ON THE THEORY OF DIVERGENCE-MEASURE FIELDS AND ITS APPLICATIONS

GUI-QIANG CHEN HERMANO FRID

Dedicated to Constantine M. Dafermos on the Occasion of His 60th Birthday

Abstract. Divergence-measure fields are extended vector fields, including vector fields in L^p and vector-valued Radon measures, whose divergences are Radon measures. Such fields arise naturally in the study of entropy solutions of nonlinear conservation laws and other areas. In this paper, a theory of divergence-measure fields is presented and analyzed, in which normal traces, a generalized Gauss-Green theorem, and product rules, among others, are established. Some applications of this theory to several nonlinear problems in conservation laws and related areas are discussed. In particular, with the aid of this theory, we prove the stability of Riemann solutions, which may contain rarefaction waves, contact discontinuities, and/or vacuum states, in the class of entropy solutions of the Euler equations for gas dynamics.

1. Introduction

In this survey paper we present and analyze a theory of divergence-measure fields established in Chen-Frid [7, 9] and discuss some of its applications. Divergence-measure fields are extended vector fields, including vector fields in L^p and vector-valued Radon measures, whose divergences are Radon measures. More precisely, we have

Definition 1.1. Let $\Omega \subset \mathbb{R}^N$ be open. For $F \in L^p(\Omega;\mathbb{R}^N)$, $1 \le p \le \infty$, or $F \in \mathcal{M}(\Omega;\mathbb{R}^N)$, set

$$|\mathrm{div}\, F|(\Omega) := \sup\{\, \langle F,\, \nabla\varphi\rangle \;:\; \varphi \in C_0^1(\Omega),\ |\varphi(x)| \le 1,\ x \in \Omega\,\}.$$

For $1 \le p \le \infty$, we say that F is an L^p divergence-measure field over Ω, i.e., $F \in \mathcal{DM}^p(\Omega)$, if

$$\|F\|_{\mathcal{DM}^p(\Omega)} := \|F\|_{L^p(\Omega;\mathbb{R}^N)} + |\mathrm{div}\, F|(\Omega) < \infty. \tag{1.1}$$

We say that F is an extended divergence-measure field over Ω, i.e., $F \in \mathcal{DM}^{ext}(\Omega)$, if

$$\|F\|_{\mathcal{DM}^{ext}(\Omega)} := |F|(D) + |\mathrm{div}\, F|(D) < \infty. \tag{1.2}$$

Date: February 9, 2002.

1991 *Mathematics Subject Classification.* Primary: 00-02, 26B20, 28C05, 35L65, 35B10, 35B35; Secondary: 26B35, 26B12, 35L67.

Key words and phrases. divergence-measure fields, normal traces, Gauss-Green theorem, product rules, Radon measures, conservation laws, Euler equations, gas dynamics, entropy solutions, entropy inequality, stability, uniqueness, vacuum, Cauchy problem, initial layers, boundary layers, initial-boundary value problems.

If $F \in \mathcal{DM}^p(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbb{R}^N$, then we say $F \in \mathcal{DM}^p_{loc}(D)$; and, if $F \in \mathcal{DM}^{ext}(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbb{R}^N$, we say $F \in \mathcal{DM}^{ext}_{loc}(D)$. We denote $F \in \mathcal{DM}(\Omega)$ either $F \in \mathcal{DM}^p(\Omega)$ or $F \in \mathcal{DM}^{ext}(\Omega)$. Here, for open sets $A, B \subset \mathbb{R}^N$, the relation $A \Subset B$ means that the closure of A, $\bar{A}$, is a compact subset of B.

These spaces under the norms (1.1) and (1.2) are Banach spaces, respectively. Such fields arise naturally in the study of entropy solutions of nonlinear conservation laws and other related areas (see §4.1).

These spaces are larger than the space of vector fields of bounded variation. The establishment of the Gauss-Green theorem, traces, and other properties of BV functions in the middle of last century (see Federer [18]) has advanced significantly our understanding of solutions of nonlinear partial differential equations and nonlinear problems in calculus of variations, differential geometry, and other areas. A natural question is whether the $\mathcal{DM}$-fields have similar properties, especially the traces and the Gauss-Green formula as for the BV functions. At a first glance, it seems unclear.

First, observe that one cannot define the traces for each component of a $\mathcal{DM}$ field over any Lipschitz boundary in general, as opposed to the case of BV fields. This fact can be easily seen through the following example.

Example 1.1: The field $F(x,y) = (\sin(\frac{1}{x-y}), \sin(\frac{1}{x-y}))$ belongs to $\mathcal{DM}^\infty(\mathbb{R}^2)$. It is impossible to define any reasonable notion of traces over the line $x = y$ for the component $\sin(\frac{1}{x-y})$.

The following example indicates that the classical Gauss-Green theorem may fail.

Example 1.2: The field $F(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ belongs to $\mathcal{DM}^1_{loc}(\mathbb{R}^2)$. As remarked in Whitney [48], for $\Omega = (0,1) \times (0,1)$,

$$\int_\Omega \operatorname{div} F dxdy = 0 \neq \int_{\partial\Omega} F \cdot \nu \, d\mathcal{H}^1 = \frac{\pi}{2},$$

if one understands $F \cdot \nu|_{\partial\Omega}$ in the classical sense, which implies that the classical Gauss-Green theorem fails.

Example 1.3: For any bounded open interval $I \subset \mathbb{R}$,

$$F(x,y) = (dx \times \mu(y), 0) \in \mathcal{DM}^{ext}(I \times \mathbb{R}).$$

A non-trivial example of such fields is provided by the Riemann solutions of the Euler equations (4.4)–(4.6) for gas dynamics, which contain vacuum states. See Section 5.

Some efforts have been made in generalizing the Gauss-Green theorem. Some results for several situations can be found in Anzellotti [1] for an abstract formulation for $F \in L^\infty$, Rodrigues [39] for $F \in L^2$, and Ziemer [50] for a related problem for $\operatorname{div} F \in L^1$ (see also Baiocchi-Capelo [2], Brezzi-Fortin [5], and Ziemer [51]). Also see Harrison [24], Harrison-Norton [23], Jurkat-Nonnenmacher [25], Nonnenmacher [36], Pfeffer [38], and Shapiro [43] for related problems and references.

In this paper, we first present and analyze a theory of divergence-measure fields established in Chen-Frid [7, 9]. Motivated by various nonlinear problems from conservation laws, a natural notion of normal traces is developed by the neighborhood information via Lipschitz deformation under which a generalized Gauss-Green theorem is shown to hold for $F \in \mathcal{DM}(\Omega)$. An explicit way is also developed to calculate the normal traces over any deformable Lipschitz surface, suitable for applications, by using the neighborhood information of the fields near the surface and the level set function of the Lipschitz deformation surfaces. Some product rules for these extended fields are also shown.

In Section 2, we show how the normal traces are developed under which a generalized Gauss-Green theorem can be established for divergence-measure fields and present several remarks and applications about the generalized Gauss-Green theorem. Their proofs require some refined properties of Radon measures and the Whitney extension theory, and the notion of the domains with Lipschitz deformable boundaries and related properties, among others.

In Section 3, we analyze some further properties of divergence-measure fields, including several product rules.

Then we discuss some applications of this theory to nonlinear hyperbolic conservation laws and degenerate parabolic equations. We first discuss a connection between divergence-measure fields and entropy solutions of hyperbolic conservation laws in Section 4. Then we show an application of this theory to the vacuum problem for the Euler equations for compressible fluids in Section 5. The initial and boundary layer problems for hyperbolic conservation laws are reviewed in Section 6. Initial-boundary value problems for hyperbolic conservation laws and nonlinear degenerate parabolic-hyperbolic equations are discussed in Sections 7 and 8, respectively.

2. Normal Traces and the Generalized Gauss-Green Theorem

We now discuss the generalized Gauss-Green theorem for $\mathcal{DM}$-fields over $\Omega \subset \mathbb{R}^N$ by introducing a suitable definition of normal traces over the boundary $\partial\Omega$ of a bounded open set with Lipschitz deformable boundary, established in Chen-Frid [7, 9].

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset. We say that $\partial\Omega$ is a deformable Lipschitz boundary, provided that

(i) $\forall\, x \in \partial\Omega$, $\exists\, r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap Q(x,r) = \{y \in \mathbb{R}^N \;:\; \gamma(y_1, \cdots, y_{N-1}) < y_N\} \cap Q(x,r),$$

where $Q(x,r) = \{y \in \mathbb{R}^N \;:\; |x_i - y_i| \le r,\ i = 1, \cdots, N\}$;

(ii) $\exists\, \Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(\omega, 0) = \omega$ for all $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Denote $\partial\Omega_s \equiv \Psi(\partial\Omega \times \{s\})$, $s \in [0,1]$, and denote Ω_s the open subset of Ω whose boundary is $\partial\Omega_s$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

Definition 2.2. We say that the Lipschitz deformation is *regular* if

$$\lim_{s \to 0+} D\Psi_s \circ \tilde{\gamma} = D\tilde{\gamma}, \qquad \text{in } L^1_{\mathrm{loc}}(B), \tag{2.1}$$

where $\tilde{\gamma}$ is a map as in Condition (i) of Definition 2.1, and Ψ_s denotes the map of $\partial\Omega$ into Ω, given by $\Psi_s(x) = \Psi(x,s)$. Here B denotes the greatest open set such that $\tilde{\gamma}(B) \subset \partial\Omega$.

Remark 2.1. It should be recognized that bounded domains with smooth boundaries (say, C^2) have always regular deformable Lipschitz boundaries. Indeed, since there is an everywhere defined unit outer normal field $\nu(r)$, one can define the deformation $\Psi(y,s) = y - \varepsilon s\nu(y)$, which satisfies all the required conditions for sufficiently small $\varepsilon > 0$.

Remark 2.2. Conditions (i)–(ii) of Definition 2.1 are also verified for both the star-shaped domains and the domains whose boundaries satisfy the cone property. For the former, there exists a point $y_0 \in \Omega$ such that, for any $y \in \partial\Omega$, one has $y + \theta(y_0 - y) \in \Omega$ for $\theta \in (0,1)$ and can then define $\Psi(y,s) = y + \frac{s}{2}(y_0 - y)$. For the latter, there exists a vector $v_0 \in \mathbb{R}^N$ such that, for any $y \in \partial\Omega$ and any $0 < s \le 1$, one has $y + sv_0 \in \Omega$ and then takes $\Psi(y,s) = y + sv_0$. In both cases, the deformation is regular.

Remark 2.3. It is also clear that, if Ω is the image through a bi-Lipschitz map of a domain $\tilde{\Omega}$ with a (regular) Lipschitz deformable boundary, then Ω itself possesses a (regular) Lipschitz deformable boundary.

We first discuss the Gauss-Green formula for fields in $\mathcal{DM}^p$ with $1 < p \le \infty$. It is more delicate for fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$, which will be addressed subsequently.

Theorem 2.1. *Let $F \in \mathcal{DM}^p(\Omega)$, $1 < p \le \infty$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional $F \cdot \nu|_{\partial\Omega}$ over $Lip\,(\partial\Omega)$ such that, for any $\phi \in Lip\,(\mathbb{R}^N)$,*

$$\langle F \cdot \nu|_{\partial\Omega}, \phi\rangle = \langle \operatorname{div} F,\, \phi\rangle + \int_\Omega \nabla\phi \cdot F\, dx. \tag{2.2}$$

Moreover, let $\nu : \Psi(\partial\Omega \times [0,1]) \to \mathbb{R}^N$ be such that $\nu(x)$ is the unit outer normal to $\partial\Omega_s$ at $x \in \partial\Omega_s$, defined for a.e. $x \in \Psi(\partial\Omega \times [0,1])$. Let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function of $\partial\Omega_s$, that is,

$$h(x) := \begin{cases} 0, & \text{for } x \in \mathbb{R}^N - \overline{\Omega}, \\ 1, & \text{for } x \in \Omega - \Psi(\partial\Omega \times [0,1]), \\ s, & \text{for } x \in \partial\Omega_s, 0 \le s \le 1. \end{cases}$$

Then, for any $\psi \in Lip\,(\partial\Omega)$,

$$\langle F \cdot \nu|_{\partial\Omega}, \psi\rangle = -\lim_{s\to 0} \frac{1}{s} \int_{\Psi(\partial\Omega \times (0,s))} \mathcal{E}(\psi)\, \nabla h \cdot F\, dx, \tag{2.3}$$

where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to all $\mathbb{R}^N$.

In the case $p = \infty$, the normal trace $F \cdot \nu|_{\partial\Omega}$ is a function in $L^\infty(\partial\Omega)$ satisfying $\|F \cdot \nu\|_{L^\infty(\partial\Omega)} \le C\|F\|_{L^\infty(\Omega)}$, for some constant C independent of F; if $\partial\Omega$ admits a regular Lipschitz deformation, then $C = 1$. Furthermore, for any field $F \in \mathcal{DM}^\infty(\Omega)$,

$$\langle F \cdot \nu|_{\partial\Omega}, \psi\rangle = ess\lim_{s\to 0} \int_{\partial\Omega_s} \psi \circ \Psi_s^{-1}\, F \cdot \nu\, d\mathcal{H}^{N-1}, \qquad \text{for any } \psi \in L^1(\Omega). \tag{2.4}$$

Finally, for $F \in \mathcal{DM}^p(\Omega)$ with $1 < p < \infty$, $F \cdot \nu|_{\partial\Omega}$ can be extended to a continuous linear functional over $W^{1-1/p,p}(\partial\Omega) \cap L^\infty(\partial\Omega)$.

Proof. Let F^ε be defined by (3.5) in Section 3 below. Since we have $\mathcal{H}^{N-1}(\partial\Omega_s) < +\infty$, Federer's extension of the Gauss-Green formula (see [18]) holds for ϕF^ε over $\partial\Omega_s$, for any $\phi \in \mathrm{Lip}\,(\mathbb{R}^N)$, and hence we have

$$\int_{\partial\Omega_s} \phi F^\varepsilon \cdot \nu \, d\mathcal{H}^{N-1} = \int_{\Omega_s} \phi \operatorname{div} F^\varepsilon dx + \int_{\Omega_s} \nabla\phi \cdot F^\varepsilon \, dx. \tag{2.5}$$

Now we integrate (2.5) in $s \in (0,\delta)$, $0 < \delta < 1$, and use the coarea formula (see, e.g. [18, 17]) in the left-hand side to obtain

$$-\int_{\Psi(\Omega\times(0,\delta))} \phi F^\varepsilon \cdot \nabla h \, dx = \int_0^\delta \left\{ \int_{\Omega_s} \phi \operatorname{div} F^\varepsilon \, dx \right\} ds + \int_0^\delta \left\{ \int_{\Omega_s} \nabla\phi \cdot F^\varepsilon \, dx \right\} ds. \tag{2.6}$$

Let $\varepsilon \to 0$. Observing that, by Proposition 3.3 in Section 3 below, the integrand of the first integral converges for a.e. $s \in (0,\delta)$ to the corresponding integral for F, we obtain

$$-\int_{\Psi(\Omega\times(0,\delta))} \phi F \cdot \nabla h \, dx = \int_0^\delta \left\{ \int_{\Omega_s} \phi \operatorname{div} F \right\} ds + \int_0^\delta \left\{ \int_{\Omega_s} \nabla\phi \cdot F \, dx \right\} ds. \tag{2.7}$$

We then divide (2.7) by δ, let $\delta \to 0$, and observe that both terms in the right-hand side converge to the corresponding integrals inside the brackets over Ω, by the dominated convergence theorem. Hence, the left-hand side also converges, which yields

$$-\lim_{\delta\to 0} \frac{1}{\delta} \int_{\Psi(\Omega\times(0,\delta))} \phi \, F \cdot \nabla h \, dx = \int_\Omega \phi \operatorname{div} F + \int_\Omega \nabla\phi \cdot F \, dx. \tag{2.8}$$

Now, for $\psi \in \mathrm{Lip}\,(\partial\Omega)$, let $\mathcal{E}(\psi) \in \mathrm{Lip}\,(\mathbb{R}^N)$ be a Lipschitz extension of ψ preserving the norm $\|\cdot\|_{\mathrm{Lip}} := \|\cdot\|_\infty + \mathrm{Lip}\,(\cdot)$ (see, e.g., [17, 18]). We then define

$$\langle F\cdot\nu|_{\partial\Omega}, \psi\rangle = -\lim_{s\to 0} \frac{1}{s} \int_{\Psi(\partial\Omega\times(0,s))} \mathcal{E}(\psi) \, \nabla h \cdot F \, dx. \tag{2.9}$$

Because the right-hand side of (2.8) does not depend on the particular deformation Ψ for $\partial\Omega$, we see that the normal trace defined by (2.9) is also independent of the deformation. We still have to prove that the normal trace as defined by (2.9) also does not depend on the specific Lipschitz extension $\mathcal{E}(\psi)$ of ψ. This will be accomplished if we prove that the right-hand side of (2.8) vanishes if $\phi|_{\partial\Omega} \equiv 0$. Denote it by $[F,\phi]_{\partial\Omega}$, that is,

$$[F,\phi]_{\partial\Omega} := \langle \operatorname{div} F, \phi\rangle_\Omega + \langle F, \nabla\mathcal{E}(\phi)\rangle_\Omega.$$

We claim that $[F,\phi]_{\partial\Omega} = 0$ if $\phi|_{\partial\Omega} \equiv 0$. In fact, we may approximate such a ϕ by a sequence $\phi^j \in C_0^\infty(\Omega)$, with $\|\phi^j\|_\infty \le \|\phi\|_\infty$, such that $\phi^j \to \phi$ locally uniformly in Ω and $\nabla\phi^j \to \nabla\phi$ in $L^q(\Omega)^N$, with $\frac{1}{p} + \frac{1}{q} = 1$. Hence, $[F,\phi]_{\partial\Omega} = \lim_{j\to\infty}[F,\phi^j] = 0$, as asserted. In particular, for $1 < p \le \infty$, the values of the normal trace, $\langle F\cdot\nu|_{\partial\Omega}, \phi|_{\partial\Omega}\rangle$, depend only on the values of ϕ over $\partial\Omega$.

In the case $p = \infty$, we can go further. Indeed, given $\psi \in \mathrm{Lip}\,(\partial\Omega)$, we can take a particular extension of ψ, $\mathcal{E}(\psi) \in \mathrm{Lip}\,(\mathbb{R}^N)$, satisfying $\mathcal{E}(\psi)(\Psi(\omega,s)) = \psi(\omega)$, for $(\omega,s) \in \partial\Omega\times[0,1]$. In this case, using the area formula (see, e.g., [18, 17]), we easily obtain, from (2.9),

$$\langle F\cdot\nu|_{\partial\Omega}, \psi\rangle \le C\|F\|_{L^\infty(\omega)} \int_{\partial\Omega} |\psi| \, d\mathcal{H}^{N-1},$$

where $C > 0$ depends only on the deformation Ψ of $\partial\Omega$, and we can take $C = 1$ if Ψ is a regular deformation. Hence, we conclude that $F \cdot \nu|_{\partial\Omega} \in L^\infty(\partial\Omega)$ and $\|F \cdot \nu|_{\partial\Omega}\|_{L^\infty(\partial\Omega)} \le C\|F\|_{L^\infty(\Omega)}$. The relation (2.4) is obtained in this special case by first taking the limit as $\varepsilon \to 0$ in (2.7), observing that the limit of the left-hand side exists for a.e. $s \in (0,1)$, by the dominated convergence, and that the limits of both integrals on the right-hand side exist by the dominated convergence and Proposition 3.3 in Section 3 below. We then consider an extension of ψ as just mentioned, and let $s \to 0$ with our observation that the right-hand side converges to the right-hand side of (2.8), again by the dominated convergence.

As for the last assertion, we recall a well-known result of Gagliardo [20] which establishes, in particular, tha, if $\partial\Omega$ is Lipschitz (that is, satisfies (i) of Definition 2.1) and $\psi \in W^{1-1/p,p}(\partial\Omega)$, then it can be extended into Ω to a function $\mathcal{E}(\psi) \in W^{1,p}(\Omega)$, and

$$\|\mathcal{E}(\psi)\|_{W^{1,p}(\Omega)} \le \tilde{C}\|\psi\|_{W^{1-1/p,p}}(\partial\Omega),$$

for some positive constant $\tilde{C}$ independent of ψ. By means of cuts, we can define $\mathcal{E}(\psi)$ so that it also satisfies $\|\mathcal{E}(\psi)\|_{L^\infty(\Omega)} \le \|\psi\|_{W^{1-1/p,p}(\partial\Omega)}$, besides (2.10). Hence, using these facts and (2.8), we easily deduce the last assertion.

□

Remark 2.4. As an example, consider the normal trace of $F(x,y)$ in Example 1.1 over the line $x = y$ where there is no reasonable notion of traces for the component $\sin(\frac{1}{x-y})$. Nevertheless, the unit normal ν_s to the line $x - y = s$ is the vector $(-1/\sqrt{2}, 1/\sqrt{2})$ so that the scalar product $F(x, x-s) \cdot \nu_s$ is identically zero over this line. Hence, we find that $F \cdot \nu \equiv 0$ over the line $x = y$ and the Gauss-Green formula implies in this case that, for any $\phi \in C_0^1(\mathbb{R}^2)$,

$$0 = \langle \operatorname{div} F|_{x>y}, \phi\rangle = -\int_{x>y} F \cdot \nabla\phi \, dxdy.$$

This identity could be also directly obtained by applying the dominated convergence theorem to the analogous identity obtained from the classical Gauss-Green formula for the domain $\{\, (x,y) \,|\, x > y + s \,\}$ when $s \to 0$.

As anticipated by Whitney's example (see Example 2.1 above), it is more delicate for fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$. Then we have to define the normal traces as functionals over the spaces $\operatorname{Lip}(\gamma, \partial\Omega)$ with $\gamma > 1$ (see the definition below). For $1 < \gamma \le 2$, the elements of $\operatorname{Lip}(\gamma, \partial\Omega)$ are $(N+1)$-components vectors, where the first component is the function itself, and the other N components are its "first-order partial derivatives". In particular, as a functional over $\operatorname{Lip}(\gamma, \partial\Omega)$, the values of the normal trace of a field in $\mathcal{DM}^1$ or $\mathcal{DM}^{ext}$ on $\partial\Omega$ will depend not only on the values of the respective functions over $\partial\Omega$, but also on the values of their first-order derivatives over $\partial\Omega$. To define the normal traces for $F \in \mathcal{DM}^1$ or $\mathcal{DM}^{ext}$, we resort to the properties of the Whitney extensions of functions in $\operatorname{Lip}(\gamma, \partial\Omega)$ to $\operatorname{Lip}(\gamma, \mathbb{R}^N)$; we recall the construction below. We first have the following analogue of Theorem 2.1 which covers fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$ (see [9] for the proof).

Theorem 2.2. *Let $F \in \mathcal{DM}^1(\Omega)$ or $F \in \mathcal{DM}^{ext}(\Omega)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional $F\cdot\nu|_{\partial\Omega}$ over Lip $(\gamma, \partial\Omega)$ for any $\gamma > 1$ such that, for any $\phi \in$ Lip $(\gamma, \mathbb{R}^N)$,*

$$\langle F \cdot \nu|_{\partial\Omega}, \phi\rangle = \langle \operatorname{div} F, \phi\rangle_\Omega + \langle F, \nabla\phi\rangle_\Omega. \tag{2.10}$$

Moreover, let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function as in Theorem 2.1. In the case that $F \in \mathcal{DM}^{ext}(\Omega)$, we also assume that $\partial_{x_i} h$ is $|F_i|$-measurable and its set of non-Lebesgue points has $|F_i|$-measure zero, $i = 1, \cdots, N$. Then, for any $\psi \in$ Lip $(\gamma, \partial\Omega)$, $\gamma > 1$,

$$\langle F \cdot \nu|_{\partial\Omega}, \psi\rangle = -\lim_{s\to 0} \frac{1}{s}\langle F, \mathcal{E}(\psi)\, \nabla h\rangle_{\Psi(\partial\Omega\times(0,s))}, \tag{2.11}$$

where $\mathcal{E}(\psi) \in$ Lip $(\gamma, \mathbb{R}^N)$ is the Whitney extension of ψ to all $\mathbb{R}^N$.

Remark 2.5. In general, for $F \in \mathcal{DM}(D)$, the normal traces $F \cdot \nu|_{\partial\Omega}$ may be no longer functions. This can be seen in Example 1.2 for $F \in \mathcal{DM}^1_{loc}(\mathbb{R}^2)$ with $\Omega = (0, 1) \times (0, 1)$, for which

$$F \cdot \nu|_{\partial\Omega} = \frac{\pi}{2}\delta_{(0,0)} - d\mathcal{H}^1|_{\partial\Omega},$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure on $\partial\Omega$.

We now recall the construction of the Whitney extension and some of its properties used in Theorem 2.2 and its proof.

Whitney Extension. Let k be a nonnegative integer and $\gamma \in (k, k + 1]$. We say that a function f, defined on C, belongs to Lip (γ, C) if there exist functions $f^{(j)}$, $0 \le |j| \le k$, defined on C, with $f^{(0)} = f$ such that, if

$$f^{(j)}(x) = \sum_{|j+l|\le k} \frac{f^{(j+l)}(y)}{l!}(x - y)^l + R_j(x, y),$$

then

$$\begin{cases} |f^{(j)}(x)| \le M, \\ |R_j(x, y)| \le M|x - y|^{\gamma-|j|}, \quad \text{for any } x, y \in C,\ |j| \le k. \end{cases} \tag{2.12}$$

Here j and l denote multi-indices $j = (j_1, \cdots, j_N)$ and $l = (l_1, \cdots, l_N)$ with $j! = j_1! \cdots j_N!$, $|j| = j_1 + j_2 + \cdots + j_N$, and $x^l = x_1^{l_1} x_2^{l_2} \cdots x_N^{l_N}$. An element of Lip (γ, C) means the collection $\{f^{(j)}(x)\}_{|j|\le k}$. The norm of an element in Lip (γ, C) is defined as the smallest M for which the inequalities in (2.12) holds. We notice that Lip (γ, C) with this norm is a Banach space. For the case $C = \mathbb{R}^N$, since the functions $f^{(j)}$ are determined by $f^{(0)}$, this collection is then identified with $f^{(0)}$.

The Whitney extension of order k is defined as follows. Let $\{f^{(j)}\}_{|j|\le k}$ be an element of Lip (γ, C). The linear mapping $\mathcal{E}_k$: Lip $(\gamma, C) \to$ Lip $(\gamma, \mathbb{R}^N)$ assigns to each collection a function $\mathcal{E}_k(f^{(j)})$ defined on $\mathbb{R}^N$ which is an extension of $f^{(0)} = f$ to $\mathbb{R}^N$. The definition of $\mathcal{E}_k$ is the following:

$$\begin{cases} \mathcal{E}_0(f)(x) = f(x), \qquad x \in C, \\ \mathcal{E}_0(f)(x) = \sum_i f(p_i)\varphi_i(x), \qquad x \in \mathbb{R}^N - C, \end{cases}$$

and, for $k \geq 1$,

$$\begin{cases} \mathcal{E}_k(f^{(j)})(x) = f^{(0)}(x), & x \in C, \\ \mathcal{E}_k(f^{(j)})(x) = \sum_i' P(x,p_i)\varphi_i(x), & x \in \mathbb{R}^N - C. \end{cases}$$

Here $P(x,y)$ denotes the polynomial in x, which is the Taylor expansion of f about the point $y \in C$:

$$P(x,y) = \sum_{|l|\leq k} \frac{f^{(l)}(y)(x-y)^l}{l!}, \qquad x \in \mathbb{R}^N, \ y \in C.$$

The functions $\{\varphi_i\}$ form a partition of unity of $\mathbb{R}^N - C$ with the following properties:

(i) $\mathrm{spt}\,(\varphi_i) \subset Q_i$ where Q_i is a cube with edges parallel to the coordinate axes and

$$c_1 \,\mathrm{diam}\,(Q_i) \leq \mathrm{dist}\,(Q_i, C) \leq c_2 \,\mathrm{diam}\,(Q_i),$$

for certain positive constants c_1 and c_2 independent of C;

(ii) each point of $\mathbb{R}^N - C$ is contained in at most N_0 cubes Q_i, for certain number N_0 depending only on the dimension N;

(iii) the derivatives of φ_i satisfy

$$\left|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N} \varphi_i(x)\right| \leq A_\alpha (\mathrm{diam}\, Q_i)^{-|\alpha|}. \tag{2.13}$$

Here $p_i \in C$ is such that $\mathrm{dist}\,(C, Q_i) = \mathrm{dist}\,(p_i, Q_i)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and the symbol $\sum'$ indicates that the summation is taken only over those cubes whose distances to C are not greater than one.

The following theorem is due to Whitney [49], whose proof can be also found in Stein [44].

Theorem 2.3. *Suppose that k is a non-negative integer, $\gamma \in (k, k+1]$, and C is a closed set. Then the mapping $\mathcal{E}_k$ is a continuous linear mapping from $Lip\,(\gamma, C)$ to $Lip\,(\gamma, \mathbb{R}^N)$ which defines an extension of $f^{(0)}$ to $\mathbb{R}^N$, and the norm of this mapping has a bound independent of C.*

The following theorem plays an important role in establishing the generalized Gauss-Green theorem for fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$; Its proof can be found in Chen-Frid [9].

Theorem 2.4. *Let C be a closed set in $\mathbb{R}^N$ and*

$$C_\delta := \{\, x \in \mathbb{R}^N \ : \ dist\,(x, C) \leq \delta \,\}, \qquad for\ \ \delta > 0.$$

Let $\mathcal{E}_k : Lip\,(\gamma, C) \to Lip\,(\gamma, \mathbb{R}^N)$ with $\gamma \in (k, k+1]$ be the Whitney extension of order k. Then, for any $\phi \in Lip\,(\gamma, \mathbb{R}^N)$ and any $\gamma' \in (k, \gamma)$,

$$\|\mathcal{E}_k(\phi|_C) - \phi\|_{Lip\,(\gamma', C_\delta)} \to 0, \quad as \ \ \delta \to 0. \tag{2.14}$$

3. Further Properties of Divergence-Measure Fields

In this section we first discuss some basic properties of divergence-measure fields in the spaces $\mathcal{DM}^p(\Omega), 1 \le p \le \infty$, and $\mathcal{DM}^{ext}(\Omega)$. Then we discuss some product rules for divergence-measure fields.

Proposition 3.1. (i) *Let $\{F_j\}$ be a sequence in $\mathcal{DM}^p(\Omega)$ such that*

$$F_j \rightharpoonup F \qquad L^p_{loc}(\Omega;\mathbb{R}^N), \quad \textit{for } 1 \le p < \infty, \tag{3.1}$$

$$F_j \overset{*}{\rightharpoonup} F \qquad L^\infty_{loc}(\Omega;\mathbb{R}^N), \quad \textit{for } p = \infty. \tag{3.2}$$

Then

$$\|F\|_{L^p(\Omega)} \le \lim\inf_{j\to\infty} \|F_j\|_{L^p(\Omega)}, \qquad |\mathrm{div}\, F|(\Omega) \le \lim\inf_{j\to\infty} |\mathrm{div}\, F_j|(\Omega).$$

(ii) *Let $\{F_j\}$ be a sequence in $\mathcal{DM}^{ext}(\Omega)$ such that*

$$F_j \rightharpoonup F \qquad \mathcal{M}_{loc}(\Omega;\mathbb{R}^N).$$

Then

$$|F|(\Omega) \le \lim\inf_{j\to\infty} |F_j|(\Omega), \qquad |\mathrm{div}\, F|(\Omega) \le \lim\inf_{j\to\infty} |\mathrm{div}\, F_j|(\Omega).$$

This proposition implies that the spaces $\mathcal{DM}^p, 1 \le p \le \infty$, and $\mathcal{DM}^{ext}(\Omega)$ are Banach spaces under the norms (1.1) and (1.2), respectively.

Proposition 3.2. *Let $\{F_j\}$ be a sequence in $\mathcal{DM}(\Omega)$ satisfying*

$$\lim_{j\to\infty} |\mathrm{div}\, F_j|(\Omega) = |\mathrm{div}\, F|(\Omega),$$

and one of the following three conditions:

$$F_j \rightharpoonup F \qquad L^p_{loc}(\Omega;\mathbb{R}^N), \ \textit{for } 1 \le p < \infty,$$
$$F_j \overset{*}{\rightharpoonup} F \qquad L^\infty_{loc}(\Omega;\mathbb{R}^N), \ \textit{for } p = \infty,$$
$$F_j \rightharpoonup F \qquad \mathcal{M}_{loc}(\Omega;\mathbb{R}^N).$$

Then, for every open set $A \subset \Omega$,

$$|\mathrm{div}\, F|(\bar{A}\cap\Omega) \ge \lim\sup_{j\to\infty} |\mathrm{div}\, F_j|(\bar{A}\cap\Omega). \tag{3.3}$$

In particular, if $|\mathrm{div}\, F|(\partial A\cap\Omega) = 0$, then

$$|\mathrm{div}\, F|(A) = \lim_{j\to\infty} |\mathrm{div}\, F_j|(A). \tag{3.4}$$

We will use the so-called positive symmetric mollifiers $\omega : \mathbb{R}^N \to \mathbb{R}$ satisfying $\omega(x) \in C_0^\infty(\mathbb{R}^N)$, $\omega(x) \ge 0$, $\omega(x) = \omega(|x|)$, $\int_{\mathbb{R}^N} \omega(x)\,dx = 1$, $\mathrm{supp}\,\omega(x) \subset B_1 \equiv \{x \in \mathbb{R}^N : |x| < 1\}$. A standard example of such mollifiers is

$$\omega(x) = \begin{cases} 0, & |x| \ge 1, \\ C\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \end{cases}$$

where C is the constant such that $\int_{\mathbb{R}^N} \omega(x)\,dx = 1$. We denote $\omega_\varepsilon(x) = \varepsilon^{-N}\omega(\frac{x}{\varepsilon})$ and $F_\varepsilon = F * \omega_\varepsilon$, that is,

$$F^\varepsilon(x) = \varepsilon^{-N}\int_{\mathbb{R}^N} F(y)\omega(\frac{x-y}{\varepsilon})\,dy = \int_{\mathbb{R}^N} F(x+\varepsilon y)\omega(y)\,dy. \tag{3.5}$$

Then $F^\varepsilon \in C^\infty(A;\mathbb{R}^N)$ for any $A \Subset \Omega$ when ϵ is sufficiently small. We will use some well-known properties of the mollifiers. In particular, we recall that, for any $f, g \in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f_\varepsilon g\, dx = \int_{\mathbb{R}^N} f g_\varepsilon\, dx. \tag{3.6}$$

The following fact for $\mathcal{DM}$ fields is analogous to a well-known property of BV functions.

Proposition 3.3. *Let $F \in \mathcal{DM}(\Omega)$. Let $A \Subset \Omega$ be open and $|\operatorname{div} F|(\partial A) = 0$. Then, for any $\varphi \in C(\Omega;\mathbb{R})$,*

$$\lim_{\varepsilon\to 0} \langle \operatorname{div} F^\varepsilon, \varphi\chi_A\rangle = < \operatorname{div} F, \varphi\chi_A > .$$

Furthermore, if $F \in \mathcal{DM}^{ext}(\Omega)$ and $|F|(\partial A) = 0$, then, for any $\varphi \in C(\Omega;\mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0} < F^\varepsilon, \varphi\chi_A > = < F, \varphi\chi_A > .$$

Now we discuss some product rules for divergence-measure fields.

Proposition 3.4. *Let $F = (F_1, \cdots, F_N) \in \mathcal{DM}(\Omega)$. Let $g \in BV \cap L^\infty(\Omega)$ be such that $\partial_{x_j} g(x)$ is $|F_j|$-integrable, for each $j = 1, \cdots, N$, and the set of non-Lebesgue points of $\partial_{x_j} g(x)$ has $|F_j|$-measure zero; and $g(x)$ is $|F| + |\operatorname{div} F|$-integrable and the set of non-Lebesgue points of $g(x)$ has $|F| + |\operatorname{div} F|$-measure zero. Then $gF \in \mathcal{DM}(\Omega)$ and*

$$\operatorname{div}(gF) = g \operatorname{div} F + \nabla g \cdot F. \tag{3.7}$$

In particular, if $F \in \mathcal{DM}^\infty(\Omega)$, $gF \in \mathcal{DM}^\infty(\Omega)$ for any $g \in BV \cap L^\infty(\Omega)$; moreover, if g is also Lipschitz over any compact set in Ω,

$$\operatorname{div}(gF) = g \operatorname{div} F + F \cdot \nabla g. \tag{3.8}$$

In fact, for $F \in \mathcal{DM}^\infty(\Omega)$, one may refine the above result to yield that (3.8) holds a.e. in a more general case, not only for local Lipschitz functions. In this case, we must take the absolutely continuous part of ∇g. For $g \in BV$, let $(\nabla g)_{\rm ac}$ and $(\nabla g)_{\rm sing}$ denote the absolutely continuous part and the singular part of the Radon measure ∇g, respectively. Then

Proposition 3.5. *Given $F \in \mathcal{DM}^\infty(\Omega)$ and $g \in BV(\Omega) \cap L^\infty(\Omega)$, the identity*

$$\operatorname{div}(gF) = \bar{g} \operatorname{div} F + \overline{F \cdot \nabla g}$$

holds in the sense of Radon measures in Ω, where $\bar{g}$ is the limit of a mollified sequence for g through a positive symmetric mollifier, and $\overline{F\cdot\nabla g}$ is a Radon measure absolutely continuous with respect to $|\nabla g|$, whose absolutely continuous part with respect to the Lebesgue measure in Ω coincides with $F \cdot (\nabla g)_{ac}$ almost everywhere in Ω.

Finally, as a corollary of the generalized Gauss-Green formula in $\mathcal{DM}^\infty$, we have

Proposition 3.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and $F_1 \in \mathcal{DM}^\infty(\Omega)$, $F_2 \in \mathcal{DM}^\infty(\mathbb{R}^N - \bar{\Omega})$. Then*

$$F(y) = \begin{cases} F_1(y), & y \in \Omega, \\ F_2(y), & y \in \mathbb{R}^N - \bar{\Omega} \end{cases} \tag{3.9}$$

belongs to $\mathcal{DM}^\infty(\mathbb{R}^N)$, *and*

$$\|F\|_{\mathcal{DM}^\infty(\mathbb{R}^N)} \le \|F_1\|_{\mathcal{DM}^\infty(\Omega)} + \|F_2\|_{\mathcal{DM}^\infty(\mathbb{R}^N-\bar{\Omega})} + \|F_1\cdot\nu - F_2\cdot\nu\|_{L^\infty(\partial\Omega)}\mathcal{H}^{N-1}(\partial\Omega).$$

4. Connection: Hyperbolic Conservation Laws and Divergence-Measure Fields

We now discuss some applications of the theory of $\mathcal{DM}$ fields to various nonlinear problems for hyperbolic conservation laws and degenerate parabolic-hyperbolic equations in Sections 5–7. We first show a connection between $\mathcal{DM}$-fields and hyperbolic conservation laws.

The $\mathcal{DM}$-fields arise naturally in the study of entropy solutions of nonlinear hyperbolic systems of conservation laws, which take the form

$$\partial_t u + \nabla_x \cdot f(u) = 0, \qquad u \in \mathbb{R}^m,\ x \in \mathbb{R}^n, \tag{4.1}$$

where $f : \mathbb{R}^m \to (\mathbb{R}^m)^n$ is a nonlinear map. The condition of hyperbolicity requires that, for any wave number $\xi \in S^{n-1}$, the matrix $\xi \cdot \nabla f(u)$ have m real eigenvalues and left (right) eigenvectors. For the one-dimensional case, system (4.1) is called strictly hyperbolic if the Jacobian $\nabla f(u)$ of f has m real and distinct eigenvalues, $\lambda_1(u) < \cdots < \lambda_m(u)$, and thus has m linearly independent right and left eigenvectors $r_j = r_j(u)$ and $l_j = l_j(u)$:

$$\nabla f(u) r_j(u) = \lambda_j(u) r_j(u), \qquad l_j(u)\nabla f(u) = \lambda_j(u) l_j(u). \tag{4.2}$$

The jth characteristic field is genuinely nonlinear or linearly degenerate in the sense of Lax [29] if

$$r_j \cdot \nabla\lambda_j \neq 0 \quad \text{or} \quad r_j \cdot \nabla\lambda_j \equiv 0. \tag{4.3}$$

That is, the jth eigenvalue changes monotonically or remains constant along the jth characteristic field for the genuinely nonlinear case or the linearly degenerate case, respectively.

One of its most important prototypes is the Euler equations for gas dynamics in Lagrangian coordinates:

$$\partial_t \tau - \partial_x v = 0, \tag{4.4}$$

$$\partial_t v + \partial_x p = 0, \tag{4.5}$$

$$\partial_t (e + \frac{v^2}{2}) + \partial_x (pv) = 0, \tag{4.6}$$

where $\tau = 1/\rho$ is the specific volume with the density ρ, and v, p, e are the velocity, the pressure, the internal energy, respectively; the other two gas dynamical variables are the temperature θ and the entropy S. For ideal polytropic gases, system (4.4)–(4.6) is closed by the following constitutive relations:

$$p\tau = R\theta, \qquad e = c_v\theta, \qquad p(\tau, S) = \kappa\tau^{-\gamma}e^{S/c_v}, \tag{4.7}$$

where c_v, R, and κ are positive constants, and $\gamma = 1 + c_v/R > 1$. For isentropic gases, the Euler equations become

$$\partial_t \tau - \partial_x v = 0, \tag{4.8}$$

$$\partial_t v + \partial_x p(\tau) = 0, \tag{4.9}$$

where $p(\tau) = \kappa\tau^{-\gamma}, \gamma > 1$.

The main feature of nonlinear hyperbolic conservation laws, especially (4.4)–(4.6), is that, no matter how smooth the initial data are, solutions may develop singularities and form shock waves in finite time. One may expect solutions in the space of functions of bounded variation. This is indeed the case by the Glimm theorem [21] which indicates that, when the initial data have sufficiently small total variation and stay away from the vacuum for (4.4)–(4.6), there exists a global entropy solution in BV satisfying the Clausius inequality:

$$S_t \geq 0 \tag{4.10}$$

in the sense of distributions. However, when the initial data are large, still away from the vacuum, the solutions may develop vacuum states in finite time, even instantaneously as $t > 0$, or approach the vacuum states indefinitely. In this case, the specific volume $\tau = 1/\rho$ may then become a Radon measure or an L^1 function, rather than a function of bounded variation (see Wagner [47] and Liu-Smoller [32]). This indicates that solutions of nonlinear hyperbolic conservation laws are generally in $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^n)$, the space of signed Radon measures, or in $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$, $1 \leq p \leq \infty$. On the other hand, the fact that (4.4)–(4.6) and (4.10) hold in the sense of distributions implies, in particular, that the divergences of the fields $(\tau, -v)$, (v, p), $(e + v^2/2, pv)$, $(S, 0)$, in the (t, x) variables, are also Radon measures, in which the first three are the trivial null measure and the last one is a nonnegative measure as a consequence of the Schwartz Lemma [40]. This motivates our study of the extended divergence-measure fields (see Definition 1.1).

For general hyperbolic conservation laws, we have

Definition 4.1. A function $\eta : \mathbb{R}^m \to \mathbb{R}$ is called an entropy of (4.1) if there exists $q : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\nabla q_k(u) = \nabla \eta(u) \nabla f_k(u), \qquad k = 1, 2, \cdots, n. \tag{4.11}$$

The function $q(u)$ is called the entropy flux associated with the entropy $\eta(u)$, and the pair $(\eta(u), q(u))$ is called an entropy pair. The entropy pair $(\eta(u), q(u))$ is called a convex entropy pair on the domain $K \subset \mathbb{R}^m$ if the Hessian matrix $\nabla^2 \eta(u) \geq 0$, for $u \in K$. The entropy pair $(\eta(u), q(u))$ is called a strictly convex entropy pair on the domain K if $\nabla^2 \eta(u) > 0$, for $u \in K$.

Consider a 2×2 strictly hyperbolic system with globally defined Riemann invariants $w_j, j = 1, 2$. The Riemann invariants $w_j : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\nabla w_j(u) \nabla f(u) = \lambda_j(u) \nabla w_j(u), \qquad j = 1, 2,$$

and hence diagonalize system (2.1), for smooth solutions, into

$$\partial_t w_j + \lambda_j \partial_x w_j = 0, \qquad j = 1, 2.$$

Lax's theorem [30] indicates for such a system that, given any bounded domain $K \Subset \mathbb{R}^2$, there exists a strictly convex entropy pair $(\eta(u), q(u))$ on the domain K. That is,

$$\nabla^2 \eta(u) \geq c_K > 0, \qquad u \in K.$$

For $m \geq 3$, system (4.11) is overdetermined, thereby generally preventing the existence of nontrivial entropies. Friedrichs-Lax [19] observed that most of the systems of conservation laws that result from continuum mechanics are endowed with a globally defined, strictly convex entropy. Systems endowed with a rich family of entropies were described by Serre [42].

Available existence theories show that the solutions $u(t,x)$ of (4.1) are in the following class of entropy solutions:

i) $u(t,x) \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^n)$, or $L^p(\mathbb{R}_+ \times \mathbb{R}^n), 1 \le p \le \infty$;

ii) $u(t,x)$ satisfies the Lax entropy inequality:

$$\partial_t \eta(u(t,x)) + \nabla_x \cdot q(u(t,x)) \le 0 \tag{4.12}$$

in the sense of distributions, for any entropy pair $(\eta, q) : \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^n$ with convex $\eta, \nabla^2\eta(u) \ge 0$, so that $(\eta(u(t,x))$ and $q(u(t,x))$ are distributional functions. If we take $\eta = \pm u$, we see that any entropy solution is a weak solution.

One of the main issues in conservation laws is to study the behavior of solutions in this class to explore all possible information of solutions, including large-time behavior, uniqueness, stability, and traces of solutions, among others. The Schwartz lemma indicates from (4.12) that the distribution

$$\partial_t \eta(u(t,x)) + \nabla_x \cdot q(u(t,x))$$

is in fact a Radon measure, that is,

$$\operatorname{div}_{(t,x)}(\eta(u(t,x)), q(u(t,x))) \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^n). \tag{4.13}$$

In particular, for $u \in L^\infty$, (4.13) is also true for any C^2 entropy pair (η, q) (η not necessarily convex) if system (4.1) has a strictly convex entropy, which implies that, for any C^2 entropy pair (η, q), the field $\eta(u(t,x)), q(u(t,x)))$ is a $\mathcal{DM}$-field.

Divergence-measure fields also arise in various nonlinear problems involving some extended vector fields whose divergences are Radon measures. For example, see Bouchitté-Buttazzo [4] for such fields in the characterization of optimal shapes and masses through the Monge-Kantorovich equation.

From the previous discussion, it is clear that understanding more properties of $\mathcal{DM}$-fields can advance our understanding of the behavior of entropy solutions for hyperbolic conservation laws and other related nonlinear equations.

In Sections 5–8, we discuss some applications of the theory of $\mathcal{DM}$-fields described in Sections 2-3 to several nonlinear problems for conservation laws and related nonlinear equations.

5. Stability of Riemann Solutions in a Class of Entropy Solutions with the Vacuum for the Euler Equations

In this section, we show how the theory of $\mathcal{DM}$ fields can be applied to establish the uniqueness and stability of Riemann solutions that may contain the vacuum for the Euler equations for gas dynamics in Lagrangian coordinates.

Denote $\mathbb{R}^2_+ = (0,\infty) \times \mathbb{R}$ and $\overline{\mathbb{R}^2_+} = [0,\infty) \times \mathbb{R}$. We consider $\tau \in \mathcal{M}^+(\mathbb{R}^2_+)$ satisfying $\tau \ge c\,\mathcal{L}^2$ for some $c > 0$, where $\mathcal{L}^k$ is the k-dimensional Lebesgue measure. Let $v \in L^\infty(\mathbb{R}^2_+)$ and $\tau_0 \in \mathcal{M}^+(\mathbb{R})$ with $\tau_0 \ge c\,\mathcal{L}^1$. We assume that τ, v, and τ_0 satisfy

$$\iint_{\mathbb{R}^2_+} (\phi_t \tau - v\,\phi_x\,dt\,dx) + \int_{\mathbb{R}} \phi(0,x)\,\tau_0(x) = 0, \tag{5.1}$$

for any $\phi \in C^1_0(\mathbb{R}^2)$.

Definition 5.1. Let τ and τ_0 be as above. We say that a function $\phi(t,x)$ defined on $\overline{\mathbb{R}^2_+}$ is a τ-test function if it satisfies the following:

(1) $\operatorname{spt}(\phi)$ is a compact subset of $\overline{\mathbb{R}^2_+}$ and ϕ is continuous on $\mathbb{R}^2_+$;

(2) ϕ_t and ϕ_x are τ-measurable; and ϕ_t is τ-integrable over $\mathbb{R}^2_+$, that is, the integrals $\iint_{\mathbb{R}^2_+}(\phi_t)_\pm \tau$ exist and at least one of them is finite;

(3) $\lim\limits_{\substack{t\to 0\\ x\to a}} \phi(t,x) = \phi(0,a)$ for τ_0–a.e. $a \in \mathbb{R}$.

Theorem 5.1. *Let τ, v, and τ_0 be as above. Then*

(1) *the nonnegative measure τ admits a slicing of the form $\tau = dt \otimes \mu_t(x)$ with $\mu_t \in \mathcal{M}^+(\mathbb{R})$ for $\mathcal{L}^1$-a.e. $t > 0$. More precisely, for any $\phi \in C_0(\mathbb{R}^2_+)$,*

$$\iint \phi(t,x)\,\tau = \int \Big(\int \phi(t,x)\,\mu_t(x)\Big)\,dt.$$

(2) *the points $(t,x) \in \mathbb{R}^2_+$ such that $\mu_t(x) > 0$, with the exception of a set of $\mathcal{H}^1$-measure zero, form a countable union of vertical line segments, called vacuum lines. In particular, $\tau(l) = 0$ for any non-vertical straight line segment l.*

(3) *the identity* (5.1) *holds for any τ-test function $\phi(t,x)$.*

As a corollary, we have

Corollary 5.1. *Let τ, v, and τ_0 be as above. Let $\bar{p}(t,x)$ be a nonnegative function over $\overline{\mathbb{R}^2_+}$, continuous on $\mathbb{R}^2_+$, such that $\phi\,\bar{p}$ is a τ-test function for any $\phi \in C_0^1(\mathbb{R}^2)$, $\bar{p}_t \le 0$, τ-a.e., and $\bar{p}_x \in L^1_{loc}(\mathbb{R}^2_+)$. Then, for any nonnegative function $\zeta \in C_0^1(\mathbb{R})$,*

$$\limsup_{t\to 0+} \int \zeta(x)\,\bar{p}(t,x)\,\mu_t(x) \le \int \zeta(x)\,\bar{p}(0,x)\,\tau_0(x). \tag{5.2}$$

We now consider the solutions of the Euler equations (4.4)–(4.6) for gas dynamics in the sense of distributions such that τ is a nonnegative Radon measure, with $\tau \ge c\mathcal{L}^2$ for some $c > 0$, and $v(t,x)$ and $S(t,x)$ are bounded τ-measurable functions, along with our understanding that the constitutive relations (4.7) for $(\tau, p, e, \theta, S)(t,x)$ hold $\mathcal{L}^2$-almost everywhere out of the vacuum lines, in the set where τ is absolutely continuous with respect to $\mathcal{L}^2$, and both $p(t,x)$ and $e(t,x)$ are defined as zero on the remaining set with measure zero in $\mathbb{R}^2_+$, including the vacuum lines.

We consider the Cauchy problem for (4.4)–(4.6):

$$(\tau, v, S)|_{t=0} = (\tau_0, v_0, S_0)(x), \tag{5.3}$$

where $\tau_0(x)$ is a nonnegative Radon measure over $\mathbb{R}$, $\tau_0 \ge c\mathcal{L}^1$ for some $c > 0$, $v_0(x)$ and $S_0(x)$ are bounded τ_0-measurable functions, and $e_0(x) = e(\tau_0(x), S_0(x))$ a.e. out of the countable points $\{x_k\}$ such that $\tau_0(x_k) > 0$, the initial vacuum set.

Set $\Pi_T = (0,T) \times \mathbb{R}$ and $\Pi_T^* = (-\infty, T) \times \mathbb{R}$ for $T > 0$. Let D and F be functions or measures over Π_T. Let D_0 be a function or a measure over $\mathbb{R}$. By *weak formulation* on Π_T for the Cauchy problem:

$$D_t + F_x = 0, \tag{5.4}$$

$$D|_{t=0} = D_0, \tag{5.5}$$

we mean that, for a suitable set of test functions $\phi(t,x)$ defined on Π_T^*,

$$\iint_{\Pi_T} (\phi_t D + \phi_x F) + \int_{\mathbb{R}} \phi(0,x) D_0 = 0. \tag{5.6}$$

Analogously, if the identity " $=$ " in (5.4) is replaced by " $\geq$ " or " $\leq$ ", the weak formulation of the corresponding problem (5.4) and (5.5) is (5.6) with " $=$ " replaced by " $\leq$ " or " $\geq$ ", respectively, for a suitable set of *nonnegative* test functions defined on Π_T^*.

Denote $W = (\tau, v, S)$, $f(W) = (-v, p(\tau, S), 0)$, $\eta(W) = e(\tau, S) + \frac{v^2}{2}$, $q(W) = vp(\tau, S)$, and

$$\alpha(W, \overline{W}) = \eta(W) - \eta(\overline{W}) - \nabla\eta(\overline{W}) \cdot (W - \overline{W}),$$
$$\beta(W, \overline{W}) = q(W) - q(\overline{W}) - \nabla\eta(\overline{W}) \cdot (f(W) - f(\overline{W})).$$

Observe that $\nabla\eta(\overline{W}) = (-\bar{p}, -\bar{v}, \bar{\theta})$.

Definition 5.2. We say that $W(t,x)$ is a distributional entropy solution of (4.4)–(4.6), and (5.3) in Π_T if τ is a Radon measure on Π_T with $\tau \geq c\mathcal{L}^2$ for some $c > 0$, v and S are bounded τ-measurable functions such that the weak formulations of (4.4)–(4.6), (4.10), and (5.3) are satisfied for all test functions in $C_0^1(\Pi_T^*)$, and $S(t, \cdot) \rightharpoonup S_0(\cdot)$, as $t \to 0$, in the weak-star topology of $L^\infty(\mathbb{R})$.

Observe that the weak formulation implies that $\mu_t \rightharpoonup \tau_0$ in $\mathcal{M}(\mathbb{R})$, and $v(t, \cdot) \rightharpoonup v_0(\cdot)$, and $E(t, \cdot) \rightharpoonup E_0(\cdot)$ in the weak-star topology of $L^\infty(\mathbb{R})$, as $t \to 0$, where $E = e + v^2/2$. We also remark that these convergences can be strengthened to the convergences in $L^1_{loc}(\mathbb{R})$ in the case that τ is a bounded measurable function, as an easy consequence of the $\mathcal{DM}^\infty$ theory in Sections 2 and 3.

As shown by Wagner [47], by means of the transformation from Eulerian to Lagrangian coordinates, bounded measurable entropy solutions of the Euler equations in Eulerian coordinates transform into distributional entropy solutions of (4.4)–(4.6) and (5.3), satisfying the additional restriction that the weak formulation of (4.4)–(4.6), and (4.10) holds for test functions with compact support in Π_T such that $\phi_t = g$, $\phi_x = h\,\tau$, where $g, h \in L^\infty(\Pi_T, \tau)$. It is also shown through an example in [47] that distributional entropy solutions without the additional restriction may have no physical meaning.

Now we consider the Riemann solution $\overline{W}(t,x)$ associated with the Riemann problem for (4.4)–(4.6) with initial condition

$$\overline{W}_0(x) = \begin{cases} W_L, & x < 0, \\ W_R, & x > 0, \end{cases} \tag{5.7}$$

where W_L and W_R are two constant states in the physical domain $\{W = (\tau, v, S) : \tau > 0\}$. First, we address the case where $\overline{W}(t,x)$ is a bounded self-similar entropy solution of (4.4)–(4.6) which consists of at most two rarefaction waves, one corresponding to the first characteristic family and the other corresponding to the third one, and possibly one contact discontinuity on the line $x = 0$. Then, $\overline{W}(t,x)$ has the following general form:

$$\overline{W}(x,t) = \begin{cases} W_L, & x/t < \xi_1, \\ R_1(x/t), & \xi_1 \leq x/t < \xi_2, \\ W_M, & \xi_2 \leq x/t < 0, \\ W_N, & 0 < x/t < \xi_3, \\ R_3(x/t), & \xi_3 \leq x/t < \xi_4, \\ W_R, & x/t \geq \xi_4. \end{cases} \tag{5.8}$$

Theorem 5.2. *Let $\overline{W}(t,x)$ be the shock-free Riemann solution* (5.8) *and $W(t,x)$ be any distributional entropy solution of* (4.4)–(4.6) *and* (5.3) *with $W_0 \in L^\infty(\mathbb{R};\mathbb{R}^3)$. Then there exist positive constants C and K_0, and a function $\omega \in L^\infty(\Pi_T)$, positive a.e. in Π_T, such that, for any $X > 0$ and a.e. $t > 0$,*

$$\int_{|x|\le X} |W_{a.c.}(t,x) - \bar{W}(t,x)|^2 \omega(t,x)\,dx \le C \int_{|x|\le X+K_0 t} |W_0(x) - \overline{W}_0(x)|^2 \omega(0,x)\,dx. \tag{5.9}$$

Sketch of Proof. Given any $X > 0$ and $t > 0$, let $t_0 \in (0,t)$ and

$$\Omega_{t_0,t} = \{(\sigma,x) \,:\, |x| < X + K_0(t-\sigma),\ t_0 < \sigma < t\},$$

with $K_0 > 0$ to be suitably chosen later. We consider the measure

$$\mu := \alpha(W,\overline{W})_t + \beta(W,\overline{W})_x.$$

Using the product rules (Proposition 3.4) and the Gauss-Green theorem (Theorem 2.2), we have

$$\mu(\Omega_{t_0,t}) = \langle(\alpha,\beta)\cdot\nu|_{\partial\Omega_{t_0,t}}, 1\rangle$$

We can show that, for a.e. t and t_0 as above,

$$\langle(\alpha,\beta)\cdot\nu|_{\partial\Omega_{t_0,t}}, 1\rangle \ge \int_{|x|\le X} |W_{a.c.}(t,x) - \overline{W}(t,x)|^2\omega(t,x)\,dx - \int_{\substack{|x|\le X+K_0(t-t_0)\\ \sigma=t_0}} \{\eta(W) - \eta(\overline{W}) - \bar{v}(v-\bar{v}) + \bar{p}(\mu_{t_0} - \bar{\tau}) - \bar{\theta}(S - \overline{S})\},$$

where we have also used that $\mu_\sigma \rightharpoonup \mu_{t_0}$ as $\sigma \to t_0 + 0$, for a.e. $t_0 > 0$, and that $\bar{p}$ is continuous on $[t_0,t]\times\mathbb{R}$.

On the other hand,

$$\mu(\Omega_{t_0,t}) = \sum_{i=1}^{4} \mu(\tilde{l}_i \cap \Omega_{t_0,t}) + \mu(l\cap\Omega_{t_0,t}) + \mu(\Omega_1\cap\Omega_{t_0,t}) + \mu(\Omega_3\cap\Omega_{t_0,t}) + \mu\big(\Omega_{t_0,t} - (\cup_{i=1}^4 \tilde{l}_i \cup l \cup \Omega_1 \cup \Omega_3)\big),$$

where Ω_1 and Ω_3 are the left and right rarefaction regions, $\tilde{l}_i, 1 \le i \le 4$, are the lines bounding the rarefaction regions Ω_1 and Ω_3, and l is the line $\{x = 0\}$ where $\overline{W}(t,x)$ has a contact discontinuity.

We first observe that, on $\Omega_{t_0,t} - (\cup_{i=1}^4 \tilde{l}_i \cup l \cup \Omega_1 \cup \Omega_3)$, the measure μ reduces to $-\bar{\theta}\partial_t S$ which is nonpositive. Now, we have

$$\mu = -\mathrm{div}\,(F_1 + F_2 + F_3),$$

where

$$F_1 = \bar{v}(v-\bar{v}, p-\bar{p}), \quad F_2 = -\bar{p}(\tau-\bar{\tau}, \bar{v}-v), \quad F_3 = \bar{\theta}(S-\overline{S}, 0),$$

and $\mathrm{div} := \mathrm{div}_{t,x}$. Applying the product rule (Proposition 3.4), we get

$$\mathrm{div}\,F_1 = \bar{v}_t(v-\bar{v}) + \bar{v}_x(p-\bar{p}), \quad \mathrm{div}\,F_2 = -\bar{p}_t(\tau-\bar{\tau}) + \bar{p}_x(v-\bar{v}).$$

Hence,

$$\mathrm{div}\ F_1(\tilde{l}_j \cap \Omega_{t_0,t}) = \mathrm{div}\ F_1(l\cap\Omega_{t_0,t}) = 0, \qquad j = 1,\cdots,4,$$

since div F_1 is absolutely continuous with respect to $\mathcal{L}^2$. Also,

$$\text{div } F_3(\tilde{l}_j \cap \Omega_{t_0,t}) \geq 0, \qquad j = 1, \cdots, 4.$$

On the other hand, since $F_3 \in \mathcal{DM}^\infty(\Omega_{t_0,t})$ and $\nu|_l = (0,1)$, we have

$$\text{div } F_3(l \cap \Omega_{t_0,t}) = [\langle F_3 \cdot \nu|_l, 1\rangle] = 0,$$

where the square-bracket denotes the difference between the normal traces from the right and the left, which make sense for $F_3 \in \mathcal{DM}^\infty$ because the normal traces of $\mathcal{DM}^\infty$ fields are functions in L^∞ over the boundaries.

Concerning F_2, we have

$$\text{div } F_2(\tilde{l}_j \cap \Omega_{t_0,t}) = 0, \qquad j = 1, \cdots, 4,$$

since $\bar{p}_t$ is τ-integrable and $\tau(\tilde{l}_j) = 0$, $j = 1, \cdots, 4$. On the other hand, $\bar{p}_t$ vanishes on l so that

$$\text{div } F_2(l \cap \Omega_{t,t_0}) = 0.$$

Finally, we have

$$\mu(\Omega_1) \leq 0, \qquad \mu(\Omega_3) \leq 0,$$

since $\bar{v}_x(t,x) \geq 0$ everywhere over Ω_1 and Ω_3.

Putting all these estimates together, we have

$$\int_{|x|\leq X} |W_{a.c.}(t,x) - \overline{W}(t,x)|^2 \omega(t,x)\, dx$$
$$\leq \int_{\substack{|x|\leq X+K_0(t-t_0)\\ \sigma=t_0}} \{\eta(W) - \eta(\overline{W}) - \bar{v}(v-\bar{v}) + \bar{p}(\mu_{t_0} - \bar{\tau}) - \bar{\theta}(S - \overline{S})\}.$$

Now, applying Corollary 5.1, we finally arrive at (5.9).

Corollary 5.2. *Let $W(t,x)$ and $\overline{W}(t,x)$ satisfy the conditions of Theorem 5.2 and $W_0(x) = \overline{W}_0(x)$. Then $\tau(t,x)$ is absolutely continuous with respect to $\mathcal{L}^2$ in Π_T and $W(t,x) = \overline{W}(t,x)$ a.e. in Π_T.*

We now consider the case that the Riemann solution, with the initial condition (5.7), has a vacuum line at $x = 0$. In this case, the Riemann solution $\bar{W}(t,x)$ has the following form:
(5.10)

$$\overline{W}(t,x) = \begin{cases} W_L, & x/t < \xi_L, \\ R_1(x/t), & \xi_L \leq x/t < 0, \\ ((\bar{v}_1+\bar{v}_2)/2, (\bar{v}_2-\bar{v}_1)tdt \otimes \delta_0(x), (\overline{S}_L+\overline{S}_R)/2), & x = 0, \\ R_3(x/t), & 0 < x/t \leq \xi_R, \\ W_R, & x/t > \xi_R. \end{cases}$$

Here $R_1(x/t)$ and $R_3(x/t)$ are as above the rarefaction waves of the first and third characteristic families, respectively, $\bar{v}_1 = \lim_{\xi\to 0_-} \bar{v}(\xi)$, $\bar{v}_2 = \lim_{\xi\to 0_+} \bar{v}(\xi)$, and $\delta_0(x)$ is the Dirac measure over $\mathbb{R}$ concentrated at 0. It is easy to check that $\overline{W}(t,x)$ is a distributional solution of (4.4)–(4.6) and (5.3). The values of $\bar{v}$ and $\overline{S}$ on the line $x = 0$ could be taken as any other constants instead of $\frac{\bar{v}_1+\bar{v}_2}{2}$ and $\frac{\overline{S}_L+\overline{S}_R}{2}$, respectively, while the formula of $\bar{\tau}$ at $x = 0$, $(\bar{v}_2 - \bar{v}_1)tdt \otimes \delta_0(x)\otimes$, is dictated by the fact that (4.4) must hold in the sense of distributions.

Using the theory of $\mathcal{DM}$ fields and following similar line of arguments as in the proof of Theorem 5.2, we can show

Theorem 5.3. *Let $\overline{W}(t,x)$ be a Riemann solution containing the vacuum as described in* (5.10). *Let $W(t,x)$ be a distributional entropy solution of* (4.4)–(4.6) *and* (5.3) *in Π_T with $W_0 \in L^\infty(\mathbb{R};\mathbb{R}^3)$. Then there exist positive constants C and K_0, and a function $\omega \in L^\infty(\Pi_T)$, positive a.e. in Π_T, such that, for all $X > 0$ and a.e. $t > 0$,*

(5.11)
$$\int_{|x|\le X} |W_{a.c.}(t,x)-\overline{W}_{a.c.}(t,x)|^2\omega(t,x)\,dx \le C\int_{|x|\le X+K_0t} |W_0(x)-\overline{W}_0(x)|^2\omega(0,x)\,dx.$$

Corollary 5.3. *Let $\overline{W}(t,x)$ and $W(t,x)$ satisfy the hypotheses of Theorem 5.3 and $W_0(x) = \overline{W}_0(x)$. Then $(v,S)(t,x) = (\bar{v},\overline{S})(t,x)$, $\mathcal{L}^2$-a.e. in Π_T, and $\tau = \bar{\tau}$ in $\mathcal{M}(\Pi_T)$.*

6. Initial Layers and Boundary Layers

We are first concerned with initial layers and uniqueness of weak entropy solutions for the Cauchy problem of scalar hyperbolic conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \tag{6.1}$$

$$u(x,0) = u_0(x). \tag{6.2}$$

The weak entropy solutions we address are defined in the following sense.

Definition 6.1. (1). A function $u(t,x) \in L^\infty$ is called a *weak entropy solution* of (6.1)-(6.2) if $u : (t,x) \to u(t,x)$ satisfies the following.

(a). u is a weak solution: For any function $\phi \in C_0^\infty(\mathbb{R}^2_+)$, $\mathbb{R}^2_+ \equiv [0,\infty)\times\mathbb{R}$,

$$\int_0^\infty\int_{-\infty}^\infty (u\,\partial_t\phi + f(u)\,\partial_x\phi)dxdt + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx = 0. \tag{6.3}$$

(b). For any nonnegative function $\phi \in C_0^\infty(\mathbb{R}^2_+ - \{t=0\})$ and any convex entropy pair $(\eta(u),q(u)), \eta''(u)\ge 0, q'(u)=\eta'(u)f'(u)$,

$$\int_0^\infty\int_{-\infty}^\infty (\eta(u)\partial_t\phi + q(u)\,\partial_x\phi)dx\,dt \ge 0. \tag{6.4}$$

(2). In contrast, a function $u(t,x) \in L^\infty$ is called a Kruzkov solution if $u(t,x)$ satisfies, besides (6.3)–(6.4), the following property of (weak) L^1-continuity in time: For any $R > 0$,

$$\frac{1}{T}\int_0^T\int_{|x|\le R} |u(t,x) - u_0(x)|\,dxdt \to 0, \quad \text{when } T\to 0. \tag{6.5}$$

(3). We say that a function $u(t,x)$ satisfies the strong entropy inequality if, for any convex entropy pair $(\eta(u),q(u))$ and any $\phi \in C_0^\infty(\mathbb{R}^2_+), \phi\ge 0$,

$$\int_0^\infty\int_{-\infty}^\infty (\eta(u)\partial_t\phi + q(u)\,\partial_x\phi)dx\,dt + \int_{-\infty}^\infty \eta(u_0)(x)\phi(x,0)dx \ge 0. \tag{6.6}$$

It is easy to check that any function $u(t,x)$ satisfying the strong entropy inequality (6.6) is a Kruzkov solution. This fact can be easily achieved with the aid of basic properties of divergence-measure fields, especially the normal traces. First, we pick a trivial entropy $\eta(u) = \pm u$ in (6.6) to conclude that u satisfies (6.3) and has a trace $u(0_+,\cdot) = u_0$ on the set $t = 0$, defined at least in the weak-star sense in L^∞. Then, using the Gauss-Green formula (Theorem 2.1), we conclude from (6.6) that, for any strictly convex entropy η, the trace $\eta(u)(0_+,\cdot)$ of $\eta(u)$ at $t = 0_+$ satisfies

$$\eta(u)(0_+,\cdot) \le \eta(u_0) = \eta(u(0_+,\cdot)).$$

Then the strict convexity of η implies that the trace of u on the set $t = 0$ is in fact defined in the strong sense in L^1, which immediately implies (6.5).

The main objective here is to establish the uniqueness and L^1 strong continuity in time of solutions satisfying only (6.3)–(6.4), provided that equation (6.1) has weakly genuine nonlinearity, that is,

$$\text{There exists no nontrivial interval on which } f \text{ is affine.} \tag{6.7}$$

Observe that the solutions defined in (6.3)–(6.4) are in general weaker than the Kruzkov solutions. It has been proved ([27, 15], see also [41] for the extension to the L^p case) that the Kruzkov solutions are *uniquely* defined.

For approximate solutions generated by either the vanishing viscosity method or a total variation diminishing (TVD) numerical scheme, e.g. a monotone conservative scheme, one can easily show that the limit function $u(t,x)$ satisfies (6.6), even if the initial data $u_0(x)$ are only in L^∞. Then, by the above arguments, there is no initial layer, which implies that the solution $u(t,x)$ is unique and stable in L^1.

However, when we consider the limit behavior of other physical regularizing effects, especially the zero relaxation limit, there is definitely an initial layer, unless the initial data are already at the equilibrium; see [11]. Therefore, the uniqueness of limit functions becomes a crucial problem, as observed in [33] (also see [26]). In this connection, we recall the following result of Chen-Rascle [12], to which we refer for the proof.

Theorem 6.1. *Assume that (6.7) is satisfied. Let $u(t,x)$ be an L^∞ weak entropy solution of the Cauchy problem (6.1)–(6.2). Then $u(t,x)$ satisfies (6.6), which implies that $u(t,x)$ is the unique Kruzkov solution.*

Remark. An interesting observation is that, under condition (6.7), even a weak solution which is *not* an entropy solution, but whose entropy production is controlled, is also *strongly* continuous in L^1 at time $t = 0$. Indeed, if one replaces the entropy condition (6.4) in Theorem 6.1 by

$$(\eta(u(t,x)), q(u(t,x))) \in \mathcal{DM}((0,\infty)\times\mathbb{R}), \tag{6.8}$$

for any C^2 entropy pair (η, q), then (6.5) still holds, but of course that does not imply that u is the Kruzkov solution, if u does *not* satisfy (6.4)!

Theorem 6.1 can be applied to clarify the initial layers and uniqueness of zero relaxation limits for various physical relaxation systems whose initial data are not at the equilibrium.

Theorem 6.1 was generalized with slightly strong nonlinearity condition by Vasseur [45], with the aid of the generalized Gauss-Green theorem and normal traces (Theorem 2.1) combined with the kinetic formulation of Lions-Perthame-Tadmor [31].

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ have a regular deformable Lipschitz boundary. Assume that $f \in C^3(\mathbb{R};\mathbb{R}^n)$ satisfies*

$$|\{\xi \mid \tau + \zeta \cdot f'(\xi) = 0\}| = 0,$$

for every $(\tau,\zeta) \in \mathbb{R}\times\mathbb{R}^n$ and $(\tau,\zeta) \neq (0,0)$. Then, for every weak solution $u \in L^\infty(\Omega)$ that satisfies the entropy inequality in the sense of distributions in Ω, there exists $u^\tau \in L^\infty(\partial\Omega)$ such that, for every $\partial\Omega$-regular Lipschitz deformation ψ and every compact set $K \Subset \partial\Omega$:

$$ess \lim_{s\to 0} \int_K |u(\psi(s,z)) - u^\tau(z)| d\mathcal{H}^n(z) = 0,$$

In particular, for every smooth function G, we have

$$[G(u)]^\tau = G(u^\tau).$$

7. Initial-Boundary Value Problems for Conservation Laws

The existence of normal traces for divergence-measure fields is a crucial property which makes $\mathcal{DM}$-fields a natural class in connection to entropy solutions in L^∞ of initial-boundary value problems for conservation laws and has greatly motivated its study. The basic question of the formulation of the way in which the boundary conditions should be interpreted is the key point in this analysis. For example, given a bounded open set $\Omega \subset \mathbb{R}^n$, we consider the following initial-boundary value problem in $Q_T = (0,T)\times\Omega$:

$$\partial_t u + \operatorname{div} f(u) = 0, \tag{7.1}$$

$$u|_{t=0} = u_0(x), \tag{7.2}$$

$$u(t,x) = u_b(t,x), \qquad (t,x) \in \Gamma_T := (0,T)\times\partial\Omega, \tag{7.3}$$

where $u : Q_T \to \mathbb{R}^m$, $f = (f_1,\cdots,f_n)$ with $f_i : \mathbb{R}^m \to \mathbb{R}^m$, for $u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty(\Gamma)$. Although the formulation of the concept of entropy solutions of (7.1)–(7.3) may be given in a general setting, an existence theory is currently available only for the cases $m = 1$ and general n (multi-D scalar conservation laws), and $n = 1$ and general m for special systems, however, the results for $m = 2$ cover almost all models of interest in applications. Here, for simplicity, we consider mainly the cylindrical case, in which the space-time domain is a Cartesian product; but the concepts and results also extend to more general non-cylindrical space-time domains in [7]. We refer to [7, 8] and the references cited therein for a more general and detailed discussion of the topic in this section.

Let Ω be a bounded open domain in $\mathbb{R}^n$ with Lipschitz deformable boundary, and let $\Phi : \partial\Omega\times[0,1] \to \overline{\Omega}$ be a regular Lipschitz deformation for $\partial\Omega$. We consider the deformations Ψ of ∂Q_T which, for any $\delta > 0$, over $\Gamma_T \cap \{t \in (\delta, T-\delta)\}$, are given by $\Psi((t,\omega);s) = \Phi(\omega,s)$ for $(\omega,s) \in \Omega\times[0,1]$. Clearly, one can define the deformations of ∂Q_T with this property. The weak formulation of the boundary condition (7.3) is prescribed with the help of parametrized entropy pairs $(\alpha(u,v),\beta(u,v))$ such that, for each fixed $v \in \mathbb{R}^m$, $(\alpha(\cdot,v),\beta(\cdot,v))$ is a convex entropy pair satisfying

$$\alpha(v,v) = \beta(v,v) = \partial_u\alpha(v,v), \tag{7.4}$$

or which are uniform limits on compact sets of C^2 pairs satisfying (7.4). We call these pairs, boundary-entropy pairs, following a denomination proposed in [37].

Examples of such pairs are the Kruzkov entropy pairs for scalar conservation laws

$$\alpha(u,v)=|u-v|, \qquad \beta_i(u,v)=\operatorname{sign}(u-v)(f_i(u)-f_i(v)), \qquad i=1,\cdots,n,$$

and the Dafermos entropy pairs, obtained from a C^2 convex entropy pair $(\eta(u),q(u))$ by

$$\alpha(u,v)=\eta(u)-\eta(v)-\nabla\eta(v)(u-v),$$
$$\beta_i(u,v)=q_i(u)-q_i(v)-\nabla\eta(v)(f_i(u)-f_i(v)), \qquad i=1,\cdots,n,$$

among others. Denoting $\Gamma_s=\Psi(\Gamma\times\{s\})$, ν_s the unit outer normal to Ω_s, the weak formulation of the boundary condition (7.3) is translated by the imposition that

$$\operatorname*{ess\,lim}_{s\to 0}\int_{\Gamma_T}\beta(u\circ\Psi_s(r),u_b(r))\cdot(\nu_s\circ\Phi_s)(r)\,\zeta(r)\,d\mathcal{H}^n(r)\ge 0, \tag{7.5}$$

for all nonnegative $\zeta\in L^1(\Gamma)$. Observe that the limit on the left-hand side of (7.5) exists due to the properties of $\mathcal{DM}^\infty$ fields (see (2.4) above). Using a weak formulation of (7.3) of form (7.5), the existence and uniqueness of L^∞ solutions of (7.1)–(7.3) was proved by Otto [37] (see also [34]) for scalar conservation laws. In [8], many existence results are given for systems in the one-dimensional case. We refer to [7, 8] for other references on this problem.

8. Nonlinear Degenerate Parabolic-Hyperbolic Equations

Here we briefly mention some applications of the theory of $\mathcal{DM}$ fields to initial-boundary value problems for nonlinear degenerate parabolic-hyperbolic scalar equations. In [35], Mascia, Porretta, and Terracina used the properties of $\mathcal{DM}^2$ fields to study the initial-boundary value problem

$$\partial_t u+\operatorname{div} f(u)=\Delta a(u), \tag{8.1}$$

$$u|_{t=0}=u_0(x), \tag{8.2}$$

$$u(t,x)=u_b(t,x), \qquad (t,x)\in\Gamma_T:=(0,T)\times\partial\Omega, \tag{8.3}$$

where $a(u)$ is assumed to be continuous and nondecreasing, possibly assuming a constant value over a non-trivial interval. The definition of entropy solutions of (8.1)-(8.3) requires that, for each fixed $v\in\mathbb{R}$, the field $(A(u(t,x),v),B(u(t,x),v))$ defined by

$$A(u(t,x),v)=|u(t,x)-v|, \tag{8.4}$$

$$B(u(t,x),v)=\operatorname{sign}(u(t,x)-v)\big(f(u(t,x))-f(v)\big)-\nabla_x|a(u(t,x))-a(v)|, \tag{8.5}$$

belongs to $\mathcal{DM}^2(Q_T)$ and satisfies

$$\int_{Q_T}(A(u(t,x),v)\phi_t+B(u(t,x),v)\cdot\nabla_x\phi)\,dx\,dt\ge 0, \tag{8.6}$$

for any nonnegative $\phi\in C_0^\infty(Q_T)$. Now, setting

$$H(u(t,x),v,u_b(t,x)):=B(u(t,x),v)+B(u(t,x),u_b(t,x))-B(u_b(t,x),v),$$

the weak formulation of the boundary condition (8.3) is then given by

$$\lim_{\delta\to 0}\int_0^\delta\left\{\int_{\Gamma_T}H\left(u\circ\Psi_s(r),v,u_b(r)\right)\cdot(\nu_s\circ\Phi_s)(r)\,\zeta(r)\,d\mathcal{H}^n(r)\right\}ds\ge 0, \tag{8.7}$$

for any $v\in\mathbb{R}$ and all nonnegative $\zeta\in L^1(\Gamma)$. Again, observe that the limit in the left-hand side of (8.7) exists by the properties of $\mathcal{DM}^2$ fields. The initial condition

(8.2) is required to be attained in the standard L^1 sense. The uniqueness of entropy solutions of (8.1)–(8.3) was proved in [35]; the existence of entropy solutions for general L^∞ initial and *boundary* data is still open.

As another example, we mention an application of the properties of $\mathcal{DM}^2$ fields in the study of a free-boundary problem for a degenerate parabolic-hyperbolic equation arising as a model of pressure filtration in [6]. In this case, the boundary conditions are of the Neumann type that are imposed on the normal traces of the field $(u, f(u) - \nabla_x a(u))$. We refer to [6] for the details.

Acknowledgments

Gui-Qiang Chen's research was supported in part by the National Science Foundation under Grants DMS-9971793, INT-9987378, and INT-9726215. Hermano Frid's research was supported in part by CNPq through the grants 352871/96-2, 46.5714/00-5, and FAPERJ through the grant E-26/151.890/2000.

References

[1] Anzellotti, G., *Pairings between measures and functions and compensated compactness*, Ann. Mat. Pura Appl. **135** (1983), 293–318.

[2] Baiocchi, C. and A. Capelo, Variational and Quasi-Variational Inequalities, Applications to Free-Boundary Problems, Vols. **1,2**, John Wiley: Chichester-New York, 1984.

[3] Bardos, C., Le Roux, A. Y., and J. C. Nedelec, *First order quasilinear equations with boundary conditions*, Comm. Partial Diff. Eqs. **4** (1979), 1017–1034.

[4] Bouchitté, G. and G. Buttazzo, *Characterization of optimal shapes and masses through Monge-Kantorovich equation*, Preprint, University of Pisa, February 2000.

[5] Brezzi, F. and M. Fortin, Mixed and Hydrid Finite Element Methods, Springer-Verlag: New York, 1991.

[6] Bürger, R., Karlsen, K. H., and H. Frid, *On a free boundary problem for a strongly degenerate quasilinear parabolic equation arising in a model for pressure filtration*, submitted for publication in December 2001.

[7] Chen, G.-Q. and H. Frid, *Divergence-measure fields and conservation laws*, Arch. Rational Mech. Anal. **147** (1999), 89-118.

[8] Chen, G.-Q. and H. Frid, *Vanishing viscosity limit for initial-boundary value problems for conservation laws*, Contemporary Mathematics, **238** (1999), 35-51.

[9] Chen, G.-Q. and H. Frid, *Extended divergence-measure fields and the Euler equations for gas dynamics*, submitted for publication in September 2001.

[10] Chen, G.-Q., Frid, H., and Y. Li, *Uniqueness and stability of Riemann solutions with large oscillation in gas dynamics*, Commun. Math. Phys. 2001 (to appear).

[11] Chen, G.-Q., Levermore, C. D., and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. **47** (1994), 787–830.

[12] Chen, G.-Q. and M. Rascle, *Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws*, Arch. Rational Mech. Anal. **153** (2000), 205-220.

[13] Dafermos, C. M., Hyperbolic Conservation Laws in Continuum Physics, Springer-Verlag: Berlin, 1999.

[14] DiPerna, R., *Convergence of the viscosity method for isentropic gas dynamics*, Comm. Math. Phys. **91** (1983), 1-30.

[15] DiPerna, R., *Measure-valued solutions to conservation laws*, Arch. Rational Mech. Anal. **88** (1985), 223–270.

[16] DiPerna, R., *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J. **28** (1979), 137-188.

[17] Evans, L. C. and R. F. Gariepy, Lecture Notes on Measure Theory and Fine Properties of Functions, CRC Press: Boca Raton, Florida, 1992.

[18] Federer, H., Geometric Measure Theory, Springer-Verlag: Berlin-Heidelberg-New York, 1969.

[19] Friedrichs, K. O. and P. D. Lax, *Systems of conservation equations with a convex extension*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 1686–1688.

[20] E. Gagliardo, *Caratterizioni delle tracce sulla frontiera relativa ad alcune classi di funzioni in n variabli*, Rend. Sem. Mat. Univ. Padova, **27** (1957), 284-305.

[21] Glimm, J., *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. **18** (1965), 95-105.

[22] Glimm, J. and P. D. Lax, *Decay of Solutions of Systems of Nonlinear Hyperbolic Conservation Laws*, Mem. Amer. Math. Soc. **101**, AMS: Providence, R.I. 1970.

[23] Harrison, J. and A. Norton, *The Gauss-Green theorem for fractal boundaries.* Duke Math. J. **67** (1992), 575–588.

[24] Harrison, J., *Stokes' theorem for nonsmooth chains*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), 235–242.

[25] Jurkat, W. B. and D. J. F. Nonnenmacher, *A generalized n-dimensional Riemann integral and the divergence theorem with singularities*, Acta Sci. Math. (Szeged) **59** (1994), 241–256.

[26] Katsoulakis, M. and A. Tzavaras, *Contractive relaxation systems and the scalar multidimensional conservation law*, Comm. Partial Diff. Eqs. **22** (1997), 195–233.

[27] Kruzkov S. N., *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.) (Russian) **81**(123) (1970), 228–255.

[28] Lax, P. D., *Hyperbolic systems of conservation laws*, Comm. Pure Appl. Math. **10** (1957), 537-566.

[29] Lax, P. D., Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, CBMS. **11**, SIAM, Philadelphia, 1973.

[30] Lax, P. D., *Shock waves and entropy*, In: Contributions to Functional Analysis, ed. E. A. Zarantonello, Academic Press, New York, 1971, pp. 603–634.

[31] Lions, P. L., Perthame, B., and E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc. **7** (1994), 169–191.

[32] Liu, T.-P. and J. Smoller, *On the vacuum state for the isentropic gas dynamics equations*, Adv. Appl. Math. **1** (1980), 345–359.

[33] Natalini, R., *Convergence to equilibrium for the relaxation approximations of conservation laws*, Comm. Pure Appl. Math. **49** (1996), 795–823.

[34] Málec, J., Nečas, J., Rokyta, M., and M. Ružička, Weak and Measure-valued Solutions to Evolutionary PDEs, Chapman & Hall: London, 1996.

[35] Mascia, C., Porretta, A., and A. Terracina, *Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations*, Arch. Rational Mech. Anal. (to appear).

[36] Nonnenmacher, D. J. F., *Sets of finite perimeter and the Gauss-Green theorem with singularities*, J. London Math. Soc. (2) **52** (1995), 335–344.

[37] Otto, F., *First order equations with boundary conditions*, Preprint no. 234, SFB 256, Univ. Bonn., 1992.

[38] Pfeffer, W. F., Derivation and Integration, Cambridge Tracts in Math. **140**, Cambridge Univ. Press: Cambridge, 2001.

[39] Rodrigues, J.-F., Obstacle Problems in Mathematical Physics, North-Holland Mathematics Studies **134**, Elsevier Science Publishers B.V. 1987.

[40] Schwartz, L., Théorie des Distributions, Actualites Scientifiques et Industrielles **1091**, **1122**, Herman: Paris, 1950-51.

[41] Szepessy, A., *An existence result for scalar conservation laws using measure valued solutions*, Commun. Partial Diff. Eqs. **14** (1989), 1329–1350.

[42] Serre, D., Systems of Conservation Laws I, II, Cambridge University Press: Cambridge, 2000.

[43] Shapiro, V., *The divergence theorem for discontinuous vector fields.* Ann. Math. (2) **68** 1958, 604-624.

[44] Stein, E., Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press: Princeton, NJ, 1970.

[45] Vasseur, A., *Strong traces for solutions to multidimensional scalar conservation laws*, Arch. Rational Mech. Anal. **160** (2001), 181-193.

[46] Volpert, A. I., *The space BV and quasilinear equations*, Mat. Sb. (N.S.) **73** (1967), 255–302, Math. USSR Sbornik, **2** (1967), 225–267 (in English).

[47] Wagner, D., *Equivalence of the Euler and Lagrange equations of gas dynamics for weak solutions*, J. Diff. Eqs. **68** (1987), 118–136.

[48] Whitney, H., Geometric Integration Theory, Princeton Univ. Press: Princeton, NJ, 1957.
[49] Whitney, H., *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63-89.
[50] Ziemer, W. P., *Cauchy flux and sets of finite perimeter*, Arch. Rational Mech. Anal. **84** (1983), 189-201.
[51] Ziemer, W. P., Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation, Springer-Verlag: New York, 1989.

(G.-Q. Chen) Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, Illinois 60208-2730, USA
E-mail address: gqchen@math.northwestern.edu

(H. Frid) Instituto de Matemática Pura e Aplicada - IMPA, Estrada Dona Castorina, 110, Rio de Janeiro, RJ 22460-320, Brazil
E-mail address: hermano@impa.br