Stability of Entropy Solutions to the Cauchy Problem

for a Class of Nonlinear Hyperbolic–Parabolic Equations

Gui-Qiang Chen[1]   Emmanuele DiBenedetto[2]

Short title for running head: **Solutions for Hyperbolic-Parabolic Equations**

Key words: **Stability, uniqueness, entropy solutions, hyperbolic-parabolic, degenerate parabolic, nonlinear equations, unbounded solutions, optimal growth, Cauchy problem**

AMS subject classifications: **35K65,35B35,35G25,35D99**

Abstract

Consider the Cauchy problem for the nonlinear hyperbolic-parabolic equation:

$$(*) \qquad u_t + \frac{1}{2}\mathbf{a}\cdot\nabla_x u^2 = \Delta u_+ \qquad \text{for } t>0,$$

where $\mathbf{a}$ is a constant vector and $u_+ = \max\{u, 0\}$. The equation is hyperbolic in the region $[u<0]$ and parabolic in the region $[u>0]$. It is shown that entropy solutions to (*), that grow at most linearly as $|x|\to\infty$, are stable in a weighted $L^1(I\!R^N)$ space, which implies that the solutions are unique. The linear growth as $|x|\to\infty$ imposed on the solutions is shown to be optimal for uniqueness to hold. The same results hold if the Burgers nonlinearity $\frac{1}{2}\mathbf{a}u^2$ is replaced by a general flux function $\mathbf{f}(u)$, provided $\mathbf{f}'(u(x,t))$ grows in x at most linearly as $|x|\to\infty$, and/or the degenerate term u_+ is replaced by a non-decreasing, degenerate, Lipschitz continuous function $\beta(u)$ defined on $I\!R$. For more general $\beta(\cdot)$, the results continue to hold for bounded solutions.

---

[1] Department of Mathematics, Northwestern University, Evanston, IL 60208–2730;
Email: gqchen@math.northwestern.edu; Phone: 847-491-5553; Fax: 847-491-8906; Partially supported by NSF Grants DMS-9971793, INT-9987378, and INT-9726215.

[2] Department of Mathematics, Vanderbilt University, Nashville, TN 60208–2730;
Email: dibe@math.vanderbilt.edu; Phone: 615-343-5906; Fax: 615-343-0215; Partially supported by NSF Grants DMS-9706388 and DMS-9708261.

1. Introduction and Results

Consider the Cauchy problem for nonlinear hyperbolic–parabolic equation:

$$
\begin{aligned}
&u_t + \tfrac{1}{2}\,\mathbf{a}\cdot\nabla_x u^2 = \Delta u_+ \qquad \text{in } \mathcal{D}'(S_T),\\
&u(\cdot,0) = u_0 \in L^\infty_{\rm loc}(I\!R^N),
\end{aligned} \tag{1.1}
$$

where $\mathbf{a}$ is a constant vector, $u_+ = \max\{u,0\}$, the set S_T is the strip

$$
S_T \;=\; I\!R^N \times (0,T] \qquad \text{for some } \; T>0,
$$

and the initial data are taken in the sense of $L^1_{\rm loc}(I\!R^N)$.

The equation in (1.1) could be regarded as a prototype model for the motion of an ideal fluid filling $I\!R^N$ and exhibiting both viscous and non–viscous phases. The set $[u>0]$ can be identified with the viscous phase, the set $[u<0]$ is the inviscid phase, and the set $[u=0]$ is the free boundary interface, separating these phases. Accordingly the equation in (1.1) is of mixed type, i.e., it is hyperbolic in the inviscid phase and parabolic in the viscous phase. It can also be regarded as degenerate parabolic.

The main issues relating to the Cauchy problem (1.1) are its unique solvability and the local behavior of its solutions. Both issues are relatively well understood if one removes the hyperbolic part $\mathbf{a}\cdot\nabla u^2$, thereby obtaining a degenerate parabolic equation (see for example the discussion and references of [5,6]). They are equally well understood if one removes the viscosity term Δu_+. This would give the N–dimensional Burgers equation (see for example [3,4]).

We establish that entropy solutions of (1.1), that grow at most linearly as $|x|\to\infty$, are stable, which implies that the solutions are unique. We also show that such a growth is optimal for uniqueness to hold.

For a smooth convex function $\eta(\cdot)$, let $\mathbf{q}(\eta;u)$ denote the flux function corresponding to the entropy $\eta(u)$, i.e.,

$$
q(\eta;u) \;=\; \int_0^u s\eta'(s)ds, \qquad \mathbf{q}(\eta;u) \equiv q(\eta;u)\,\mathbf{a}. \tag{1.2}
$$

A function $u\in L^\infty_{\rm loc}(S_T)$ is an entropy solution of the Cauchy problem (1.1) if,

$$
u_+ \;\in\; L^2\big(0,T;W^{1,2}_{\rm loc}(I\!R^N)\big), \tag{1.3}
$$

and if, for every convex function $\eta\in C^2(I\!R)$, for any non–negative testing function $\psi\in C^1\big([0,T);C^2_0(I\!R^N)\big)$ with $\psi|_{t\ge T}=0$,

$$
\begin{aligned}
&\iint_{S_T}\big\{\eta(u)\,\psi_t \,+\, \mathbf{q}(\eta;u)\cdot\nabla_x\psi - \eta'(u)\nabla_x u_+\cdot\nabla_x\psi\big\}\,dxdt\\
&-\iint_{S_T}\eta''(u)\,|\nabla_x u_+|^2\,\psi\,dxdt + \int_{I\!R^N}\eta(u_0)\,\psi(x,0)\,dx \ge 0.
\end{aligned} \tag{1.4}
$$

Entropy solutions are distributional solutions. Since $u \in L^\infty_{\text{loc}}(S_T)$, an adaptation of the results of [2] implies that $u_+ \in C_{\text{loc}}(S_T)$.

MAIN THEOREM. *Let u and v be two entropy solutions of (1.1) in the sense of (1.2)–(1.4) for initial data u_0 and v_0. Assume in addition that they satisfy the growth condition*

$$|u(x,t)| + |v(x,t)| \le \gamma(1+|x|), \tag{1.5}$$

for almost all $(x,t) \in I\!R^N \times [0,T]$, for some positive constant γ. Then, there exists a smooth, positive weight $w(x,t)$, that can be determined a–priori only in terms of γ and satisfying

$$w(x,t)\,\big(1+|x|\big) \in L^1(I\!R^N) \qquad \textit{uniformly in } t \in (0,T),$$

such that

$$\int_{I\!R^N} w(x,t)\,\big|u(x,t)-v(x,t)\big|\,dx \le \int_{I\!R^N} w(x,0)\,\big|u_0(x)-v_0(x)\big|\,dx, \tag{1.6}$$

for a.e. $t \in (0,T]$.

The theorem continues to hold if the Burgers nonlinearity $\frac{1}{2}\mathbf{a}u^2$ is replaced by a more general flux function $\mathbf{f}(u)$, provided $|\mathbf{f}'(u(x,t))|$ grows in x at most linearly as $|x| \to \infty$ for any fixed $t \in [0,T)$. Also the degenerate term u_+ can be replaced by a non–decreasing, degenerate (possibly identically zero), Lipschitz continuous function $\beta(u)$ defined in $I\!R$. These generalizations follow, mutatis mutandis, from the arguments in § 3–7. The Lipschitz requirement on $\beta(\cdot)$ would exclude degeneracies of the type $\beta(u) = u_+^m$ for some $m > 1$. For such a $\beta(\cdot)$, the theorem continues to hold for bounded solutions.

The stability theorem and the techniques we develop here may be useful to make error estimates for various approximate solutions, especially numerical methods, and to establish similar results for more general hyperbolic-parabolic equations, which model a wide variety of phenomena, ranging from porous media flow, via flow of glaciers and sedimentation processes, to traffic flow.

In Section 2, we make some remarks to show that the linear growth as $|x| \to \infty$ imposed on the solutions in our main theorem is optimal; and the solutions in general are local, and if they exist, they blow up in finite time.

In order to prove the main theorem, in Section 3, we first use the definition (1.4) of entropy solutions to derive two integral inequalities for a class of C^2 entropy-entropy flux pairs, depending on a parameter $\varepsilon > 0$. These two inequalities are written for two entropy solutions u and v of (1.1) with initial data u_0 and v_0, respectively.
In Section 4, we first add these two integral inequalities and choose appropriate testing functions depending upon a parameter $h > 0$. These testing functions have a mollifier-type effect and serve to handle the possible irregularity of the solutions. Then we introduce a change of variables and employ the new variables in Sections 5 and 6 to transform these integral inequalities into a form suitable to study the limits as $\varepsilon \to 0$ and then $h \to 0$ in the indicated order. This will produce a further more stringent integral inequality for u and v involving testing functions still to be chosen. In Section 7, such testing functions are chosen to identify the weight $w(x,t)$ and the stability result (1.6).

2. Remarks on the Stability

Distributional solutions, that are not entropy solutions, are not unique and hence are not stable. For example, non-positive solutions of (1.1) would be solutions of the inviscid Burgers equation, for which uniqueness fails outside the class of entropy solutions [3,4].

The growth condition (1.5) is optimal for uniqueness (hence stability) to hold. For this, consider locally bounded, non–negative solutions of (1.1) for $N = 1$, i.e.,

$$\begin{aligned} &u_t + u\,u_x = u_{xx} && \text{in } \mathcal{D}'(S_T), \\ &u(\cdot,0) = u_0 \geq 0 && \text{and smooth in } I\!R. \end{aligned} \tag{2.1}$$

Solutions of (2.1) are smooth and positive in S_T. Set

$$v(x,t) = \int_\alpha^x u(y,t)\,dy \qquad \text{for some } \ \alpha \in I\!R.$$

Then $u = v_x$ and u is a solution of (2.1) if and only if v is a classical solution of

$$\begin{aligned} &v_t + \tfrac{1}{2}(v_x)^2 - v_{xx} = c_\alpha(t), \\ &v(x,0) = \int_\alpha^x u_0(y)\,dy, \end{aligned} \tag{$2.1)_v$}$$

where

$$c_\alpha(t) = \tfrac{1}{2}u^2(\alpha,t) - u_x(\alpha,t).$$

The Hopf–like transformation

$$w(x,t) = \exp\left\{-\frac{1}{2}v + \frac{1}{2}\int_0^t c_\alpha(\tau)\,d\tau\right\}$$

transforms $(2.1)_v$ into the equivalent formulation

$$\begin{aligned} &w_t - w_{xx} = 0 && \text{in } S_T, \\ &w(x,0) = \exp\left\{-\tfrac{1}{2}v(x,0)\right\}. \end{aligned} \tag{$2.1)_w$}$$

Therefore, well-posedness for (2.1) is equivalent to well-posedness for $(2.1)_w$. The latter is well-posed if and only if

$$w(x,t) \leq \exp\left\{C\,\left(1 + x^2\right)\right\},$$

for some positive constant C and for all $t \in [0,T]$ (see [1], Chap. 5). The constant C and the time T are linked by $4CT < 1$. By the Tychonov counterexample ([1] page 237), a faster growth would not guarantee uniqueness.

In terms of $u = -2(e^w)_x$, this implies that the stability theorem would be false for solutions growing faster than linearly as $|x| \to \infty$. This also implies that solutions of (1.1) are, in general, not global in time.

If the initial datum u_0 is bounded in $I\!R^N$ and in $BV(I\!R^N)$, then solutions of (1.1) can be constructed as in Volpert–Hudjaev [6]. For such initial data, the authors also proved uniqueness. If the initial datum u_0 is bounded in $I\!R^N$ but not necessarily regular (i.e., not in $BV(I\!R^N)$), existence of entropy solutions can be established as in Kruzhkov [3], as indicated also in [5]. The same construction also implies $\nabla u_+ \in L^2_{\rm loc}(S_T)$. Attention to the uniqueness problem for parabolic–hyperbolic equations such as (1.1) has been drawn in [5].

If the initial datum is unbounded, then, by the previous remarks, solutions in general are local, and if they exist, they blow up in finite time. An existence theorem would have to involve an estimation of the blow–up time.

3. Proof of the Stability Theorem–(i)

In this section, we first use the definition (1.4) of entropy solutions to derive two integral inequalities for a class of C^2 entropy-entropy flux pairs, depending on a parameter $\epsilon > 0$.

To choose suitable entropy functions $\eta(u)$ in (1.4), introduce the regularizations of the Heaviside function:

$$H_\varepsilon(s) \equiv \begin{cases} 1 & \text{if } s > \varepsilon, \\ \sin\left(\dfrac{\pi}{2\varepsilon}\, s\right) & \text{if } |s| \le \varepsilon, \\ -1 & \text{if } s < -\varepsilon. \end{cases}$$

Then, for each $k \in I\!R$, the functions

$$u \longrightarrow \eta_\varepsilon(u-k) \equiv \int_0^{u-k} H_\varepsilon(s)\, ds, \qquad 0 < \varepsilon \ll 1, \tag{3.1}$$

are convex and of class C^2 in $I\!R$. Moreover,

$$\eta_\varepsilon(u-k) \longrightarrow |u-k|, \qquad \text{as } \varepsilon \to 0. \tag{3.1$'$}$$

For the corresponding flux functions defined in (1.2), we set

$$\mathbf{Q}_\varepsilon(u;k) \;=\; \mathbf{q}(\eta_\varepsilon;u) \;\equiv\; \left(\int_k^u s\, H_\varepsilon(s-k)\, ds\right) \mathbf{a}. \tag{3.2}$$

One verifies that

$$\mathbf{Q}_\varepsilon(u;k) \longrightarrow \tfrac{1}{2}\left|u^2 - k^2\right| \mathbf{a}, \qquad \text{as} \quad \varepsilon \to 0. \tag{3.2$'$}$$

In (1.4) we choose non–negative testing functions $(x,t;y,\tau) \to \psi(x,t;y,\tau)$ depending upon two pairs of variables (x,t) and (y,τ) such that

$$\begin{aligned} (x,t) \;\to\; \psi(x,t;y,\tau) \;\in\; C^1\big([0,T); C_0^2(I\!R^N)\big) \quad & \text{uniformly in } (y,\tau), \\ (y,\tau) \;\to\; \psi(x,t;y,\tau) \;\in\; C^1\big([0,T); C_0^2(I\!R^N)\big) \quad & \text{uniformly in } (x,t). \end{aligned}$$

We also require $\psi(x,t;y,\tau) = 0$ for $t \geq T$ or $\tau \geq T$. With such a choice, (1.4) can be written interchangeably in terms of either pair of variables.

In (1.4) written in terms of (x,t), choose the entropy function

$$(3.1)_{(x,t)} \qquad (x,t) \longrightarrow \eta_\varepsilon\big(u(x,t) - v(y,\tau)\big), \qquad (y,\tau) \in S_T \text{ fixed.}$$

For the choice of $k = v(y,\tau)$, the flux function introduced in (3.2) takes the form

$$(3.2)_{(x,t)} \qquad \mathbf{Q}_\varepsilon\big(u(x,t); v(y,\tau)\big) = \left(\int_{v(y,\tau)}^{u(x,t)} s\, H_\varepsilon\big(s - v(y,\tau)\big)\, ds\right) \mathbf{a}.$$

We put these choices in (1.4) and then integrate over S_T in $dyd\tau$ to obtain

$$(3.3)_{(x,t)} \qquad \begin{aligned} \iint_{S_T}\iint_{S_T} \Big\{ &\eta_\varepsilon\big(u(x,t) - v(y,\tau)\big)\psi_t + \mathbf{Q}_\varepsilon\big(u(x,t); v(y,\tau)\big) \cdot \nabla_x \psi \\ &- H_\varepsilon\big(u(x,t) - v(y,\tau)\big) \nabla_x u_+(x,t) \cdot \nabla_x \psi \\ &- H'_\varepsilon\big(u(x,t) - v(y,\tau)\big) |\nabla_x u_+(x,t)|^2\, \psi \Big\} dxdt\, dyd\tau \\ &+ \iint_{S_T}\int_{\mathbb{R}^N} \eta_\varepsilon\big(u_0(x) - v(y,\tau)\big)\, \psi(x,0;y,\tau)\, dx\, dyd\tau \;\geq\; 0. \end{aligned}$$

Next we write the weak formulation (1.1) for the solution $(y,\tau) \to v(y,\tau)$ and in the resulting expression take the entropy function:

$$(3.1)_{(y,\tau)} \qquad (y,\tau) \longrightarrow \eta_\varepsilon\big(v(y,\tau) - u(x,t)\big), \qquad (x,t) \in S_T \text{ fixed,}$$

and the corresponding flux function:

$$(3.2)_{(y,\tau)} \qquad \mathbf{Q}_\varepsilon\big(v(y,\tau); u(x,t)\big) = \left(\int_{u(x,t)}^{v(y,\tau)} s\, H_\varepsilon\big(s - u(x,t)\big)\, ds\right) \mathbf{a}.$$

Integrating over S_T in $dxdt$ yields

$$(3.3)_{(y,\tau)} \qquad \begin{aligned} \iint_{S_T}\iint_{S_T} \Big\{ &\eta_\varepsilon\big(v(y,\tau) - u(x,t)\big)\psi_\tau + \mathbf{Q}_\varepsilon\big(v(y,\tau); u(x,t)\big) \cdot \nabla_y \psi \\ &- H_\varepsilon\big(v(y,\tau) - u(x,t)\big) \nabla_y v_+(y,\tau) \cdot \nabla_y \psi \\ &- H'_\varepsilon\big(v(y,\tau) - u(x,t)\big) |\nabla_y v_+(y,\tau)|^2\, \psi \Big\} dxdt\, dyd\tau \\ &+ \iint_{S_T}\int_{\mathbb{R}^N} \eta_\varepsilon\big(v_0(y) - u(x,t)\big)\, \psi(x,t;y,0) dy\, dxdt \;\geq\; 0. \end{aligned}$$

4. Proof of the Stability Theorem–(ii)

In this section, we first add the inequalities $(3.3)_{(x,t)}$–$(3.3)_{(y,\tau)}$ and choose appropriate testing functions depending upon a parameter $h > 0$, and then we introduce a change of variables to transform these integral inequalities into a form suitable to study the limits as ϵ and h tend to zero.

We now add the inequalities $(3.3)_{(x,t)}$–$(3.3)_{(y,\tau)}$ and use the fact that η_ε and η''_ε are even functions and η'_ε is odd to obtain

$$I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} \geq 0, \tag{4.1}$$

where we have set

$$\begin{aligned} I_{1,\varepsilon} = \iint\limits_{S_T}\iint\limits_{S_T} \Big\{ &\eta_\varepsilon\big(u(x,t) - v(y,\tau)\big)\,(\psi_t + \psi_\tau) \\ &+ \mathbf{Q}_\varepsilon\big(u(x,t); v(y,\tau)\big)\cdot\nabla_x\psi \\ &+ \mathbf{Q}_\varepsilon\big(v(y,\tau); u(x,t)\big)\cdot\nabla_y\psi \Big\}\, dxdt\, dyd\tau, \end{aligned} \tag{4.2}$$

$$\begin{aligned} I_{2,\varepsilon} = -\iint\limits_{S_T}\iint\limits_{S_T} \Big\{ &H_\varepsilon\big(u(x,t) - v(y,\tau)\big) \\ &\times \big[\nabla_x u_+(x,t)\cdot\nabla_x\psi - \nabla_y v_+(y,\tau)\cdot\nabla_y\psi\big] \\ &+ H'_\varepsilon\big(u(x,t) - v(y,\tau)\big) \\ &\times \big[|\nabla_x u_+(x,t)|^2 + |\nabla_y v_+(y,\tau)|^2\big]\psi \Big\}\, dxdt\, dyd\tau, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} I_{3,\varepsilon} = &\iint\limits_{S_T}\int\limits_{\mathbb{R}^N} \eta_\varepsilon\big(u_0(x) - v(y,\tau)\big)\psi(x,0;y,\tau)\,dx\, dyd\tau \\ &+ \iint\limits_{S_T}\int\limits_{\mathbb{R}^N} \eta_\varepsilon\big(v_0(y) - u(x,t)\big)\psi(x,t;y,0)\,dy\, dxdt. \end{aligned} \tag{4.4}$$

Next we choose the function $(x,t;y,\tau) \to \psi(x,t;y,\tau)$ of the form

$$\psi(x,t;y,\tau) = \varphi\left(\tfrac{1}{2}(x+y); \tfrac{1}{2}(t+\tau)\right)\, j_h\left(\tfrac{1}{2}(x-y); \tfrac{1}{2}(t-\tau)\right),$$

where $\varphi(\cdot;\cdot) \in C_0^\infty(S_T)$ is nonnegative and

$$j_h\left(\tfrac{1}{2}(x-y); \tfrac{1}{2}(t-\tau)\right) = \omega_h\left(\tfrac{1}{2}|x-y|\right)\,\omega_h\left(\tfrac{1}{2}(t-\tau)\right).$$

Here ω denotes the standard, symmetric mollifying kernel in $I\!R$, and

$$\omega_h(s) \;=\; \frac{1}{h}\,\omega\left(\frac{s}{h}\right), \qquad \text{and} \qquad \omega_h(s) \;=\; 0 \qquad \text{for } |s| \ge h.$$

Consider the change of variables:

$$\begin{array}{llll} \xi \;=\; \frac{1}{2}(x+y), & \zeta \;=\; \frac{1}{2}(x-y), & s \;=\; \frac{1}{2}(t+\tau), & \sigma \;=\; \frac{1}{2}(t-\tau); \\ x \;=\; \xi+\zeta, & y \;=\; \xi-\zeta, & t \;=\; s+\sigma, & \tau \;=\; s-\sigma, \end{array} \tag{4.5}$$

whose Jacobian 4. As $(x,t;y,\tau)$ range over $S_T \times S_T$, the new variables (ξ, s) range over S_T and (ζ,σ) range over S_T', where

$$S_T' \;=\; I\!R^N \times \left(-\tfrac{1}{2}T\,,\,\tfrac{1}{2}T\right)\,.$$

Therefore, the change of variables in (4.5) maps $S_T \times S_T$ into $S_T \times S_T'$. In terms of the new variables, we compute

$$\begin{aligned} \psi(x,t;y,\tau) &= \varphi(\xi;s)\, j_h(\zeta;\sigma), \\ \nabla_x \psi &= \tfrac{1}{2}\left\{\nabla_\xi \varphi\, j_h \;+\; \varphi\, \nabla_\zeta j_h\right\}, \\ \nabla_y \psi &= \tfrac{1}{2}\left\{\nabla_\xi \varphi\, j_h \;-\; \varphi\, \nabla_\zeta j_h\right\}, \\ \nabla_x u_+(x,t) &= \tfrac{1}{2}\left\{\nabla_\xi u_+(\xi+\zeta\,,\, s+\sigma) \;+\; \nabla_\zeta u_+(\xi+\zeta\,,\, s+\sigma)\right\} \\ &= \nabla_\xi u_+(\xi+\zeta\,,\, s+\sigma), \\ \nabla_y v_+(y,\tau) &= \tfrac{1}{2}\left\{\nabla_\xi v_+(\xi-\zeta\,,\, s-\sigma) \;+\; \nabla_\xi v_+(\xi-\zeta\,,\, s-\sigma)\right\} \\ &= \nabla_\xi v_+(\xi-\zeta\,,\, s-\sigma), \\ \nabla_x j_h \;+\; \nabla_y j_h &= 0, \qquad\quad j_{h,t} \;+\; j_{h,\tau} \;=\; 0, \\ \nabla_x \psi \;+\; \nabla_y \psi &= \nabla_\xi \varphi\, j_h, \qquad\quad \psi_t \;+\; \psi_\tau \;=\; \varphi_s\, j_h, \\ \varphi_{\zeta_i} &= 0, \qquad j_{h,\xi_i} = 0, \quad i = 1,2,\cdots,N. \end{aligned}$$

In view of the dependence of u upon $(\xi+\zeta)$ and of v upon $(\xi-\zeta)$,

$$\begin{aligned} \nabla_\xi u_+(\xi+\zeta\,,\, s-\sigma) \;&=\; \nabla_\zeta u_+(\xi+\zeta\,,\, s-\sigma), \\ \nabla_\xi v_+(\xi-\zeta\,,\, s-\sigma) \;&=\; -\nabla_\zeta v_+(\xi-\zeta\,,\, s-\sigma). \end{aligned} \tag{4.6}$$

Next, using these new variables, we transform the various integrals in (4.1).

5. Transformation and Limits of $I_{2,\varepsilon}$

We first use the new variables introduced in Section 4 to transform $I_{2,\varepsilon}$ into a form suitable to study the limits as $\varepsilon \to 0$ and then $h \to 0$ in the indicated order, and we then estimate the limits.

With the indicated choices and change of variables, we have

$$
\begin{aligned}
-\tfrac{1}{2}I_{2,\varepsilon} = &\iint\limits_{S'_T}\iint\limits_{S_T} H_\varepsilon\big(u(\xi+\zeta,s+\sigma)-v(\xi-\zeta,s-\sigma)\big)\\
&\times\Big\{\nabla_\xi u_+(\xi+\zeta,s+\sigma)\cdot(\nabla_\xi\varphi\, j_h+\varphi\nabla_\zeta j_h)\\
&\quad-\nabla_\xi v_+(\xi-\zeta,s-\sigma)\,(\nabla_\xi\varphi\, j_h-\varphi\nabla_\zeta j_h)\Big\}d\xi ds\, d\zeta d\sigma\\
&\quad+2\iint\limits_{S'_T}\iint\limits_{S_T} H'_\varepsilon\big(u(\xi+\zeta,s+\sigma)-v(\xi-\zeta,s-\sigma)\big)\\
&\qquad\qquad\times\left\{|\nabla_\xi u_+|^2+|\nabla_\xi v_+|^2\right\}\,\varphi\, j_h\, d\xi ds\, d\zeta d\sigma\\
=&\iint\limits_{S'_T}\iint\limits_{S_T} H_\varepsilon(u-v)\nabla_\xi(u_+-v_+)\cdot\nabla_\xi\varphi\, j_h\, d\xi ds\, d\zeta d\sigma\\
&\quad+\iint\limits_{S'_T}\iint\limits_{S_T} H_\varepsilon(u-v)\nabla_\xi(u_++v_+)\cdot\varphi\,\nabla_\zeta j_h\, d\xi ds\, d\zeta d\sigma\\
&\quad+2\iint\limits_{S'_T}\iint\limits_{S_T} H'_\varepsilon\big(u(\xi+\zeta,s+\sigma)-v(\xi-\zeta,s-\sigma)\big)\\
&\qquad\qquad\times\left\{|\nabla_\xi u_+|^2+|\nabla_\xi v_+|^2\right\}\,\varphi\, j_h\, d\xi ds\, d\zeta d\sigma\\
=&\,I^{(1)}_{2,\varepsilon}+I^{(2)}_{2,\varepsilon}+I^{(3)}_{2,\varepsilon}.
\end{aligned}
$$

In transforming these integrals, we make use of the integrability of ∇u_+. In particular, $|\nabla u_+|$ vanishes a.e. on the set $[u\le 0]$, and a similar fact holds for v_+. Then, for a.e. $(\xi,s;\zeta,\sigma)\in S_T\times S'_T$, we write

$$
\begin{aligned}
H_\varepsilon(u-v)\nabla_\xi(u_+-v_+) =&H_\varepsilon(u_+-v_+)\nabla_\xi(u_+-v_+)\\
&+\nabla_\xi u_+\left\{H_\varepsilon(u_+-v)-H_\varepsilon(u_+-v_+)\right\}\\
&+\nabla_\xi v_+\left\{H_\varepsilon(u_+-v_+)-H_\varepsilon(u-v_+)\right\}.
\end{aligned}
$$

As $\varepsilon\to 0$, the last two terms tend to zero a.e. on every compact subset of $S_T\times S'_T$ and their modulus is dominated, uniformly in ε, by a locally integrable function. Thus, when they are put in the expression of $I^{(1)}_{2,\varepsilon}$ and, after we take the limit as $\varepsilon\to 0$, they give no contribution. This process in $I^{(1)}_{2,\varepsilon}$ gives

$$\begin{aligned}
\lim_{\varepsilon\to 0} I^{(1)}_{2,\varepsilon} &= \lim_{\varepsilon\to 0} \iint\limits_{S'_T}\iint\limits_{S_T} \nabla_\xi \left(\int_0^{u_+-v_+} H_\varepsilon(\theta)d\theta\right) \cdot \nabla_\xi\varphi\, j_h\, d\xi ds\, d\zeta d\sigma \\
&= -\lim_{\varepsilon\to 0} \iint\limits_{S'_T}\iint\limits_{S_T} \eta_\varepsilon(u_+-v_+)\, \Delta_\xi\varphi\, j_h\, d\xi ds\, d\zeta d\sigma \\
&= -\iint\limits_{S'_T}\iint\limits_{S_T} |u_+-v_+|\, \Delta_\xi\varphi\, j_h\, d\xi ds\, d\zeta d\sigma.
\end{aligned}$$

In transforming $I^{(2)}_{2,\varepsilon}$, we first assume that u_+ and v_+ are regular and proceed formally. By a repeated formal integration by parts,

$$\begin{aligned}
I^{(2)}_{2,\varepsilon} &= -\iint\limits_{S'_T}\iint\limits_{S_T} H'_\varepsilon(u_+-v_+)\, \nabla_\xi(u_++v_+)\cdot\nabla_\zeta(u_+-v_+)\,\varphi\, j_h\, d\xi ds\, d\zeta d\sigma \\
&\quad - \iint\limits_{S'_T}\iint\limits_{S_T} H_\varepsilon(u_+-v_+)\, \mathrm{div}_\zeta\nabla_\xi(u_++v_+)\,\varphi\, j_h\, d\xi ds\, d\zeta d\sigma \\
&= -\iint\limits_{S'_T}\iint\limits_{S_T} H'_\varepsilon(u_+-v_+)\Big\{\nabla_\xi(u_++v_+)\cdot\nabla_\zeta(u_+-v_+) \\
&\qquad\qquad - \nabla_\xi(u_+-v_+)\cdot\nabla_\zeta(u_++v_+)\Big\}\,\varphi\, j_h\, d\xi ds\, d\zeta d\sigma \\
&\quad + \iint\limits_{S'_T}\iint\limits_{S_T} H_\varepsilon(u_+-v_+)\, \nabla_\zeta(u_++v_+)\cdot\nabla_\xi\varphi\, j_h\, d\xi ds\, d\zeta d\sigma.
\end{aligned} \tag{5.1}$$

From this, taking into account the differentiation formulae (4.6) and performing a further integration by parts,

$$\begin{aligned}
I^{(2)}_{2,\varepsilon} &= \iint\limits_{S'_T}\iint\limits_{S_T} H'_\varepsilon(u_+-v_+)\big\{|\nabla_\xi(u_+-v_+)|^2 \\
&\qquad\qquad - |\nabla_\xi(u_++v_+)|^2\big\}\,\varphi\, j_h\, d\xi ds\, d\zeta d\sigma \\
&\quad - \iint\limits_{S'_T}\iint\limits_{S_T} \left(\int_0^{(u_+-v_+)} H_\varepsilon(\theta)d\theta\right) \Delta_\xi\varphi\, j_h\, d\xi ds\, d\zeta d\sigma.
\end{aligned} \tag{5.2}$$

These calculations can be made rigorous by the following procedure. Denote by $u_{+,\nu}$ and $v_{+,\nu}$ the mollifications of u_+ and v_+ with respect to the variables ξ and ζ. Then,

$$I^{(2)}_{2,\varepsilon} = o_\nu(1) + \iint\limits_{S'_T}\iint\limits_{S_T} H_\varepsilon(u_{+,\nu}-v_{+,\nu})\nabla_\xi(u_{+,\nu}+v_{+,\nu})\cdot\varphi\,\nabla_\zeta j_h\, d\xi ds\, d\zeta d\sigma,$$

where, for $\varepsilon > 0$ fixed, $o_\nu(1) \to 0$ as $\nu \to 0$. We perform integrations by parts in the integrals involving $u_{+,\nu}$ and $v_{+,\nu}$, to arrive to a formula analogous to (5.2). We then let $\nu \to 0$ to obtain (5.2), the various limits being justified, since ∇u_+ and ∇v_+ are in $L^2_{\mathrm{loc}}(S_T)$. Finally, letting $\varepsilon \to 0$ gives

$$\begin{aligned}\liminf_{\varepsilon \to 0} I^{(2)}_{2,\varepsilon} = &- \iint\limits_{S'_T} \iint\limits_{S_T} |u_+ - v_+| \, \Delta_\xi \varphi \, j_h \, d\xi ds \, d\zeta d\sigma \\ &- 4 \liminf_{\varepsilon \to 0} \iint\limits_{S'_T} \iint\limits_{S_T} H'_\varepsilon(u_+ - v_+) \, \nabla_\xi u_+ \cdot \nabla_\xi v_+ \, \varphi \, j_h \, d\xi ds \, d\zeta d\sigma.\end{aligned}$$

We now combine these calculations in the expression of $I_{2,\varepsilon}$ and let $\varepsilon \to 0$ to obtain

$$\begin{aligned}\limsup_{\varepsilon \to 0} I_{2,\varepsilon} = &4 \iint\limits_{S'_T} \iint\limits_{S_T} |u_+ - v_+| \, \Delta_\xi \varphi \, j_h \, d\xi ds \, d\zeta d\sigma \\ &- 4 \limsup_{\varepsilon \to 0} \iint\limits_{S'_T} \iint\limits_{S_T} H'_\varepsilon(u_+ - v_+) |\nabla_\xi (u_+ - v_+)|^2 \varphi \, j_h \, d\xi ds \, d\zeta d\sigma \\ \leq &4 \iint\limits_{S'_T} \iint\limits_{S_T} |u_+ - v_+| \, \Delta_\xi \varphi \, j_h \, d\xi ds \, d\zeta d\sigma,\end{aligned}$$

since $\varphi \geq 0$ and $j_h \geq 0$. Finally we let $h \searrow 0$ by following the same arguments as in Kruzhkov [3] to obtain

$$\lim_{h \to 0} \limsup_{\varepsilon \to 0} I_{2,\varepsilon} \leq 4 \iint_{S_T} \big| u_+(x,t) - v_+(x,t) \big| \, \Delta_x \varphi(x,t) \, dx \, dt.$$

6. Transformation and Limits of $I_{1,\varepsilon}$ and $I_{3,\varepsilon}$

Now we continue to perform the change of variables to transform $I_{1,\varepsilon}$ and $I_{3,\varepsilon}$ into a form suitable to study the limits as $\varepsilon \to 0$ and then $h \to \varepsilon$.

In transforming $I_{1,\varepsilon}$, we use the definitions (3.1)–(3.1)$'$ of the entropy and the definitions (3.2)–(3.2)$'$ of the corresponding flux functions. Taking into account (4.5), we compute

$$\begin{aligned}\lim_{\varepsilon \to 0} I_{1,\varepsilon} = &4 \iint\limits_{S'_T} \iint\limits_{S_T} \Big\{ |u(\xi + \zeta, s + \sigma) - v(\xi - \zeta, s - \sigma)| \, \varphi_s(\xi, s) \, j_h(\zeta, \sigma) \\ &+ \tfrac{1}{2} \big| u^2(\xi + \zeta, s + \sigma) - v^2(\xi - \zeta, s - \sigma) \big| \, \mathbf{a} \cdot \nabla_\xi \varphi(\xi, s) \, j_h(\zeta, \sigma) \Big\} d\xi ds \, d\zeta d\sigma.\end{aligned}$$

Letting now $h \to 0$, we find

$$\lim_{\varepsilon \to 0} I_{1,\varepsilon} = 4 \iint_{S_T} \Big\{ |u(x,t) - v(x,t)| \varphi_t(x,t) + \tfrac{1}{2} \left| u^2(x,t) - v^2(x,t) \right| \mathbf{a} \cdot \nabla_x \varphi(x,t) \Big\} \, dx \, dt.$$

In transforming $I_{3,\varepsilon}$ we perform the change of variables (4.5) involving only the space variables, whereas the time variables are left unchanged. The Jacobian of the transformation is 2. Analogous arguments and limiting processes yield

$$\begin{aligned} \lim_{\varepsilon \to 0} I_{3,\varepsilon} =& 2 \iint_{S_T} \int_{\mathbb{R}^N} |u_0(\xi + \zeta) - v(\xi - \zeta, \tau)| \varphi\left(\xi, \tfrac{1}{2}\tau\right) j_h\left(\zeta, -\tfrac{1}{2}\tau\right) d\xi \, d\zeta d\tau \\ &+ 2 \iint_{S_T} \int_{\mathbb{R}^N} \left| v_0(\xi - \zeta) - u(\xi + \zeta, t) \right| \varphi\left(\xi, \tfrac{1}{2}t\right) j_h\left(\zeta, \tfrac{1}{2}t\right) d\xi \, d\zeta dt. \end{aligned}$$

Letting now $h \to 0$ gives

$$\lim_{h \to 0} \lim_{\varepsilon \to 0} I_{3,\varepsilon} = 4 \int_{\mathbb{R}^N} \left| u_0(x) - v_0(x) \right| \varphi(x, 0) \, dx.$$

7. Proof of the Stability Theorem-(iii)

By combining these calculations in (4.1) and after taking the limit first for $\varepsilon \searrow 0$ and then for $h \searrow 0$, we arrive at

$$\begin{aligned} \iint_{S_T} \Big\{ & \left| u(x,t) - v(x,t) \right| \varphi_t(x,t) + \tfrac{1}{2} \left| u^2(x,t) - v^2(x,t) \right| \mathbf{a} \cdot \nabla_x \varphi(x,t) \\ & + \left| u_+(x,t) - v_+(x,t) \right| \Delta_x \varphi(x,t) \Big\} dx dt \\ & + \int_{\mathbb{R}^N} \left| u_0(x) - v_0(x) \right| \varphi(x,0) dx \geq 0. \end{aligned} \tag{7.1}$$

In this more stringent integral inequality, the non-negative testing function φ is still to be chosen.

In this section, we choose the testing function to identify the weight $w(x,t)$ and the stability result (1.6).

First we rewrite (7.1) in the form

$$\begin{aligned} \iint_{S_T} & \left| u(x,\tau) - v(x,\tau) \right| \left\{ \varphi_\tau + \mathbf{A} \cdot \nabla_x \varphi + b \, \Delta_x \varphi \right\} dx \, d\tau \\ & + \int_{\mathbb{R}^N} \left| u_0(x) - v_0(x) \right| \varphi(x,0) \, dx \geq 0, \end{aligned} \tag{7.2}$$

where we have set

$$\mathbf{A}(x,\tau) \equiv \tfrac{1}{2}\frac{\left|u^2(x,\tau) - v^2(x,\tau)\right|}{|u(x,\tau) - v(x,\tau)|}\,\mathbf{a} \;=\; \tfrac{1}{2}\,|u(x,\tau)+v(x,\tau)|\;\mathbf{a},$$

$$b(x,\tau) \equiv \frac{\left|u_+(x,\tau) - v_+(x,\tau)\right|}{\left|u(x,\tau) - v(x,\tau)\right|},$$

if $u \neq v$, and $A = b = 0$ if $u = v$. In (7.2) we choose the testing functions

$$\varphi_{\varepsilon,\delta}(x,\tau) \;=\; h_\delta(\tau)\,w(x,\tau)\,\zeta(x).$$

Here, for $0 < \delta \ll 1$, we have set

$$h_\delta(\tau) \;=\; \frac{1}{\delta}\int_{\tau-t}^{\infty} \omega\left(\frac{s}{\delta}\right) ds, \qquad 0 < t \leq T - \delta,$$

where $\omega(\cdot)$ is the standard, symmetric mollifier in $I\!R$. Moreover

$$w(x,\tau) \;=\; e^{-\,(1+\lambda_1\,\tau)\,|x|^2\,-\,\lambda_2\,\tau},$$

for positive constants λ_1 and λ_2 to be chosen. Finally, $x \to \zeta(x)$ is a standard, non–negative cutoff function in the ball $\{|x| < 2R\}$, satisfying

$$\begin{cases} \zeta(x) = 1 & \text{for } \ |x| < R, \\ |\nabla\zeta| \;\leq\; \dfrac{1}{R} & \text{for all } \ |x| < 2R, \\ |\Delta\zeta| \;\leq\; \dfrac{\text{const}}{R^2} & \text{for all } \ |x| < 2R. \end{cases}$$

These testing functions are admissible since both h_δ and $w\zeta$ are non–negative, regular and, by the properties of the mollifiers, $h_\delta(\tau) = 0$ for $\tau \geq t + \delta$. We compute

$$\begin{aligned} w_\tau(x,\tau) \;&=\; -\left(\lambda_2 \,+\, \lambda_1\,|x|^2\right)\, w(x,\tau), \\ \nabla_x w(x,\tau) \;&=\; -2\;(1\,+\lambda_1\,\tau)\;x\,w(x,\tau), \\ \Delta_x w(x,\tau) \;&=\; \;2\,(1\;+\;\lambda_1\,\tau)\left\{2\,(1\;+\;\lambda_1\,\tau)\,|x|^2\;-\;N\right\}w(x,\tau). \end{aligned}$$

By the structure of $x \to w(x,\tau)$ and $\zeta(x)$,

$$\lim_{R\to\infty}\int_{\{R<|x|<2R\}} \left\{w\,(\zeta + |\nabla\zeta| + |\Delta\zeta|) + |\nabla_x w \cdot \nabla\zeta|\right\} dx \;=\; 0,$$

uniformly in $\tau \in (0,T)$. Putting this testing function in (7.2) and letting $R \to \infty$ give

$$\begin{aligned} & -\iint_{S_T} |u(x,\tau) - v(x,\tau)|\,\frac{1}{\delta}\,\omega\left(\frac{\tau - t}{\delta}\right)\, w(x,\tau)\,dx\,d\tau \\ (7.3) \qquad & +\iint_{S_T} |u(x,\tau) - v(x,\tau)|\,h_\delta(\tau)\,\{w_\tau + \mathbf{A}\cdot\nabla_x w + b\Delta_x w\}\;dx\,d\tau \\ & +\int_{I\!R^N} |u_0(x) - v_0(x)|\,h_\delta(0)\,w(x,0)\,dx \;\geq\; 0. \end{aligned}$$

The growth condition (1.5) implies that there exists a positive constant C, depending only upon γ, such that

$$\mathbf{A} \cdot x \leq C\left(1+|x|^2\right), \quad \text{for all } x \in I\!R^N.$$

Then

$$w_\tau + \mathbf{A} \cdot \nabla_x w + b \Delta_x w \leq w\left\{C - \lambda_2 + \left[C + 4(1+\lambda_1 \tau)^2 - \lambda_1\right]|x|^2\right\}.$$

Choose $\lambda_2 = C$ and $T_0 = \lambda_1^{-1}$. Then select λ_1 from

$$C + 4(1+\lambda_1 \tau)^2 - \lambda_1 \leq C + 16 - \lambda_1 \leq 0.$$

For such choices, we discard the second integral in (7.3) and let $\delta \to 0$ to obtain

$$\int_{I\!R^N} w(x,t)|u(x,t) - v(x,t)|\, dx \leq \int_{I\!R^N} w(x,0)|u_0 - v_0|\, dx,$$

for a.e. $t \in (0, T_0)$. Since the weight w depends only upon γ, the process can be repeated to exhaust, in a finite number of steps, the time interval of existence.

In the case of general non–linearities $\mathbf{f}(u)$ and $\beta(u)$, the weight w can be constructed only depending on the Lipschitz constant of $\beta(u)$ and $\sup_{(x,t)\in S_T} |\mathbf{f}'(u(x,t))|/(1+|x|)$. ■

Add in Proofs. After we submitted this paper, we were aware of Carrillo's paper [7] dealing with bounded solutions for a similar problem with bounded domains.

References

[1] E. DiBenedetto, *Partial Differential Equations*, Birkhäuser, Boston, 1995.

[2] E. DiBenedetto, Continuity of Weak Solutions to Certain Singular Parabolic Equations, *Ann. Mat. Pura Appl.* **130***(1982), 131–176.*

[3] S. Kruzhkov, First Order Quasilinear Equations with Several Space Variables, *Mat. Sbornik,* **123***, (1970), 228–255*; Engl. Transl. *Math. USSR Sb.* **10** *(1970), 217–273.*

[4] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMS, **11**, SIAM, 1973.

[5] P.-L. Lions, B. Perthame, and E. Tadmor, A Kinetic Formulation of Multidimensional Scalar Conservation Laws and Related Equations, *J. Amer. Math. Soc.* **7***(1) (1994), 169–191.*

[6] A. I. Volpert and S. I. Hudjaev, Cauchy Problem for Degenerate, Second Order Quasilinear Parabolic Equations, *Mat. Sbornik,* **78(120)** *(1969), 374-396*; Engl. Transl. *Math. USSR Sb.* **7(3)** *(1969), 365-387.*

[7] J. Carrillo, Entropy Solutions for Nonlinear Degenerate Problems, Preprint.