DIVERGENCE-MEASURE FIELDS AND HYPERBOLIC CONSERVATION LAWS

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Abstract. We analyze a class of L^∞ vector fields, called divergence-measure fields. We establish the Gauss-Green formula, the normal traces over subsets of Lipschitz boundaries, and the product rule for this class of L^∞ fields. Then we apply this theory to analyze L^∞ entropy solutions of initial-boundary value problems for hyperbolic conservation laws and to study the ways in which the solutions assume their initial and boundary data. The examples of conservation laws include multidimensional scalar equations, the system of nonlinear elasticity, and a class of $m \times m$ systems with affine characteristic hypersurfaces. The analysis in L^∞ also extends to L^p.

1. Introduction

In this paper we analyze a class of L^∞ vector fields, called divergence-measure fields, defined in some domain of $\mathbf{R}^N$. This class of L^∞ vector fields especially includes the class of BV vector fields and shares some important features with it, although there are essential differences between them. Our main motivation for studying the divergence-measure fields is from hyperbolic conservation laws to analyze the behavior of weak entropy solutions in L^∞, determined by the Lax entropy inequality for convex entropy-entropy flux pairs in the sense of distributions. For this purpose, we establish the Gauss-Green formula, the normal traces, and the product rule for the divergence-measure fields. Then we apply this theory to establish a general framework for L^∞ entropy solutions of initial-boundary value problems for hyperbolic systems of conservation laws in several space variables, and to study the ways in which the solutions assume their initial and boundary data. We also discuss some examples of conservation laws and show that L^∞ weak entropy solutions satisfying (4.7)-(4.9) behave as one expects, when approaching the boundary. The existence of normal traces of the divergence-measure fields and the way through which we obtain the normal traces are the decisive tools for our applications given here. This theory has been also applied to studying the asymptotic behavior and uniqueness of L^∞ entropy solutions of conservation laws in [4].

Some results in this paper were announced in [3].

Definition 1.1. Let $D \subset \mathbf{R}^N$ be an open set. For $F \in L^\infty(D; \mathbf{R}^N)$, set

$$(1.1) \qquad |\operatorname{div} F|(D) = \sup\Big\{ \int_D F \cdot \nabla\phi \, dx \;\mid\; \phi \in C_0^1(D; \mathbf{R}),\ |\phi(x)| \le 1,\ x \in D \Big\}.$$

1991 *Mathematics Subject Classification.* Primary: 35L65, 46E30, 35L50, 26B20; Secondary: 35L67, 26B12.

Key words and phrases. Divergence-measure fields, hyperbolic conservation laws, Gauss-Green Theorem, normal traces, product rule, initial-boundary value problems.

Definition 1.2. We say that F is a *divergence-measure field* over D if

$$F \in L^\infty(D;\mathbf{R}^N), \qquad |\operatorname{div} F|(D) < \infty. \tag{1.2}$$

We define $\mathcal{DM}(D)$ as the space of divergence-measure fields over D. In particular, if $F \in \mathcal{DM}(D)$, then $\operatorname{div} F$ is a Radon measure over D.

Remark 1.1. This class of vector fields was first considered by Anzellotti [1]. We emphasize that the notion of normal traces introduced here has a different point of view from that in [1], in which a related notion of normal traces was introduced as a representation function of a linear functional, in a more abstract fashion. However, the extended Gauss-Green formula (Theorem 2.2 below) coincides, which means that both notions are consistent. Our new notion has the advantage of providing essential information about the normal trace in certain hypersurface from the knowledge of the normal traces in its neighboring hypersurfaces. This advantage is made possible by introducing deformations, which are important not only for our definition of the normal traces, but also for all the applications of this theory here and in [4]. Compare the discussions here with those in Anzellotti [1].

Remark 1.2. The L^∞ assumption on F in Definitions 1.1-1.2 can be relaxed. Many results in this paper also hold for $F \in L^p$ replacing $F \in L^\infty$.

Theorem 1.1. *Let $D \subset \mathbf{R}^N$ be an open set. Let $\{F_j\}$ be a sequence of fields in $\mathcal{DM}(D)$ converging in $L^1_{\mathrm{loc}}(D;\mathbf{R}^N)$ to a field F. Then*

$$|\operatorname{div} F|(D) \le \liminf_{j\to\infty} |\operatorname{div} F_j|(D).$$

Proof. Let $\phi \in C_0^1(D)$, $|\phi(x)| \le 1$. Then

$$\int_D F\cdot\nabla\phi\,dx = \lim_{j\to\infty}\int_D F_j\cdot\nabla\phi\,dx \le \liminf_{j\to\infty}|\operatorname{div} F_j|(D).$$

□

Corollary 1.1. *Under the norm*

$$\|F\|_{\mathcal{DM}} = \|F\|_{L^\infty} + |\operatorname{div} F|(D),$$

$\mathcal{DM}(D)$ is a Banach space.

Proof. Let $\{F_j\}$ be a Cauchy sequence in $\mathcal{DM}(D)$. Since L^∞ is complete, there exists a field $F \in L^\infty(D;\mathbf{R}^N)$ such that $F_j \to F$ in L^∞. By Theorem 1.1, one has

$$|\operatorname{div} F|(D) \le \liminf_{j\to\infty}|\operatorname{div} F_j|(D),$$

which implies $F \in \mathcal{DM}(D)$. Furthermore, for any $\varepsilon > 0$, we use Theorem 1.1 again and the fact that $|\operatorname{div}(F_j - F_k)|(D) \to 0$, when $j,\, k \to \infty$, to find

$$|\operatorname{div}(F_j - F)|(D) \le \liminf_{k\to\infty}|\operatorname{div}(F_j - F_k)|(D) < \varepsilon,$$

for sufficiently large j. Therefore, $F_j \to F$ in $\mathcal{DM}(D)$. □

Another consequence of Theorem 1.1 is given in the following proposition, which is in the line of analogous results known in the theory of BV functions (compare with Proposition 1.13 of [15]).

Proposition 1.1. *Suppose $\{F_j\}$ is a sequence in $\mathcal{DM}(D)$ such that $F_j \to F$ in $L^1_{loc}(D)$ and*

$$\lim_{j\to\infty} \int_D |\operatorname{div} F_j| = \int_D |\operatorname{div} F|.$$

Then, for every open set $A \subset D$,

$$\int_{\bar{A}\cap D} |\operatorname{div} F| \geq \limsup \int_{\bar{A}\cap D} |\operatorname{div} F_j|. \tag{1.3}$$

In particular, if $\int_{\partial A\cap D} |\operatorname{div} F| = 0$, then

$$\int_A |\operatorname{div} F| = \lim_{j\to\infty} \int_A |\operatorname{div} F_j|. \tag{1.4}$$

Proof. Let $B = D - \bar{A}$. Then, by Theorem 1.1,

$$\int_A |\operatorname{div} F| \leq \liminf_{j\to\infty} \int_A |\operatorname{div} F_j|, \qquad \int_B |\operatorname{div} F| \leq \liminf_{j\to\infty} \int_B |\operatorname{div} F_j|.$$

On the other hand, we have

$$\begin{aligned}
\int_{\bar{A}\cap D} |\operatorname{div} F| + \int_B |\operatorname{div} F| &= \int_D |\operatorname{div} F| = \lim_{j\to\infty} \int_D |\operatorname{div} F_j| \\
&\geq \limsup_{j\to\infty} \int_{\bar{A}\cap D} |\operatorname{div} F_j| + \liminf_{j\to\infty} \int_B |\operatorname{div} F_j| \\
&\geq \limsup_{j\to\infty} \int_{\bar{A}\cap D} |\operatorname{div} F_j| + \int_B |\operatorname{div} F|,
\end{aligned}$$

and then (1.3) follows. ☐

We will use the so-called positive symmetric mollifiers $\omega : \mathbf{R}^N \to \mathbf{R}$ satisfying $\omega(x) \in C_0^\infty(\mathbf{R}^N)$, $\omega(x) \geq 0$, $\omega(x) = \omega(|x|)$, $\int_{\mathbf{R}^N} \omega(x)\,dx = 1$, $\operatorname{supp}\omega(x) \subset B_1 \equiv \{x \in \mathbf{R}^N \mid |x| < 1\}$. A standard example of such mollifiers is:

$$\omega(x) = \begin{cases} 0, & |x| \geq 1, \\ C\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \end{cases}$$

where C is the constant such that $\int_{\mathbf{R}^N} \omega(x)\,dx = 1$. We denote $\omega_\varepsilon(x) = \varepsilon^{-N}\omega(\frac{x}{\varepsilon})$ and $F_\varepsilon = \omega_\varepsilon * F$, that is,

$$F_\varepsilon(x) = \varepsilon^{-N} \int_{\mathbf{R}^N} \omega\Big(\frac{x-y}{\varepsilon}\Big) F(y)\,dy = \int_{\mathbf{R}^N} \omega(y) F(x+\varepsilon y)\,dy.$$

We will use some well-known properties of the mollifiers. In particular, we recall that, for any $f(x)$, $g(x) \in L^1(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} f_\varepsilon g\,dx = \int_{\mathbf{R}^N} f g_\varepsilon\,dx. \tag{1.5}$$

We now establish a fact for $\mathcal{DM}$ fields, which is analogous to a well-known property of BV functions.

Theorem 1.2 (Approximation by C^∞ functions)**.** *Let $F \in \mathcal{DM}(D)$. Then there exists a sequence $\{F_j\}$ in $C^\infty(D;\mathbf{R}^N)$ such that*

$$\lim_{j\to\infty} \int_D |F_j - F|\,dx = 0, \qquad \lim_{j\to\infty} \int_D |\operatorname{div} F_j|\,dx = |\operatorname{div} F|(D).$$

Proof. The proof is similar to the one for BV functions (e.g. [15]). Let $\varepsilon > 0$. There exists an $M > 0$ such that

$$|\mathrm{div}\, F|(D - D_k) < \varepsilon,$$

where $k = 0, 1, 2, \cdots$, and

$$D_k = \left\{ x \in D \mid \mathrm{dist}(x, \partial D) > \frac{1}{M+k} \right\} \bigcap \{\, x \in \mathbf{R}^N \mid |x| < M + k \,\}.$$

Consider the sets $\Omega_1 = D_2$ and $\Omega_i = D_{i+1} - \bar{D}_{i-1}$, $i = 2, 3, \cdots$. Let $\{\varphi_i\}$ be a partition of the unity subordinate to the covering $\{\Omega_i\}$, that is,

$$\varphi_i \in C_0^\infty(\Omega_i), \quad 0 \le \varphi_i \le 1, \quad \sum_{i=1}^\infty \varphi_i = 1.$$

Let ω be a positive symmetric mollifier. For every index i, one can choose $\varepsilon_i \in (0, \varepsilon)$ such that

$$\mathrm{supp}\ \omega_{\varepsilon_i} * (F\varphi_i) \subset D_{i+2} - \bar{D}_{i-2}, \quad (D_{-1} = \emptyset) \tag{1.6}$$

$$\int_D |\omega_{\varepsilon_i} * (F\varphi_i) - F\varphi_i|\, dx < \varepsilon/2^i, \tag{1.7}$$

$$\int_D |\omega_{\varepsilon_i} * (F \cdot \nabla\varphi_i) - F \cdot \nabla\varphi_i|\, dx < \varepsilon/2^i. \tag{1.8}$$

Finally, let

$$F_\varepsilon = \sum_{i=1}^\infty \omega_{\varepsilon_i} * (F\varphi_i).$$

It follows from (1.6) that the sum defining F_ε is locally finite. Hence $F_\varepsilon \in C^\infty(D; \mathbf{R}^N)$, and $\{F_\varepsilon\}$ is uniformly bounded. Moreover, since $F = \sum_{i=1}^\infty F\varphi_i$, one has

$$\int_D |F_\varepsilon - F|\, dx \le \sum_{i=1}^\infty \int_D |\omega_{\varepsilon_i} * (F\varphi_i) - F\varphi_i|\, dx < \varepsilon,$$

that is, $F_\varepsilon - F \to 0$ in $L^1(D; \mathbf{R}^N)$ when $\varepsilon \to 0$. Theorem 1.1 indicates

$$|\mathrm{div}\, F|(D) \le \liminf_{\varepsilon \to 0} |\mathrm{div}\, F_\varepsilon|(D). \tag{1.9}$$

Let now $\phi \in C_0^1(D)$, $|\phi| \le 1$. One has

$$\int_D F_\varepsilon \cdot \nabla\phi\, dx = \sum_{i=1}^\infty \int_D \omega_{\varepsilon_i} * (\varphi_i F) \cdot \nabla\phi\, dx = \sum_{i=1}^\infty \int_D \varphi_i F \cdot \nabla(\omega_{\varepsilon_i} * \phi) dx.$$

Thus

$$\begin{aligned}\int_D F_\varepsilon \cdot \nabla\phi\, dx = &\int_D F \cdot \nabla(\varphi_1 \omega_{\varepsilon_1} * \phi)\, dx + \sum_{i=2}^\infty \int_D F \cdot \nabla(\varphi_i \omega_{\varepsilon_i} * \phi)\, dx \\ &- \sum_{i=1}^\infty \int_D \big(\omega_{\varepsilon_i} * (F \cdot \nabla\varphi_i) - F \cdot \nabla\varphi_i\big) \phi\, dx,\end{aligned}$$

from the identity $\sum_{i=1}^{\infty} \nabla\varphi_i = 0$. Since $|\varphi_i\omega_{\varepsilon_i} * \phi| \le 1$, one obtains

$$\int_D F \cdot \nabla(\varphi_1\omega_{\varepsilon_1} * \phi)\, dx \le |\mathrm{div}\, F|(D).$$

The fact that the intersection of more than any three of the sets D_k is empty yields

$$\sum_{i=2}^{\infty} \int_D F \cdot \nabla(\varphi_i\omega_{\varepsilon_i} * \phi)\, dx \le 3|\mathrm{div}\, F|(D - D_0) < 3\varepsilon.$$

Therefore, we obtain $\int_D F_\varepsilon \cdot \nabla\phi\, dx \le |\mathrm{div}\, F|(D) + 4\varepsilon$, and hence

$$|\mathrm{div}\, F_\varepsilon|(D) \le |\mathrm{div}\, F|(D) + 4\varepsilon. \tag{1.10}$$

Thus (1.9) and (1.10) give the desired result.

□

2. Gauss-Green Formula and Normal Traces

The main difference between $\mathcal{DM}$ fields and BV fields is that the former may have much worse behavior. One obvious example of such a $\mathcal{DM}$ field in $\mathbf{R}^2$ is $(v(x-y), v(x-y))$, where v is any function in $L^\infty(\mathbf{R})$.

Our motivation for studying this class of fields comes from the entropy inequality in distributional sense, characterizing admissible solutions for hyperbolic systems of conservation laws. In this case, applying Schwartz's lemma on nonnegative distributions [26], one concludes that, if $u(t,x) \in L^\infty((0,T)\times\mathbf{R}^n;\mathbf{R}^m)$ is an entropy solution of a hyperbolic system of conservation laws in $(0,T)\times\mathbf{R}^n$, then, for any entropy-entropy flux pair (η, q), $q = (q^1, \cdots, q^n)$, of the system, the field $(\eta(u(t,x)), q(u(t,x)))$ is in $\mathcal{DM}((0,T)\times V)$, for any bounded open set $V \subset \mathbf{R}^n$. As we will see, for such a field, one can define its normal traces through Lipschitz hypersurfaces and extend the Gauss-Green Theorem.

Given $F \in \mathcal{DM}(D)$, we fix a precise representative F^* for F. We choose this precise representative as follows. By the definition of the sequence F_j in the proof of Theorem 1.2, it is clear that the C^∞ field sequence F_j converges a.e. in D. Actually, given an element in the class F (characterized by equality almost everywhere), they converge pointwise to this member of the class F at all its Lebesgue points. Denote by F a specific member in this class.

Let $\mathcal{N}$ be a Borel set of measure zero containing the set of all points that are not Lebesgue points of this specific member F. We define

$$F^*(x) = \begin{cases} F(x), & x \in D - \mathcal{N}, \\ 0, & x \in \mathcal{N}. \end{cases}$$

In particular, F^* is Borel measurable, since it is a pointwise limit of C^∞ functions in $D - \mathcal{N}$. We will drop $*$ and simply denote $F^* = F$.

Definition 2.1. Let Ω be an open subset in $\mathbf{R}^N$. We say that $\partial\Omega$ is a *deformable Lipschitz boundary* provided that the following hold.

- **(i):** For each $x \in \partial\Omega$, there exist $r > 0$ and a Lipschitz mapping $\gamma : \mathbf{R}^{N-1} \to \mathbf{R}$ such that, upon rotating and relabeling the coordinate axes if necessary,

$$\Omega \cap Q(x,r) = \{\, y \in \mathbf{R}^N \mid \gamma(y_1, \cdots, y_{N-1}) < y_N \,\} \cap Q(x,r),$$

  where $Q(x,r) = \{\, y \in \mathbf{R}^N \mid |y_i - x_i| \le r,\ i = 1, \cdots, N \,\}$. We denote by $\tilde{\gamma}$ the map $y \mapsto (y, \gamma(y))$.

(ii): There exists a map $\Psi : \partial\Omega \times [0,1] \to \bar{\Omega}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(\cdot,0) \equiv I$, where I is the identity map over $\partial\Omega$.

Denote $\partial\Omega_\tau \equiv \Psi(\partial\Omega \times \{\tau\})$, $\tau \in [0,1]$, and denote Ω_τ the open subset of Ω whose boundary is $\partial\Omega_\tau$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

Definition 2.2. We say that the Lipschitz deformation is *regular* if

$$\lim_{\tau\to 0+} D\Psi_\tau \circ \tilde{\gamma} = D\tilde{\gamma}, \qquad \text{in } L^1_{\text{loc}}(B), \tag{2.1}$$

where $\tilde{\gamma}$ is a map as in condition (i) of Definition 2.1, and Ψ_τ denotes the map of $\partial\Omega$ into Ω, given by $\Psi_\tau(x) = \Psi(x,\tau)$. Here B denotes the greatest open set such that $\tilde{\gamma}(B) \subset \partial\Omega$.

Remark 2.1. It should be recognized that bounded domains with smooth boundaries (say, C^2) have always regular deformable Lipschitz boundaries. Indeed, since there is an everywhere defined (say, outward) normal unit field $\nu(r)$, one can define the deformation $\Psi(y,\tau) = y - \varepsilon\tau\nu(y)$, which satisfies all the required conditions for sufficiently small $\varepsilon > 0$.

Remark 2.2. Conditions (i)-(ii) of Definition 2.1 are also verified for the star-shaped domains and for the domains whose boundaries satisfy the cone property. For the former, there exists a point $y_0 \in \Omega$ such that, for any $y \in \partial\Omega$, one has $y+\theta(y_0-y) \in \Omega$ for $\theta \in (0,1)$ and can then define $\Psi(y,\tau) = y + \frac{\tau}{2}(y_0 - y)$. For the latter, there exists a vector $v_0 \in \mathbf{R}^N$ such that, for any $y \in \partial\Omega$ and any $0 < s \le 1$, one has $y + sv_0 \in \Omega$ and then takes $\Psi(y,\tau) = y + \tau v_0$. In both cases, the deformation is regular.

Remark 2.3. It is also clear that, if Ω is the image through a bi-Lipschitz map of a domain $\bar{\Omega}$ with a (regular) Lipschitz deformable boundary, then Ω itself possesses a (regular) Lipschitz deformable boundary.

Lemma 2.1. *Let $F \in \mathcal{DM}(D)$. Assume that $\{F_j\}$ is the sequence in $C^\infty(D;\mathbf{R}^N)$ given by Theorem 1.2. Let $\Omega \subset D$ be an open set with a deformable Lipschitz boundary and a Lipschitz deformation Ψ of $\partial\Omega$. Then there exists a set $\mathcal{T} \subset [0,1]$ with meas $([0,1]-\mathcal{T}) = 0$ such that, for all $\tau \in \mathcal{T}$, F_j converges to F, $\mathcal{H}^{N-1}$-almost everywhere in $\partial\Omega_\tau$.*

Proof. Let $\mathcal{N}$ be the null set in the definition of the precise representative of F and $\mathcal{V} = \Psi(\partial\Omega \times [0,1])$. Let $h : \mathcal{V} \to [0,1]$ be the Lipschitz function given by $h(y) = \tau$ if $y \in \partial\Omega_\tau$. We extend h to all $\mathbf{R}^N$ by setting $h(y) = 0$, for $y \notin \Omega$, and $h(y) = 1$, for $y \in \Omega - \mathcal{V}$. Let $Jh(y)$ denote the Jacobian of h at y. By the Coarea Formula for Lipschitz functions (see [13, 14]), one has

$$0 = \int_{\mathbf{R}^N} \chi_{\mathcal{N}\cap\mathcal{V}}(y) Jh(y)\, dy = \int_0^1 d\tau \int_{h^{-1}(\tau)} \chi_{\mathcal{N}}(\omega)\, d\mathcal{H}^{N-1}(\omega).$$

Thus, for almost all $\tau \in (0,1)$, one must have

$$\mathcal{H}^{N-1}(\mathcal{N} \cap \partial\Omega_\tau) = \int_{h^{-1}(\tau)} \chi_{\mathcal{N}}(\omega)\, d\mathcal{H}^{N-1}(\omega) = 0.$$

$\square$

Lemma 2.2. *Let $F \in \mathcal{DM}(D)$. Let $\Omega \subset D$ be an open set with a deformable Lipschitz boundary and a Lipschitz deformation Ψ of $\partial\Omega$. Then, there exists a countable set $\mathcal{J} \subset (0,1)$ such that $|\mathrm{div}\, F|(\partial\Omega_\tau) = 0$, for every $\tau \in (0,1) - \mathcal{J}$.*

Proof. Since $\partial\Omega_{\tau_1} \cap \partial\Omega_{\tau_2} = \emptyset$, if $\tau_1 \neq \tau_2$, $\tau_1, \tau_2 \in (0,1)$, then, for each $n \in \mathbf{N}$, the cardinality of the set

$$\{\, \tau \in (0,1) \,:\, |\mathrm{div}\, F|(\partial\Omega_\tau \cap B(0;n)) > 1/n \,\}$$

must be finite, because $|\mathrm{div}\, F|$ is a Radon measure.

□

Theorem 2.1. *Let $F \in \mathcal{DM}(D)$. Let $\Omega \subset D$ be an open set with deformable Lipschitz boundary. Let Ψ be a Lipschitz deformation of $\partial\Omega$. Let $\mathcal{T}, \mathcal{J} \subset (0,1)$ be as in Lemmas 2.1-2.2, and $\mathcal{T}^* = \mathcal{T} - \mathcal{J}$. Then, for every $\tau \in \mathcal{T}^*$ and all $\phi \in C_0^1(\mathbf{R}^N)$,*

$$\langle \mathrm{div}\, F|_{\Omega_\tau}, \phi \rangle = \int_{\partial\Omega_\tau} \phi(\omega) F(\omega) \cdot \nu_\tau(\omega)\, d\mathcal{H}^{N-1}(\omega) - \int_{\Omega_\tau} F(y) \cdot \nabla\phi(y)\, dy, \tag{2.2}$$

where ν_τ is a unit outward normal field defined $\mathcal{H}^{N-1}$-almost everywhere in $\partial\Omega_\tau$.

Proof. It suffices to consider the case where Ω is bounded. Take the sequence $F_j \in C^\infty(D; \mathbf{R}^N)$ given by Theorem 1.2. Let $\tau \in \mathcal{T}^*$ and take $n \in \mathbf{N}$ such that $\tau > 1/n$. The classical Gauss-Green formula gives

$$\int_{\Omega_\tau} \phi\, \mathrm{div}\, F_j\, dy = \int_{\partial\Omega_\tau} \phi(\omega) F_j(\omega) \cdot \nu_\tau(\omega)\, d\mathcal{H}^{N-1}(\omega) - \int_{\Omega_\tau} F_j(y) \cdot \nabla\phi(y)\, dy, \tag{2.3}$$

for any $\phi \in C_0^1(\mathbf{R}^N)$. Since $\tau \in \mathcal{T}$, the right-hand side of (2.3) converges to the right-hand side of (2.2) when $j \to \infty$, where Lemma 2.1 is used for the convergence of the first term. Now, for j large enough, one has

$$F_j(x) = \omega_j * (F\varphi_j)(x), \qquad \text{for } x \in \Omega_{1/n}, \tag{2.4}$$

where $\omega_j(x) = \varepsilon_j^{-N} \omega(\frac{x}{\varepsilon_j})$, ω is a symmetric mollifier, $\varepsilon_j \to 0$ as $j \to \infty$, and $\varphi_j \in C_0^\infty(D)$ satisfy $\varphi_j(x) = 1$, for $x \in \Omega_{1/n}$. We denote by μ^j the divergence of the right-hand side of (2.4). Then $\mu^j = \mathrm{div}\, F_j$ over $C_0(\Omega_{1/n})$, and

$$\mu^j \rightharpoonup \mathrm{div}\, F, \qquad \text{as } j \to \infty, \text{ in } \mathcal{M}(\Omega_{1/n}). \tag{2.5}$$

Now, viewed as (signed) Radon measures over $\mathbf{R}^N$, it is clear that μ^j is a sequence uniformly bounded in $\mathcal{M}(\mathbf{R}^N)$. Consider a Jordan decomposition of μ^j as $\mu^j = \mu_+^j - \mu_-^j$, with nonnegative Radon measures μ_+^j and μ_-^j. That is,

$$\mu_\pm^j \left(\mathrm{supp}\, \mu_-^j \cap \mathrm{supp}\, \mu_+^j \right) = 0, \qquad |\mu^j| = \mu_+^j + \mu_-^j.$$

Since μ_+^j and μ_-^j are uniformly bounded in $\mathcal{M}(\mathbf{R}^N)$, passing to a subsequence if necessary, we may assume that there exist $\mu_+, \mu_- \in \mathcal{M}(\mathbf{R}^N)$ such that

$$\mu_+^j \rightharpoonup \mu_+, \quad \mu_-^j \rightharpoonup \mu_-, \qquad \text{in } \mathcal{M}(\mathbf{R}^N).$$

In particular, we must have $\mu_+ - \mu_- = \mathrm{div}\, F$ in $\mathcal{M}(\Omega_{1/n})$, because of (2.5). Let $\mu = \mu_+ + \mu_-$. We claim that $\mu(\partial\Omega_\tau) = 0$. Indeed, let $A_{i\delta} = \Omega_{\tau - i\delta} - \Omega_{\tau + i\delta}$, $i = 1, 2$, with $1/n < \tau - 2\delta < \tau + 2\delta < 1$. Hence,

$$\begin{aligned}\mu_{\pm}(A_\delta) &\le \limsup_{j\to\infty} \mu_{\pm}^j(A_\delta)\\ &\le \limsup_{j\to\infty} |\operatorname{div} F_j|(A_\delta)\\ &\le |\operatorname{div} F|(\bar{A}_\delta) \qquad \text{(by Proposition 1.1)}\\ &\le |\operatorname{div} F|(A_{2\delta}) \to 0, \qquad \text{as } \delta \to 0.\end{aligned}$$

Therefore, for $\tau \in \mathcal{T}^*$, we have (see, *e.g.*, [13])

$$\mu_+^j(\Omega_\tau) \to \mu_+(\Omega_\tau), \quad \mu_-^j(\Omega_\tau) \to \mu_-(\Omega_\tau), \quad \text{as } j \to \infty.$$

Consequently, for $\tau \in \mathcal{T}^*$,

$$\operatorname{div} F_j(\Omega_\tau) = \mu_+^j(\Omega_\tau) - \mu_-^j(\Omega_\tau) \longrightarrow \mu_+(\Omega_\tau) - \mu_-(\Omega_\tau) = \operatorname{div} F(\Omega_\tau),$$

as $j \to \infty$. More generally, the same arguments lead to

$$(\phi \operatorname{div} F_j)(\Omega_\tau) \to (\phi \operatorname{div} F)(\Omega_\tau), \qquad \text{as } j \to \infty, \tag{2.6}$$

for $\tau \in \mathcal{T}^*$ and any $\phi \in C_0^1(\mathbf{R}^N)$. This means that the left-hand side of (2.3) converges as $j \to \infty$ to the left-hand side of (2.2), for $\tau \in \mathcal{T}^*$.

□

Now we use (2.2) to define the normal trace of F through $\partial\Omega$ so that (2.2) holds for $\tau = 0$, that is, the Gauss-Green formula holds for any open set $\Omega \subset D$ with Lipschitz deformable boundary. Specifically, given an outward unit normal field ν defined $\mathcal{H}^{N-1}$-almost everywhere in $\partial\Omega$, we define $F \cdot \nu$ as a Radon measure over $\partial\Omega$ (actually an element of $L^\infty(\partial\Omega)$) in the following way. Using a Lipschitz deformation Ψ for $\partial\Omega$, we may regard any function $\phi \in C_0(\partial\Omega)$ as an element of $C_0(\partial\Omega_\tau)$ through the mapping $\phi \mapsto \phi \circ \Psi_\tau^{-1}$. And, conversely, we may regard any element of $C_0(\partial\Omega_\tau)$ as an element of $C_0(\partial\Omega)$ through the inverse mapping $\phi \mapsto \phi \circ \Psi_\tau$. Now, since $F \cdot \nu_\tau$ is defined $\mathcal{H}^{N-1}$-almost everywhere on $\partial\Omega_\tau$, for $\tau \in \mathcal{T}^*$ with $\mathcal{T}^*$ given by Theorem 2.1, we may regard $F \cdot \nu_\tau$ as either a Radon measure over $\partial\Omega_\tau$ or a Radon measure over $\partial\Omega$. We then define

$$F \cdot \nu|_{\partial\Omega} = \text{w-}\lim_{\tau\to 0\ \tau\in\mathcal{T}^*} F \cdot \nu_\tau, \qquad \text{in } \mathcal{M}(\partial\Omega). \tag{2.7}$$

We justify (2.7) in the following theorem.

Theorem 2.2 (Gauss-Green Formula for $\mathcal{DM}$ Fields)**.** *Let $F \in \mathcal{DM}(D)$ and $\Omega \subset D$ be an open set with deformable Lipschitz boundary. The limit in* (2.7) *exists when $F \cdot \nu_\tau$ are regarded as Radon measures on $\partial\Omega$ through the formula*

$$\langle F \cdot \nu_\tau, \phi \rangle \equiv \int_{\partial\Omega_\tau} \phi(\Psi_\tau^{-1}(\omega)) F(\omega) \cdot \nu_\tau(\omega)\, d\mathcal{H}^{N-1}(\omega), \tag{2.8}$$

where $\Psi_\tau : \partial\Omega \to \partial\Omega_\tau$ is given by $\Psi_\tau(\omega) = \Psi(\omega, \tau)$. This definition for $F \cdot \nu$ over $\partial\Omega$ yields the following Gauss-Green formula:

$$\langle \operatorname{div} F|_\Omega, \phi \rangle = \int_{\partial\Omega} \phi(\omega) F(\omega) \cdot \nu(\omega)\, d\mathcal{H}^{N-1}(\omega) - \int_\Omega F(y) \cdot \nabla\phi(y)\, dy, \tag{2.9}$$

for any $\phi \in C_0^1(\mathbf{R}^N)$, where, in the first integral, we use the formal notation $F(\omega) \cdot \nu(\omega)\, d\mathcal{H}^{N-1}(\omega) \equiv F \cdot \nu$ for the normal trace measure justified in (i) below.

The normal trace measure $F \cdot \nu$ has the following properties:

(i): *$F \cdot \nu$ does not depend on the particular Lipschitz deformation for $\partial\Omega$ and is absolutely continuous with respect to $\mathcal{H}^{N-1}|_{\partial\Omega}$;*

(ii): *If $\partial\Omega \subset D$ and $|\operatorname{div} F|(\partial\Omega) = 0$, the density of $F \cdot \nu$ coincides with the function $F \cdot \nu$, $\mathcal{H}^{N-1}$- a.e. in $\partial\Omega$, whenever $\mathcal{H}^{N-1}(\partial\Omega \cap \mathcal{N}) = 0$, where $\mathcal{N}$ is the null set in the definition of the precise representative of F;*

(iii): *Denote also $F \cdot \nu$ the corresponding density. Then $F \cdot \nu \in L^\infty(\partial\Omega)$ and, for some $C > 0$,*

$$\|F \cdot \nu\|_{L^\infty(\partial\Omega)} \leq C\|F\|_{L^\infty(\Omega)}. \tag{2.10}$$

If there exists a regular deformation Ψ of $\partial\Omega$, then we may take $C = 1$ and

$$F \cdot \nu = w^* - \operatorname*{ess\,lim}_{\tau \to 0+} (F \cdot \nu_\tau) \circ \Psi_\tau, \qquad \text{in } L^\infty(\partial\Omega). \tag{2.11}$$

Proof. Let ϕ be any function in $C_0^1(\mathbf{R}^N)$. Take $\tau \to 0$ in (2.2), with $\tau \in \mathcal{T}$ given in the proof of Theorem 2.1. Using the Dominated Convergence Theorem in the left-hand side for the measure $\operatorname{div} F$ and the sequence of functions $\chi_{\Omega_\tau}\phi$, converging pointwise to $\chi_\Omega\phi$, and in the right-hand side for the Lebesgue measure and the sequence $\chi_{\Omega_\tau}\nabla\phi \cdot F$, one sees that the first term on the right-hand side of (2.2) must converge as $\tau \to 0$. Since $\phi|_{\partial\Omega_\tau}$ in (2.2) can be substituted by $\phi|_{\partial\Omega} \circ \Psi_\tau^{-1}$ with an error that goes to 0 when $\tau \to 0$, one sees that the limit in (2.7) exists if ϕ is the restriction of a function in $C_0^1(\mathbf{R}^N)$. Since the set of such functions is dense in $C_0(\partial\Omega)$ and the measures in $\partial\Omega$, given by $F \cdot \nu_\tau$ as in (2.8), are uniformly bounded, this limit exists for all $\phi \in C_0(\partial\Omega)$. Now, we can take $\tau \to 0$ in (2.2) to obtain the Gauss-Green formula (2.9).

For assertion (i), the fact that $F \cdot \nu$ does not depend on the particular deformation for $\partial\Omega$ is a direct consequence of (2.9), since the latter does not involves the deformation. For the remaining part of (i), we must prove that $|F \cdot \nu|(A) = 0$ provided that $A \subset \partial\Omega$ is a Borel set such that $\mathcal{H}^{N-1}(A) = 0$. Since $|F \cdot \nu|$ is a Radon measure over $\partial\Omega$, it suffices to prove this fact in the case where A is compact. Given $\varepsilon > 0$, we can cover A with a finite number, J, of balls $B_i = B(x_i; r_i)$ with radius $r_i < \varepsilon$ such that $\mathcal{H}^{N-1}\left(\cup_{i=1}^J B_i \bigcap \partial\Omega\right) < \varepsilon$. Now, for any $\phi \in C_0\big(\cup_{i=1}^J B_i \cap \partial\Omega\big)$,

$$\int_{\partial\Omega_\tau} \phi(\Psi_\tau^{-1}(\omega))F(\omega) \cdot \nu_\tau(\omega)\, d\mathcal{H}^{N-1}(\omega) \leq \varepsilon\|\phi\|\,\|F\|\operatorname{Lip}(\Psi)^{N-1}.$$

Thus $\langle F \cdot \nu, \phi\rangle \leq C\varepsilon\|\phi\|\,\|F\|$, with $C = \operatorname{Lip}(\Psi)^{N-1}$, and then

$$|F \cdot \nu|(A) \leq |F \cdot \nu|(\cup_{i=1}^J B_i \cap \partial\Omega) \leq C\varepsilon\|F\|.$$

Taking ε to 0 gives the expected result.

Regarding assertion (ii), if $\partial\Omega \subset D$, $|\operatorname{div} F|(\partial\Omega) = 0$, and $\mathcal{H}^{N-1}(\mathcal{N} \cap \partial\Omega) = 0$, using the same arguments as in the proof of Theorem 2.1, we can prove that the Gauss-Green formula holds in the usual sense, *i.e.*, with $F \cdot \nu$ given by the very scalar product of the restriction of F over $\partial\Omega$ and the outward unit field ν normal to $\partial\Omega$, both defined $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$. Indeed, taking the sequence of C^∞ vector fields F_j approaching F in D, given by Theorem 1.2, for j large enough we have

$$F_j(x) = \omega_j * (F\varphi_j)(x), \qquad \text{for } x \in \Omega, \tag{2.12}$$

where $\omega_j(x) = \varepsilon_j^{-N}\omega(\frac{x}{\varepsilon_j})$, ω is a symmetric mollifier, $\varepsilon_j \to 0$ as $j \to \infty$, and $\varphi_j \in C_0^\infty(D)$ satisfy $\varphi_j(x) = 1$, for $x \in \Omega$. Defining μ_j and μ as in the proof of Theorem 2.1 and following the same lines, this time taking $A_\delta = \{x \in D : \operatorname{dist}(x, \partial\Omega) < \delta\}$, for δ sufficiently small, we arrive at

$$(\phi \operatorname{div} F_j)(\Omega) \to (\phi \operatorname{div} F)(\Omega), \qquad \text{as } j \to \infty,$$

and so we can get (2.9) from the classical Gauss-Green formula for F_j, with $F \cdot \nu$ satisfying the assertion.

Let us consider assertion (iii). For any $\omega_0 \in \partial\Omega$ and $r > 0$, sufficiently small, out of a countable set, one has

$$\begin{aligned}
&\left| \frac{1}{\mathcal{H}^{N-1}\left(B(\omega_0; r) \cap \partial\Omega\right)} \int\limits_{B(\omega_0;r)\cap\partial\Omega} F \cdot \nu \, d\mathcal{H}^{N-1}(\omega) \right| \\
&\qquad = \lim_{\tau \to 0} \left| \frac{1}{\mathcal{H}^{N-1}\left(B(\omega_0; r) \cap \partial\Omega\right)} \int\limits_{\Psi_\tau(B(\omega_0;r)\cap\partial\Omega)} F \cdot \nu_\tau \, d\mathcal{H}^{N-1}(\omega) \right| \\
&\qquad \le \|F\|_\infty \operatorname{Lip}(\Psi)^{N-1}.
\end{aligned} \tag{2.13}$$

Then, from the Radon-Nikodym Theorem on the differentiation of measures [13], we get the first part of assertion (iii). Identity (2.11) follows from the area formula (see [14]). Indeed, if $\phi \in L^1(\partial\Omega)$ and $B = \operatorname{supp} \phi \subset \operatorname{graph} \gamma$, with γ as in condition (i) of Definition 2.1, then

$$\begin{aligned}
\int_{\partial\Omega} F(\omega) \cdot \nu(\omega) \phi(\omega) \, d\mathcal{H}^{N-1}(\omega) &= \lim_{\tau \to 0+} \int_{\partial\Omega_\tau} F \cdot \nu_\tau \, \phi \circ \Psi_\tau^{-1} \, d\mathcal{H}^{N-1}(\omega) \\
= \lim_{\tau \to 0+} \int_{\tilde\gamma^{-1}(B)} &(F \cdot \nu_\tau) \circ (\Psi_\tau \circ \tilde\gamma) \, \phi \circ \tilde\gamma \, J\Psi_\tau \circ \tilde\gamma \, d\mathcal{L}^{N-1}(\omega) \\
= \lim_{\tau \to 0+} \int_B &(F \cdot \nu_\tau) \circ \Psi_\tau \, \phi \, d\mathcal{H}^{N-1}(\omega),
\end{aligned}$$

which is the desired conclusion, where we have used the area formula [14] and (2.1). Here, Jf denotes the Jacobian of the Lipschitz map f (see [13, 14]). The fact that we can take $C = 1$ in inequality (2.10), in this case, follows immediately from (2.11). This concludes the proof. □

Remark 2.4. It is important to observe that, in general, one cannot define the trace for each one of the components of a $\mathcal{DM}$ field over any Lipschitz boundary, as opposed to the case of BV fields. This fact can be easily seen through the example provided by the $\mathcal{DM}$ field $F(x,y) = (\sin \frac{1}{x-y}, \sin \frac{1}{x-y})$; it is impossible to define any reasonable notion of trace over the line $x = y$ for the component $\sin \frac{1}{x-y}$. Nevertheless, the unit normal ν_τ to the line $x - y = \tau$ is the vector $(-1/\sqrt{2}, 1/\sqrt{2})$ so that the scalar product $F(x, x-\tau) \cdot \nu_\tau$ is identically zero over this line. Hence, we find that $F \cdot \nu \equiv 0$ over the line $x = y$ and the Gauss-Green formula implies in this case that, for any $\phi \in C_0^1(\mathbf{R}^2)$,

$$0 = \langle \operatorname{div} F|_{x>y}, \phi \rangle = -\int_{x>y} F \cdot \nabla\phi \, dxdy.$$

This identity could be also directly obtained by applying the Dominated Convergence Theorem to the analogous identity obtained from the classical Gauss-Green formula for the domain $\{\, (x,y) \mid x > y + \tau \,\}$ when $\tau \to 0$.

As a corollary of the extended Gauss-Green formula, we have the following proposition.

Proposition 2.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary and $F_1 \in \mathcal{DM}(\Omega)$, $F_2 \in \mathcal{DM}(\mathbf{R}^N - \bar{\Omega})$. Then*

$$F(y) = \begin{cases} F_1(y), & y \in \Omega, \\ F_2(y), & y \in \mathbf{R}^N - \bar{\Omega} \end{cases} \tag{2.14}$$

belongs to $\mathcal{DM}(\mathbf{R}^N)$, and

$$\|F\|_{\mathcal{DM}(\mathbf{R}^N)} \le \|F_1\|_{\mathcal{DM}(\Omega)} + \|F_2\|_{\mathcal{DM}(\mathbf{R}^N-\bar{\Omega})} + \|F_1\cdot\nu - F_2\cdot\nu\|_{L^\infty(\partial\Omega)}\mathcal{H}^{N-1}(\partial\Omega).$$

Proof. Obviously, $F \in L^\infty(\mathbf{R}^N;\mathbf{R}^N)$ and

$$\|F\|_{L^\infty(\mathbf{R}^N)} \le \|F_1\|_{L^\infty(\Omega)} + \|F_2\|_{L^\infty(\mathbf{R}^N-\bar{\Omega})}.$$

Now, choosing $\phi \in C_0^1(\mathbf{R}^N)$ such that $|\phi| \le 1$ and using the Gauss-Green formula, we have

$$\begin{aligned}\int_{\mathbf{R}^N} F\cdot\nabla\phi\,dy &= \int_\Omega F_1\cdot\nabla\phi\,dy + \int_{\mathbf{R}^N-\bar{\Omega}} F_2\cdot\nabla\phi\,dy \\ &= -<\operatorname{div} F_1|_\Omega,\phi> - <\operatorname{div} F_2|_{\mathbf{R}^N-\bar{\Omega}},\phi> + \int_{\partial\Omega}\{F_1\cdot\nu - F_2\cdot\nu\}\phi\,d\mathcal{H}^{N-1}(\omega) \\ &\le |\operatorname{div} F_1|(\Omega) + |\operatorname{div} F_2|(\mathbf{R}^N-\bar{\Omega}) + \|F_1\cdot\nu - F_2\cdot\nu\|_{L^\infty(\partial\Omega)}\mathcal{H}^{N-1}(\partial\Omega).\end{aligned}$$

Hence, by the definition of the $\mathcal{DM}$ norm in Corollary 1.1 and that of $|\operatorname{div} F|(\mathbf{R}^N)$ in Definition 1.1, we conclude the desired result. □

3. Product Rule

Theorem 3.1. *Let $g \in BV(D)\cap L^\infty(D)$ and $F \in \mathcal{DM}(D)$. Then $gF \in \mathcal{DM}(D)$. Moreover, if g is also Lipschitz over any compact set in D,*

$$\operatorname{div}(gF) = g\operatorname{div} F + F\cdot\nabla g. \tag{3.1}$$

Proof. Let F_j be as in Theorem 2.1. Let g_j be the analogous sequence for g. We have

$$\begin{aligned}\int_D |\operatorname{div}(g_jF_j)|dx &= \sup\{\int_D g_jF_j\cdot\nabla\phi dx \mid \phi \in C_0^1(D), |\phi|\le 1\} \\ &\le 3\|g\|_\infty \sup\{\int_D F_j\cdot\nabla\phi dx \mid \phi\in C_0^1(D),\ |\phi|\le 1\} \\ &\quad + 3\|F\|_\infty \sup\{\int_D \nabla g_j\cdot\phi dx \mid \phi\in C_0^1(D;\mathbf{R}^N),\ |\phi|\le 1\} \\ &\le 3\{\|g\|_\infty|\operatorname{div} F_j|(D) + \|F\|_\infty\|\nabla g_j\|_\infty\}.\end{aligned}$$

Here in the first inequality we just divided g_j by its L^∞ norm, which is less than $3\|g\|_\infty$ by the construction, and then we considered $g_j\phi/\|g_j\|_\infty$ as a test function. We similarly handle F_j to obtain the second term in the right-hand side of the same inequality. Hence, for any $\phi \in C_0^1(D)$ with $|\phi| \le 1$, one has

$$\int_D gF\cdot\nabla\phi\,dx = \lim\int_D g_jF_j\cdot\nabla\phi\,dx \le 3\{\|g\|_\infty|\operatorname{div} F|(D) + \|F\|_\infty\|\nabla g\|_\infty\},$$

and

$$|\mathrm{div}\,(gF)|(D) < \infty.$$

Since $gF \in L^\infty(D;\mathbf{R}^N)$, one has $gF \in \mathcal{DM}(D)$. Now Theorem 1.2 implies that $\mathrm{div}\,F_j \rightharpoonup \mathrm{div}\,F$ as Radon measures over D. Hence, if g is Lipschitz over all compact sets contained in D, then

$$g\mathrm{div}\,F_j + F_j \cdot \nabla g \rightharpoonup g\mathrm{div}\,F + F \cdot \nabla g, \qquad \text{in } \mathcal{M}(D).$$

On the other hand, clearly $\mathrm{div}\,(gF_j) \rightharpoonup \mathrm{div}\,(gF)$ in the sense of distributions. Taking the limit in the identity

$$\mathrm{div}\,(gF_j) = g\mathrm{div}\,F_j + F_j \cdot \nabla g$$

in the sense of distributions and using the fact that $C_0^\infty(D)$ is dense in $C_0(D)$, we obtain (3.1). □

In fact, one may refine the above result and prove that (3.1) holds a.e. in the general case, not only for local Lipschitz functions. In this case, we must take the absolutely continuous part of ∇g. To show this, we establish a preliminary result, which is of interest in itself.

Proposition 3.1. *Let $F \in \mathcal{DM}(D)$ and $A \subset D$ be a Borel measurable set such that $\mathcal{H}^{N-1}(A) = 0$. Then $|\mathrm{div}\,F|(A) = 0$.*

Proof. Since there are Borel measurable sets D_+ and D_-, $D_+ \cup D_- = D$, such that $\mathrm{div}\,F$ is a nonnegative measure over D_+ and a nonpositive measure over D_-, one may assume $A \subset D_+$ and hence $|\mathrm{div}\,F|(A) = (\mathrm{div}\,F)_+(A) = \mathrm{div}\,F(A)$. Also, since $(\mathrm{div}\,F)_+$ is a Radon measure, it suffices to prove the assertion for any compact A. Now, for any $\varepsilon > 0$, we can find a finite number, J, of balls of radius less than ε such that $A \subset \cup_{i=1}^J B(x_i; r_i)$ and $\sum_{i=1}^J r_i^{N-1} < \varepsilon$, since $\mathcal{H}^{N-1}(A) = 0$. Now we may apply the Gauss-Green formula (2.9) with $\Omega = \Omega_\varepsilon \equiv \cup_{i=1}^J B(x_i; r_i)$ and any function ϕ in $C_0^1(\mathbf{R}^N)$ that identically equals one over $\bar{\Omega}_\varepsilon$. Then

$$|\mathrm{div}\,F(\Omega_\varepsilon)| \le \|F\|_\infty \mathcal{H}^{N-1}(\partial\Omega_\varepsilon) \le C\|F\|_\infty \sum_{i=1}^J r_i^{N-1} \le \varepsilon C \|F\|_\infty.$$

Now, since $\chi_{\Omega_\varepsilon} \to \chi_A$ pointwise in D as $\varepsilon \to 0$ (recall that A is compact), one has $|\mathrm{div}\,F|(A) = \mathrm{div}\,F(A) = 0$. This completes the proof. □

For $g \in BV$, let $(\nabla g)_{\mathrm{ac}}$ and $(\nabla g)_{\mathrm{sing}}$ denote the absolutely continuous part and the singular part of the Radon measure ∇g, respectively. We can now state the refinement of Theorem 3.1.

Theorem 3.2. *Given $F \in \mathcal{DM}(D)$ and $g \in BV(D) \cap L^\infty(D)$, the identity*

$$\mathrm{div}\,(gF) = \bar{g}\mathrm{div}\,F + \overline{F \cdot \nabla g}$$

holds in the sense of Radon measures in D, where $\bar{g}$ is the limit of a mollified sequence for g through a positive symmetric mollifier, and $\overline{F \cdot \nabla g}$ is a Radon measure absolutely continuous with respect to $|\nabla g|$, whose absolutely continuous part with respect to the Lebesgue measure in D coincides with $F \cdot (\nabla g)_{ac}$ almost everywhere in D.

Proof. Let $g_\delta = \omega_\delta * g$, where $\omega_\delta(x) = \delta^{-N}\eta(\frac{x}{\delta})$ with a positive symmetric mollifier ω. From Theorem 3.1, we have

$$\mathrm{div}\,(g_\delta F) = g_\delta \mathrm{div}\, F + F \cdot \nabla g_\delta.$$

Now, it is well known that g_δ converges to a Borel function $\bar{g}$, $\mathcal{H}^{N-1}$-a.e. in D (this function equals g a.e. in D). Then, using Proposition 3.1, we get

$$g_\delta \mathrm{div}\, F \rightharpoonup \bar{g}\mathrm{div}\, F, \qquad \text{in } \ \mathcal{M}(D),$$

as a consequence of the Dominated Convergence Theorem applied to the measure $\mathrm{div}\, F$. On the other hand, as in the proof of Theorem 3.1, one may easily show that $\{\mathrm{div}\,(g_\delta F)\}$ is uniformly bounded in $\mathcal{M}(D)$.

Now, this sequence converges to $\mathrm{div}\,(gF)$ in the sense of distributions over D. Then, one must have $\mathrm{div}\,(g_\delta F) \rightharpoonup \mathrm{div}\,(gF)$ in $\mathcal{M}(D)$. Hence,

$$F \cdot \nabla g_\delta \rightharpoonup \overline{F \cdot \nabla g} \equiv \mathrm{div}\,(gF) - \bar{g}\mathrm{div}\, F.$$

Now we pass to the proof of the assertions about $\overline{F \cdot \nabla g}$. Let $A \subset D$ be such that $|\nabla g|(A) = 0$. We are going to prove that $|\overline{F \cdot \nabla g}|(A) = 0$. Again, it suffices to consider any compact set A with $|\nabla g|(A) = 0$. Given $\varepsilon > 0$, we can cover A by a finite number, J, of balls so that

$$A \subset \cup_{i=1}^J B(x_i; r_i), \quad r_i < \varepsilon; \qquad |\nabla g|\big(\cup_{i=1}^J B(x_i; r_i)\big) < \varepsilon.$$

We may assume without loss of generality that $|\nabla g|(\partial B(x_i; r_i)) = 0$, $i = 1, \cdots, J$. Let $\phi \in C_0(\cup_{i=1}^J B(x_i; r_i))$. Thus

$$\begin{aligned}
\langle \overline{F \cdot \nabla g}, \phi \rangle &= \lim_{\delta \to 0} \int \phi(x)\, F(x) \cdot \nabla g_\delta(x)\, dx \\
&\le \|\phi\|_\infty \|F\|_\infty \limsup_{\delta \to 0} |\nabla g_\delta|\big(\cup_{i=1}^J B(x_i; r_i)\big) \\
&= \|\phi\|_\infty \|F\|_\infty |\nabla g|\big(\cup_{i=1}^J B(x_i; r_i)\big) \le \varepsilon \|\phi\|_\infty \|F\|_\infty,
\end{aligned}$$

from the fact that $|\nabla g_\delta|(B) \to |\nabla g|(B)$, for all open sets $B \subset D$ with $|\nabla g|(\partial B) = 0$ (see [15]). Hence, we obtain

$$|\overline{F \cdot \nabla g}|(A) \le |\overline{F \cdot \nabla g}|\bigg(\cup_{i=1}^J B(x_i; r_i)\bigg) \le \varepsilon \|F\|_\infty.$$

Taking $\varepsilon \to 0$, we obtain the desired result.

We now prove the last assertion about $\overline{F \cdot \nabla g}$. Let $x \in D$ be such that the limit

$$\lim_{r \to 0} \frac{1}{\alpha(N) r^N} \int_{B(x;r)} \overline{F \cdot \nabla g}(y) dy$$

exists, and

$$\lim_{r \to 0} \frac{1}{\alpha(N) r^N} \int_{B(x;r)} |(\nabla g)_{\mathrm{sing}}(y)| dy = 0, \tag{3.2}$$

$$\lim_{r \to 0} \frac{1}{\alpha(N) r^N} \int_{B(x;r)} |F(y) \cdot (\nabla g)_{\mathrm{ac}}(y) - F(x) \cdot (\nabla g)_{\mathrm{ac}}(x)|\, dy = 0. \tag{3.3}$$

Almost every $x \in D$ has this property. Fix $r > 0$ such that

$$|(\nabla g)_{\mathrm{sing}}|(\partial B(x; r)) = 0. \tag{3.4}$$

One has

$$\nabla(\omega_\delta * g) = \omega_\delta * \nabla g = \omega_\delta * (\nabla g)_{\mathrm{ac}} + \omega_\delta * (\nabla g)_{\mathrm{sing}}.$$

Hence, for any $\phi \in C_0(B(x;r))$, we use $|\omega_\delta * (\nabla g)_{\rm sing}(y)| \le \omega_\delta * |(\nabla g)_{\rm sing}(y)|$ to find

$$\begin{aligned}
&\left|\frac{1}{\alpha(N)r^N}\int_{B(x;r)}\phi(y)\{F(y)\cdot\omega_\delta*(\nabla g)(y)-F(x)\cdot(\nabla g)_{\rm ac}(x)\}dy\right|\\
&\le \frac{1}{\alpha(N)r^N}\int_{B(x;r)}|\phi(y)\{F(y)\cdot\omega_\delta*(\nabla g)_{\rm ac}(y)-F(x)\cdot(\nabla g)_{\rm ac}(x)\}|\,dy\\
&\quad+\frac{\|\phi\|_\infty\|F\|_\infty}{\alpha(N)r^N}\int_{B(x;r)}\omega_\delta*|(\nabla g)_{\rm sing}(y)|dy.
\end{aligned}$$

Now, the last term of the right-hand side in the inequality converges to

$$\frac{\|\phi\|_\infty\|F\|_\infty}{\alpha(N)r^N}\int_{B(x;r)}|(\nabla g)_{\rm sing}(y)|dy,$$

since $\omega_\delta * |(\nabla g)_{\rm sing}| \rightharpoonup |(\nabla g)_{\rm sing}|$ and (3.4) holds. Then we take $\delta \to 0$ to obtain

$$\begin{aligned}
&\left|\frac{1}{\alpha(N)r^N}\int_{B(x;r)}\phi(y)\{\overline{F\cdot\nabla g}(y)-F(x)\cdot(\nabla g)_{\rm ac}(x)\}dy\right|\\
&\le \frac{\|\phi\|_\infty\|F\|_\infty}{\alpha(N)r^N}\int_{B(x;r)}|(\nabla g)_{\rm sing}(y)|dy\\
&\quad+\frac{1}{\alpha(N)r^N}\int_{B(x;r)}|\phi(y)\{F(y)\cdot(\nabla g)_{\rm ac}(y)-F(x)\cdot(\nabla g)_{\rm ac}(x)\}|\,dy.
\end{aligned}$$

Now, taking $\phi \to 1$ pointwise in $B(x;r)$ so that $\|\phi\| \le 1$, we get

$$\begin{aligned}
&\left|\frac{1}{\alpha(N)r^N}\int_{B(x;r)}\left(\overline{F\cdot\nabla g}(y)-F(x)\cdot(\nabla g)_{\rm ac}(x)\right)dy\right|\\
&\le \frac{\|F\|_\infty}{\alpha(N)r^N}\int_{B(x;r)}|(\nabla g)_{\rm sing}(y)|dy\\
&\quad+\frac{1}{\alpha(N)r^N}\int_{B(x;r)}|F(y)\cdot(\nabla g)_{\rm ac}(y)-F(x)\cdot(\nabla g)_{\rm ac}(x)|\,dy, \qquad (3.5)
\end{aligned}$$

by the Dominated Convergence Theorem. Finally, we take $r \to 0$ in (3.5) and use (3.2)-(3.3) to get the desired result.

□

4. Applications to Initial-Boundary Value Problems for Hyperbolic Conservation Laws

In this section we apply the theory developed in §1-§3 to establish a general framework for L^∞ weak entropy solutions of initial-boundary value problems for hyperbolic conservation laws. Let $Q \subset \mathbf{R}^{n+1}$ be a domain, whose points are denoted by (t,x), with $\Omega \equiv Q \cap \{t=0\} \ne \emptyset$, $\Gamma = \partial Q$. Consider

$$\partial_t u + \nabla_x \cdot f(u) = 0, \qquad \text{in } Q\cap\{t>0\}, \tag{4.1}$$

$$u|_{t=0} = u_0, \tag{4.2}$$

$$u|_{\Gamma\cap\{t>0\}} = u_b. \tag{4.3}$$

Here $u \in U \subset \mathbf{R}^m$, $f \in C^1(U;(\mathbf{R}^m)^n)$, for some domain $U \subset \mathbf{R}^m$. We assume that the initial-boundary data satisfy

$$u_0 \in L^\infty(\Omega;\mathbf{R}^m), \qquad u_b \in L^\infty(\Gamma\cap\{t>0\};\mathbf{R}^m). \tag{4.4}$$

We will consider the domain Q of the following form. Let $\mathcal{L} : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be a bi-Lipschitz map:

$$\mathcal{L}(t,x) = (t, y(t,x)), \qquad \text{for all } (t,x) \in \mathbf{R}^{n+1},$$

where $y : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is a certain Lipschitz map satisfying $y(t,x) = x$, for $t \le 0$. Let $\Omega \subset \mathbf{R}^n$ be an open domain with regularly deformable Lipschitz boundary $\partial\Omega$. Set

$$Q = \mathcal{L}(\mathbf{R} \times \Omega), \qquad \Gamma = \mathcal{L}(\mathbf{R} \times \partial\Omega),$$
$$Q_T = \mathcal{L}((0,T) \times \Omega), \qquad \Gamma_T = \mathcal{L}((0,T) \times \partial\Omega).$$

Assume that $\Theta : \partial\Omega \times [0,1] \to \bar{\Omega}$ is a regular Lipschitz deformation for $\partial\Omega$. We fix a standard regular Lipschitz deformation $\Psi : \Gamma \times [0,1] \to Q$ for Γ, given by

$$\Psi(r,s) = \mathcal{L}\left(\pi_1(r), \Theta(\pi_2 \circ \mathcal{L}^{-1}(r), s)\right),$$

where $\pi_1 : \mathbf{R}^{n+1} \to \mathbf{R}$ and $\pi_2 : \mathbf{R}^{n+1} \to \mathbf{R}^n$ are given by $\pi_1(t,x) = t$, $\pi_2(t,x) = x$. We denote $\Psi_s(r) = \Psi(r,s)$, $\Gamma^s = \Psi_s(\Gamma)$, and $\Gamma_T^s = \Psi_s(\Gamma_T)$. We denote $A \Subset B$ that A is an open subset of B and its closure $\bar{A}$ is compact subset of B.

The discussion which follows is partly motivated by the analysis carried out by Otto [25] for the initial-boundary value problem for multidimensional scalar conservation laws in cylindric domains of the form $(0,T) \times \Omega$, Ω with smooth boundary. In [25], the existence and uniqueness of entropy solutions of the initial-boundary value problem were established for L^∞ initial and boundary conditions. This result extends the earlier one for BV initial and boundary conditions, by Bardos, Leroux, and Nedelec [2]. Other attempts to extend partially in various ways the results in [2] have been made in some references such as in [12, 7, 18, 28] and those cited therein.

4.1. General Framework

Definition 4.1. We say that $\eta \in C^1(\mathbf{R}^m)$ is an *entropy* for (4.1), with associated *entropy flux* $q = (q^1, \cdots, q^n) \in C^1(\mathbf{R}^m; \mathbf{R}^n)$, if

$$\nabla q_i(u) = \nabla\eta(u)\nabla f_i(u), \qquad i = 1, \cdots, n. \tag{4.5}$$

We call $F(u) = (\eta(u), q(u))$ an *entropy pair*. If $\eta(u)$ is convex, we say $F(u)$ is a convex entropy pair. An entropy pair $\mathcal{F}(u,v) = (\alpha(u,v), \beta(u,v))$ is called a *boundary entropy pair* if, for each fixed $v \in \mathbf{R}^m$, $\alpha(u,v)$ is convex with respect to u, and

$$\alpha(v,v) = \beta(v,v) = \partial_u\alpha(v,v) = 0. \tag{4.6}$$

We say that $\mathcal{F}(u,v) = (\alpha(u,v), \beta(u,v))$ is a *generalized boundary entropy pair* if it is the uniform limit of a sequence of boundary entropy pairs over compact sets.

In this section, with the aid of the theory of divergence-measure fields, we focus on any weak entropy solution $u \in L^\infty(Q_T; \mathbf{R}^m)$ of (4.1)-(4.3) in Q_T, in the sense given by the following definition.

Definition 4.2. We say that $u(t,x) \in L^\infty(Q_T; \mathbf{R}^m)$ is a weak entropy solution of (4.1)-(4.3) in Q_T if it satisfies the following.

- Conservation Laws (4.1): For all $\phi \in C_0^\infty(Q_T)$, $\phi \geq 0$, and any convex entropy pair (η, q),

$$\iint_{Q_T} (\eta(u)\phi_t + q(u) \cdot \nabla_x \phi) \, dx \, dt \geq 0; \tag{4.7}$$

- Initial Condition (4.2): For any $\tilde{\Omega} \Subset \Omega$,

$$\operatorname*{ess\,lim}_{t \to 0+} \int_{\tilde{\Omega}} |u(t,x) - u_0(x)| \, dx = 0; \tag{4.8}$$

- Boundary Condition (4.3): For any $\gamma \in L^1(\Gamma_T)$, $\gamma \geq 0$ $\mathcal{H}^n$-a.e., and any boundary entropy flux $\mathcal{F} = (\alpha, \beta)$,

$$\operatorname*{ess\,lim}_{s \to 0+} \int_{\Gamma_T} \mathcal{F}(u \circ \Psi_s(r), u_b(r)) \cdot \nu_s(\Psi_s(r))\gamma(r) \, d\mathcal{H}^n(r) \geq 0, \tag{4.9}$$

where ν_s is the outward unit normal field defined $\mathcal{H}^n$-a.e. in Γ_T^s.

The main focus in this section is to investigate classes of equations and the boundary data for which the solutions satisfying (4.7)–(4.9) assume their boundary data in the usual sense, with the aid of the theory of divergence-measure fields.

For convenience of presentation, we extend the weak entropy solution $u(t,x) \in L^\infty(Q_T; \mathbf{R}^m)$ to $u(t,x) \in L^\infty(Q; \mathbf{R}^m)$ setting

$$u(t,x) = 0, \qquad (t,x) \in Q - Q_T.$$

We observe from Proposition 2.1 that, if u_j and f_j denote the jth component of u and the jth row of the $m \times n$ matrix f, respectively, $j = 1, \cdots, m$, and $(\eta(u), q(u))$ is any convex entropy pair for (4.1), (4.7) implies that the fields

$$E_j(u) \equiv \big(u_j, f_j(u)\big) \in \mathcal{DM}(D), \quad F(u) \equiv \big(\eta(u), q(u)\big) \in \mathcal{DM}(D)$$

for any bounded open set $D \subset Q$ as a consequence of the Schwartz lemma on nonnegative distributions [26]. In particular, the normal traces $E^j(u) \cdot \nu|S$, $j = 1, \cdots, n$, and $F(u) \cdot \nu|S$ are defined for any open subset S of the Lipschitz boundary of any open set $D \Subset Q$.

We next establish the first important fact about the solutions of (4.1)–(4.3) in the sense of (4.7)–(4.9).

Theorem 4.1. *Assume that* (4.1) *is endowed with a strictly convex entropy. A function $u(t,x) \in L^\infty(Q_T; \mathbf{R}^m)$ satisfies* (4.7)–(4.9) *if and only if it satisfies the following.*

1. *Equation* (4.1) *holds in Q_T in the sense of the distributions;*
2. *Given any boundary entropy pair $(\alpha(u,v), \beta(u,v))$, there is a constant $M > 0$ such that, for any nonnegative $\phi(t,x) \in C_0^\infty\big((-\infty, T) \times \mathbf{R}^n\big)$,*

$$\begin{aligned} &\iint_{Q_T} (\alpha(u(t,x),v)\phi_t + \beta(u(t,x),v) \cdot \nabla_x \phi) \, dx \, dt \\ &\qquad + \int_\Omega \alpha(u_0(x),v)\phi(0,x) \, dx + M \int_{\Gamma_T} |u^b(r) - v| \phi(r) \, d\mathcal{H}^n(r) \geq 0. \end{aligned} \tag{4.10}$$

Proof. Given any convex entropy pair (η, q),

$$\alpha(u,v) = \eta(u) - \eta(v) - \nabla\eta(v)(u - v), \tag{4.11}$$

$$\beta(u,v) = q(u) - q(v) - \nabla\eta(v)(f(u) - f(v)) \tag{4.12}$$

form a boundary entropy pair. Since u satisfies (4.1) in the sense of distributions, one easily deduces (4.7) from (4.10).

We now prove (4.8). We consider α and β as in (4.11)-(4.12) for a strictly convex entropy η. Let $\delta < T$ be small enough. Choose $\phi(t,x) = \zeta(t)\xi(x)$, with $\zeta \in C_0^\infty(-\infty,\delta)$, $\xi \in C_0^\infty(\Omega)$, $\zeta \geq 0$, $\xi \geq 0$. We get

$$\int_0^\delta \zeta'(t) \int_\Omega \alpha(u,v)\xi(x)\,dx\,dt + C\int_0^\delta \zeta(t)\,dt + \int_\Omega \alpha(u_0(x),v)\xi(x)\,dx \geq 0.$$

Hence, choosing $\zeta = \chi_{(-\delta,\delta)}$ (after mollifying and passing to the limit) and making $\delta \to 0$, we get

$$\operatorname*{ess\,lim}_{t\to 0+} \int_\Omega \alpha(u(t,x),v)\xi(x)\,dx \leq \int_\Omega \alpha(u_0(x),v)\xi(x)\,dx,$$

where the limit on the left-hand side exists because of Theorem 2.2. Proceeding as above, we conclude from the last inequality that, for any $\xi, v_0 \in L^1(\Omega)$, one has

$$\operatorname*{ess\,lim}_{t\to 0+} \int_\Omega \alpha(u(t,x),v_0(x))\xi(x)\,dx \leq \int_\Omega \alpha(u_0(x),v_0(x))\xi(x)\,dx.$$

Therefore, choosing $v_0 = u_0$, and using the strict convexity of η and the fact that u and u_0 are uniformly bounded, we arrive at (4.8).

We finally prove (4.9). Let $h : \mathbf{R}^{n+1} \to [0,1]$ be defined by setting $h(t,x) = s$, if $(t,x) \in \Gamma^s$; $h(t,x) = 0$, if $(t,x) \notin Q$; and $h(t,x) = 1$, otherwise. In (4.10), we choose

$$\phi(t,x) = \gamma(\Psi_{h(t,x)}^{-1}(t,x))\zeta(h(t,x)), \quad \text{for } (t,x) \in \mathrm{Image}(\Psi),$$

where $\gamma \in \mathrm{L}ip(\Gamma)$, $\operatorname{supp}\gamma \subset \Gamma_T$, $\gamma \geq 0$, $\zeta \in C_0^\infty(-\infty,1)$, and set $\phi(t,x) = 0$ for $(t,x) \in Q - \mathrm{Image}(\Psi)$. We then extend ϕ to all $\mathbf{R}^{n+1}$ as a Lipschitz function with compact support contained in $(0,T) \times \mathbf{R}^n$. With this choice of ϕ in (4.10), using the coarea formula [13, 14], one obtains

$$\begin{aligned}\int_0^1 \int_{\Gamma^s} &\mathcal{F}(u(r),v)\cdot\nu_s(r)\gamma(\Psi_s^{-1}(r))\,d\mathcal{H}^n(r)\zeta'(s)\,ds \\ &+ C\int_0^1 \zeta(s)\,ds + M\int_{\Gamma_T} |u_b(r)-v|\gamma(r)\,d\mathcal{H}^n(r)\zeta(0) \geq 0.\end{aligned}$$

Choosing $\zeta(s) = \chi_{(-\delta,\delta)}$, $0<\delta<1$ (mollifying and passing to the limit), and then making $\delta \to 0$, we get

$$\begin{aligned}\operatorname*{ess\,lim}_{s\to 0+} \int_{\Gamma^s} &\mathcal{F}(u(r),v)\cdot\nu_s(r)\gamma(\Psi_s^{-1}(r))\,d\mathcal{H}^n(r) \\ &\geq -M\int_{\Gamma_T} |u_b(r)-v|\gamma(r)\,d\mathcal{H}^n(r),\end{aligned}$$

where we used Theorem 2.2 to ensure the existence of the limit on the left-hand side. By approximation, we conclude that the above inequality holds for any $\gamma \in L^1(\Gamma_T), \gamma \geq 0, \mathcal{H}^n-$ a.e. Using the area formula [14], one obtains

$$\begin{aligned}\operatorname*{ess\,lim}_{s\to 0+} \int_{\Gamma_T} &\mathcal{F}(u\circ\Psi_s(r),v)\cdot\nu_s(\Psi_s(r))\gamma(r)\,d\mathcal{H}^n(r) \\ &\geq -M\int_{\Gamma_T} |u_b(r)-v|\gamma(r)\,d\mathcal{H}^n(r).\end{aligned}$$

Now, considering first simple functions $v_b(r)$ and using a standard approximation argument again, one then deduces from the last inequality that

$$\begin{aligned}\operatorname*{ess\,lim}_{s\to 0+}\int_{\Gamma_T} &\mathcal{F}(u\circ\Psi_s(r), v_b(r))\cdot\nu_s(\Psi_s(r))\gamma(r)\,d\mathcal{H}^n(r)\\ &\geq -M\int_{\Gamma_T}|u_b(r)-v_b(r)|\gamma(r)\,d\mathcal{H}^n(r),\end{aligned}$$

for any $v_b\in L^1(\Gamma_T)$. Taking $v_b=u_b$, we then recover (4.9).

Conversely, from (4.7) with $\eta(u)=\pm u$, we deduce that u is a solution of (4.1) in the sense of distributions. Also, (4.7) and (4.8) imply that, for any $\psi\in C_0^\infty(Q\cap\{t<T\})$ with $\psi\geq 0$, one has

$$\iint_{Q_T}(\eta(u)\psi_t+q(u)\cdot\nabla_x\psi)\;dx\,dt+\int_\Omega\eta(u_0(x))\psi(0,x)\,dx\geq 0. \tag{4.13}$$

Now we choose ψ in (4.13) as $\psi(t,x)=\phi(t,x)(1-\zeta(h(t,x)))$, with $\phi\in C_0^\infty((-\infty,T)\times\mathbf{R}^n)$, $\phi\geq 0$, ζ and h as above. Again, choosing $\zeta(s)=\chi_{(-\delta,\delta)}(s)$, $0<\delta<1$, making $\delta\to 0$, and arguing as above, one arrives at

$$\begin{aligned}\iint_{Q_T}(\eta(u)\phi_t+q(u)\cdot\nabla_x\phi)\;dx\,dt&+\int_\Omega\eta(u_0(x))\phi(0,x)\,dx\\ -\operatorname*{ess\,lim}_{s\to 0+}\int_{\Gamma_T}&\mathcal{F}(u\circ\Psi_s(r),v)\cdot\nu_s(\Psi_s(r))\phi(r)\,d\mathcal{H}^n(r)\geq 0.\end{aligned}\tag{4.14}$$

Observe that, for some $M>0$, depending on $\mathcal{F}$ and an L^∞ bound for u,

$$|\mathcal{F}(u\circ\Psi_s(r),v)-\mathcal{F}(u\circ\Psi_s(r),u_b(r)|\leq M|u_b(r)-v|.$$

Thus, using (4.9) and the above inequality, we finally obtain (4.10) from (4.14). □

We now investigate some cases at which the boundary data turn out to be assumed in a way closer to the usual sense. We start with a result for the scalar case $m=1$, which extends the one in [25] to the present context of non-cylindric domains with Lipschitz boundary.

For $m=1$, we denote by Γ_{act} the subset of Γ_T given by

$$\Gamma_{act}=\{\,r\in\Gamma_T\mid u\mapsto E(u)\cdot\nu(r)\text{ is decreasing}\,\},$$

where $E(u)=(u,f(u))$, and $\nu(r)$ is the outward unit normal field defined $\mathcal{H}^n$-a.e. in Γ_T.

Proposition 4.1. *If $u\in L^\infty(Q_T)$ and $u_b\in L^\infty(\Gamma_T)$ satisfy* (4.9) *for any boundary entropy pairs associated with* (4.1)*, then*

$$\operatorname*{ess\,lim}_{s\to 0+}\int_{\Gamma_{act}}|u\circ\Psi_s(r)-u_b(r)|\,d\mathcal{H}^n(r)=0.$$

Proof. The entropy pair

$$\mathcal{F}(u,v)=(|u-v|,\operatorname{sign}(u-v)(f(u)-f(v)))$$

is the uniform limit of boundary entropy pairs. Hence, (4.9) implies

$$\operatorname*{ess\,lim}_{s\to 0+}\int_{\Gamma_T}\mathcal{F}(u\circ\Psi_s(r),u_b(r))\cdot\nu_s(\Psi_s(r))\gamma(r)\,dr\geq 0,$$

for all $\gamma \in L^1(\Gamma_T)$, $\gamma \geq 0$, $\mathcal{H}^n$-a.e. Now, choose γ as the characteristic function of Γ_{act} and observe that

$$\mathcal{F}(u,v) \cdot \nu(r) = -|E(u) \cdot \nu(r) - E(v) \cdot \nu(r)|,$$

for all $r \in \Gamma_{act}$. The regularity of the deformation Ψ implies that $\nu_s(\Psi_s(r)) \to \nu(r)$, as $s \to 0+$, for $\mathcal{H}^n$ almost all $r \in \Gamma_T$. Hence, the Dominated Convergence Theorem gives

$$\operatorname{ess\,lim}_{s\to 0+} \int_{\Gamma_{act}} |E(u \circ \Psi_s(r)) \cdot \nu(r) - E(u_b(r)) \cdot \nu(r)|\, d\mathcal{H}^n(r) = 0.$$

Now, given any sequence $\{s_l\}$ in $[0,1]$, converging to 0, which is not contained in a certain fixed set of measure zero, one may extract a subsequence (still denoted s_l) such that

$$\lim_{l\to\infty} E(u \circ \Psi_{s_l}(r)) \cdot \nu(r) = E(u_b(r)) \cdot \nu(r), \qquad \mathcal{H}^n - \text{a.e. } r \in \Gamma_{act}.$$

By the definition of Γ_{act}, we deduce that $\lim_{l\to\infty} u \circ \Psi_{s_l}(r) = u_b(r)$, for $\mathcal{H}^n$ −a.e. $r \in \Gamma_{act}$. By the Dominated Convergence Theorem, we then conclude that

$$\lim_{l\to\infty} \int_{\Gamma_{act}} |u \circ \Psi_{s_l}(r) - u_b(r)|\, d\mathcal{H}^n(r) = 0,$$

which gives the desired result. □

We now look for an analogue of Proposition 4.1 in the general case of systems, $m > 1$. For each $v \in \mathbf{R}^m$, let $(w_1(u,v), \cdots, w_m(u,v))$ denote a certain change of coordinates for $\mathbf{R}^m$. Let $\mathcal{F}^j(u,v)$, $j = 1, \cdots, m$, be a certain fixed generalized boundary entropy pair associated with (4.1). We also assume that there are given certain functions

$$\rho_j \in L^\infty\left(\Gamma_T; C(\mathbf{R}^2; [0,\infty))\right),$$

satisfying $\rho_j(r)(\lambda,\mu) > 0$, if $\lambda \neq \mu$, and $\rho_j(r)(\lambda,\lambda) = 0$, $\mathcal{H}^n$ a.e. $r \in \Gamma_T$, $j = 1, \cdots, m$. Corresponding to w_j, $\mathcal{F}^j$, and ρ_j, we define

$$\Gamma_{act}{}^j = \left\{\, r \in \Gamma_T \mid \mathcal{F}^j(u,v) \cdot \nu(r) \leq -\rho^j(r)\left(w_j(u,v), w_j(v,v)\right) \right\},$$

$j = 1, \cdots, m$, and, for any subset $\{j_1, \cdots, j_k\} \subset \{1, 2, \cdots, m\}$, we set

$$\Gamma_{act}{}^{j_1,\cdots,j_k} = \Gamma_{act}{}^{j_1} \cap \cdots \cap \Gamma_{act}{}^{j_k}.$$

The following proposition is the result analogous to Proposition 4.1. The proof is entirely similar and so we omit it.

Proposition 4.2. *If $u \in L^\infty(Q_T)$ and $u_b \in L^\infty(\Gamma_T)$ satisfy* (4.9) *for all boundary entropy pairs associated with* (4.1)*, then*

$$\operatorname{ess\,lim}_{s\to 0+} \int_{\Gamma_{act}{}^j} |w_j(u \circ \Psi_s(r), u_b(r)) - w_j(u_b(r), u_b(r))|\, d\mathcal{H}^n(r) = 0,$$

$j = 1, \cdots, m$.

We now give some examples of applications of Proposition 4.2.

4.2. Hyperbolic Systems with Space-like Boundary Data

First we analyze whether the boundary condition (4.9) can recover the intuitive notion for space-like boundaries, in the context of one-dimensional systems of hyperbolic conservation laws. In other words, suppose the space domain at each time t is a bounded interval $\Omega(t)$, whose, say, left-hand extreme $y_0(t)$ moves with speed less than the minimum of the slowest characteristic speed, in the region of the phase-space where the solution assumes its value, for t in a certain subset of $(0,\infty)$. Then one expects that the boundary condition will be assumed in a strong sense, say, that of convergence in L^1_{loc}.

Consider a one-dimensional $m\times m$ hyperbolic system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \tag{4.15}$$

endowed with a strictly convex entropy pair $(\eta(u), q(u))$. Let $\lambda_1(u)\le\cdots\le\lambda_m(u)$ be the eigenvalues of $\nabla f(u)$, which is diagonalizable for all u, and $r_1(u),\dots,r_m(u)$ be corresponding eigenvectors forming a basis of $\mathbf{R}^m$. We take $\Omega=[0,1]$, consider $\mathcal{L}$ as above, and denote $Q_\infty=\mathcal{L}((0,\infty)\times\Omega)$, $\Gamma^-=\mathcal{L}((0,\infty)\times\{0\})$, $\Gamma^+=\mathcal{L}((0,\infty)\times\{1\})$. We form the initial-boundary value problem for (4.15) with initial data:

$$u|_{t=0} = u_0(x) \tag{4.16}$$

and the boundary data:

$$\begin{cases} u|_{x=y_0(t)} = u_\flat(t), \\ u|_{x=y_1(t)} = u_\sharp(t), \end{cases} \tag{4.17}$$

where $y_0(t)=y(t,0)$ and $y_1(t)=y(t,1)$. Then we have

Theorem 4.2. *Let U be a bounded domain in $\mathbf{R}^m$ and $M>0$ be such that $|\beta(u,v)|\le M\alpha(u,v)$, for $u,v\in U$, where α,β are given by* (4.11)-(4.12) *for given strictly convex entropy η. Suppose $u\in L^\infty(Q_\infty)$ satisfies* (4.17) *in the sense of* (4.9) *and takes its values in U. Assume that B is a Borelian subset of* $(0,\infty)$ *such that $y_0'(t)\le -M-\delta_0$, for a.e. $t\in B$, for some $\delta_0>0$. Then*

$$\operatorname*{ess\,lim}_{s\to 0+}\int_{\mathcal{L}(B\times\{0\})} |u\circ\Psi_s(r) - u_\flat\circ\pi_1(r))|\, d\mathcal{H}^1(r) = 0. \tag{4.18}$$

Furthermore, let $\lambda^-=\inf_{u\in U}\lambda_1(u)$. Then, given $\bar u\in U$, for any $\varepsilon>0$, there exits $\delta>0$ such that, if $y_0'(t)<\lambda^- -\varepsilon$, for a.e. $t\in B$, and

$$\|u-\bar u\|_\infty,\ \|u_\flat-\bar u\|_\infty,\ \|u_\sharp-\bar u\|_\infty<\delta, \tag{4.19}$$

then (4.18) *holds.*

Proof. Set $\mathcal{F}(u,v)=(\alpha(u,v),\beta(u,v))$. For the first part, we observe that

$$\sqrt{1+y_0'(t)^2}\,\mathcal{F}(u,v)\cdot\nu(r) = y_0'(t)\alpha(u,v)-\beta(u,v)\le -\delta\alpha(u,v)\le -c\delta|u-v|^2,$$

where $t=\pi_1(r)$. In particular, this shows that $\mathcal{L}(B\times\{0\})\subset{\Gamma_{\mathrm{act}}}^{1,\dots,m}$, if we take $\mathcal{F}^j(u,v)=\mathcal{F}(u,v)$, $w_j=u_j$, $\rho_j(r)(\lambda,\mu)=(\lambda-\mu)^2$, $j=1,\dots,m$. Hence Proposition 4.2 applies. For the second part, we first observe that

$$\sqrt{1+y_0'(t)^2}\,\mathcal{F}(u,v)\cdot\nu(r)\le(\lambda^- -\varepsilon)\alpha(u,v)-\beta(u,v).$$

Proceeding as in [11], pp.155, we show that, for $p(u,v)=(\lambda^- -\varepsilon)\alpha(u,v)-\beta(u,v)$, we have $p(u,v)\le -\frac{\varepsilon}{2}\alpha(u,v)$, provided that $|u-v|$ is sufficiently small. Hence, as in the proof of the first part, we can apply Proposition 4.2 to get (4.18). □

Remark 4.1. The analogous statements for the right-hand boundary curve are obtained through changing x by $-x$ and $f(u)$ by $-f(u)$ and applying the result for the left-hand boundary curve.

4.3. Systems with Affine Characteristic Hypersurfaces

Now we apply Proposition 4.2 to initial-boundary value problems for the one-dimensional $m \times m$ systems introduced in Temple [31], which are strictly hyperbolic and endowed with a complete set of independent Riemann invariants w_j, $j = 1, \cdots, m$, whose level sets are affine hyperplanes.

Again, let $\Omega = [0,1]$, and let $\mathcal{L}$, Q_∞, and $\Gamma^\pm$ be as before. Assume now that (4.15) is a system with affine characteristic hypersurfaces. We consider again the boundary value problem (4.15)–(4.17). Let $\lambda_j(u)$ be the eigenvalues of $\nabla f(u)$, $l_j(u)$ the associated left-eigenvectors, and $w_j(u)$ the associated Riemann invariants, $j = 1, \cdots, m$, respectively. For the purpose of this section, it will be convenient to assume that $l_j(u) = \nabla w_j(u)$ without loss of generality by normalization. Then the following are entropy pairs (see, *e.g.* [27, 17]):

$$\begin{cases} \alpha_j(u,v) = |l_j(v) \cdot (u - v)|, \\ \beta_j(u,v) = \mathrm{sign}(l_j(v) \cdot (u-v))(l_j(v) \cdot (f(u) - f(v))), \end{cases} \tag{4.20}$$

$$\begin{cases} \alpha_j^+(u,v) = (l_j(v) \cdot (u - v))_+, \\ \beta_j^+(u,v) = H(l_j(v) \cdot (u-v))(l_j(v) \cdot (f(u) - f(v))), \end{cases} \tag{4.21}$$

$$\begin{cases} \alpha_j^-(u,v) = (l_j(v) \cdot (u - v))_-, \\ \beta_j^-(u,v) = H(l_j(v) \cdot (v-u))(l_j(v) \cdot (f(u) - f(v))), \end{cases} \tag{4.22}$$

where $(s)_+ = \max\{s, 0\}$, $(s)_- = \max\{-s, 0\}$, and $H(s) = \chi_{(0,\infty)}(s)$ denotes the Heaviside function.

We will need the following lemma.

Lemma 4.1. *Let $\mathcal{F}^j \equiv (\alpha_j, \beta_j)$ and $\mathcal{F}^j_\pm \equiv (\alpha_j^\pm, \beta_j^\pm)$. Then $\mathcal{F}^j$ and $\mathcal{F}^j_\pm$ are generalized boundary entropy pairs.*

Proof. Indeed, let us consider first the case of $\mathcal{F}^j_+$. Let $\zeta : \mathbf{R}^m \to \mathbf{R}$ denote a symmetric mollifier. Define

$$\begin{aligned} \alpha_j^{+\varepsilon}(u,v) &= \int \left(l_j(U(w)) \cdot (u - U(w))\right)_+ \zeta_\varepsilon(w - W(v + c\varepsilon l_j(v)))\, dw, \\ \beta_j^{+\varepsilon}(u,v) &= \int H\left(l_j(U(w)) \cdot (u - U(w))\right) \\ &\qquad \times l_j(U(w)) \cdot (f(u) - f(U(w)))\, \zeta_\varepsilon(w - W(v + c\varepsilon l_j(v)))\, dw, \end{aligned}$$

where $u = U(w)$ and $w = W(u)$ are the changes of coordinates from w to u and vice-versa, and c will be suitably chosen. We denote $\mathcal{F}^{j\varepsilon}_+ = (\alpha_j^{+\varepsilon}, \beta_j^{+\varepsilon})$. Since W is a bi-Lipschitz map, there exists $\delta > 0$ such that $|W(v) - W(\tilde{v})| \geq \delta |v - \tilde{v}|$. Taking $\tilde{v}$ as $v + c\varepsilon l_j(v)$, we see that it is possible to choose c as a positive constant independent of ε such that $|W(z) - W(v + c\varepsilon l_j(v))| \geq \varepsilon$, for any point z with $w_j(z) \leq w_j(v)$. Here we have used the fact that $W(v)$ is the point of the image of the hyperplane $w_j(z) = w_j(v)$ to which has the smallest distance from $v + c\varepsilon l_j(v)$. Therefore, $\alpha_j^{+\varepsilon}(v,v) = \beta_j^{+\varepsilon}(v,v) = 0$ for any $\varepsilon > 0$.

We now pass to the verification of the second condition that $\nabla_u \alpha_j^{+\varepsilon}(v,v) = 0$. Notice that

$$\alpha_j^{+\varepsilon}(u,v) = \int_{w_j \leq w_j(u)} l_j(U(w)) \cdot (u - U(w)) \zeta_\varepsilon(w - W(v + c\varepsilon l_j(v)))\, dw,$$

and so we have

$$\nabla_u \alpha_j^{+\varepsilon}(v,v) = \int_{w_j \le w_j(v)} l_j(U(w))\zeta_\varepsilon(w - W(v + c\varepsilon l_j(v)))\, dw,$$

where we used that $l_j(U(w)) \cdot (v - U(w))$ vanishes identically on the hyperplane $w_j = w_j(v)$. By the reasoning above, it follows that the right hand side of the last identity is zero, so the second condition is verified.

Finally, the fact that the Hessian $\nabla_u^2 \alpha_j^{+\varepsilon}(u,v)$ is a nonnegative matrix follows from the formula

$$\nabla_u^2 \alpha_j^{+\varepsilon}(u,v)\{\xi,\xi\} = \int_{w_j = w_j(v)} (l_j(U(w)) \cdot \xi)^2 \zeta_\varepsilon(w - W(v + c\varepsilon l_j(v)))\, d\mathcal{H}^{m-1}(w),$$

for any $\xi \in \mathbf{R}^m$. Hence, $\mathcal{F}_+^{j\varepsilon}$ are boundary entropy pairs, for any $\varepsilon > 0$, $j = 1, \cdots, m$. It is now evident that $\mathcal{F}_+^{j\varepsilon}$ converges uniformly over compact sets to $\mathcal{F}_+^j$, as $\varepsilon \to 0+$, for $j = 1, \cdots, m$. So we have proved that $\mathcal{F}_+^j$ are generalized boundary entropy pairs.

Analogously, define

$$\begin{aligned}
\alpha_j^{-\varepsilon}(u,v) &= \int \left(l_j(U(w)) \cdot (u - U(w))\right)_- \zeta_\varepsilon(w - W(v - c\varepsilon l_j(v)))\, dw, \\
\beta_j^{-\varepsilon}(u,v) &= \int H\left(l_j(U(w)) \cdot (U(w) - u)\right) \\
&\qquad \times l_j(U(w)) \cdot (f(u) - f(U(w)))\, \zeta_\varepsilon(w - W(v - c\varepsilon l_j(v)))\, dw,
\end{aligned}$$

and $\mathcal{F}_-^{j\varepsilon} = (\alpha_j^{-\varepsilon}, \beta_j^{-\varepsilon})$. We can show that $\mathcal{F}_-^{j\varepsilon}$ are boundary entropy pairs and, consequently, that $\mathcal{F}_-^j$ are generalized boundary entropy pairs. Finally, defining $\mathcal{F}^{j\varepsilon} = \mathcal{F}_-^{j\varepsilon} + \mathcal{F}_+^{j\varepsilon}$, we conclude that $\mathcal{F}^j$ are also generalized boundary entropy pairs, $j = 1, \ldots, m$. $\square$

The existence of solutions of the initial-boundary problem (4.15)-(4.17), satisfying (4.7)-(4.9), for such a system can be established as in [5], for $u_\flat$, $u_\sharp \in L^\infty(\mathbf{R}_+)$, $u_0 \in L^\infty(0,1)$, in non-cylindric domains with Lipschitz boundaries. We are now ready to establish our application of Proposition 4.2.

Theorem 4.3. *Assume that $u \in L^\infty(Q_\infty; \mathbf{R}^2)$, and $u_\flat$, $u_\sharp \in L^\infty(\Gamma^\pm; \mathbf{R}^2)$ satisfy (4.17) in the sense of (4.9). Let u, $u_\flat$, $u_\sharp$ assume their values in a domain $\mathbf{O}$ of the form*

$$\mathbf{O} = \{\, u \in \mathbf{R}^m \,:\, |w_j(u) - w_j(\bar u)| < M_j,\ j = 1, \cdots, m \,\},$$

where $\bar u$ is a given constant state and M_j, $j = 1, \cdots, m$, are given positive constants, in which

$$\lambda_1(u) \le \cdots \le \lambda_k(u) \le \kappa_- < 0 < \kappa_+ \le \lambda_{k+1}(u) \le \cdots \le \lambda_m(u)$$

for certain positive constants $\kappa_\pm$. Then there exists a positive constant δ such that, if $\|y_t(\,\cdot\,,0)\|_\infty,\ \|y_t(\,\cdot\,,1)\|_\infty < \delta$,

$$\operatorname*{ess\,lim}_{s\to 0+}\int_{\Gamma^-}|l_l(u_\flat\circ\pi_1(r))\cdot\big(u\circ\Psi_s(r)-u_\flat\circ\pi_1(r)\big)|\,d\mathcal{H}^1(r)=0,\qquad \textit{for } l=k+1,\cdots,m, \tag{4.23}$$

$$\operatorname*{ess\,lim}_{s\to 0+}\int_{\Gamma^+}|l_j(u_\sharp\circ\pi_1(r))\cdot\big(u\circ\Psi_s(r)-u_\sharp\circ\pi_1(r)\big)|\,d\mathcal{H}^1(r)=0,\qquad \textit{for } j=1,\cdots,k. \tag{4.24}$$

Proof. Again our task is to find suitable (generalized) boundary entropy pairs $\mathcal{F}^j$, corresponding to which Γ^-, Γ^+ are ${\Gamma_{act}}^{k+1,\cdots,m}$, ${\Gamma_{act}}^{1,\cdots,k}$, respectively. We use a lemma in Heibig [17] which asserts that, if (4.15) is a Temple system, then there exists a smooth matrix-valued function $M:\mathbf{R}^m\times\mathbf{R}^m\to M_{m\times m}(\mathbf{R})$, with the following properties:

1. for $u,v\in\mathbf{R}^m$, $f(u)-f(v)=M(u,v)(u-v)$ and $M(u,u)=\nabla f(u)$;
2. $M(u,v)$ and $\nabla f(v)$ have the same (left and right) eigenvectors.

Here $M_{m\times m}(\mathbf{R})$ denotes the space of $m\times m$ real matrices. Furthermore, a lemma in [4] states that the eigenvalues of $M(u,v)$, $\bar\lambda_j(u,v)$, satisfy

$$\min_{z\in\mathbf{O}}\lambda_j(z)\le\bar\lambda_j(u,v)\le\max_{z\in\mathbf{O}}\lambda_j(z),\qquad\text{for all } u,v\in\mathbf{O},\quad j=1,\cdots,m.$$

In particular, one has $\bar\lambda_j(u,v)\le\kappa_-$, for $j=1,\cdots,k$, and $\bar\lambda_l(u,v)\ge\kappa_+$, for $l=k+1,\cdots,m$, for all $u,v\in\mathbf{O}$. Now, let $\mathcal{F}^j=(\alpha_j,\beta_j)$ be the generalized boundary entropy pairs given in (4.20). Then

$$\beta_j(u,v)=\bar\lambda_j(u,v)|l_j(v)\cdot(u-v)|\le\kappa_-|l_j(v)\cdot(u-v)|,\quad j=1,\cdots,k,$$
$$\beta_l(u,v)=\bar\lambda_l(u,v)|l_l(v)\cdot(u-v)|\ge\kappa_+|l_l(v)\cdot(u-v)|,\quad l=k+1,\cdots,m.$$

Now it is easy to deduce that, if $\|y_t(\,\cdot\,,0)\|_\infty,\ \|y_t(\,\cdot\,,1)\|_\infty<\delta$, for $\delta<\kappa_0=\min\{|\kappa_-|,\kappa_+\}$, then

$$\mathcal{F}^l(u,v)\cdot\nu(r)\le -c|l_l(v)\cdot(u-v)|,\quad\text{for } r\in\Gamma^-,\ l=k+1,\cdots,m,$$
$$\mathcal{F}^j(u,v)\cdot\nu(r)\le -c|l_j(v)\cdot(u-v)|,\quad\text{for } r\in\Gamma^+,\ j=1,\cdots,k,$$

where c may be taken as $(\kappa_0-\delta)/\sqrt{1+\delta^2}$. Hence, for the changes of coordinates $w_j(u,v)=l_j(v)\cdot u$, $j=1,\cdots,m$, the boundary entropy pairs $\mathcal{F}^j(u,v)$ given in (4.20), and $\rho_j(r)(\mu,\lambda)=c|\mu-\lambda|$, $j=1,\cdots,m$, Proposition 4.2 is applicable. $\square$

4.4. Remarks on 2×2 Systems of Conservation Laws

One cannot expect to extend Theorem 4.3 to general systems endowed with globally defined Riemann invariant coordinates. The following simple example serves to illustrate this fact.

Let (4.15) be any 2×2 strictly hyperbolic system of conservation laws. We consider the initial-boundary value problem in the domain $x>0$, $t>0$, with boundary data given by a constant state u_L and initial data by a constant state u_R. Assume that u_L can be connected to u_R by an admissible 1-shock with speed $\sigma<0$. Then it is desirable that $u(t,x)\equiv u_R$, $t,x>0$, is a solution of this problem satisfying (4.7)-(4.9). This fact can be shown for quite general cases. Then, except in the case of a system for which the 1-shock and 1-rarefaction curves coincide, one would be able to choose u_L and u_R such that $w_2(u_R) \neq w_2(u_L)$, in contradiction with (4.23).

Nevertheless, it has been shown by Lax [20], in the case where the first characteristic field is genuinely nonlinear, that $|w_2(u_L) - w_2(u_R)| = O(|u_L - u_R|^3)$, if $|u_L - u_R|$ is small. This prompts us with the question whether, for general boundary and initial data, condition (4.9) implies a certain smallness of $|w_2(u) - w_2(u_\flat)|$, near the boundary $x = 0$, comparing with the oscillation of the solution. This question can be generalized to more general domains.

Consider the problem (4.15)–(4.17) satisfying

$$\lambda_1(u) < -k_0 < 0 < k_0 < \lambda_2(u) \tag{4.25}$$

for u in some domain $U \subset \mathbf{R}^2$ and certain positive constant k_0.

A typical example is the 2×2 system of nonlinear elasticity:

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = 0, \\ \partial_t u_2 - \partial_x \sigma(u_1) = 0, \end{cases} \tag{4.26}$$

where $\sigma \in C^2(\mathbf{R})$ satisfies $\sigma'(\kappa) > 0$, for all $\kappa \in \mathbf{R}$, and $\kappa\sigma''(\kappa) > 0$, for $\kappa \neq 0$.

For (4.26), one has $f(u) = (-u_2, -\sigma(u_1))^\top$. For each $u \in \mathbf{R}^2$, the Jacobian matrix of f, $\nabla f(u)$, has two eigenvalues

$$\lambda_1(u) = -\sqrt{\sigma'(u_1)}, \qquad \lambda_2(u) = \sqrt{\sigma'(u_1)}.$$

The functions

$$w_1(u) = u_2 + \int^{u_1} \sqrt{\sigma'(\kappa)}\, d\kappa, \quad w_2(u) = u_2 - \int^{u_1} \sqrt{\sigma'(\kappa)}\, d\kappa$$

are two independent smooth Riemann invariants satisfying

$$\nabla w_i(u) \nabla f(u) = \lambda_i(u) \nabla w_i(u), \qquad i = 1, 2.$$

The function

$$\eta(u) = u_2^2 + \int^{u_1} \sigma(\kappa)\, d\kappa \tag{4.27}$$

is a strictly convex entropy for (4.26) with associated entropy flux

$$q(u) = u_2 \sigma(u_1). \tag{4.28}$$

The existence of a solution of (4.26) and (4.16)-(4.17) satisfying (4.7)-(4.9) can be established as in [5] for $u_\flat$, $u_\sharp \in L^\infty(\mathbf{R}_+)$, $u_0 \in L^\infty(0,1)$, even in non-cylindric domains with Lipschitz boundaries. Regarding the question mentioned above, we have the following result.

Proposition 4.3. *Assume* (4.15) *is a* 2×2 *strictly hyperbolic system satisfying* (4.25). *Suppose that* $u \in L^\infty(Q_\infty; \mathbf{R}^2)$ *and* $u_\flat$, $u_\sharp \in L^\infty(\Gamma^\pm; \mathbf{R}^2)$ *satisfy* (4.9), *and* $B \subset (0,\infty)$ *is a bounded Borelian set such that* $\|y_0'(t)\|_\infty \leq k_* < k_0$, *for a.e.* $t \in B$. *Given any constant state* $\bar{u} \in U$, *there exists a positive constant* δ_0 *such that, for any* $\delta < \delta_0$, *if*

$$\|u - \bar{u}\|_\infty, \ \|u_\flat - \bar{u}\|_\infty, \ \|u_\sharp - \bar{u}\|_\infty < \delta,$$

then

$$\operatorname*{ess\,lim\,sup}_{s \to 0+} \int_{\mathcal{L}(B \times \{0\})} |w_2(u \circ \Psi_s(r)) - w_2(u_\flat \circ \pi_1(r))|^2 \, d\mathcal{H}^1(r) < C\delta^3, \tag{4.29}$$

where $C > 0$ *depends only on* f, k_*, *and* B.

Proof. We first construct a boundary entropy pair, $\mathcal{F}^1$, associated with Γ^-. We recall from [11] that, for 2×2 strictly hyperbolic systems, in a sufficiently small neighborhood of a constant state $\bar{u}$, there exist the entropy pairs $\eta_j(u, w_k(v))$, $q_j(u, w_k(v))$, $j,k=1,2$, $j\neq k$, with the following properties:

$$c_1(w_k(u)-w_k(v))^2 \le \eta_j(u,w_k(v)) \le c_2(w_k(u)-w_k(v))^2, \quad k\neq j; \tag{4.30}$$

$$\nabla_u^2\eta_j(\bar{u},w_k(\bar{u}))\ge 0, \quad j=1,2, \quad k\neq j; \tag{4.31}$$

$$q_1(u,w_2(v))\ge c_3\eta_1(u,w_2(v)), \quad q_2(u,w_1(v))\le -c_4\eta_2(u,w_1(v)), \tag{4.32}$$

for certain positive constants c_l, $l=1,\cdots,4$. Although the pairs

$$\mathcal{G}_j(u,v)=(\tilde{\alpha}_j(u,v),\tilde{\beta}_j(u,v))\equiv(\eta_j(u,w_k(v)),q_j(u,w_k(v))), \quad j,k=1,2, \quad j\neq k, \tag{4.33}$$

fail to be boundary entropy pairs since $\tilde{\alpha}_j$, $j=1,2$, are not convex functions of u, property (4.30) indicates that the other conditions for a boundary entropy pair are satisfied by $\mathcal{G}_j$, $j=1,2$. Let $(\alpha(u,v),\beta(u,v))$ be obtained from $(\eta(u),q(u))$, by (4.11)-(4.12). Property (4.31) ensures that there exists $\delta_0>0$ such that, when $\delta<\delta_0$, the entropies

$$\alpha_j(u,v)=\tilde{\alpha}_j(u,v)+c\delta\alpha(u,v), \quad j=1,2,$$

are convex functions of u, for some positive constant c depending only on f, so that $\mathcal{F}^j(u,v)=(\alpha_j(u,v),\beta_j(u,v))$, $j=1,2$, are boundary entropy pairs, where

$$\beta_j(u,v)=\tilde{\beta}_j(u,v)+c\delta\beta(u,v), \quad j=1,2.$$

Now, by the assumption on y_0', one easily sees that

$$\mathcal{F}^1(u,v)\cdot\nu(r)\le -c_5|w_2(u)-w_2(v)|^2+c_6\delta^3, \quad \text{for } r\in\mathcal{L}(B\times\{0\}),$$

where c_5 and c_6 are positive constants depending only on f and k_*. Hence, using (4.9) and the regularity of the deformation to assure that $\nu(\Psi_s(r))\to\nu(r)$, in $L^1_{\rm loc}(\Gamma^\pm)$ as $s\to 0+$, one obtains (4.29). □

Remark 4.2. An entirely similar statement holds at the right-hand boundary curve concerning the first Riemann invariant.

Acknowledgments

Gui-Qiang Chen's research was supported in part by the National Science Foundation grants DMS-9623203 and DMS-9708261, and by an Alfred P. Sloan Foundation Fellowship. Hermano Frid's research was supported in part by CNPq-Brazil, proc. 352871/96-2.

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