t)| dx \le C \int_{|x| \le \bar{N} + Lt} (|w_0(x) - \bar{w}_0(x)| + |p_0(x) - q_0(x)|) dx, \tag{3.30}$$

for any $N > 0$, *where* $L = \max(\sup_{\Pi_T} |\lambda(y, t; w_0(y))|, \sup_{\Pi_T} |\lambda(y, t; \bar{w}_0(y))|)$, $\bar{N} = \max(\sup_{[-N,N]\times[0,T]} |p_0(x)|, \sup_{[-N,N]\times[0,T]} |q_0(x)|, N)$, *and* C *is the constant independent of* N *and* L. *In particular, if*

$$\begin{cases} w_0(y) = \bar{w}_0(y), \qquad a.e. \\ p_0(x) = q_0(x), \qquad a.e. \end{cases} \tag{3.31}$$

then

$$p(x,t) = q(x,t), \qquad a.e.\ t \in [0, T].$$

Proof. First we have from (3.28)-(3.29) that

$$(p - q)_t + (\int_q^p \lambda(\xi, t; w_0(\xi)) d\xi)_x + (\lambda(q, t; w_0(q)) - \lambda(q, t; \bar{w}_0(q))) q_x = 0.$$

Therefore we conclude, for almost all $(x, t) \in \Pi_T$,

$$|p - q|_t + (\int_q^p \lambda(\xi, t; w_0(\xi)) d\xi \mathrm{sign}(p - q))_x + \mathrm{sign}(p - q)(\lambda(q, t; w_0(q)) - \lambda(q, t; \bar{w}_0(q))) q_x = 0.$$

Finally, we can obtain from the boundness of the dependent domain of $p - q$ that, for almost all $t \in [0, T]$,

$$\begin{aligned}
&\textstyle\int_{|x|\le N} |p(x,t) - q(x,t)| dx \\
&\le C \left\{ \begin{array}{l} \int_{|x|\le N+Lt} |p_0(x) - q_0(x)| dx \\ + \int_0^t \int_{|x|\le N} |\lambda(q(x,s), s; w_0(q(x,s))) - \lambda(q(x,s), s; \bar{w}_0(q(x,s)))| dx ds \end{array} \right\} \\
&\le C \textstyle\int_0^t \int_{|x|\le \bar{N}+Ls} |\lambda(x, s; w_0(x)) - \lambda(x, s; \bar{w}_0(x))| dx ds + \int_{|x|\le N+Lt} |p_0(x) - q_0(x)| dx \\
&\le C \textstyle\int_{|x|\le \bar{N}+Lt} (|w_0(x) - \bar{w}_0(x)| + |p_0(x) - q_0(x)|) dx.
\end{aligned}$$

This completes the proof.

A direct corollary follows.

Corollary 3.2 *Suppose that there exists an* L^∞ *function* $\lambda(y, t)$ *such that*

$$\begin{cases} \lambda_1(x,t) = \lambda(p(x,t), t), \\ \lambda_2(x,t) = \lambda(q(x,t), t), \end{cases} \tag{3.32}$$

and suppose the function $a_0(w) = |w|^m$, $m > 0$, *and* $P_1(x,0) = P_2(x,0)$ *in the Framework(A). Then the weighted* L^m*-stability with weight* $(P_1P_2)(x,t)$ *for functions* $w(x,t)$ *and* $z(x,t)$ *holds; that is, for almost all* $t \in [0,T)$,

$$\int_{|x|\leq N} |w(x,t)-z(x,t)|^m P_1(x,t)P_2(x,t)dx \leq \|P_1\|_{L^\infty}\|P_2\|_{L^\infty} \int_{|x|\leq K_0(N+Lt)} |w_0(x)-z_0(x)|^m dx, \tag{3.33}$$

where $L = \max_{i=1,2} \sup_{\Pi_T} |\lambda_i(x,t)|$.

Furthermore, we also obtain a uniqueness theorem from Theorem 3.2 and Theorem 3.3.

Theorem 3.4 *(Uniqueness). Suppose that the functions* $w(x,t)$ *and* $z(x,t)$ *satisfy Framework (A) with (3.27), (3.32), and suppose* $a_0(0) = 0$ *and* $a_0(w) > 0$ *for* $w \neq 0$. *Then* $w_0(x) = z_0(x)$ *leads to*

$$w(x,t) = z(x,t), \qquad a.e.\ \Pi_T - \{(x,t) : P_1(x,t) = P_2(x,t) = 0\}.$$

Remark. Theorem 3.4 shows that the solution of the Cauchy problem

$$\begin{cases} \partial_t(a_0(w-k)P(x,t)) + \partial_x(a_0(w-k)\lambda(x,t)P(x,t)) \leq 0, & \text{for any } k \in (-\infty,\infty), \\ w(x,0) = w_0(x), \end{cases}$$

with $a_0(0) = 0$, $a_0(w) > 0$ for $w \neq 0$, is independent of the choice of the function $P(x,t) > 0$ with the same initial data $P(x,0)$. This finding is essential to solve the uniqueness and stability problems of solutions to systems of conservation laws in Sections 4-6.

c. Framework for the Limit Behavior of Approximate Solutions

We shall establish a framework for the limit dynamics behavior of approximate solutions to conservation laws. We recall from Section 2 that with each point $(x,t) \in \Pi_T$ is associated a Young measure over the range space $\mathbb{R}^m$, which corresponds to a sequence of functions $w^\epsilon(x,t)$ and describes the oscillation behavior of w^ϵ at (x,t). For this reason, we shall establish the framework for a family of Radon measures.

Consider a family of Radon measures $\{\mu_{x,t} : \Pi_T \to \mathrm{Prob}(\mathbb{R})\}$, supp $\mu_{x,t} \subset\subset (-\infty,\infty)$, with a family of the initial Radon measures $\{\mu_{x,0}\}$, satisfying the following Framework (B):
B_1: There exists a continuous state variable set Λ such that, for any $a(\lambda) \in \Lambda$,

$$\begin{cases} \partial_t(<\mu_{x,t}, a(\lambda)> P(x,t)) + \partial_x(<\mu_{x,t}, a(\lambda)> \lambda(x,t)P(x,t)) \leq 0, \\ \mu_{x,t}|_{t=0} = \mu_{x,0}, \end{cases} \tag{3.34}$$

in the generalized sense of distribution in Π_T.
B_2: The set Λ is sufficiently large such that, for any function of bounded variation $g(\lambda)$,

$$\int_\alpha^\beta a(\lambda)dg(\lambda) \leq 0 \tag{3.35}$$

implies that $g(\lambda)$ is a monotonically decreasing function.
Here the vector field $(P(x,t), \lambda(x,t)P(x,t))$, $P(x,t) \geq 0$, is divergence free in the (x,t)-space

$$\operatorname{div}(P(x,t), \lambda(x,t)P(x,t)) = 0, \tag{3.36}$$

and

$$\int_{-\infty}^{0} P(x,0)dx = \int_{0}^{\infty} P(x,0)dx = \infty. \tag{3.37}$$

Then we have the following theorem.

Theorem 3.5 *Suppose that a family of Radon measures* $\{\mu_{x,t} : \Pi_T \to Prob(\mathbb{R})\}$, *supp* $\mu_{x,t} \subset\subset (-\infty, \infty)$, *with a family of the initial Radon measures* $\{\mu_{x,0}\}$, *satisfies Framework (B). Then*

$$\mu_{x,t} \leq \mu_{T_0^{-1}(p(x,t)),0}, \qquad a.e.\ \{(x,t) \in \Pi_T : P(x,t) > 0\}, \tag{3.38}$$

where $p : \Pi_T \to \mathbb{R}^1$ *is determined by the quasidecoupling transformation, that is,*

$$\begin{cases} \frac{\partial p(x,t)}{\partial x} = P(x,t), \\ \frac{\partial p(x,t)}{\partial t} = -\lambda(x,t)P(x,t), \\ p(x,0) = \int_0^x P(\xi,0)d\xi. \end{cases} \tag{3.39}$$

Remark 1. Suppose that the conclusion in B_2 is that $g(\lambda)$ is a monotonically increasing function. Then the inequality (3.38) of Theorem 3.5 is changed into

$$\mu_{x,t} \geq \mu_{T_0^{-1}(p(x,t)),0}, \qquad \text{a.e. } \{(x,t) \in \Pi_T : P(x,t) > 0\}. \tag{3.40}$$

Remark 2. The formulae (3.38) and (3.40) express that, along the dynamic curve

$$C: \qquad \frac{dx}{dt} = \lambda(x,t), \tag{3.41}$$

the value of the measures $\mu_{x,t}$ is smaller or larger than the value of the initial measures $\mu_{x,0}$. The formula (3.38) indicates that the Radon measures $\mu_{x,t}$ are Dirac measures, provided that $\mu_{x,0}$ are Dirac measures, which shows convergence of the corresponding approximate solutions in the strong topology in $\{(x,t) \in \Pi_T : P(x,t) > 0\}$. The formula (3.40) indicates that the Radon measures $\mu_{x,t}$ are not Dirac measures, provided that $\mu_{x,0}$ are not Dirac measures, which shows that the initial oscillations propagate along the curve (3.41). This framework is useful for studying the limit behavior of approximate solutions to conservation laws, especially the dynamic behavior of the initial oscillations (see Sections 4–6).

Proof of Theorem 3.5. First we have from (3.34) that

$$\int\int_{\Pi_T} < \mu_{x,t}, a(\lambda) > P(x,t)\{\phi_t + \lambda(x,t)\phi_x\}dxdt + \int_{-\infty}^{\infty} < \mu_{x,0}, a(\lambda) > P(x,0)\phi(x,0)dx \geq 0, \tag{3.42}$$

for any $a \in \Lambda$ and any nonnegative function $\phi \in BV(\Pi_T) \cap C_0^1(\Pi_T - \{(x,t) : P(x,t) = 0\})$. Notice that the vector field $(P(x,t), \lambda(x,t)P(x,t))$ is divergence free in the (x,t)-space and

$$\int_{-\infty}^{0} P(x,0)dx = \int_{0}^{\infty} P(x,0)dx = +\infty.$$

Make the quasidecoupling transformation $T : (x,t) \to (p,t)$ in the integral (3.42). We conclude from Lemma 3.1 that the transformation is proper and onto in Π_T. We obtain from (3.42) that

$$\int_0^T \int_{-\infty}^{\infty} < \mu_{T^{-1}(p,t)}, a(\lambda) > \phi(p)\psi'(t)dpdt + \psi(0) \int_{-\infty}^{\infty} < \mu_{T_0^{-1}(p),0}, a(\lambda) > \phi(p)dp \geq 0, \forall a \in \Lambda,$$

for any $\phi(p) \in C_0^1(-\infty,\infty)$, $\psi(t) \in C_0^1[0,T)$, $\phi(p)\psi(t) \geq 0$.

Using Lemma 3.2, we have

$$\int_{-\infty}^{\infty} < \mu_{T^{-1}(p,t)}, a(\lambda) > \phi(p)dp \leq \int_{-\infty}^{\infty} < \mu_{T_0^{-1}(p),0}, a(\lambda) > \phi(p)dp;$$

that is,

$$\int_{-\infty}^{\infty} < \mu_{T^{-1}(p,t)} - \mu_{T_0^{-1}(p),0}, a(\lambda) > \phi(p)dp \leq 0,$$

for any nonnegative $\phi \in C_0^1(-\infty,\infty)$.

Using Corollary 3.1, we have

$$\int_{-\infty}^{\infty} < \mu_{x,t} - \mu_{T_0^{-1}(p(x,t)),0}, a(\lambda) > \phi(p(x,t))P(x,t)dx \leq 0,$$

for almost all $t \in (0,T)$. Therefore, we obtain

$$< \mu_{x,t} - \mu_{T_0^{-1}(p(x,t)),0}, a(\lambda) > \leq 0, \qquad \text{a.e. } \{(x,t) \in \Pi_T : P(x,t) > 0\}. \tag{3.43}$$

Notice that Λ is sufficiently large such that the assumption (B_2) holds. Thus we obtain from (3.43)

$$\mu_{x,t} \leq \mu_{T_0^{-1}(p(x,t)),0}, \qquad \text{a.e. } \{(x,t) : P(x,t) > 0\}.$$

Therefore, one arrives at (3.38).

4 Systems with Fully Contact Fields

In this section we consider how the quasidecoupling method can be applied to the Cauchy problem of $n \times n$ systems with n independent contact fields, that is, linearly degenerate fields, which are described by Serre [50]:

$$\begin{cases} u_t + f(u)_x = 0, & u \in \mathbb{R}^n, \\ u|_{t=0} = u_0(x), \end{cases} \tag{4.1}$$

with

$$\partial_{w_i}\lambda_i \equiv 0, \qquad \text{for all } 1 \le i \le n, \tag{4.2}$$

and

$$\partial_{w_k}((\lambda_j-\lambda_i)^{-1}\partial_{w_j}\lambda_i) = \partial_{w_j}((\lambda_k-\lambda_i)^{-1}\partial_{w_k}\lambda_i), \tag{4.3}$$

for distinct i, j, and k, where λ_i , $1 \le i \le n$, are eigenvalues of the Jacobian $\nabla f(u)$, and w_i, $1 \le i \le n$, are corresponding Riemann invariants. For simplicity, we assume that

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n, \qquad \lambda_i \neq 0, \qquad 1 \le i \le n, \tag{4.4}$$

and the Riemann invariants w_i, $1 \le i \le n$, are smooth functions of u.

The entropy-entropy flux fields for such systems are of the following general representation from (2.17) (see also [50]):

$$\begin{cases} \eta_a = \sum_{i=1}^m N_i a_i(w_i), \\ q_a = \sum_{i=1}^m \lambda_i N_i a_i(w_i), \end{cases} \tag{4.5}$$

for any continuous vector function

$$a(w) = (a_1(w_1), \ldots, a_m(w_m)), \tag{4.6}$$

where

$$N_i = \exp\left\{\int^{w_j} \frac{\partial_{w_j}\lambda_i}{\lambda_j-\lambda_i} dw_j\right\} \tag{4.7}$$

is independent of j, $j \neq i$, because of the equality (2.5) for all distinct i, j, and k, and

$$N_i(u) \ge c_0 > 0, \tag{4.8}$$

on the parallelipiped $K = \prod_i [\inf w_i^0, \sup w_i^0]$ provided (4.4) holds.

A simple 2×2 example of such systems is (see [52])

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} -u_1^2 - u_2 \\ u_1^3 + u_1 u_2 \end{pmatrix}_x = 0.$$

Serre [50] proved a global existence and uniqueness theorem of a smooth solution for the Cauchy problem (4.1) by using a hierarchy of transport equations involving higher derivatives of w_i, and the local existence and uniqueness theorem of Kato [28].

Theorem 4.1 *Assume that λ_i, $1 \le i \le n$, are smooth functions of $w = (w_1, \ldots, w_n)$. Then there exists a unique smooth solution $u(x,t)$ of the Cauchy problem (4.1), satisfying the following regularity: $w \in C^k(\mathbb{R}_+^2; K) \equiv \{w \in C^k(\mathbb{R}_+^2) : w \in K\}$ if and only if $w^0 \in C^k(\mathbb{R}_+^2; K)$.*

This result shows that the smooth regularity of the solution for the Cauchy problem (4.1) is the same as the smooth regularity of the initial data because of the degeneracy of the systems. With the aid of the quasidecoupling method, we shall now study the well-posedness of generalized solutions for the Cauchy problem with the initial data $w_0 \in K$ and $w_0 \in BV(\mathbb{R}^2_+; K)$: existence, uniqueness and stability as well as the dynamic behavior of the initial oscillations for the generalized solutions.

Definition 4.1. A bounded measurable vector function $u(x,t) \in \mathbb{R}^n$ is called a generalized solution of the Cauchy problem (4.1) in Π_T if the equality

$$\int\int_{\Pi_T} \{\eta_a(u(x,t))\phi_t + q_a(u(x,t))\phi_x\}dxdt + \int_{-\infty}^{\infty} \eta_a(u_0(x))\phi(x,0)dx = 0 \tag{4.9}$$

holds for any continuous vector function $a(w)$ and any smooth function $\phi \in C_0^1(\Pi_T)$.

a. Propagation of Oscillations and Existence of Generalized Solutions

We first study the dynamic behavior of the initial oscillations, which are of $O(1)$ amplitude with high frequency, as time evolves. That is, we study the limit behavior of the smooth solutions $u^\epsilon(x,t)$ in Theorem 4.1 that corresponds to the oscillatory sequence of large initial data:

$$\begin{cases} u_t + f(u)_x = 0, \\ u|_{t=0} = u_0^\epsilon(x) \in C^k(\mathbb{R}^2_+; K), \qquad k \geq 0. \end{cases} \tag{4.10}$$

We have the following theorem.

Theorem 4.2 *Consider a hyperbolic system of conservation laws with fully contact fields satisfying (4.3) and (4.4). Suppose the initial data sequence* $u_0^\epsilon(x) \in C^k(\mathbb{R}^2_+; K)$ *and the bounded set* K *independent of* ϵ*. Then the corresponding* C^k *solutions* $u^\epsilon(x,t)$ *have the following limit behavior.*

(i) Let one of the corresponding initial Riemann invariants

$$w_{i0}^\epsilon(x) = w_i(u_0^\epsilon(x))$$

be a highly oscillatory sequence. Then the initial oscillations will propagate along the corresponding linearly degenerate field.

(ii) Let the initial data sequence $u_0^\epsilon(x)$ *be compact in the strong topology of* L^∞*. Then there exists a subsequence converging pointwise a.e. to an* L^∞ *generalized solution to the Cauchy problem (4.1) with the Cauchy data* $u_0(x) = w^* - \lim u_0^\epsilon(x)$.

Remark. Theorem 4.2 shows that the initial oscillations can propagate along the corresponding linearly degenerate field, which is not affected by other fields. It is consistent with theoretical results of propagation of the initial oscillations for 2 × 2 systems in [4, 5] and with the evidence of oscillatory solutions given in [48, 2]. The phenomenon differs from those in [34, 15, 3].

Proof of Theorem 4.2. We first obtain from Theorem 4.1

$$u^\epsilon(x,t) \in K \qquad (K \text{ is independent of } \epsilon) \tag{4.11}$$

and

$$\eta_a(u^\epsilon(x,t))_t + q_a(u^\epsilon(x,t))_x = 0, \tag{4.12}$$

for any continuous vector function $a(w)$. Using Theorem 2.1, we conclude that there exists a family of Young measures $\nu_{x,t} : \mathbb{R}^2_+ \to \mathrm{Prob}(\mathbb{R}^n)$, supp $\nu_{x,t} \subset K$, uniquely determined by the approximate solutions $u^\epsilon(x,t)$, satisfying

$$\partial_t < \nu_{x,t}, N_i a_i(w_i) > +\partial_x < \nu_{x,t}, \lambda_i N_i a(w_i) >= 0, \qquad 1 \le i \le n, \tag{4.13}$$

and, for any continuous vector functions $a(w)$ and $b(w)$,

$$< \nu_{x,t}, \eta_a >< \nu_{x,t}, q_b > - < \nu_{x,t}, \eta_b >< \nu_{x,t}, q_a >= 0. \tag{4.14}$$

We conclude from (4.14) that there exists an L^∞ function $\bar\lambda_i(x,t)$ such that

$$< \nu_{x,t}, \lambda_i N_i a_i(w_i) >= \bar\lambda_i(x,t) < \nu_{x,t}, N_i a_i(w_i) >, \tag{4.15}$$

for any continuous vector function $a(w)$. Therefore, (4.13) can be written into the following form:

$$\partial_t < \nu_{x,t}, N_i a_i(w_i) > +\partial_x(\bar\lambda_i(x,t) < \nu_{x,t}, N_i a_i(w_i) >) = 0, \qquad 1 \le i \le n. \tag{4.16}$$

In particular,

$$\partial_t < \nu_{x,t}, N_i > +\partial_x(\bar\lambda_i(x,t) < \nu_{x,t}, N_i >) = 0, \qquad 1 \le i \le n, \tag{4.17}$$

and

$$< \nu_{x,t}, N_i >\ge c_0 > 0.$$

For fixed i, $1 \le i \le n$, we define a family of probability measures:

$$\mu^i_{x,t} = \frac{N_i \nu_{x,t}}{< \nu_{x,t}, N_i >}.$$

Then $\mu^i_{x,t}$ satisfies

$$\partial_t(< \mu^i_{x,t}, a(w_i) >< \nu_{x,t}, N_i >) + \partial_x(< \mu^i_{x,t}, a(w_i) > \bar\lambda_i(x,t) < \nu_{x,t}, N_i >) = 0. \tag{4.18}$$

We conclude from (4.17)-(4.18) that the family of probability measures satisfies Framework (B) and the set Λ is the space of continuous functions. Therefore we have from Theorem 3.5

$$\mu^i_{x,t} = \mu^i_{T_0^{-1}(p_i(x,t)),0}, \tag{4.19}$$

where $p_i(x,t)$ is determined by

$$\begin{cases} \frac{\partial p_i(x,t)}{\partial x} = < \nu_{x,t}, N_i > \geq c_0 > 0, \\ \frac{\partial p_i(x,t)}{\partial t} = -\bar{\lambda}_i(x,t) < \nu_{x,t}, N_i >, \\ p_i(x,0) = \int_0^x < \nu_{\xi,0}, N_i > d\xi. \end{cases}$$

If the composite sequence of approximate initial data $w^\epsilon_{i0}(x) = w_i(u^\epsilon_0(x))$ is a highly oscillatory function sequence, then $\mu^i_{x,0}$ is not a point measure; that is, $\mu^i_{x,t}$ is not a point measure from (4.19). Therefore $\nu_{x,t}$ is not a point measure. This fact means that the initial oscillations propagate along the “average” eigendirection:

$$\frac{dx}{dt} = \bar{\lambda}_i(x,t) \equiv \frac{< \nu_{x,t}, \lambda_i N_i >}{< \nu_{x,t}, N_i >}.$$

If the initial data sequence $u^\epsilon_0(x)$ is compact in the strong topology of L^∞, then there exists a subsequence (still labeled) $u^\epsilon_0(x)$ such that all composite sequences w^ϵ_{i0} converge strongly to $w_{i0}(x)$, a.e.,

$$w^\epsilon_{i0}(x) = w_i(u^\epsilon_0(x)) \to w_{i0}(x), \qquad \text{a.e.} \qquad 1 \leq i \leq n,$$

and $\mu^i_{x,0}, 1 \leq i \leq n$, is a Dirac measure. Therefore $\mu^i_{x,t}, 1 \leq i \leq n$, is a Dirac measure from (4.19). Finally, we conclude that $\nu_{x,t}$ is a Dirac measure. We know from Section 2 that there exists a subsequence (still labeled) $u^\epsilon(x,t)$ converging pointwise a.e. and that the limit function is a generalized solution of the corresponding Cauchy problem (4.1) with the Cauchy data $u_0(x) = w^* - \lim_{\epsilon\to 0} u^\epsilon_0(x)$. This completes the proof of Theorem 4.2.

A direct corollary follows.

Corollary 4.1 *For the Cauchy problem (4.1) with fully contact fields, the solution operator* $S_t : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ *with* $S_t(u^0) = u(\cdot, t)$ *is not compact.*

As a byproduct, we have the following theorem.

Theorem 4.3 *There exists an* L^∞ *generalized solution* $u(x,t) \in K$ *to the Cauchy problem (4.1) with the* L^∞ *Cauchy data* $u_0(x) \in K$*, that satisfies*

$$\eta_a(u(x,t))_t + q_a(u(x,t))_x = 0, \tag{4.20}$$

for any continuous vector function $a(w)$ *in the sense of distribution.*

Proof. Note that, for any $u_0(x) \in L^\infty$, there exists an approximate sequence $u_0^\epsilon(x) \in C^\infty(\mathbb{R}; K)$ of $u_0(x)$ such that, for any $N > 0$,

$$\|u_0^\epsilon - u_0\|_{L^p(-N,N)} \to 0, \qquad \epsilon \to 0. \tag{4.21}$$

In fact, we can take

$$u_0^\epsilon(x) = (J_\epsilon * u_0)(x),$$

where J_ϵ is a standard compactly supported nonnegative mollifier:

$$J_\epsilon = \epsilon J(x/\epsilon), \quad J(x) \geq 0, \quad \int J(x)dx = 1, \quad \text{supp } J(x) \subset [-1, 1].$$

Then we consider the Cauchy problem (4.1) with the Cauchy initial data $u_0^\epsilon(x)$. We obtain from Theorem 4.1 that there exists a unique solution $u^\epsilon(x,t)$ for this Cauchy problem. Using Theorem 4.1, we conclude that there exists a subsequence $u^{\epsilon_k}(x,t)$ converging a.e. to an L^∞ solution $u(x,t)$ for the Cauchy problem (4.1) satisfying (4.20).

Finally, we establish a regularity theorem for the generalized solutions of the Cauchy problem (4.1) in Theorem 4.3 by using the quasidecouplind method.

Theorem 4.4 *Consider the generalized solutions of the Cauchy problem (4.1) satisfying (4.20). If the initial data* $u_0(x) \in BV(\mathbb{R}^1; K)$, *then the corresponding generalized solution* $u(x,t)$ *is also a bounded variation function,*

$$u(x,t) \in BV(\mathbb{R}_+^2; K), \tag{4.22}$$

satisfying (4.20).

Proof. Suppose that $w_i, 1 \leq i \leq n$, are Riemann invariants corresponding to i-th characteristic fields, respectively. Choose

$$P(x,t) = N_i(u(x,t)), \qquad \lambda(x,t) = \lambda_i(u(x,t)),$$

and

$$a_0(w - k) = |w_i - k|$$

in Framework (A). We immediately obtain from Theorem 3.2 that, for almost all $t \in [0, T)$ and $x \in (-\infty, \infty)$,

$$w_i(u(x,t)) = w_i(u_0(T_{0i}^{-1}(p_i(x,t)))), \tag{4.23}$$

where the Lipschitz function $T_{0i}^{-1}(p)$ is the inverse map of T_{0i}:

$$x \to T_{0i}(x) = \int_0^x N_i(u_0(\xi))d\xi,$$

and the Lipschitz function $p_i(x,t), 1 \le i \le n$, are determined by the quasidecoupling transformations corresponding to the vector fields $(N_i(u), \lambda(u)N_i(u))$, respectively. Therefore, we conclude from (4.23) that

$$w(u(x,t)) \in BV(\mathbb{R}^2_+; K), \qquad 1 \le i \le n,$$

and (4.22) follows.

b. Uniqueness and Stability of Generalized Solutions

We first study the stability of L^∞ generalized solutions for the Cauchy problem (4.1) established in Subsection 4a.

Theorem 4.5 *(Stability of L^∞ Generalized Solution). Suppose that L^∞ functions $u(x,t)$ and $v(x,t)$ are generalized solutions of the Cauchy problem (4.1) with initial data $u_0(x)$ and $v_0(x)$, respectively. Then, for almost all $t \in [0,T)$ and any positive constant m,*

$$w_i(u(x,t)) = w_i(u_0(T_{0i}^{-1}(p_i(x,t)))), \qquad w_i(v(x,t)) = w_i(v_0(\bar{T}_{0i}^{-1}(q_i(x,t)))), \qquad a.e., \tag{4.24}$$

$$\begin{cases} \int_{|x|\le N} |w_i(u(x,t)) - w_i(v(\bar{T}_{ti}^{-1}T_{ti}(x),t))|^m dx \\ \quad \le C\int_{|x|\le K_0(N+Lt)} |w(u_0(x)) - w(v_0(\bar{T}_{0i}^{-1}T_{0i}(x)))|^m dx, \\ \int_{|x|\le N} |w_i(u(T_{ti}^{-1}\bar{T}_{ti}(x),t)) - w_i(v(x,t))|^m dx \\ \quad \le C\int_{|x|\le K_0(N+Lt)} |w(u_0(T_{0i}^{-1}\bar{T}_{0i}(x))) - w(v_0(x))|^m dx, \end{cases} \tag{4.25}$$

$$\begin{cases} \int_{|x|\le N} |w_i(u(x,t)) - w_i(v(\bar{T}_{ti}^{-1}\bar{T}_{0i}T_{0i}^{-1}T_{ti}(x),t))|^m dx \\ \quad \le C\int_{|x|\le K_0(N+Lt)} |w(u_0(x)) - w(v_0(x))|^m dx, \\ \int_{|x|\le N} |w_i(u(T_{ti}^{-1}T_{0i}\bar{T}_{0i}^{-1}\bar{T}_{ti}(x),t)) - w_i(v(x,t))|^m dx \\ \quad \le C\int_{|x|\le K_0(N+Lt)} |w(u_0(x)) - w(v_0(x))|^m dx, \end{cases} \tag{4.26}$$

where $L = \max_{1\le i\le n}(\sup|\lambda_i(u)|, \sup|\lambda_i(v)|)$; C is a constant independent of m, L, and N; $K_0 = \max_{1\le i\le n}(\sup_\Pi N_i(u(x,t)), \sup_\Pi N_i(v(x,t)))/c_0$; $w = (w_1, w_2, ..., w_n)$ with Riemann invariants $w_i, 1 \le i \le n$, corresponding to the i-th characteristic field; and $p_i(x,t) = T_{ti}(x)$ and $q_i(x,t) = \bar{T}_{ti}(x)$ are determined by the quasidecoupling transformations corresponding to the vector fields $(N_i(u(x,t)), \lambda(u(x,t))N_i(u(x,t)))$ and $(N_i(v(x,t)), \lambda(v(x,t))N_i(v(x,t)))$.

Proof. Choose

$$\begin{aligned} P_1(x,t) &= N_i(u(x,t)), & \lambda_1(x,t) &= \lambda_i(u(x,t)), \\ P_2(x,t) &= N_i(v(x,t)), & \lambda_2(x,t) &= \lambda_i(v(x,t)), \end{aligned}$$

and

$$a_0(w-k) = |w_i - k|^m$$

in Framework (A). We immediately obtain from Theorem 3.2 that, for almost all $t \in [0,T)$ and $x \in (-\infty,\infty)$,

$$w_i(u(x,t)) = w_i(u_0(T_{0i}^{-1}(p_i(x,t)))), \qquad w_i(v(x,t)) = w_i(v_0(\bar{T}_{0i}^{-1}(q_i(x,t)))),$$

and then (4.25)–(4.26) follows for any $N > 0$.

Now we consider the deviation between $(p_i(x,t),t) \equiv (p_i(x,t;w(u(x,t))),t) = T_i(x,t)$ and $(q_i(x,t),t) \equiv (q_i(x,t;w(v(x,t))),t) = \bar{T}_i(x,t)$ for the systems consisting of two equations (n=2).

Theorem 4.6 *Suppose that L^∞ functions $u(x,t)$ and $v(x,t)$ are generalized solutions of the Cauchy problem (4.1) with initial data $u_0(x)$ and $v_0(x)$, respectively. Then, for almost all $t \in [0,T)$,*

$$\begin{aligned} &\textstyle\int_{|x|\le N} |(\bar{T}_{ti}^{-1}T_{ti} - I)(x)|dx \\ &\quad \le C \textstyle\int_{|x|\le K_0(N+\bar{L}t)} \{|w(u_0(x)) - w(v_0(x))| + |w(u_0(x)) - w(v_0(\bar{T}_{0i}^{-1}T_{0i}(x)))|\}dx, \qquad i = 1,2, \end{aligned} \tag{4.27}$$

where $L = \max_{i=1,2}(\sup|\lambda_i(u)|, \sup|\lambda_i(v)|)$; $\bar{L} = L + 1 + \max_{i=1,2}(\sup_{[0,T]}(|\bar{T}_{t_i}^{-1}T_{t_i}(\pm N)| - N)/t)$; $K_0 = \max_{i=1,2}(\sup_\Pi(\lambda_2(u) - \lambda_1(u))^{-1}, \sup_\Pi(\lambda_2(v) - \lambda_1(v))^{-1})/P_0$; C is a constant independent of m, L, and N; $w = (w_1,w_2)$ with Riemann invariants $w_i, 1 \le i \le 2$, corresponding to the i-th characteristic field; and $p_i(x,t) = T_{ti}(x)$ and $q_i(x,t) = \bar{T}_{ti}(x)$ are determined by the quasidecoupling transformations corresponding to the vector fields $((\lambda_2-\lambda_1)^{-1}, \lambda_i(\lambda_2-\lambda_1)^{-1})(u(x,t))$ and $((\lambda_2-\lambda_1)^{-1}, \lambda_i(\lambda_2-\lambda_1)^{-1})(v(x,t))$.

Proof. Using Theorem 4.5, we know that the Lipschitz functions $p_i(x,t)$ and $q_i(x,t)$, determined by the quasidecoupling transformations, satisfy the following Cauchy problems:

$$\begin{cases} p_i(x,t)_t + \lambda_i(w_j(u(x,t)))p_i(x,t)_x = 0, \qquad j \ne i, \\ p_i(x,0) = \int_0^x P_1(\xi,0)d\xi, \end{cases} \tag{4.28}$$

and

$$\begin{cases} q_i(x,t)_t + \lambda_i(w_j(v(x,t)))q_i(x,t)_x = 0, \qquad j \ne i, \\ q_i(x,0) = \int_0^x P_2(\xi,0)d\xi. \end{cases} \tag{4.29}$$

Notice that, under the transformation $T_i(x,t) = (p_i(x,t),t)$, $w_j(u(T_i^{-1}(p_i,t)))(i \ne j)$ satisfies the following Cauchy problem:

$$\begin{cases} |w_j - k|_t + |w_j - k|_{p_i} = 0, \\ w_j|_{t=0} = w_j(u_0(T_{0i}^{-1}(p_i))). \end{cases} \tag{4.30}$$

Furthermore, under the transformation $\bar{T}_i(x,t) = (q_i(x,t),t)$, $w_j(v(\bar{T}_i^{-1}(q_i,t)))(i \ne j)$ satisfies the following Cauchy problem:

$$\begin{cases} |w_j - k|_t + |w_j - k|_{q_i} = 0, \\ w_j|_{t=0} = w_j(v_0(\bar{T}_{0i}^{-1}(q_i))). \end{cases} \tag{4.31}$$

Therefore, we can deduce that, for almost all $(x,t) \in \Pi_T$,

$$w_j(u(x,t)) = w_j(u_0(T_{0i}^{-1}(p_i(x,t)-t))), \qquad w_j(v(x,t)) = w_j(v_0(\bar{T}_{0i}^{-1}(q_i(x,t)-t))). \tag{4.32}$$

Substitute (4.32) into (4.28)-(4.29). We have, for almost all $t \in (0,T)$,

$$\begin{cases} p_i(x,t)_t + \lambda_i(w_j(u_0(T_{0i}^{-1}(p_i(x,t)-t))))p_i(x,t)_x = 0, \qquad j \neq i, \\ p_i(x,0) = \int_0^x P_1(\xi,0)d\xi, \end{cases} \tag{4.33}$$

and

$$\begin{cases} q_i(x,t)_t + \lambda_i(w_j(v_0(\bar{T}_{0i}^{-1}(q_i(x,t)-t))))q_i(x,t)_x = 0, \qquad j \neq i, \\ q_i(x,0) = \int_0^x P_2(\xi,0)d\xi. \end{cases} \tag{4.34}$$

Notice that

$$\begin{cases} \int_{|x|\le N+Lt} |p_i(x,0) - q_i(x,0)|dx \le C\int_{|x|\le N+Lt} |w(u_0(x)) - w(v_0(x))|dx, \\ \int_{|x|\le \|P_1\|_{L^\infty} N+Lt} |w_j(u_0(T_{0i}^{-1}(x-t))) - w_j(v_0(\bar{T}_{0i}^{-1}(x-t)))|dx \\ \quad \le C\int_{|x|\le \|P_1\|_{L^\infty} N+(L+1)t} |w_j(u_0(T_{0i}^{-1}(x))) - w_j(v_0(\bar{T}_{0i}^{-1}(x)))|dx \\ \quad \le C\int_{|x|\le K_0(N+(L+1)t)} |w_j(u_0(x)) - w_j(v_0(\bar{T}_{0i}^{-1}T_{0i}(x)))|dx. \end{cases} \tag{4.35}$$

Using Theorem 3.3, and (4.33)-(4.35), we finally obtain

$$\begin{aligned} &\int_{|x|\le N} |p_i(x,t) - q_i(x,t)|dx \\ &\le C\int_{|x|\le K_0(N+(L+1)t)} (|w(u_0(x)) - w(v_0(x))| + |w(u_0(x)) - w(v_0(\bar{T}_{0i}^{-1}T_{0i}(x)))|)dx. \end{aligned}$$

Thus (4.27) follows. This completes the proof of Theorem 4.6.

Remark. Theorem 4.5 and Theorem 4.6 show that, for the systems with fully contact fields ($n = 2$), the generalized solution of the Cauchy problem (4.1) is L^m-stable in the sense of shift, for any positive constant m.

As a corollary, we obtain the following theorem.

Theorem 4.7 *(Uniqueness). The L^∞ generalized solution of the Cauchy problem (4.1) ($n = 2$) is unique.*

Finally, we consider the systems (4.1)-(4.3) consisting of arbitrary n equations. For simplicity, we assume that

$$\sup_{w\in K}\lambda_1(w) < \inf_{w\in K}\lambda_2(w) < \sup_{w\in K}\lambda_2(w) < \dots < \sup_{w\in K}\lambda_{n-1}(w) < \inf_{w\in K}\lambda_n(w),$$

where K is the parallelipiped $K = \prod_i[\inf w_i(u_0), \sup w_i(u_0)]$. Then we have the following stability result.

Theorem 4.8 *(Stability of BV Generalized Solution). Suppose that the vector functions $u(x,t)$ and $v(x,t)$ are generalized solutions of the Cauchy problem (4.1) in Π_T with initial data $u_0(x)$ and $v_0(x)$ in $BV(\mathbb{R}^2_+;K)$, respectively. If $TV_{-\infty}^{\infty}(u_0(x))$ is sufficiently small and $u_0 - v_0 \in L^1(-\infty,\infty)$, then*

$$\int_{-\infty}^{\infty} |u(x,t) - v(x,t)|dx \le C \int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx, \tag{4.36}$$

for almost all $t \in (0,T)$, where the constant $C = C(TV_{-\infty}^{\infty}(u_0(x)))$ with $C(s) \in C[0,s_0)$ and $\lim_{s\to s_0} C(s) = +\infty$ for some $s_0 > 0$ only depending on the parallelipiped K.

Proof. We first prove that the Lipschitz functions $p_i(x,t)$ and $q_i(x,t)$, $1 \le i \le n$, satisfy

$$\sup_{\Pi_T} \sum_{i=1}^{n} |T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))| \le C \int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx, \tag{4.37}$$

where the constant $C = C(TV_{-\infty}^{\infty}(u_0(x)))$, with $C(s) \in C[0,s_0)$ and $\lim_{s\to s_0} C(s) = +\infty$ for some $s_0 > 0$ only depending on the parallepiped K; the functions $p_i(x,t)$ and $q_i(x,t)$ are determined by the quasidecoupling transformations corresponding to the vector functions $u(x,t)$ and $v(x,t)$, and satisfy

$$\begin{cases} p_i(x,t)_t + \lambda_i(u(x,t))p_i(x,t)_x = 0, \\ p_i(x,0) = T_{0i}(x) \equiv \int_0^x P_1(\xi,0)d\xi, \end{cases} \tag{4.38}$$

and

$$\begin{cases} q_i(x,t)_t + \lambda_i(v(x,t))q_i(x,t)_x = 0, \\ q_i(x,0) = \bar{T}_{0i}(x) \equiv \int_0^x P_2(\xi,0)d\xi, \end{cases} \tag{4.39}$$

for almost all $t \in (0,T)$.

Then we have

$$|T_{0i}^{-1}(p_i)-\bar{T}_{0i}^{-1}(q_i)|_t+\lambda_i(u)|T_{0i}^{-1}(p_i)-\bar{T}_{0i}^{-1}(q_i)|_x = \mathrm{sign}(T_{0i}^{-1}(p_i)-\bar{T}_{0i}^{-1}(q_i))(\lambda_i(u)-\lambda_i(v))\bar{T}_{0i}^{-1}(q_i)_x. \tag{4.40}$$

Multiply (4.40) by $N_i(u)\phi(p_i)$, for any $\phi(x) \in C_0^1, \phi(x) \ge 0$, and note that

$$(N_i(u)\phi(p_i))_t + (\lambda_i(u)N_i(u)\phi(p_i))_x = 0.$$

We obtain

$$\begin{aligned} &(N_i(u)\phi(p_i)|T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i)|)_t + (\lambda_i(u)N_i(u)\phi(p_i)|T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i)|)_x \\ &= \mathrm{sign}(T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i))(\lambda_i(u) - \lambda_i(v))N_i(u)\phi(p_i)\bar{T}_{0i}^{-1}(q_i)_x. \end{aligned} \tag{4.41}$$

Integrate (4.41) over the domain $(-\infty,\infty) \times [0,t]$. We have

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} N_i(u)\phi(p_i)|T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i)|dx \\ &\le \textstyle\int_{-\infty}^{\infty}\int_0^t |\lambda_i(u) - \lambda_i(v)|N_i(u)\phi(p_i)|\bar{T}_{0i}^{-1}(q_i)_x|dxd\tau, \end{aligned}$$

that is,

$$\int_{-\infty}^{\infty}\phi(p_i(x,t))|T_{0i}^{-1}(p_i(x,t))-\bar{T}_{0i}^{-1}(q_i(x,t))|dx \leq C\int_{-\infty}^{\infty}\int_0^t|\lambda_i(u)-\lambda_i(v)|\phi(p_i)dxd\tau, \quad (4.42)$$

for any $\phi \in C_0^1(-\infty,\infty), \phi(x) \geq 0$.

Since $p_i(x,t)_x \geq c_0 > 0$, we obtain by changing variable $(x,t) \to (p_i,t)$ in the integrations,

$$\begin{array}{l} \int_{-\infty}^{\infty}\phi(p_i)|T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i(x(p_i,t),t))|dp_i \\ \leq C\{\int_{-\infty}^{\infty}\int_0^t|\lambda_i(u(x(p_i,\tau),\tau)) - \lambda_i(v(x(p_i,\tau),\tau))|\phi(p_i)dp_id\tau\}, \end{array}$$

for any $\phi \in C_0^1(-\infty,\infty), \phi \geq 0$. Therefore, we have

$$|T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i(x(p_i,t),t))| \leq C\int_0^t|\lambda_i(u(x(p_i,\tau),\tau)) - \lambda_i(v(x(p_i,\tau),\tau))|d\tau. \quad (4.43)$$

As (4.21), for any $u_0(x) \in L^\infty \cap BV$, there exists an approximate sequence $u_0^\epsilon(x) \in C^\infty(\mathbb{R};K)$ of $u_0(x)$ such that

$$\left\{\begin{array}{l} \|u_0^\epsilon - u_0\|_{L^1(-\infty,\infty)} \leq \epsilon CTV_{-\infty}^{\infty}(u_0(x)) \to 0, \\ TV_{-\infty}^{\infty}(u_0^\epsilon(x)) \leq TV_{-\infty}^{\infty}(u_0(x)). \end{array}\right.$$

Furthermore, we have from Theorem 4.5 that

$$\left\{\begin{array}{ll} \lambda_i(u(x,t)) = \lambda_i(w_j(u_0(T_{0i}^{-1}(p_i(x,t))))), & j \neq i, \\ \lambda_i(v(x,t)) = \lambda_i(w_j(v_0(\bar{T}_{0i}^{-1}(q_i(x,t))))), & j \neq i, \end{array}\right.$$

where $\lambda_i(w_j), j \neq i$, is the compact notation of $\lambda_i(w_1,...,w_{i-1},w_{i+1},...,w_n)$. Then we have

$$\begin{array}{l} |T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i(x(p_i,t),t))| \\ \leq C\{\int_0^t|\lambda_i(w_j(u_0^\epsilon(T_{0j}^{-1}(p_j(x(p_i,\tau),\tau))))) - \lambda_i(w_j(u_0^\epsilon(\bar{T}_{0j}^{-1}(q_j(x(p_i,\tau),\tau)))))|d\tau \\ \qquad + \int_0^t|\lambda_i(w_j(u_0(\bar{T}_{0j}^{-1}(q_j(x(p_i,\tau),\tau))))) - \lambda_i(w_j(v_0(\bar{T}_{0j}^{-1}(q_j(x(p_i,\tau),\tau)))))|d\tau \\ \qquad + \int_0^t|\lambda_i(w_j(u_0^\epsilon(T_{0j}^{-1}(p_j(x(p_i,\tau),\tau))))) - \lambda_i(w_j(u_0(T_{0j}^{-1}(p_j(x(p_i,\tau),\tau)))))|d\tau \\ \qquad + \int_0^t|\lambda_i(w_j(u_0^\epsilon(\bar{T}_{0j}^{-1}(q_j(x(p_i,\tau),\tau))))) - \lambda_i(w_j(u_0(\bar{T}_{0j}^{-1}(q_j(x(p_i,\tau),\tau)))))|d\tau\} \\ \leq C\sum_{j\neq i}\{\int_0^t\int_0^1|u_{0x}^\epsilon(s_j(p_i,\tau;\theta))||T_{0j}^{-1}(p_j(x(p_i,\tau),\tau)) - \bar{T}_{0j}^{-1}(q_j(x(p_i,\tau),\tau))|d\tau d\theta \\ \qquad + \int_0^t(|u_0^\epsilon(\bar{s}_j(p_i,\tau)) - u_0(\bar{s}_j(p_i,\tau))| + |u_0^\epsilon(\tilde{s}_j(p_i,\tau)) - u_0(\tilde{s}_j(p_i,\tau))|)d\tau \\ \qquad + \int_0^t|u_0(\tilde{s}_j(p_i,\tau)) - v_0(\tilde{s}_j(p_i,\tau))|d\tau\} \end{array}$$

for $j \neq i$, where

$$\left\{\begin{array}{l} s_j(p_i,t;\theta) = \theta T_{0j}^{-1}(p_j(x(p_i,t),t)) + (1-\theta)T_{0j}^{-1}(q_j(x(p_i,t),t)), \\ \bar{s}_j(p_i,t) = T_{0j}^{-1}(p_j(x(p_i,t),t)), \\ \tilde{s}_j(p_i,t) = \bar{T}_{0j}^{-1}(q_j(x(p_i,t),t)). \end{array}\right.$$

Since

$$\left\{\begin{array}{l} |T_{0i}^{-1}(p_i(x,0)) - \bar{T}_{0i}^{-1}(q_i(x,0))| = 0, \\ |\partial_t|T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))|| \leq M_0 < \infty, \end{array}\right.$$

for all i, $1 \le i \le n$, we conclude that the functions $|T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))|, 1 \le i \le n$, are uniformly bounded,

$$|T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))| \le M_0 T, \qquad 1 \le i \le n.$$

Moreover, one can check that, for $j \ne i$,

$$\begin{aligned} &|s_j(p_i,t)_t| \ge c_0 > 0, \\ &|\bar{s}_j(p_i,t)_t| \ge c_0 > 0, \\ &|\tilde{s}_j(p_i,t)_t| \ge c_0 > 0, \end{aligned}$$

and the transformations

$$\begin{aligned} &(p_i,t) \to (p_i,s_j), \\ &(p_i,t) \to (p_i,\bar{s}_j), \\ &(p_i,t) \to (p_i,\tilde{s}_j), \end{aligned}$$

are one-to-one for $j \ne i$, respectively.

Make these transformations in the corresponding integrations, we can immediately obtain

$$\begin{aligned} &|T_{0i}^{-1}(p_i) - \bar{T}_{0i}^{-1}(q_i(x(p_i,t),t))| \\ &\le C\{\sup_{\Pi_T} \textstyle\sum_{j\ne i} |T_{0j}^{-1}(p_j(x,t)) - \bar{T}_{0j}^{-1}(q_j(x,t))| \int_{-\infty}^{\infty} |u_{0x}^{\epsilon}(x)|dx \\ &\qquad + \textstyle\int_{-\infty}^{\infty} |u_0^{\epsilon}(x) - u_0(x)|dx + \int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx\} \\ &\le C\{TV_{-\infty}^{\infty}(u_0(x)) \sup_{\Pi_T} \textstyle\sum_{j\ne i} |T_{0j}^{-1}(p_j(x,t)) - \bar{T}_{0j}^{-1}(q_j(x,t))| + \int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx + \epsilon\}. \end{aligned} \tag{4.44}$$

Set $\epsilon \to 0$ and then substitute $p_j(x,t)$ into p_j in (4.44). We obtain

$$\begin{aligned} &|T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))| \\ &\le C\{\textstyle\int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx + TV_{-\infty}^{\infty}(u_0(x)) \sup_{\Pi_T} \sum_{j\ne i} |T_{0j}^{-1}(p_j(x,t)) - \bar{T}_{0j}^{-1}(q_j(x,t))|\}. \end{aligned}$$

This implies

$$\begin{aligned} &\sup_{\Pi_T} \textstyle\sum_{i=1}^{n} |T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))| \\ &\le C\{\textstyle\int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx + TV_{-\infty}^{\infty}(u_0(x)) \sup_{\Pi_T} \sum_{i=1}^{n} |T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))|\}. \end{aligned} \tag{4.45}$$

If $TV_{-\infty}^{\infty}(u_0(x))$ is sufficiently small such that $CTV_{-\infty}^{\infty}(u_0(x)) < 1$, we obtain

$$\sup_{\Pi_T} \sum_{i=1}^{n} |T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))| \le C \int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx, \tag{4.46}$$

Notice that

$$w_i(u(x,t)) = w_i(u_0(T_{0i}^{-1}(p_i(x,t)))), \qquad w_i(v(x,t)) = w_i(v_0(\bar{T}_{0i}^{-1}(q_i(x,t)))).$$

Similarly, we have

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} |u(x,t) - v(x,t)|dx \\ &\le C \textstyle\int_{-\infty}^{\infty} \sum_{i=1}^{n} |w_i(u(x,t)) - w_i(v(x,t))|dx \\ &\le C \textstyle\int_{-\infty}^{\infty} \sum_{i=1}^{n} |u_0(T_{0i}^{-1}(p_i(x,t))) - v_0(\bar{T}_{0i}^{-1}(q_i(x,t)))|dx \\ &\le C \textstyle\int_{-\infty}^{\infty} \sum_{i=1}^{n} \{|u_0(T_{0i}^{-1}(p_i(x,t))) - u_0(\bar{T}_{0i}^{-1}(q_i(x,t)))| + |u_0(\bar{T}_{0i}^{-1}(q_i(x,t))) - v_0(\bar{T}_{0i}^{-1}(q_i(x,t)))|\}dx \\ &\le C\{TV_{-\infty}^{\infty}(u_0(x)) \sup_{\Pi_T} \textstyle\sum_{i=1}^{n} |T_{0i}^{-1}(p_i(x,t)) - \bar{T}_{0i}^{-1}(q_i(x,t))| + \int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx\} \\ &\le C \textstyle\int_{-\infty}^{\infty} |u_0(x) - v_0(x)|dx. \end{aligned}$$

This completes the proof of Theorem 4.8.

As a corollary, we obtain the following uniqueness theorem.

Theorem 4.9 *(Uniqueness). The generalized solution of the Cauchy problem (4.1) with the initial data* $u_0(x) \in BV(\mathbb{R}^1; K)$ *is unique provided that* $TV_{-\infty}^{\infty}(u_0(x))$ *is sufficiently small.*

5 Systems with One Contact Field and One Line Field

In this section, with the aid of the quasidecoupling method, we discuss systems with one contact field and one line field, for which the shock wave curves and rarefaction wave curves coincide (see TEMPLE [52]). Such systems arise in the fields of elasticity theory (cf. [30, 1]), enhanced oil recovery (cf. [27, 53]), and magnetohydrodynamics (cf. [20, 1]).

We note that a 2 × 2 system of conservation laws

$$u_t + f(u)_x = 0, \tag{5.1}$$

has one contact field and one line field if and only if the flux function

$$f(u) = u\phi(u),$$

for some smooth function ϕ of u, where ϕ with $\phi_u \neq 0$ is both the wave speed (i.e., $\phi = \lambda_1$ is an integral curve of R_1) and the Riemann invariant for the contact field. The Riemann invariant for the line field is

$$w = \frac{u_2}{u_1};$$

that is, $w = const.$, is a line of $w = \frac{du_2}{du_1}$.

Using the polar coordinates (r, θ), we have that the eigenvalues of the system (5.1) are

$$\begin{cases} \lambda_1 = \phi, \\ \lambda_2 = (r\phi)_r, \end{cases}$$

respectively. The corresponding Riemann invariants are

$$\begin{cases} w = \theta, \\ z = \phi(r, \theta), \end{cases}$$

respectively.

There are two families of entropy pairs—entropy pairs (E_a, F_a):

$$\begin{cases} E_a = \Phi_0 a(w) = r a(\theta), \\ F_a = \lambda_1 \Phi_0 a(w) = r\phi(r, \theta) a(\theta), \end{cases} \tag{5.2}$$

and entropy pairs (G_b, H_b):

$$\begin{cases} G_b = \Phi_0 \int_{\bar{\phi}}^{\phi(r,\theta)} \frac{b'(\eta)}{(\lambda_2 - \lambda_1)(\theta, \eta)} \exp(-\int_{\bar{\phi}}^{\eta} \frac{1}{\lambda_2(\theta,\xi) - \lambda_1(\xi)} d\xi) d\eta, \\ \quad = r\phi_r(r(\bar{\phi}, \theta), \theta) \int_{\bar{\phi}}^{\phi(r,\theta)} b'(\eta) \exp(-\int_{\bar{\phi}}^{\eta} \frac{\lambda_{2\phi}(\theta,\xi)}{\lambda_2(\theta,\xi) - \lambda_1(\xi)} d\xi) d\eta, \\ H_b = \lambda_1 G_b + b(\phi(r, \theta)), \end{cases} \tag{5.3}$$

where a and b are two arbitrary continuous functions.

(E_a, F_a) is a convex entropy pair if and only if $a''(\theta) + a(\theta) \geq 0$. Furthermore, if $|r\phi_r| \geq c_0 > 0$ and $\|u\|_{L^\infty} \leq C$, then there exists a family of C^2 entropy pairs (G_{b_k}, H_{b_k}) that are convex with respect to ϕ, with $b_k(\phi) = e^{k\phi}$, $k \gg 1$.

One can verify that any admissible solution to the system (5.1) (see [4]) satisfies

$$E_a(u)_t + F_a(u)_x = 0, \tag{5.4}$$

for an arbitrary continuous function $a(\theta)$. Moreover, if the system (5.1) is strictly hyperbolic on the solution domain under consideration, then on the contact discontinuity $C = (x(t), t)$,

$$\begin{cases} \phi(u(x(t)-0,t)) = \phi(u(x(t)+0,t)), \\ \phi\ (G_{b_k}(u(x(t)+0)) - G_{b_k}(u(x(t)-0,t))) = H_{b_k}(u(x(t)+0,t)) - H_{b_k}(u(x(t)-0,t)), \end{cases} \tag{5.5}$$

and, on the second kind of shock wave $S = (x(t), t)$ with the slope s,

$$\begin{cases} \theta(u(x(t)-0,t)) = \theta(u(x(t)+0,t)), \\ s(G_{b_k}(u(x(t)+0,t)) - G_{b_k}(u(x(t)-0,t))) - (H_{b_k}(u(x(t)+0,t)) - H_{b_k}(u(x(t)-0,t))) \geq 0, \end{cases} \tag{5.6}$$

for almost all $t \in (0, \infty)$.

We consider the Cauchy problem

$$\begin{cases} u_t + (u\phi(u))_x = 0, \\ u|_{t=0} = u_0(x). \end{cases} \tag{5.7}$$

Definition 5.1. A BV vector function $u(x,t)$, $r(u(x,t)) \geq r_0 > 0$ is called a generalized solution for the Cauchy problem (5.7) in Π_T if

(a) $u(x,t)$ is a weak solution of the Cauchy problem (5.7).

(b) $u(x,t)$ satisfies the following entropy conditions:

(i) For any function $a \in C_{\text{loc}}(-\infty, \infty)$ and $\psi \in C_0^1(\Pi_T)$,

$$\int\int_{\Pi_T} (ra(\theta)\psi_t + r\phi a(\theta)\psi_x)dxdt + \int_{-\infty}^{\infty} r_0(x)a(\theta_0(x))\psi(x,0)dx = 0. \tag{5.8}$$

(ii) On the discontinuity wave curves with $\theta = \mathit{const.}$, and with left state $(r_-, \theta) = (r(\phi_-, \theta), \theta)$ and right state $(r_+, \theta) = (r(\phi_+, \theta), \theta)$,

$$s(r_+, r_-; \theta) \leq s(r, r_-; \theta), \qquad r \text{ between } r_- \text{ and } r_+, \tag{5.9}$$

where

$$s(r_+, r_-; \theta) = \frac{r_+\phi(r_+, \theta) - r_-\phi(r_-, \theta)}{r_+ - r_-}.$$

(iii) On the discontinuity wave curves with $\phi(r, \theta) = \mathit{const.}$, and with left state $(r_-, \theta_-) = (r(\phi, \theta_-), \theta_-)$ and right state $(r_+, \theta_+) = (r(\phi, \theta_+), \theta_+)$,

$$(r(\phi, \theta_-) - r(l, \theta_-))(r(\phi, \theta_+) - r(l, \theta_+)) \geq 0, \tag{5.10}$$

where l is any constant and $r(l,\theta)$ is any branch such that the curve $\phi(r,\theta) = l$ passes both $(r(l,\theta_-),\theta_-)$ and $(r(l,\theta_+),\theta_+)$.

Remark 1. The condition (5.9) is equivalent to

$$\begin{aligned}&s(|r_- - k| - |r_+ - k|)\\ &-(r_-\phi(r_-,\theta) - k\phi(k,\theta))\text{sign}(r_- - k) + (r_+\phi(r_-,\theta) - k\phi(k,\theta))\text{sign}(r_+ - k) \le 0,\end{aligned} \tag{5.11}$$

for any constants k and θ; equivalently,

$$\begin{aligned}&s(|r(\phi_-,\theta) - r(l,\theta)| - |r(\phi_+,\theta) - r(l,\theta)|) - (\phi_-r(\phi_-,\theta) - lr(l,\theta))\text{sign}(r(\phi_-,\theta) - r(l,\theta))\\ &+(\phi_+r(\phi_+,\theta) - lr(l,\theta))\text{sign}(r(\phi_+,\theta) - lr(l,\theta)) \le 0,\end{aligned} \tag{5.12}$$

for the constants l and θ such that $r(l,\theta) = k$, for any constants k and θ.

If the system (5.1) is strictly hyperbolic, $|r\phi_r| \ge c_0 > 0$, then the condition (5.10) naturally holds. For some nonstrictly hyperbolic systems, for example, systems arising in magnetohydrodynamics and elasticity (see Subsection 5.b) and a system arising in oil recovery (see Subsection 5.c), the condition (5.10), together with (5.8) and (5.9), rules out the unstable solutions for the Riemann problem.

Remark 2. If the function ϕ satisfies

$$\begin{cases} |r(r\phi)_{rr}| \ge c_0 > 0, \\ |r\phi_r| \ge c_0 > 0, \end{cases}$$

then the condition (5.9) or (5.11) is equivalent to the Lax shock criterion:

$$\lambda_2(+) < s(+,-) < \lambda_2(-), \tag{5.13}$$

where (+,-) denotes (u_+,u_-), which is a pair of the right state and left state of the second kind of shock waves. Furthermore, the condition (5.9) is also equivalent to the entropy criterion: for some entropy (G_{b_0}, H_{b_0}) strictly convex with respect to ϕ,

$$G_{b_0}(u)_t + H_{b_0}(u)_x \le 0, \tag{5.14}$$

in the sense of distribution by using (5.8).

Remark 3. The condition $r(u(x,t)) \ge r_0 > 0$ in Definition 5.1 can be removed by following the process we shall introduce and Wagner's arguments [56].

We recall that there exist generalized solutions to the Cauchy problem (5.7) for the cases of physical significance (see [4, 53]). Now we study the uniqueness and stability of the generalized solutions for the Cauchy problem (5.7) with the aid of the quasidecoupling method introduced in Section 3.

a. Uniqueness and Stability of Generalized Solutions

Theorem 5.1 *(Stability). Suppose that BV functions $u(x,t)$ and $v(x,t)$ are generalized solutions of the Cauchy problem (5.7) in Π_T with initial data $u_0(x)$ and $v_0(x)$, respectively. Then, for almost all $t \in [0,T)$ and for any positive constant m, N,*

$$\text{(i)} \begin{cases} \theta(u(x,t)) = \theta(u_0(T_0^{-1}(p(x,t)))), \qquad \theta(v(x,t)) = \theta(v_0(\bar{T}_0^{-1}(q(x,t)))), \\ \int_{|x|\le N} |\theta(u(x,t)) - \theta(v(\bar{T}_t^{-1}T_t(x),t))|^m dx \\ \quad \le C\int_{|x|\le K_0(N+Lt)} |\theta(u_0(x)) - \theta(v_0(\bar{T}_0^{-1}T_0(x)))|^m dx, \\ \int_{|x|\le N} |\theta(u(x,t)) - \theta(v(\bar{T}_t^{-1}\bar{T}_0T_0^{-1}T_t(x),t))|^m dx \\ \quad \le C\int_{|x|\le K_0(N+Lt)} |\theta(u_0(x)) - \theta(v_0(x))|^m dx, \end{cases} \tag{5.15}$$

$$\text{(ii)} \begin{cases} \int_{|x|\le N} |r(u(x,t)) - r(v(\bar{T}_t^{-1}T_t(x),t))|dx \le C\int_{|x|\le K_0(N+Lt)} |u_0(x) - v_0(\bar{T}_0^{-1}T_0(x))|dx, \\ \int_{|x|\le N} |(\bar{T}_t^{-1}T_t - I)(x)|dx \le C\int_{|x|\le K_0(N+\bar{L}t)} (|u_0(x) - v_0(x)| + |u_0(x) - v_0(\bar{T}_0^{-1}T_0(x))|)dx, \end{cases} \tag{5.16}$$

provided that

$$\theta(u_0(T_0^{-1}(x))) = \theta(v_0(\bar{T}_0^{-1}(x))), \qquad a.e.,$$

where

$$\begin{aligned} &L = \max\left(\sup_{\Pi_T}(|\phi(u)| + |(r^{-2}\phi_r)(u)|), \sup_{\Pi_T}(|\phi(v)| + |(r^{-2}\phi_r)(v)|)\right), \\ &\bar{L} = L + \max(\sup_{[0,T]}(|\bar{T}_t^{-1}T_t(\pm N)| - N)/t), \\ &K_0 = \max(\sup_{\Pi_T}|r(u(x,t))|, \sup_{\Pi}|r(v(x,t))|)/r_0, \end{aligned}$$

C is a constant independent of m, and $(p,t) = (T_t(x),t)$ and $(q,t) = (T_t(x),t)$ are the quasidecoupling transformations with

$$\begin{cases} \frac{\partial p}{\partial x} = r(u(x,t)), \\ \frac{\partial p}{\partial t} = -r(u(x,t))\phi(u(x,t)), \\ p(x,0) = \int_0^x r(u_0(\xi))d\xi, \end{cases} \tag{5.17}$$

and

$$\begin{cases} \frac{\partial q}{\partial x} = r(v(x,t)), \\ \frac{\partial q}{\partial t} = -r(v(x,t))\phi(v(x,t)), \\ q(x,0) = \int_0^x r(v_0(\xi))d\xi, \end{cases} \tag{5.18}$$

respectively.

The proof consists of several lemmas. Using the quasidecoupling method, we immediately obtain the following lemma.

Lemma 5.1 *For the generalized solutions $u(x,t)$ and $v(x,t)$ in Theorem 5.1, the functions $\theta(u(x,t))$ and $\theta(v(x,t))$ satisfy (5.15).*

Proof. Since $u(x,t)$ and $v(x,t)$ are the generalized solutions with initial data $u_0(x)$ and $v_0(x)$, we have, for any positive constant m,

$$\left\{\begin{array}{l} r(u(x,t))_t + ((r\phi)(u(x,t)))_x = 0, \\ (r(u(x,t))|\theta(u(x,t)) - k|^m)_t + ((r\phi)(u(x,t))|\theta(u(x,t)) - k|^m)_x = 0, \quad \text{for any } k \in (-\infty,\infty), \end{array}\right.$$

and

$$\left\{\begin{array}{l} r(v(x,t))_t + ((r\phi)(v(x,t)))_x = 0, \\ (r(v(x,t))|\theta(v(x,t)) - l|^m)_t + ((r\phi)(v(x,t))|\theta(v(x,t)) - l|^m)_x = 0, \quad \text{for any } l \in (-\infty,\infty). \end{array}\right.$$

Using Theorem 3.2 (Framework (A)), we immediately obtain (5.15).

Remark. This lemma holds for the L^∞ weak solutions $u(x,t)$ and $v(x,t)$ of the Cauchy problem (5.7) satisfying (5.8) with initial data $u_0(x)$ and $v_0(x)$, respectively. We did not use the assumption that both $u(x,t)$ and $v(x,t)$ are BV functions.

Denote $\tau_1(x,t) = r(u(x,t))^{-1}$ and $\tau_2(x,t) = r(v(x,t))^{-1}$. We have the following lemma.

Lemma 5.2 *If* $\theta(u_0(T_0^{-1}(x))) = \theta(v_0(\bar{T}_0^{-1}(x))) \equiv \theta_0(x)$, *a.e., then*

$$\left\{\begin{array}{l} \tau_1(x,t) = \tau_1(p(x,t),t;r(u_0(T_0^{-1}(p(x,t))))^{-1}), \\ \tau_2(x,t) = \tau_2(q(x,t),t;r(v_0(\bar{T}_0^{-1}(q(x,t))))^{-1}), \end{array}\right. \tag{5.19}$$

where both $\tau_i(y,t;\tau_0(y)), i = 1,2,$ *are solutions of the Cauchy problem*

$$\left\{\begin{array}{l} \tau_t - \phi(\tau,\theta_0(y))_y = 0, \\ \tau|_{t=0} = \tau_0(y), \end{array}\right. \tag{5.20}$$

satisfying the following entropy condition: On the discontinuity wave curve with $\theta_0(y) =$ *const., and with left state* $\tau_- = \tau(\phi_-,\theta_0(y))$ *and right state* $\tau_+ = \tau(\phi_+,\theta_0(y))$,

$$\sigma(\tau_+,\tau_-;\theta_0(y)) \le \sigma(\tau,\tau_-;\theta_0(y)), \qquad \textit{for any } \tau \textit{ between } \tau_- \textit{ and } \tau_+, \tag{5.21}$$

and on the discontinuity wave curve with $\phi =$ *const., and with left state* $\tau_- = \tau(\phi,\theta_-)$ *and right state* $\tau_+ = \tau(\phi,\theta_+)$,

$$(\tau(\phi,\theta_-) - \tau(l,\theta_-))(\tau(\phi,\theta_+) - \tau(l,\theta_+)) \ge 0, \tag{5.22}$$

where l *is any constant and* $\tau(l,\theta)$ *is any branch such that the curve* $\phi(r,\theta) = l$ *passes both* $(r(l,\theta_-),\theta_-)$ *and* $(r(l,\theta_+),\theta_+)$, *and*

$$\sigma(\tau_+,\tau_-;\theta_0(y)) = -\frac{\phi(\tau_+,\theta_0(y)) - \phi(\tau_-,\theta_0(y))}{\tau_+ - \tau_-}.$$

Proof. Using the quasidecoupling transformations (5.17) and (5.18), we can immediately verify that both $\tau_1(T^{-1}(p,t))$ and $\tau_2(\bar{T}^{-1}(q,t))$ are weak solutions of the Cauchy problem (5.20) with the initial data $r(u_0(T_0^{-1}(p)))^{-1}$ and $r(v_0(\bar{T}_0^{-1}(q)))^{-1}$, respectively.

Now we prove the entropy condition (5.21). For concreteness, we prove only that $\tau_1(x,t) = \tau_1(T^{-1}(p,t))$ satisfies (5.21). In fact, on the shock wave curve of $u(x,t)$, i.e., $\theta = \theta(u_0(T_0^{-1}(p))) = const.$,

$$s(r_+, r_-; \theta(u_0(T_0^{-1}(p)))) \le s(r, r_-; \theta(u_0(T_0^{-1}(p)))),$$

for any r between r_- and r_+, where

$$s(r_+, r_-; \theta(u_0(T_0^{-1}(p)))) = \frac{r_+\phi(r_+, \theta(u_0(T_0^{-1}(p)))) - r_-\phi(r_-, \theta(u_0(T_0^{-1}(p))))}{r_+ - r_-}.$$

The quasidecoupling transformation $T_1 : (x,t) \longmapsto (p,t)$ transforms the shock wave $S = (x(t; r_+, r_-), t)$ with $\theta(u_0(T_0^{-1}(p))) = const.$ and the slope $s(r_+, r_-; \phi(u_0(T_0^{-1}(p))))$ in the (x, t)-plane to the shock wave $S' = (p(t; \tau_+, \tau_-), t)$ with $\theta(u_0(T_0^{-1}(p))) = const.$ and the slope

$$\begin{aligned} \sigma(\tau_+, \tau_-; \theta(u_0(T_0^{-1}(p)))) &= r_+(s(r_+, r_-; \theta(u_0(T_0^{-1}(p)))) - \phi_+) \\ &= r_-(s(r_+, r_-; \theta(u_0(T_0^{-1}(p)))) - \phi_-). \end{aligned}$$

Therefore, on the shock wave curve,

$$\begin{cases} \sigma(\tau - \tau_-) = \phi(\tau_-, \theta(u_0(T_0^{-1}(p)))) - \phi(\tau, \theta(u_0(T_0^{-1}(p)))), \\ \theta(u_0(T_0^{-1}(p))) = const., \end{cases}$$

equivalently,

$$\begin{cases} s(r - r_-) = r\phi(r, \theta(u_0(T_0^{-1}(p)))) - r_-\phi(r_-, \theta(u_0(T_0^{-1}(p)))), \\ \theta(u_0(T_0^{-1}(p))) = const., \end{cases}$$

we have

$$\begin{aligned} \sigma(\tau_+, \tau_-; \theta(u_0(T_0^{-1}(p)))) &= r_-(s(r_+, r_-; \theta(u_0(T_0^{-1}(p)))) - \phi_-) \\ &\le r_-(s(r, r_-; \theta(u_0(T_0^{-1}(p)))) - \phi_-) \\ &= r(s(r, r_-; \theta(u_0(T_0^{-1}(p)))) - \phi) \\ &= \sigma(\tau, \tau_-; \theta(u_0(T_0(p)))). \end{aligned}$$

Similarly, we can prove that $\tau_2(x,t) = \tau_2(\bar{T}^{-1}(q,t))$ satisfies (5.21). The entropy condition (5.22) can directly be verified from (5.10).

Lemma 5.3 *For any positive constant* N_1,

$$\int_{|y|\le N_1} |\tau(y,t;\tau_0(y)) - \tau(y,t;\bar{\tau}_0(y))|dy \le C \int_{|y|\le N_1 + L_1 t} |\tau_0(y) - \bar{\tau}_0(y)|dy,$$

where $L_1 = \sup_{\{|\tau| \le sup_{\Pi_T}|\tau|, y\in R^1\}} |\phi_\tau(\tau, \theta_0(y))|$.

Proof. First we note that the entropy condition (5.21) is equivalent to the condition:

$$\begin{aligned}&\sigma(|\tau_- - k| - |\tau_+ - k|)\\ &+(\phi(\tau_-,\theta) - \phi(k,\theta))\mathrm{sign}(\tau_- - k) - (\phi(\tau_+,\theta) - \phi(k,\theta))\mathrm{sign}(\tau_+ - k) \le 0,\end{aligned} \tag{5.23}$$

for any constants k and θ; equivalently,

$$\begin{aligned}&\sigma(|\tau(\phi_-,\theta) - \tau(l,\theta)| - |\tau(\phi_+,\theta) - \tau(l,\theta)|)\\ &+(\phi_- - l)\mathrm{sign}(\tau(\phi_-,\theta) - \tau(l,\theta)) - (\phi_+ - l)\mathrm{sign}(\tau(\phi_+,\theta) - \tau(l,\theta)) \le 0,\end{aligned} \tag{5.24}$$

for the constants l and θ such that $\tau(l,\theta) = k$.

Next we consider the deviation between $\tau(y,t) \equiv \tau(y,t;\tau_0(y))$ and $\bar{\tau}(y,t) \equiv \tau(y,t;\bar{\tau}_0(y))$ in the norm of L^1. For this purpose, we associate with the pair $(\tau,\bar{\tau})$ a Radon measure $B = B_{(\tau,\bar{\tau})}$ defined on Π_T as follows: B is the divergence of the vector field $(|\tau-\bar{\tau}|, (\phi(\tau,\theta_0) - \phi(\bar{\tau},\theta_0))\mathrm{sign}(\tau - \bar{\tau}))$:

$$B = |\tau - \bar{\tau}|_t - ((\phi(\tau,\theta_0) - \phi(\bar{\tau},\theta_0))\mathrm{sign}(\tau - \bar{\tau}))_y. \tag{5.25}$$

We recall that with each function τ in BV, there is associated a set of points $\Gamma(\tau)$ at which τ experiences an approximate jump discontinuity [55]. We refer to $\Gamma(\tau)$, $\Gamma(\bar{\tau})$, $\Gamma(\phi)$, $\Gamma(\bar{\phi})$, and $\Gamma(\theta_0)$ as the jump set of the functions τ, $\bar{\tau}$, ϕ, $\bar{\phi}$, and θ_0, respectively. Note that if $\tau(y,t)$, $\bar{\tau}(y,t)$ and $\theta_0(y)$ are continuous functions, then

$$B_{(\tau,\bar{\tau})}(\Pi_T) = 0.$$

Then we can conclude that the measure $B_{(\tau,\bar{\tau})}$ is concentrated on $\Gamma = \Gamma(\phi) \cup \Gamma(\bar{\phi}) \cup \Gamma(\theta_0)$:

$$B(E) = B(E \cap \Gamma), \tag{5.26}$$

for any Borel set $E \subset \Pi_T$.

According to the generalized Green's theorem for measures [19], the B-measure is expressed by

$$B_{(\tau,\bar{\tau})}(E) = \int_{E\cap\Gamma} (n_t[|\tau - \bar{\tau}|] - n_y[(\phi(\tau,\theta_0) - \phi(\bar{\tau},\theta_0))\mathrm{sign}(\tau - \bar{\tau})])dH_1. \tag{5.27}$$

Here H_1 denotes the one-dimensional Hausdorff measure, $\vec{n} = (n_y, n_t)$ denotes the unit normal to $E \cap \Gamma$ in the sense of FEDERER [19], and the bracket denotes the jump of the enclosed quantity across E in the direction of $\vec{n}$:

$$[|\tau - \bar{\tau}|] = [|\tau - \bar{\tau}|](P) = |\tau_+ - \bar{\tau}_+| - |\tau_- - \bar{\tau}_-|,$$

where $(\tau_\pm, \bar{\tau}_\pm)(P) = (\tau(\phi_\pm, \theta_{0\pm}), \tau(\bar{\phi}_\pm, \theta_{0\pm}))$ denote the approximate limits of $(\tau,\bar{\tau})$ at $P \in E \cap \Gamma$ with the half-planes

$$H_{\pm\vec{n}}(P) = \{Q :< Q - P, \pm\vec{n} >\ge 0, n_y < 0\}.$$

We note that the set E_0 of irregular points and points on which $n_y \neq 0$ for $(\tau, \bar\tau) \in BV(\Pi_T)$ is the set of zero one-dimensional Hansdorff measure. Therefore we can write (5.27) as

$$B_{(\tau,\bar\tau)}(E) = \int_{E\cap E_0^c\cap\Gamma} (\sigma[|\tau-\bar\tau|] + [(\phi-\bar\phi)\mathrm{sign}(\tau-\bar\tau)])dt, \tag{5.28}$$

where $\sigma \equiv -n_t(P)/n_y(P)$ denotes the speed of propagation of Γ at $P \in E \cap E_0^c \cap \Gamma$ and $dt = -n_y dH_1$. Then for any Borel set E, the Borel measure

$$B_{(\tau,\bar\tau)}(E) \leq 0.$$

In fact, if $E \subset (\Gamma(\phi) \cap \Gamma(\bar\phi) \cap \Gamma(\theta_0))^c$, we use (5.23) and (5.28) and immediately obtain

$$B_{(\tau,\bar\tau)}(E) \leq 0;$$

If $E \subset \Gamma(\phi) \cap (\Gamma(\bar\phi) \cap \Gamma(\theta_0))^c$, or $\Gamma(\bar\phi) \cap (\Gamma(\phi) \cap \Gamma(\theta_0))^c$, or $\Gamma(\theta_0) \cap (\Gamma(\phi) \cap \Gamma(\bar\phi))^c$, we use (5.22), (5.23) and (5.28), and also deduce

$$B_{(\tau,\bar\tau)}(E) \leq 0;$$

if $E \subset \Gamma(\phi) \cap \Gamma(\bar\phi) \cap \Gamma(\theta_0)^c$, then for any point $P \in E \cap E_0^c$, we have from (5.24) that

$$\begin{aligned}&\sigma(|\tau(\phi_-,\theta_0) - \tau(\bar\phi_-,\theta_0)| - |\tau(\phi_+,\theta_0) - \tau(\bar\phi_-,\theta_0)|)\\ &+(\phi_- - \bar\phi_-)\mathrm{sign}(\tau(\phi_-,\theta_0) - \tau(\bar\phi_-,\theta_0)) - (\phi_+ - \bar\phi_-)\mathrm{sign}(\tau(\phi_+,\theta_0) - \tau(\bar\phi_-,\theta_0)) \leq 0,\end{aligned}$$

and

$$\begin{aligned}&\sigma(|\tau(\bar\phi_-,\theta_0) - \tau(\phi_+,\theta_0)| - |\tau(\bar\phi_+,\theta_0) - \tau(\phi_+,\theta_0)|)\\ &+(\bar\phi_- - \phi_+)\mathrm{sign}(\tau(\bar\phi_-,\theta_0) - \tau(\phi_+,\theta_0)) - (\bar\phi_+ - \phi_+)\mathrm{sign}(\tau(\bar\phi_+,\theta_0) - \tau(\phi_+,\theta_0)) \leq 0,\end{aligned}$$

and, therefore, we obtain

$$\begin{aligned}&\sigma(|\tau(\phi_-,\theta_0) - \tau(\bar\phi_-,\theta_0)| - |\tau(\phi_+,\theta_0) - \tau(\bar\phi_+,\theta_0)|)\\ &+(\phi_- - \bar\phi_-)\mathrm{sign}(\tau(\phi_-,\theta_0) - \tau(\bar\phi_-,\theta_0)) - (\phi_+ - \bar\phi_+)\mathrm{sign}(\tau(\phi_+,\theta_0) - \tau(\bar\phi_+,\theta_0)) \leq 0,\end{aligned}$$

and, equivalently,

$$\begin{aligned}&\sigma(|\tau_- - \bar\tau_-| - |\tau_+ - \bar\tau_+|)\\ &+(\phi(\tau_-,\theta_0) - \phi(\bar\tau_-,\theta_0))\mathrm{sign}(\tau_- - \bar\tau_-) - (\phi(\tau_+,\theta_0) - \phi(\bar\tau_+,\theta_0))\mathrm{sign}(\tau_+ - \bar\tau_+) \leq 0,\end{aligned}$$

which means from (5.28) that

$$B_{(\tau,\bar\tau)}(E) \leq 0;$$

If $E \subset \Gamma(\phi) \cap \Gamma(\theta_0) \cap \Gamma(\bar\phi)^c$, or $\Gamma(\bar\phi) \cap \Gamma(\theta_0) \cap \Gamma(\phi)^c$, for example, the former case, then

$$\begin{cases} \sigma = 0, \\ (\phi_- - \bar\phi)\mathrm{sign}(\tau(\phi_-,\theta_{0-}) - \tau(\bar\phi,\theta_{0-})) - (\phi_+ - \bar\phi)\mathrm{sign}(\tau(\phi_+,\theta_{0-}) - \tau(\bar\phi,\theta_{0-})) \leq 0, \\ (\tau(\phi_\pm,\theta_-) - \tau(\bar\phi,\theta_-))(\tau(\phi_\pm,\theta_+) - \tau(\bar\phi,\theta_+)) \geq 0, \end{cases}$$

and, therefore, we have

$$\begin{cases} \sigma = 0, \\ (\phi_- - \bar{\phi})\mathrm{sign}(\tau(\phi_-, \theta_{0-}) - \tau(\bar{\phi}, \theta_{0-})) - (\phi_+ - \bar{\phi})\mathrm{sign}(\tau(\phi_+, \theta_{0+}) - \tau(\bar{\phi}, \theta_{0+})) \le 0, \end{cases}$$

and, equivalently,

$$\begin{cases} \sigma = 0, \\ (\phi(\tau_-, \theta_{0-}) - \phi(\bar{\tau}_-, \theta_{0-}))\mathrm{sign}(\tau_- - \bar{\tau}_-) - (\phi(\tau_+, \theta_{0+}) - \phi(\bar{\tau}_+, \theta_{0+}))\mathrm{sign}(\tau_+ - \bar{\tau}_+) \le 0, \end{cases}$$

which means from (5.28) that

$$B_{(\tau,\bar{\tau})}(E) \le 0;$$

If $E \subset \Gamma(\phi) \cap \Gamma(\bar{\phi}) \cap \Gamma(\theta_0)$, then

$$\begin{cases} \sigma = 0, \\ (\phi_- - \bar{\phi}_-)\mathrm{sign}(\tau(\phi_-, \theta_{0-}) - \tau(\bar{\phi}_-, \theta_{0-})) - (\phi_+ - \bar{\phi}_-)\mathrm{sign}(\tau(\phi_+, \theta_{0-}) - \tau(\bar{\phi}_-, \theta_{0-})) \le 0, \\ (\phi_+ - \bar{\phi}_-)\mathrm{sign}(\tau(\phi_+, \theta_{0-}) - \tau(\bar{\phi}_-, \theta_{0-})) - (\phi_+ - \bar{\phi}_+)\mathrm{sign}(\tau(\phi_+, \theta_{0-}) - \tau(\bar{\phi}_+, \theta_{0-})) \le 0, \\ (\tau(\phi_\pm, \theta_{0-}) - \tau(\bar{\phi}_\pm, \theta_{0-}))(\tau(\phi_\pm, \theta_{0+}) - \tau(\bar{\phi}_\pm, \theta_{0+})) \ge 0, \end{cases}$$

and, therefore, we obtain

$$\begin{cases} \sigma = 0, \\ (\phi_- - \bar{\phi}_-)\mathrm{sign}(\tau(\phi_-, \theta_{0-}) - \tau(\bar{\phi}_-, \theta_{0-})) - (\phi_+ - \bar{\phi}_+)\mathrm{sign}(\tau(\phi_+, \theta_{0+}) - \tau(\bar{\phi}_+, \theta_{0+})) \le 0, \end{cases}$$

and, equivalently,

$$\begin{cases} \sigma = 0, \\ (\phi(\tau_-, \theta_{0-}) - \phi(\bar{\tau}_-, \theta_{0-}))\mathrm{sign}(\tau_- - \bar{\tau}_-) - (\phi(\tau_+, \theta_{0+}) - \phi(\bar{\tau}_+, \theta_{0+}))\mathrm{sign}(\tau_+ - \bar{\tau}_+) \le 0, \end{cases}$$

which means from (5.28) that

$$B_{(\tau,\bar{\tau})}(E) \le 0.$$

Fix $N > 0$, and consider domains of the form

$$D = \{(y, s) \mid |y| \le N + L(T - s), t_1 \le s \le t\}, \quad 0 < t_1 < t \le T,$$

$$S_s = \{y \mid |y| \le N + L(T - s)\}, \quad t_1 \le s \le t,$$

where the constant L will be chosen below.

It follows from the generalized Green's theorem for measures that

$$\begin{aligned} B_{(\tau,\bar{\tau})}(D) &= \int_{\partial D} \{n_t|\tau - \bar{\tau}| - n_y(\phi(\tau, \theta_0) - \phi(\bar{\tau}, \theta_0))\mathrm{sign}(\tau - \bar{\tau})\} dH_1 \\ &= \int_{\gamma} \{n_t|\tau - \bar{\tau}| - n_y(\phi(\tau, \theta_0) - \phi(\bar{\tau}, \theta_0))\mathrm{sign}(\tau - \bar{\tau})\} dH_1 \\ &\quad + \int_{S_t} |\tau - \bar{\tau}| dx - \int_{S_{t_1}} |\tau - \bar{\tau}| dx, \end{aligned}$$

where γ is the lateral boundary.

Note that, on γ,

$$\begin{aligned}&n_t|\tau-\bar{\tau}|-n_y(\phi(\tau,\theta_0)-\phi(\bar{\tau},\theta_0))\mathrm{sign}(\tau-\bar{\tau})\\&=|\tau-\bar{\tau}|(n_t-\tfrac{(\phi(\tau,\theta_0)-\phi(\bar{\tau},\theta_0))\mathrm{sign}(\tau-\bar{\tau})}{|\tau-\bar{\tau}|}n_y)\\&\geq|\tau-\bar{\tau}|(L-\max(\sup_{\Pi_T}|\phi_\tau(\tau,\theta_0)|,\sup_{\Pi_T}|\phi_\tau(\bar{\tau},\theta_0)|))=0,\end{aligned}$$

provided

$$L=\max(\sup_{\Pi}|\phi_\tau(\tau,\theta_0)|,\sup_{\Pi_T}|\phi_\tau(\bar{\tau},\theta_0)|).$$

Therefore, we obtain from Lemma 5.3,

$$\int_{S_t}|\tau(y,t)-\bar{\tau}(y,t)|dy\leq\int_{S_{t_1}}|\tau(y,t_1)-\bar{\tau}(y,t_1)|dy.$$

Notice that $\tau(x,t)$ and $\bar{\tau}(y,t)$ are BV functions. Letting $t_1\to 0$, we obtain

$$\int_{S_t}|\tau(y,t)-\bar{\tau}(y,t)|dy\leq\int_{S_0}|\tau_0(y)-\bar{\tau}_0(y)|dy.$$

This completes the proof of Lemma 5.3.

Therefore, we have the following lemma.

Lemma 5.4 *The solution of the Cauchy problem (5.20) with (5.21) and (5.22) is unique. In particular,*

$$\tau_1(y,t;\tau_0(y))=\tau_2(y,t;\tau_0(y)),\qquad a.e.$$

Lemma 5.5 *For the generalized solutions $u(x,t)$ and $v(x,t)$ with $\theta(u_0(T_0^{-1}(x)))=\theta(v_0(\bar{T}_0^{-1}(x)))$ in Theorem 5.1, the functions $r(u(x,t))$ and $r(v(x,t))$ satisfy (5.16).*

Proof. We conclude from Lemma 5.2 and Lemma 5.3 that

$$\begin{cases}\tau(u(x,t))=\tau(u(T_t^{-1}(p),t))=\tau(p,t;r(u_0(T_0^{-1}(p)))^{-1}),\\\tau(v(x,t))=\tau(v(\bar{T}_t^{-1}(q),t))=\tau(q,t;r(v_0(\bar{T}_0^{-1}(q)))^{-1}),\end{cases}$$

and both $\tau(u(x,t))=\tau(u(T_t^{-1}(p),t))$ and $\tau(v(\bar{T}_t^{-1}T_t(x),t))=\tau(v(\bar{T}_t^{-1}(p),t))$ satisfy the Cauchy problem (5.19) and (5.20) with initial data $r(u_0(T_0^{-1}(p)))^{-1}$ and $r(v_0(\bar{T}_0^{-1}(p)))^{-1}$, respectively. Using Lemma 5.3, we obtain

$$\int_{|p|\leq N_1}|\tau(u(T_t^{-1}(p),t))-\tau(v(\bar{T}_t^{-1}(p),t))|dp\leq C\int_{|p|\leq N_1+L_1t}|r(u_0(T_0^{-1}(p)))^{-1}-r(v_0(\bar{T}_0^{-1}(p)))^{-1}|dp,$$

where $L_1=\max(\sup_{\Pi_T}|(r^{-2}\phi_r)(u(x,t))|,\sup_{\Pi_T}|(r^{-2}\phi_r)(v(x,t))|)$.

Therefore, for any positive constant N, we have

$$\int_{|x|\leq N}|r(u(x,t))-r(v(\bar{T}_t^{-1}T_t(x),t))|dx\leq C\int_{|x|\leq K_0(N+Lt)}|r(u_0(x))-r(v_0(\bar{T}_0^{-1}T_0(x)))|dx.$$

Lemma 5.6 *If $\theta(u_0(T_0^{-1}(x))) = \theta(v_0(\bar{T}_0^{-1}(x)))$, a.e., then there exists a constant $C > 0$ such that, for almost all $t \in [0, T]$,*

$$\int_{|x|\leq N} |(\bar{T}_t^{-1}T_t - I)(x)|dx \leq C \int_{|x|\leq K_0(N+\bar{L}t)} (|r(u_0(x)) - r(v_0(x))| + |r(u_0(x)) - r(v_0(\bar{T}_0^{-1}T_0(x)))|)dx,$$

for any constant $N > 0$, where $\bar{L} = L + \max(\sup_{[0,T]}(|\bar{T}_t^{-1}T_t(\pm N)| - N)/t)$, and $L = \max(\sup_{\Pi_T}(|\phi(u)| + |(r^{-2}\phi_r)(u)|), \sup_{\Pi_T}(|\phi(v)| + |(r^{-2}\phi_r)(v)|))$.

Proof. It suffices to prove that, for any constant $N > 0$, there exists a constant $C > 0$ such that the Lipschitz functions $p(x, t)$ and $q(x, t)$, determined by (5.17)-(5.18), satisfy

$$\int_{|x|\leq N} |p(x,t) - q(x,t)|dx \leq C \int_{|x|\leq K_0(N+Lt)} (|r(u_0(x)) - r(v_0(x))| + |r(u_0(x)) - r(v_0(\bar{T}_0^{-1}T_0(x)))|)dx.$$

In fact, $p(x, t)$ and $q(x, t)$ satisfy

$$\begin{cases} p_t + \phi(\tau(p, t; u_0(T_0^{-1}(p))), \theta(u_0(T_0^{-1}(p))))p_x = 0, & \text{a.e.} \\ p|_{t=0} = \int_0^x r(u_0(\xi))d\xi, \end{cases}$$

and

$$\begin{cases} q_t + \phi(\tau(q, t; v_0(\bar{T}_0^{-1}(q))), \theta(v_0(\bar{T}_0^{-1}(q))))q_x = 0, & \text{a.e.} \\ q|_{t=0} = \int_0^x r(v_0(\xi))d\xi. \end{cases}$$

Using Theorem 3.3 and Lemma 5.3, we obtain

$$\begin{aligned} &\textstyle\int_{|x|\leq N} |p(x,t) - q(x,t)|dx \\ &\leq C \textstyle\int_{|x|\leq N} |\tau(q, t; u_0(T_0^{-1}(q))) - \tau(q, t; v_0(\bar{T}_0^{-1}(q)))|dx + \int_{|x|\leq N+Lt} |r(u_0(x)) - r(v_0(x))|dx \\ &\leq \textstyle\int_{|x|\leq K_0(N+Lt)} (|u_0(x) - v_0(x)| + |u_0(x) - v_0(\bar{T}_0^{-1}T_0(x))|)dx. \end{aligned}$$

This completes the proof of Lemma 5.6.

We can immediately obtain Theorem 5.1 from Lemma 5.1, Lemma 5.4, and Lemma 5.6. As a corollary, we have the following theorem.

Theorem 5.2 *(Uniqueness). The BV generalized solution of the Cauchy problem (5.7) is unique.*

Propagation and cancellation of initial oscillations for such systems (5.1) have been studied by CHEN [4, 5].

Now we discuss some applications of Theorem 5.1 and Theorem 5.2 to several important systems arising in the field of elasticity theory, enhanced oil recovery, and magnetohydrodynamics.

b. Systems Arising in Magnetohydrodynamics and Elasticity

Consider the 2×2 systems with symmetry that are discussed in [4]. We assume that $\phi(u)$ satisfies the following symmetry assumptions:

S_1: $\phi(u) > 0$.

S_2: $\phi(u) = \text{const.} > 0$ is a simple closed curve enclosing the origin $u = 0$.

This class of systems contains the elastic string model and the magnetohydrodynamics model [30, 37],

$$\begin{cases} \phi = \phi(r) > 0, \\ \phi''(r) > 0. \end{cases}$$

In particular, $\phi = \phi(r) = 1 + \delta\frac{(r-1)^2}{r}$, $\delta > 0$ constant, models a nonlinear stress-strain relation. On the region $K \supset \{u : r(u) = 1\}$, the system is nonstrictly hyperbolic.

Theorem 5.3 *If the function* $\phi(u)$ *satisfies*

$$\begin{cases} r\phi_r \geq c_0 > 0, \\ r(r\phi)_{rr} \geq c_0 > 0, \end{cases}$$

besides the assumptions $S_1 - S_2$, *and the initial data* $u_0(x) \in \{u : \phi_1 \leq \phi(u) \leq \phi_2 < \infty\}$, $\phi_1 > \phi(0)$, *then there exists a unique* BV *generalized solution* $u(x,t)$ *of the Cauchy problem (5.7) satisfying (5.8) and (5.14).*

Proof. Following [4], we first obtain a global BV weak solution $u(x,t)$ satisfying (5.8). It is easy to check that, for any C^2 strictly convex entropy (G_b, H_b) on K, the solution $u(x,t)$ also satisfies (5.14), which is equivalent to (5.9) for such systems. The result follows. We omit the details.

Furthermore, we have the following theorem.

Theorem 5.4 *If the function* $\phi(u) = \phi(r)$ *satisfies* $mes\{r : \phi''(r) = 0\} = 0$, *and the initial data* $u_0(x) \in \{u : 0 < r_1 \leq r(u) \leq r_2 \leq \infty\}$, *then there exists a unique* L^∞ *solution* $u(x,t)$ *of the Cauchy problem (5.7) satisfying the following entropy conditions:*

(i) For any function $a \in C_{loc}(-\infty, \infty)$ *and* $\psi \in C_0^1(\Pi_T)$,

$$\int\int_{\Pi_T} (ra(\theta)\psi_t + r\phi a(\theta)\psi_x)dxdt + \int_{-\infty}^{\infty} r_0(x)a(\theta_0(x))\psi(x,0)dx = 0. \tag{5.29}$$

(ii) For any constant $k \in R^1$ *and any nonnegative function* $\psi \in C_0^1(\Pi_T - \{t = 0\})$,

$$\int\int_{\Pi_T} (|r-k|\psi_t + (r\phi(r) - k\phi(k))sign(r-k)\psi_x)dxdt \leq 0. \tag{5.30}$$

Proof. Following [4], we first obtain a global BV weak solutions $u(x,t)$ satisfying (5.29)-(5.30). Using the framework established in Theorem 3.1 in [5], we conclude that there exists a global L^∞ solution of the Cauchy problem (5.7) satisfying (5.29)-(5.30). Then using Kruzkov's theorem, we immediately conclude the uniqueness of r. Finally, using the Theorem 5.1, we obtain the uniqueness of θ. This completes the proof of Theorem 5.4.

c. A System Arising in Oil Recovery

Consider

$$\phi(u_1, u_2) = \frac{a(u_1, \frac{u_2}{u_1})}{u_1}, \tag{5.31}$$

where $u_1 \in [0,1]$, $u_2 \in [0, u_1]$, are the concentration of water and the overall concentration of a polymer at any x and t, respectively, and $a(u_1, c)$ is a smooth function such that $a(u_1, c)$ increases from zero to one with one inflection point where c is constant and such that $a(u_1, c)$ decreases with increasing c for fixed u_1 (see [53]). The essential feature of the system is that strictly hyperbolicity fails along a certain curve in state space.

TEMPLE [53] obtained a global weak solution to the system (5.1) and (5.31) with the Cauchy data $u_0(x) \in BV$. One can check that Temple's solution satisfy (5.8).

Theorem 5.5 *Suppose that BV functions $u(x,t)$ and $v(x,t)$ are solutions of the Cauchy problem (5.7) satisfying (5.8) in Π_T with BV initial data $u_0(x) \geq c_0 > 0$ and $v_0(x)$, respectively. Then for almost all $t \in (0,T)$,*

$$\begin{cases} \int_{|x|\leq N} |\theta(u(x,t)) - \theta(v(\bar{T}_t^{-1}T_t(x),t))|dx \leq C\int_{|x|\leq K_0(N+Lt)} |\theta(u_0(x)) - \theta(v_0(\bar{T}_0^{-1}T_0(x)))|dx, \\ \int_{|x|\leq N} |\theta(u(x,t)) - \theta(v(\bar{T}_t^{-1}\bar{T}_0T_0^{-1}T_t(x),t))|dx \leq C\int_{|x|\leq K_0(N+Lt)} |\theta(u_0(x)) - \theta(v_0(x))|dx, \end{cases}$$

and

$$\begin{cases} \int_{|x|\leq N} |r(u(x,t)) - r(v(\bar{T}_t^{-1}T_t(x),t))|dx \leq \int_{|x|\leq K_0(N+Lt)} |r(u_0(x)) - r(v_0(\bar{T}_0^{-1}T_0(x)))|dx, \\ \int_{|x|\leq N} |(\bar{T}_t^{-1}T_t - I)(x)|dx \\ \quad \leq C\int_{|x|\leq K_0(N+\bar{L}t)}(|r(u_0(x)) - r(v_0(x))| + |r(u_0(x)) - r(v_0(\bar{T}_0^{-1}T_0(x)))|)dx, \end{cases}$$

provided that (5.9)-(5.10) and $\theta(u_0(\bar{T}_0^{-1}(x))) = \theta(v_0(T_0^{-1}(x)))$ a.e. also hold, where $T(x,t) = (T_t(x),t)$ and $\bar{T}(x,t) = (\bar{T}_t(x),t)$ are the quasidecoupling transformations corresponding to the vector fields $(r(u(x,t)),(r\phi)(u(x,t)))$ and $(r(v(x,t)),(r\phi)(v(x,t)))$. Furthermore, the BV generalized solution of the Cauchy problem (5.7) satisfying (5.8)-(5.10) is unique in Π_T.

In fact, for this system, the condition (5.10) is equivalent to the condition that the left state and the right state of any contact discontinuity lie in the same side of the degenerate curve $\{(u,v) : \phi_r = 0\}$, which is used to ensure the uniqueness of solutions for the Riemann problem as in [27] and [53]. It would be interesting to study the regularity of Temple's solution $u(x,t)$ of the Cauchy problem (5.7) and (5.31) with the Cauchy data $u_0(x) \in BV$, especially with BV regularity. In this connection we refer the work of TVEITO AND WINTHER [54] for a stability result within the solution space $B^0 = \{(u_1, u_2) : (u_1, (\frac{u_2}{u_1})_x) \in BV, \frac{u_2}{u_1} \in Lip \cap BV\}$.

6 The System of Electromagnetic Plane Waves

In this section we present an example of how to reduce important problems for large systems with linearly degenerate fields into the problems for corresponding smaller systems by using the quasidecoupling method. We are concerned with the Maxwell equations

$$\begin{cases} B_t + \text{curl } E = 0, \\ D_t - \text{curl } H = 0, \end{cases} \tag{6.1}$$

where B, D, E, and H are the magnetic induction, the electric induction, the electric intensity, and the magnetic intensity in $\mathbb{R}^3$, respectively. The constitutive laws can be of the following form in the electromagnetic field (see COLEMAN AND DILL [6]):

$$E = \frac{\partial W}{\partial D}, \quad H = \frac{\partial W}{\partial B}. \tag{6.2}$$

Here $W(B, D)$ is a convex electromagnetic energy density, i.e., $\nabla^2 W \geq 0$, satisfying

$$W_t + \text{div}(\text{E} \times \text{H}) = 0, \tag{6.3}$$

for smooth electromagnetic fields.

As described in [49], the system is translationally invariant and admits plane waves depending only on t and one space variable $x = x_3$. Then the components B_3 and D_3 can be constant, and (6.1) reduces to four equations:

$$\begin{cases} \frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}(\phi u_4) = 0, \\ \frac{\partial u_2}{\partial t} - \frac{\partial}{\partial x}(\phi u_3) = 0, \\ \frac{\partial u_3}{\partial t} - \frac{\partial}{\partial x}(\phi u_2) = 0, \\ \frac{\partial u_4}{\partial t} + \frac{\partial}{\partial x}(\phi u_1) = 0, \end{cases} \tag{6.4}$$

with the axisymmetry assumption

$$W(u) = W(r), \qquad r = \|u\|,$$

the notation $(u_1, u_2, u_3, u_4) = (D_1, D_2, B_1, B_2)$, and

$$\phi(r) = \frac{W'(r)}{r}. \tag{6.5}$$

The system (6.4) is nonstrictly hyperbolic because two of four eigenvalues coincide: $\lambda_1 = \lambda_2 = \phi(r)$. The eigenvalues $\lambda_1 = \lambda_2$ are linearly degenerate, and the other two eigenvalues λ_3, λ_4 are genuinely nonlinear.

We shall first focus on how to reduce the uniqueness problem for the system (6.4) to the uniqueness problem for a 2×2 system with the aid of the quasidecoupling method.

Introduce a nonlinear transformation $u \to (\rho, \sigma, \alpha, \theta) \in \mathbb{R}^2_+ \times [0, 2\pi]^2$:

$$\begin{cases} u_1 = \frac{1}{2}(\rho \cos\alpha + \sigma \cos\theta), \\ u_2 = \frac{1}{2}(\rho \cos\alpha - \sigma \cos\theta), \\ u_3 = \frac{1}{2}(\rho \sin\alpha + \sigma \sin\theta), \\ u_4 = \frac{1}{2}(\rho \sin\alpha - \sigma \sin\theta). \end{cases} \tag{6.6}$$

Then one can verify (see [49]) that there exists an L^∞ global solution $u(x,t)$, $(\rho(u), \sigma(u)) \geq 0$, $\|u\|_{L^\infty} \leq C(\|u_0\|_{L^\infty})$, of the Cauchy problem

$$\begin{cases} (6.4), \\ u|_{t=0} = u_0(x), \end{cases} \tag{6.7}$$

satisfying, for any continuous functions $a(\alpha)$ and $b(\theta)$,

$$\begin{cases} (\rho a(\alpha))_t + (\rho\phi a(\alpha))_x = 0, \\ (\sigma b(\theta))_t - (\sigma\phi b(\theta))_x = 0, \end{cases} \tag{6.8}$$

and

$$2W_t + ((\rho^2 - \sigma^2)\phi^2)_x \leq 0, \tag{6.9}$$

in the generalized sense of distribution under certain physical conditions on ϕ or W:

$$W'(r) > 0, \quad \left(\frac{W'(r)}{r}\right)' > 0, \quad W'''(r) > 0, \quad \left(\frac{rW''(r)}{W'(r)}\right)' > 0, \qquad \forall r > 0, \tag{6.10}$$

where $r = \sqrt{\rho^2 + \sigma^2}$.

Using Lemma 3.1 and Corollary 3.3, we can immediately conclude the following stability behavior of the solutions.

Theorem 6.1 *(Stability). Suppose that $u(x,t)$ and $v(x,t)$ are L^∞ weak solutions satisfying (6.8) of the Cauchy problem (6.7) with initial data $u_0(x)$ and $v_0(x)$ in Π_T, respectively, and*

$$c_0 = \min(\inf_{\mathbf{R}^1} \rho(u_0(x)), \inf_{\mathbf{R}^1} \sigma(u_0(x)), \inf_{\mathbf{R}^1} \rho(v_0(x)), \inf_{\mathbf{R}^1} \sigma(v_0(x))) > 0. \tag{6.11}$$

Then for almost all $t \in (0,T)$ and any positive constant m,

$$\begin{cases} \int_{|x|\leq N} |\alpha(u(x,t)) - \alpha(v(\bar{T}_{t1}^{-1}\bar{T}_{01}T_{01}^{-1}T_{t1}(x),t))|^m \rho(u(x,t))\rho(v(\bar{T}_{t1}^{-1}\bar{T}_{01}T_{01}^{-1}T_{t1}(x),t))dx \\ \quad \leq \|\rho\|^2_{L^\infty} \int_{|x|\leq K_0(N+Lt)} |\alpha(u_0(x)) - \alpha(v_0(x))|^m dx, \\ \int_{|x|\leq N} |\theta(u(x,t)) - \theta(v(\bar{T}_{t2}^{-1}\bar{T}_{02}T_{02}^{-1}T_{t2}(x),t))|^m \sigma(u(x,t))\sigma(v(\bar{T}_{t2}^{-1}\bar{T}_{02}T_{02}^{-1}T_{t2}(x),t))dx \\ \quad \leq \|\sigma\|^2_{L^\infty} \int_{|x|\leq K_0(N+Lt)} |\theta(u_0(x)) - \theta(v_0(x))|^m dx, \end{cases} \tag{6.12}$$

and

$$\left\{\begin{array}{l}\int_{|x|\le N}|\alpha(u(T_{t1}^{-1}T_{01}\bar{T}_{01}^{-1}\bar{T}_{t1}(x),t))-\alpha(v(x,t))|^m\rho(u(T_{t1}^{-1}T_{01}\bar{T}_{01}^{-1}\bar{T}_{t1}(x),t))\rho(v(x,t))dx\\ \le\|\rho\|_{L^\infty}^2\int_{|x|\le K_0(N+Lt)}|\alpha(u_0(x))-\alpha(v_0(x))|^m dx,\\ |\int_{|x|\le N}|\theta(u(T_{t2}^{-1}T_{02}\bar{T}_{02}^{-1}\bar{T}_{t2}(x),t))-\theta(v(x,t))|^m\sigma(u(T_{t2}^{-1}T_{02}\bar{T}_{02}^{-1}\bar{T}_{t2}(x),t))dx\\ \le\|\sigma\|_{L^\infty}^2\int_{|x|\le K_0(N+Lt)}|\theta(u_0(x))-\theta(v_0(x))|^m dx,\end{array}\right. \tag{6.13}$$

where $T_1(x,t)=(T_{t1}(x),t),\bar{T}_1(x,t)=(\bar{T}_{t1}(x),t),T_2(x,t)=(T_{t2}(x),t)$, *and* $\bar{T}_2(x,t)=(\bar{T}_{t2}(x),t)$ *are the quasidecoupling transformations corresponding to the vector fields* $(\rho(u),(\rho\phi)(u))$, $(\rho(v),(\rho\phi)(v))$, $(\sigma(u),(\sigma\phi)(u))$ *and* $(\sigma(v),(\sigma\phi)(v))$, *respectively, and*

$$\begin{array}{l}L=\max(\sup_{\Pi_T}|\phi(u(x,t))|,\sup_{\Pi_T}|\phi(v(x,t))|),\\ K_0=\max(\sup_{\Pi_T}|\sigma(u)|,\sup_{\Pi_T}|\sigma(v)|,\sup|\rho(u)|,\sup|\rho(v)|)/c_0.\end{array}$$

In fact, we can choose in Framework (A)

$$\left\{\begin{array}{ll}a_0(w)=|\alpha(u)-k|^m, & (P_1,\lambda_1P_1)=(\rho(u),\rho(u)\phi(u)),\\ a_0(z)=|\alpha(v)-l|^m, & (P_2,\lambda_2P_2)=(\rho(v),\rho(v)\phi(v)),\end{array}\right.$$

and

$$\left\{\begin{array}{ll}a_0(w)=|\theta(u)-k|^m, & (P_1,\lambda_1P_1)=(\sigma(u),\sigma(u)\phi(u)),\\ a_0(z)=|\theta(v)-l|^m, & (P_2,\lambda_2P_2)=(\sigma(v),\sigma(v)\phi(v)),\end{array}\right.$$

for any $0<m<\infty$ and any constants k, $l\in(-\infty,\infty)$. The result (6.12) follows.

Then we have the following theorem.

Theorem 6.2 *The* L^∞ *weak solution of the Cauchy problem (6.7) and (6.11) satisfying (6.8)-(6.9) is unique if and only if the weak solution of the Cauchy problem*

$$\left\{\begin{array}{l}\rho_t+(\rho\phi(r))_x=0,\\ \sigma_t-(\sigma\phi(r))_x=0,\end{array}\right. \tag{6.14}$$

$$(\rho,\sigma)|_{t=0}=(\rho(u_0(x)),\sigma(u_0(x))), \tag{6.15}$$

satisfying (6.9) is unique.

Proof. Suppose that $u(x,t)$ and $v(x,t)$ are two solutions of the Cauchy problem (6.7) and (6.11) satisfying (6.8)-(6.9). If the weak solution of the Cauchy problem (6.14)-(6.15) satisfying (6.9) is unique, then

$$\rho(u)=\rho(v),\qquad\sigma(u)=\sigma(v),$$

and both $(\alpha(u),\theta(u))$ and $(\alpha(v),\theta(v))$ satisfy (6.12). Corresponding quasidecoupling transformations $(T_1(x,t);T_2(x,t))=((p(x,t),t);(q(x,t),t))=((p(x,t;u),t);(q(x,t;u),t))$ and $(\bar{T}_1(x,t);\bar{T}_2(x,t))=((\bar{p}(x,t),t);(\bar{q}(x,t),t))=((p(x,t;v),t);(q(x,t;v),t))$ satisfy

$$\begin{cases} p_t + \phi(x,t) p_x = 0, \\ \bar{p}_t + \phi(x,t)\bar{p}_x = 0, \\ p|_{t=0} = \bar{p}|_{t=0}, \end{cases}$$

and

$$\begin{cases} q_t + \phi(x,t) q_x = 0, \\ \bar{q}_t + \phi(x,t)\bar{q}_x = 0, \\ q|_{t=0} = \bar{q}|_{t=0}, \end{cases}$$

where $\phi(x,t) = \phi(r(x,t)), r = \sqrt{\rho(u)^2 + \sigma(u)^2} = \sqrt{\rho(v)^2 + \sigma(v)^2}$.

Notice that all $p, \bar{p}, q$, and $\bar{q}$ are Lipschitz functions. We immediately conclude that

$$p(x,t) = \bar{p}(x,t), \qquad q(x,t) = \bar{q}(x,t), \qquad \text{a.e.}$$

Then we complete the proof by using Theorem 6.1.

Thus we can transfer the uniqueness problem of Cauchy problem (6.7) to the uniqueness problem of the Cauchy problem (6.14)-(6.15) for the 2×2 system (6.14) with the aid of the quasidecoupling method.

Let $PL = PL\{\Pi_T\}$ be the class of piecewise Lipschitz functions; that is, for each t in $[0,T)$ there exists a set of isolated points $x_j = x_j(t)$ such that the restriction $u(\cdot,t)$ of u to each interval (x_j, x_{j+1}) is a Lipschitz function of x; the dependence of the Lipschitz constant on the interval (x_j, x_{j+1}) as well as the dependence of the partition points x_j on t is arbitrary. Note that the system (6.14) is genuinely nonlinear in the sense of Lax [33]. Then combining Theorem 6.2 with DiPerna's theorem [14], we can conclude the following corollary.

Corollary 6.1 *Suppose that $u(x,t)$ and $v(x,t)$ are L^∞ weak solutions of the Cauchy problem (6.7) and (6.11) satisfying (6.8)-(6.9) in Π_T such that*

$$\begin{aligned} (\alpha(u), \theta(u), \rho(u), \sigma(u)) &\in (L^\infty)^2 \times (BV)^2\{\Pi_T\}, \\ (\alpha(v), \theta(v), \rho(v), \sigma(v)) &\in (L^\infty)^2 \times (PL)^2\{\Pi_T\}, \end{aligned}$$

and

$$u(x,0) = v(x,0), \qquad \textit{for almost all } x.$$

Then

$$u(x,t) = v(x,t),$$

for almost all (x,t) in $\prod_T$.

Finally, we conclude the following theorem by using Lemma 3.1 and Theorem 3.5, and following the proof of Theorem 4.2 for systems with fully contact fields.

Theorem 6.3 *The initial oscillations will propagate along the linearly degenerate fields for the system (6.4).*

A direct corollary follows.

Corollary 6.2 *For the Cauchy problem (6.7), the solution operator* $S_t : L^\infty(\mathbb{R}^4) \to L^1_{\mathrm{loc}}(\mathbb{R}^4)$ *with* $S_t(u^0) = u(\cdot, t)$ *is not compact.*

We omit the details of the proof of Theorem 6.3.

7 Remarks

The quasidecoupling method and ideas developed in the previous sections can be applied to solving some nonlinear problems for more general systems.

a. Propagation and Cancellation of Initial Oscillations. Using the same arguments as the proof of Theorem 4.2 with the aid of Framework (B), we can extend the results virtually without modification to strictly hyperbolic systems (1.1) endowed with (1.6) and (2.5). If there exists a linearly degenerate characteristic field for such systems, then the entropy-entropy fluxes (2.17) of the same form can be obtained, which corresponds to this field. In fact, Framework (B) is useful to solve oscillation problems for systems of conservation laws endowed with the entropy-entropy fluxes of the form (2.17).

b. Generalized Characteristics and the Structure of Solutions. Theorem 3.1 and Framework (A)-(B) in Section 3 should be applicable to clarify the generalized characteristics and the structure of solutions to systems of conservation laws endowed with the entropy-entropy fluxes of the form (2.17). In particular, they can provide insight into how the interaction between large linear waves and large nonlinear waves is, and provide a useful approach to understand large time asymptotic behavior of solutions for systems of conservation laws with such mixing linear characteristic fields and nonlinear characteristic fields. In this connection we refer GLIMM AND LAX [23], DAFERMOS [11] and the references cited therein on the generalized characteristics and the structure of solutions for conservation laws.

c. Uniqueness and Existence of Generalized Solutions. From discussions in the previous sections, it should be evident that Framework (A) and ideas developed here should be useful to solve the uniqueness and existence problems for more general systems by combining them with other techniques.

d. Equivalence of the Euler and Lagrangian Equations of Gas Dynamics for Entropy Solutions. WAGNER [56] verified the equivalent of existence of entropy discontinuous solutions between the Euler equations and the Lagrange equations of gas dynamics. As a corollary of Theorem 3.3, we can conclude the equivalent of uniqueness of entropy discontinuous solutions between the Euler equations and the Lagrange equations of gas dynamics because of the uniqueness of the transformation between the Eulerian coordinates and the Lagrangian coordinates with the same initial data from Theorem 3.3.

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