UNIQUENESS AND ASYMPTOTIC STABILITY OF RIEMANN SOLUTIONS FOR THE COMPRESSIBLE EULER EQUATIONS

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Abstract. We prove the uniqueness of Riemann solutions in the class of entropy solutions in $L^\infty \cap BV_{loc}$ for the 3×3 system of compressible Euler equations, under usual assumptions on the equation of state for the pressure which imply strict hyperbolicity of the system and genuine nonlinearity of the first and third characteristic families. In particular, if the Riemann solutions consist of at most rarefaction waves and contact discontinuities, we show the global L^2-stability of the Riemann solutions even in the class of entropy solutions in L^∞ with arbitrarily large oscillation for the 3×3 system. We apply the framework established in [4, 5] to show that the uniqueness of Riemann solutions implies their inviscid asymptotic stability under L^1 perturbation of the Riemann initial data, as long as the corresponding solutions are in L^∞ and have local bounded total variation satisfying a natural condition on its growth with time. No specific reference to any particular method for constructing the entropy solutions is made. Our uniqueness result for Riemann solutions can easily be extended to entropy solutions $U(x, t)$, piecewise Lipschitz in x, for any $t > 0$.

Descriptive Title: Entropy Solutions for the Compressible Euler Equations

1. Introduction

We are concerned with the large-time behavior of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$, $\mathbb{R}^2_+ = \mathbb{R}\times[0, \infty)$, for the 3×3 system of Euler equations for compressible fluids with a general pressure law, whose initial data are an L^1 perturbation of Riemann initial data. More specifically, for any entropy solution $U(x, t) \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ which represents the evolution through the Euler equations of perturbed Riemann initial data corresponding to the Riemann solution $R(x/t)$, the problem is whether $U(\xi t, t) \to R(\xi)$ in $L^1_{loc}(\mathbb{R})$ as $t \to \infty$. As shown in [4, 5], this problem can be reduced to proving the uniqueness of the Riemann solution in the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$, provided that $U(x, t)$ satisfies certain natural growth condition for its local total variation (see (1.23)), which guarantees the L^1_{loc} compactness of the associated scaling sequence, $U^T(x, t) = U(Tx, Tt)$.

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The Euler system for compressible fluids in Lagrangian coordinates reads

$$\begin{cases} \partial_t u + \partial_x p = 0, \\ \partial_t v - \partial_x u = 0, \\ \partial_t (e + \frac{u^2}{2}) + \partial_x (pu) = 0, \end{cases} \tag{1.1}$$

where u, p, v, and e represent the velocity, the pressure, the specific volume ($v = 1/\rho$, ρ the density), and the internal energy of the fluids, respectively. Other important physical variables are the temperature θ and the entropy S. To close system (1.1), one needs the basic law of thermodynamics which translates into the differential equation:

$$de = \theta\, dS - p\, dv. \tag{1.2}$$

It follows from relation (1.2) that any two of the variables v, p, e, θ, and S determine the other three, through the constitutive relations. We then choose v and S as the independent variables to obtain the equations of state $p = p(v, S)$, $\theta = \theta(v, S)$, and $e = e(v, S)$. These functions must be compatible with (1.2):

$$\begin{cases} \partial_v e(v, S) = -p(v, S), \\ \partial_S e(v, S) = \theta(v, S), \\ \partial_v \theta(v, S) = -\partial_S p(v, S). \end{cases} \tag{1.3}$$

For ideal polytropic gases, $pv = R\theta$, $e = c_v \theta$, where R and c_v are positive constants, and the choice of v and S as the independent variables implies

$$p(v, S) = \kappa e^{S/c_v} v^{-\gamma}, \quad \gamma = 1 + \frac{R}{c_v} > 1. \tag{1.4}$$

Using (1.2), any smooth solution of (1.1) is also a smooth solution of the system

$$\begin{cases} \partial_t u + \partial_x p = 0, \\ \partial_t v - \partial_x u = 0, \\ \partial_t S = 0. \end{cases} \tag{1.5}$$

However, for discontinuous solutions of (1.1), the last equation of (1.5) no longer holds, even in the weak sense, and it must be replaced by the so called Clausius inequality

$$\partial_t a(S) \geq 0 \tag{1.6}$$

in the sense of distributions for any C^1 function $a(S)$ with $a'(S) \geq 0$.

System (1.1) can be written in the general form:

$$\partial_t U + \partial_x F(U) = 0, \tag{1.7}$$

by setting $U = (u, v, E)$, with $E = \frac{1}{2}u^2 + e$, and $F(U) = (p, -u, pu)$, with p as a function of (u, v, E).

Recall that the Riemann problem for a general system (1.7) is a special Cauchy problem with initial data:

$$U|_{t=0} \equiv R_0(x) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0, \end{cases} \tag{1.8}$$

where U_L and U_R are constant states. We are interested in the large-time behavior of solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of the Cauchy problem for (1.1) with initial data:

$$(u, v, E)|_{t=0} = (u_0, v_0, E_0), \tag{1.9}$$

which are an L^1 perturbation of Riemann initial data. That is,

$$U|_{t=0} = U_0(x) \equiv R_0(x) + P_0(x), \qquad \text{with } P_0(x) \in L^1(\mathbb{R}). \tag{1.10}$$

To avoid ambiguities, we henceforth denote $W = (u, v, S)$ to distinguish from $U = (u, v, E)$.

We recall that an entropy-entropy flux pair for (1.7) is a pair $(\eta(U), q(U))$ of C^1 functions satisfying

$$\nabla\eta(U)\nabla F(U) = \nabla q(U). \tag{1.11}$$

Clearly, $(a(S), 0), a(S) \in C^1$, is an entropy-entropy flux pair for (1.1). Throughout this paper, the following entropy-entropy flux pair for (1.1) plays an important role:

$$\eta_*(W) = \frac{u^2}{2} - \int_{v_0}^{v} p(\sigma, S)\, d\sigma + K(S), \tag{1.12}$$

$$q_*(W) = up(v, S), \tag{1.13}$$

where $K(S)$, a C^2 function, and $v_0 > 0$ are to be suitably chosen. As usual, we assume that the function $p = p(v, S)$ satisfies

$$p_v(v, S) < 0, \qquad p_{vv}(v, S) > 0, \qquad \nabla^2_{v,S} p(v, S) \geq 0, \tag{1.14}$$

in the domain $v > 0$. The polytropic gases clearly satisfy (1.14). These assumptions imply that the system is strictly hyperbolic and that the first and third characteristic families are genuinely nonlinear in the sense of Lax [14]. They also imply the uniqueness of the Riemann solution in the class of the self-similar piecewise smooth solutions consisting of shocks, rarefaction waves, and contact discontinuities (cf. [14]).

At the onset, we require that $K(S)$ and v_0 be such that $\eta_*(W)$ *is a strictly convex function* in the bounded domain $\mathcal{V} \subset \{v > 0\}$ where the solutions to be considered take their values. This amounts to assume

$$K''(S) > \int_{v_0}^{v} p_{SS}(\sigma, S)\, d\sigma - \frac{p_S^2(v, S)}{p_v(v, S)}, \qquad (u, v, S) \in \mathcal{V}, \tag{1.15}$$

where (1.14) has been used.

Definition 1.1. A bounded measurable function $U(x, t) = (u(x, t), v(x, t), E(x, t))$ is an entropy solution of (1.1) and (1.9) in $\Pi_T = \mathbb{R} \times [0, T)$ if $U(x, t)$ takes its values in the physical region $\{(v, u, E) \,|\, v > 0\}$ and satisfies the following.

(i). Equations (1.1) hold in the weak sense in Π_T, *i.e.*, for all $\phi \in C_0^1(\Pi_T)$,

$$\int_{\Pi_T} \{U\partial_t\phi + F(U)\partial_x\phi\}\, dx\, dt + \int_{-\infty}^{\infty} U_0(x)\phi(x, 0)dx = 0, \tag{1.16}$$

with $U = (u, v, E)$ and $F(U) = (p, -u, pu)$.

(ii). The Clausius inequality holds in the sense of distributions in Π_T, *i.e.*, for all nonnegative $\phi \in C_0^1(\Pi_T)$,

$$\int_{\Pi_T} a(S)\partial_t\phi\, dx\, dt + \int_{-\infty}^{\infty} a(S_0(x))\phi(x, 0)dx \leq 0, \tag{1.17}$$

for any $a(S) \in C^1$ and $a'(S) \geq 0$.

Observe that (1.16) and (1.17) imply that any entropy solution $W(x,t)$ satisfies the entropy inequality

$$\int_{\Pi_T} \{\eta_*(W)\partial_t\psi + q_*(W)\partial_x\psi\}\,dx\,dt + \int_{-\infty}^{\infty} \eta_*(W_0(x))\psi(x,0)\,dx \geq 0, \tag{1.18}$$

for any nonnegative $\psi \in C_0^1(\Pi_T)$, where (η_*, q_*) is defined in (1.12)-(1.13), provided that $K(S)$ and $v_0 > 0$ are chosen such that

$$K'(S) < \theta(v_0, S), \quad \text{i.e.} \quad \partial_S\eta_*(W) < \theta(v,S). \tag{1.19}$$

Indeed, by (1.3) and (1.12), we have

$$\eta_*(W) = E + a_*(S), \qquad a_*(S) = K(S) - e(v_0, S).$$

Now, using (1.3) again and (1.19), we have $a_*'(S) \leq 0$, which allows to conclude (1.18) from (1.16) and (1.17).

Throughout this paper, we choose $K(S)$ and $v_0 > 0$ such that, besides (1.15), both (1.19) and

$$\partial_S\eta_*(W) \geq 0, \quad \text{i.e.} \quad K'(S) \geq \theta(v_0,S) - \theta(v,S) \tag{1.20}$$

hold. That is, we assume (1.15) and

$$\theta(v_0,S) - \theta(v,S) \leq K'(S) < \theta(v_0,S), \tag{1.21}$$

in the bounded physical region $\mathcal{V} \subset \{(u,v,S)\,|\,v>0\} \subset \mathbb{R}^3$ under consideration.

It is easy to check that, for the ideal polytropic gases (*i.e.* the γ-law gases), for any compact subset $\mathcal{V} \subset \{v > 0\}$, (1.15) and (1.21) hold, by choosing, for instance, $v_0 > 0$ arbitrary and $K(S) = \frac{\kappa c_v}{R} v_0^{-\gamma+1} e^{S/c_v} - \delta S$ with $0 < \delta \leq \min\theta(v,S)$. More generally, if the equations of state are such that $\theta(v,S)$ and $p(v,S)$ satisfy $\theta_S(v,S) + \frac{p_S^2(v,S)}{p_v(v,S)} > 0$ in $\{v > 0\}$, then (1.15) and (1.21) hold in any compact subset of $\{v > 0\}$ with $v_0 > 0$ arbitrary and $K(S)$ given by $K'(S) = \theta(v_0, S) - \delta S$.

Our main goal in this paper is to prove the uniqueness of the classical Riemann solution $R(\xi)$, corresponding to the Riemann data (1.8), which in turn implies that, for any entropy solution $U(x,t) \in L^\infty \cap BV_{loc}(\mathbb{R}_+^2)$ of (1.1) and (1.10), whose local total variation satisfies a certain natural growth condition (*cf.* (1.23) below), one has

$$ess \lim_{t\to\infty} \int_{-L}^{L} |U(\xi t, t) - R(\xi)| d\xi = 0, \qquad \text{for any } L > 0. \tag{1.22}$$

By the framework established in Chen-Frid [4, 5], for any entropy solution of (1.1) and (1.10), $U(x,t) \in L^\infty \cap BV_{loc}(\mathbb{R}_+^2)$, the asymptotic stability problem can be reduced to the problem of the uniqueness of the Riemann solution in the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}_+^2)$, provided the local total variation of $U(x,t)$ satisfies a natural growth condition:

$$\begin{cases} \text{There exists } c_0 > 0 \text{ such that, for all } c \geq c_0, \text{ there is } C > 0, \\ \text{depending only on } c, \text{ such that } TV(U|\mathcal{K}_{c,T}) \leq CT, \quad \text{for any } T > 0, \end{cases} \tag{1.23}$$

where

$$\mathcal{K}_{c,T} = \{(x,t) \in \mathbb{R}_+^2 \,|\, |x| < ct,\ t \in (0,T)\}. \tag{1.24}$$

Actually, in [4], the growth condition is imposed on the total variation over the cylinders $\{|x| < cT\} \times (0,T)$ instead of the cones $\mathcal{K}_{c,T}$. But, in this way, condition (1.23) is more general and does not require any modification in the arguments.

We recall that the growth condition (1.23), which is necessary only for the study of the asymptotic behavior, is natural since any solution obtained by the Glimm method or related methods satisfies (1.23). In particular, (1.23) is verified for BV solutions satisfying the Glimm-Lax condition [13], namely,

$$TV_x(U(\cdot,t)|(-L,L)) \le C_0 \frac{L}{t},$$

for any $t > 0$ and $L > c_0 t$, for some fixed $C_0, c_0 > 0$, where TV_x denotes the total variation in x, for fixed $t > 0$.

Hence, in the main part of this paper, we focus on the uniqueness problem for solutions in $L^\infty \cap BV_{loc}$ of (1.1) and (1.8) requiring *only* (1.16)-(1.17) (see Sections 3-4). To achieve this goal we extend the method developed by DiPerna [9] for the 2×2 case to our 3×3 system. We remark that our uniqueness result for Riemann solutions can easily be extended to the uniqueness of entropy solutions of (1.1) which are piecewise Lipschitz in x, for any $t > 0$, in the same spirit as DiPerna's theorem in [9] for the 2×2 case. We also remark that all the results in this paper have a straightforward equivalent version for system (1.1) written in Eulerian coordinates, namely,

$$\begin{cases} \partial_\tau \rho + \partial_y(\rho u) = 0, \\ \partial_\tau(\rho u) + \partial_y(p + \rho u^2) = 0, \\ \partial_\tau(\rho(\frac{1}{2}u^2 + e)) + \partial_y(\rho u(\frac{1}{2}u^2 + e) + pu) = 0, \end{cases} \tag{1.25}$$

where $\rho = 1/v$ is the density and

$$\tau = t, \quad y = \int_0^x v(s,t)\,ds + \int_0^t u(0,\sigma)\,d\sigma.$$

In order to avoid repetitions, we will not state the corresponding results for (1.25) which can are obtained using the well-known equivalence between (1.1) and (1.25) (see, *e.g.*, [25]).

Finally, we recall recent fundamental results for $m \times m$ systems in Bressan-Crasta-Piccoli [1] (also see [2]), Bressan-Liu-Yang [3], and Liu-Yang [19] on the L^1-stability of entropy solutions in $L^\infty \cap BV$ obtained by the Glimm scheme and the wave front-tracking method, or satisfying an additional regularity, with small total variation in x for all $t > 0$. We remark that the uniqueness results in this paper cannot be directly obtained from those stability results since the results presented here neither impose smallness restrictions on the total variation and additional regularity of the solutions, nor need specific reference to any particular method for constructing the entropy solutions. In this connection we also recall that, for special cases of system (1.1), including the one for polytropic gases, there are many existence results of solutions in $L^\infty \cap BV_{loc}$ with non-small total variation (see, *e.g.*, [13, 17, 22, 20, 23]). We also refer the reader to Dafermos [8] for the stability of classical solutions of hyperbolic systems of conservation laws.

This paper is organized as follows. In Section 2, we discuss the relation between the uniqueness of Riemann solutions in the class of entropy solutions in $L^\infty \cap BV_{loc}$ and the asymptotic stability of Riemann solutions with respect to L^1 initial perturbation. In Section 3, in order to make our analysis clearer, we address separately the case when the Riemann solution is shock-free, *i.e.*, it may contain only rarefaction waves of the first and third characteristic families and, possibly, a contact discontinuity of the second family. Then, in Section 4, we consider the general case in which the Riemann solution may contain shock waves.

2. Asymptotic Stability and Uniqueness of Riemann Solutions

One of the main features of hyperbolic systems of conservation laws is the invariance under self-similar scaling. That is, if $U(x,t)$ solves (1.7), then

$$U^T(x,t) = U(Tx, Tt) \tag{2.1}$$

also solves (1.7). In particular, since the Riemann initial data are also self-similar scaling invariant, a natural question is whether any entropy solution of the Riemann problem is self-similar invariant. This question is answered in the present situation by our uniqueness results proved in Sections 3-4.

The following proposition establishes the relationship among uniqueness of entropy solutions of the Riemann problem, compactness of the scaling sequence $U^T(x,t)$, and asymptotic stability in the sense of (1.22).

Proposition 2.1. *Assume that the Cauchy problems* (1.7)-(1.10) *satisfy the following.*

(i). *System* (1.7) *is endowed with a strictly convex entropy;*
(ii). *The Riemann solution $R(\xi), \xi = x/t$, is piecewise Lipschitz in the variable ξ and is unique in the class of entropy solutions of* (1.7)-(1.8) *in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$;*
(iii). *Given an entropy solution $U(x,t) \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of* (1.7)-(1.8)*, the sequence $U^T(x,t)$ is compact in $L^1_{loc}(\mathbb{R}^2_+)$, and any limit function of its subsequences is still in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$.*

Then the Riemann solution $R(x/t)$ is asymptotically stable in the sense of (1.22) *in the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ corresponding to initial perturbations $P_0(x) \in L^1$.*

Proof. This fact can be seen as follows. Take any subsequence $\{U^{T_k}(x,t)\}_{k=1}^\infty \subset \{U^T(x,t)\}_{T>0}$. Condition (iii) implies that there exists a further subsequence converging in L^1_{loc} to $\tilde{U}(x,t) \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ satisfying the same initial data as $R(x/t)$. Condition (ii) then ensures that $\tilde{U}(x,t) = R(x/t)$ a.e., which is unique. This indicates that the whole sequence $\{U^T(x,t)\}_{T>0}$ converges to the Riemann solution $R(x/t)$ in $L^1_{loc}(\mathbb{R}^2_+)$. For any $0 < r < \infty$, we have

$$\begin{aligned}
&\frac{1}{T^2}\int_0^T \int_{|\xi|\le r} |U(\xi t, t) - R(\xi)| t\,d\xi dt = \frac{1}{T^2}\int_0^T \int_{|x|\le rt} |U(x,t) - R(x/t)| dxdt \\
&= \int_0^1 \int_{|x|\le rt} |U^T(x,t) - R(x/t)| dxdt \to 0, \qquad \text{when} \quad T \to \infty,
\end{aligned}$$

which implies

$$\frac{1}{T}\int_0^T \int_{|\xi|\le r} |U(\xi t, t) - R(\xi)|\, d\xi dt \to 0, \qquad \text{when} \quad T \to \infty.$$

In view of condition (i) and the piecewise Lipschitz continuity of $R(\xi)$ given by condition (ii), we can use Theorem 3.1 of Chen-Frid [5] (also see [4]) to conclude that the Riemann solution is asymptotically stable in the sense of (1.22) with respect to the initial perturbation $P_0(x)$. □

For entropy solutions in $L^\infty \cap BV_{loc}$ satisfying condition (1.23), the compactness of the scaling sequence is guaranteed by the following observation.

Proposition 2.2. *Assume that $U(x,t) \in L^\infty \cap BV_{\rm loc}(\mathbb{R}\times(0,\infty))$ satisfies that, for some fixed $c_0 > 0$, for any $c \ge c_0$, there exists $C > 0$, depending only on c, such that*

$$TV(U|\mathcal{K}_{c,T_*}) \le CT_*, \qquad \text{for all } T_* > 0. \tag{2.2}$$

Then, for each $T > 0$, $U^T(x,t)$, defined by (2.1), also satisfies (2.2) with the same constant C, depending on c. Furthermore, the sequence U^T is compact in $L^1_{\rm loc}(\mathbb{R}^2_+)$.

Proof. The proof of the first part is immediate from a scaling argument. Then, the compactness in L^1_{loc} of U^T is a consequence of the well-known compactness in L^1_{loc} of bounded sets in BV (*cf.* [10]). □

As remarked above, condition (1.23) holds for the entropy solutions constructed by Glimm's method (see [12, 13, 17, 22, 23]). Therefore, according to Proposition 2.2, condition (iii) in Proposition 2.1 is verified in all these cases.

As we indicated above, for solutions in $L^\infty \cap BV_{loc}$, the asymptotic problem reduces to the uniqueness problem of Riemann solutions in the class of entropy solutions in $L^\infty \cap BV_{loc}$ of (1.7)-(1.8). In what follows we focus on the uniqueness problem.

3. Uniqueness of Shock-Free Riemann Solutions

In this section we consider the uniqueness of classical Riemann solutions of (1.1) and (1.8), which are shock-free. We recall that (1.1) has the eigenvalues

$$\lambda_1 = -\sqrt{-p_v}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{-p_v},$$

and the corresponding right eigenvectors r_1, r_2, r_3. The first and third families of (1.1) are genuinely nonlinear, *i.e.*, $\nabla\lambda_i \cdot r_i \ne 0$, $i = 1,3$, and the second family is linearly degenerate, *i.e.*, $\nabla\lambda_2 \cdot r_2 = 0$. For the eigenvectors, we have

$$r_i = \nabla g \tilde{r}_i, \qquad i = 1,2,3,$$

where $U = g(W)$ is the transformation which takes (1.1) into (1.5) and

$$\tilde{r}_1 = c(W)(-\lambda_1, 1, 0)^\top, \quad \tilde{r}_2 = (0, p_S, -p_v)^\top, \quad \tilde{r}_3 = c(W)(-\lambda_1, -1, 0)^\top.$$

Here $c(W) = 2\sqrt{-p_v}/p_{vv}$ is a normalizing factor such that $\nabla_W \lambda_i \cdot \tilde{r}_i = 1$, which is equivalent to $\nabla_U \lambda_i \cdot r_i = 1$, $i = 1,3$.

Under conditions (1.14), which guarantee the strict hyperbolicity of (1.1) and the genuine nonlinearity of the first and third characteristic families, it follows (see, *e.g.*, [14, 21]) that a classical shock-free Riemann solution generally has the form:

$$R(x/t) \equiv \begin{cases} U_L, & x/t \le \lambda_1(U_L), \\ R_1(x/t), & \lambda_1(U_L) < x/t < \lambda_1(U_M), \\ U_M, & \lambda_1(U_M) \le x/t < 0, \\ U_N, & 0 < x/t \le \lambda_3(U_N), \\ R_3(x/t), & \lambda_3(U_N) < x/t < \lambda_3(U_R), \\ U_R, & x/t \ge \lambda_3(U_R), \end{cases} \tag{3.1}$$

where $R_1(\xi)$ and $R_3(\xi)$ are the solutions of the boundary value problems

$$\begin{cases} \frac{d}{d\xi}R_1(\xi) = r_1(R_1(\xi)), & \lambda_1(U_L) < \xi, \\ R_1(\lambda_1(U_L)) = U_L, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \frac{d}{d\xi}R_3(\xi) = r_3(R_3(\xi)), & \xi < \lambda_3(U_R), \\ R_3(\lambda_3(U_R)) = U_R. \end{cases} \tag{3.3}$$

The states U_M and U_N are also completely determined by (3.2) and (3.3). The best way to see this fact is first to recall that S is constant over rarefaction curves of the first and third families, since S is a Riemann invariant of the first and third families (see [21]). Similarly, u and p are both constant over the wave curves of the second (linearly degenerate) family. Hence, in the space (u, p, S), we can project the curves R_1 and R_3 in the plane (u, p), find the intersection of these projected curves, and immediately obtain the two points in the planes $S = S(R_1)$ and $S = S(R_3)$.

Let $W(x,t) = (u(x,t), v(x,t), S(x,t))$ be a certain entropy solution of the Riemann problem (1.1) and (1.8). Denote by $\overline{W}(x,t) = (\overline{u}(x,t), \overline{v}(x,t), \overline{S}(x,t))$ the classical self-similar solution of this problem, that is, $\overline{W}(x,t)$ equals $R(x/t)$ written in the coordinates (u, v, S). We use the quadratic entropy-entropy flux pairs obtained from (η_*, q_*) through a procedure introduced by Dafermos (*cf.* [8, 9])

$$\alpha(W, \overline{W}) = \eta_*(W) - \eta_*(\overline{W}) - \nabla\eta_*(\overline{W})(W - \overline{W}), \tag{3.4}$$

$$\beta(W, \overline{W}) = q_*(W) - q_*(\overline{W}) - \nabla\eta_*(\overline{W})(f(W) - f(\overline{W})), \tag{3.5}$$

where $f(W) = (p(v,S), -u, 0)^\top$.

Motivated by [9], we consider the measures

$$\theta = \partial_t\eta_*(W(x,t)) + \partial_x q_*(W(x,t)) - \partial_S\eta_*(\overline{W}(x,t))\partial_t S, \quad (x,t) \in \mathbb{R}^2_+ - \{x = 0\},$$
$$\gamma = \partial_t\alpha(W(x,t), \overline{W}(x,t)) + \partial_x\beta(W(x,t), \overline{W}(x,t)).$$

Hypothesis (1.20) also guarantees that $\theta \le 0$, as a Radon measure in $\{x \ne 0\}$, and that $\gamma \le 0$, in any domain in which $\overline{W}$ is constant, due to (1.16) and (1.18).

The first important fact is that $\gamma\{\ell\} = 0$, where $\ell = \{(0,t) \,|\, t > 0\}$, since $\gamma\{\ell\} = \int_\ell [\beta(W, \overline{W})]\, d\mathcal{H}^1$ and $[\beta(W, \overline{W})] = 0$, $\mathcal{H}^1$-a.e. over ℓ. The latter follows from $\beta(W, \overline{W}) = (u - \bar{u})(p - \bar{p})$ and the fact that u, p, $\bar{u}$, $\bar{p}$ cannot change across the jump discontinuities of W and $\overline{W}$ over ℓ, because of the Rankine-Hugoniot relations. Here and in what follows we use the notation $[H] = H_- - H_+$, where H_- and H_+ are the values of the function H on the left-hand side and the right-hand side of the discontinuity curve, respectively.

Set

$$\Omega_1 = \{\, (x,t) \,|\, \lambda_1(U_L) < x/t < \lambda_1(U_M),\ t > 0 \,\},$$
$$\Omega_3 = \{\, (x,t) \,|\, \lambda_3(U_N) < x/t < \lambda_j(U_R),\ t > 0 \,\},$$

the rarefaction wave regions of the classical Riemann solution. Over these regions, γ satisfies

$$\begin{aligned}\gamma =&\partial_t\alpha(W,\overline{W}) + \partial_x\beta(W,\overline{W})\\ =&\partial_t\eta_*(W) + \partial_x q_*(W) - \partial_S\eta_*(\overline{W})\partial_t S\\ &- \nabla^2\eta_*(\overline{W})(\partial_t\overline{W}, W-\overline{W}) - \nabla^2\eta_*(\overline{W})(\partial_x\overline{W}, f(W)-f(\overline{W}))\\ =&\theta - \nabla^2\eta_*(\overline{W})(\partial_x\overline{W}, Qf(W,\overline{W})),\end{aligned} \tag{3.6}$$

where we used the fact that $\nabla^2\eta_*\nabla f$ is symmetric, and $Qf(W,\overline{W}) = f(W) - f(\overline{W}) - \nabla f(\overline{W})(W-\overline{W})$ is the quadratic part of f at $\overline{W}$. Recall that

$$\nabla^2\eta(W)(\tilde{r}_i(W),\tilde{r}_j(W)) = 0, \qquad i \neq j, \tag{3.7}$$

for any entropy η. We notice that, because of (3.7),

$$\tilde{l}_j(\overline{W}) = \tilde{r}_j(\overline{W})\nabla^2\eta_*(\overline{W}) \tag{3.8}$$

is a left eigenvector of $\nabla f(\overline{W})$ corresponding to the eigenvalue $\lambda_j(\overline{W})$, $j = 1, 3$. We also easily see that, for $(x,t) \in \Omega_j$, one has

$$\frac{\partial\overline{W}(x,t)}{\partial x} = \frac{1}{t}\tilde{r}_j(\overline{W}(x,t)), \qquad j = 1, 3.$$

Then, by (3.1) and (3.6), for any Borel set $E \subset \Omega_j$, $j = 1, 3$, we have

$$\gamma(E) = \theta(E) - \int_E \frac{1}{t}\tilde{l}_j(\overline{W})Qf(W,\overline{W})\,dxdt. \tag{3.9}$$

As a direct consequence of (3.9), we have the following result.

Theorem 3.1. *Let $R(x/t)$ be the classical shock-free solution of the Riemann problem* (1.1) *and* (1.8)*, given by* (3.1)*. Let $U(x,t)$ be any entropy solution in $L^\infty \cap BV_{loc}(\mathbb{R}\times(0,T))$ of* (1.1) *and* (1.8) *in $\Pi_T = \mathbb{R}\times(0,T)$. Assume* (1.14)-(1.15) *and* (1.21) *are satisfied. Then $U(x,t) = R(x/t)$, a.e. in Π_T.*

Proof. Let Π_t denote the strip $\{\,(x,s)\,|\,x\in\mathbb{R},\ 0<s<t\}$ and $\Omega_j(t) = \Omega_j \cap \Pi_t, j = 1, 3$, $\ell_t = \ell\cap\Pi_t$. By the Gauss-Green formula for BV functions (cf. [10, 24]) and the finiteness of propagation speed of the solution (see [9]), we have

$$\gamma\{\Pi_t\} = \int_{-\infty}^{\infty}\alpha(W(x,t),\overline{W}(x,t))\,dx. \tag{3.10}$$

On the other hand, we have from (3.9)

$$\gamma\{\Omega_j(t)\} = \theta\{\Omega_j(t)\} - \int_{\Omega_j(t)}\frac{1}{s}\tilde{l}_j(\overline{W})Qf(W,\overline{W})\,dx\,ds, \qquad j = 1, 3.$$

Since $\overline{W}(x,t)$ is constant in each component of $\Pi_t - \{\Omega_1(t)\cup\ell_t\cup\Omega_3(t)\}$, one has

$$\gamma\{\Pi_t\} = \gamma\{\ell_t\} + \theta\{\Pi_t - \{\ell_t\}\} - \sum_{j=1,3}\int_{\Omega_j(t)}\frac{1}{s}\tilde{l}_j(\overline{W})Qf(W,\overline{W})\,dxds. \tag{3.11}$$

Then, since $\theta \leq 0$ and $\gamma\{\ell_t\} = 0$, it suffices to prove that $\tilde{l}_j(\overline{W})Qf(W,\overline{W}) \geq 0$. Now, one can easily check that $\tilde{l}_j(\overline{W})$ is a positive multiple of

$$\left(1, \pm\sqrt{-p_v(\overline{v},\overline{S})}, \mp p_S(\overline{v},\overline{S})/\sqrt{-p_v(\overline{v},\overline{S})}\right),$$

and

$$Qf(W,\overline{W}) = (p(v,S) - p(\overline{v},\overline{S}) - p_v(\overline{v},\overline{S})(v-\overline{v}) - p_S(\overline{v},\overline{S})(S-\overline{S}), 0, 0)^\top.$$

Hence, by (1.14), we actually have $\tilde{l}_j(\overline{W})Qf(W,\overline{W}) \geq 0$, $j = 1, 3$. This completes the proof. $\square$

Remark 3.1. From the proof of Theorem 3.1 above, it is easy to see that the same arguments lead to an even stronger result. Actually, it is immediate to deduce from the above proof that the classical shock-free Riemann solution is in fact globally stable in L^2 with respect to its initial perturbations. More specifically, if $U(x,t)$ is an entropy solution in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of (1.1) and (1.10), then there exists a constant $C > 0$ such that

$$\int_{|x|\leq L} |U(x,t) - R(x/t)|^2\, dx \leq C \int_{|x|\leq L+Mt} |U_0(x) - R_0(x)|^2\, dx, \tag{3.12}$$

where $C, L, M > 0$ are independent of t. Moreover, if the classical Riemann solution consists of at most rarefaction waves and contact discontinuities, similar arguments to those in the above proof yield (3.12) for any L^∞ entropy solution of (1.1) and (1.10), with the aid of the theory of the divergence-measure fields developed in [6].

Combining Theorem 3.1 with Proposition 2.1, we conclude

Theorem 3.2. *Let $R(x/t)$ be the classical shock-free solution of the Riemann problem* (1.1) *and* (1.8)*, given by* (3.1)*. Let $U(x,t)$ be any entropy solution in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of* (1.1) *and* (1.10) *satisfying* (1.23)*. Assume* (1.14)-(1.15) *and* (1.21) *are satisfied. Then $U(x,t)$ asymptotically tends to $R(x/t)$ in the sense of* (1.22).

4. Uniqueness of General Riemann Solutions

In this section we prove the uniqueness of entropy solutions of the Riemann problem (1.1) and (1.8) for the general case in which the corresponding self-similar classical Riemann solution may also contain shock waves. For concreteness, we assume that the classical Riemann solution has the following generic form

$$R(x/t) = \begin{cases} U_L, & x/t < \sigma_1, \\ U_M, & \sigma_1 < x/t < 0, \\ U_N, & 0 < x/t \leq \lambda_3(U_N), \\ R_3(x/t), & \lambda_3(U_N) < x/t < \lambda_3(U_R), \\ U_R, & x/t \geq \lambda_3(U_R), \end{cases} \tag{4.1}$$

where $\sigma_1 = \sigma_1(U_L, U_M)$ is the shock speed, determined by the Rankine-Hugoniot relation,

$$\sigma_1(U_L - U_M) - (F(U_L) - F(U_M)) = 0, \tag{4.2}$$

and $R_3(\xi)$ is the solution of the boundary value problem (3.3). The 1-shock wave connecting U_L and U_M satisfies the Lax entropy condition:

$$\lambda_1(U_M) < \sigma < \lambda_1(U_L) < 0. \tag{4.3}$$

To deal with shock waves, we use the concept of generalized characteristics introduced by Dafermos (cf. [7]) as in [9]. A generalized j-characteristic associated with a solution $U(x,t) \in L^\infty \cap BV_{loc}$ of (1.1) is defined as a trajectory of the equation

$$\dot{x}(t) = \lambda_j(U(x(t),t)), \tag{4.4}$$

where (4.4) is interpreted in the sense of Filippov [11]. Thus, a j-characteristic is a Lipschitz continuous curve $(x(t),t)$ whose speed of propagation $\dot{x}(t)$ satisfies

$$\dot{x}(t) \in \left[m_x\{\lambda_j(U(x(t),t))\}, M_x\{\lambda_j(U(x(t),t))\}\right], \tag{4.5}$$

where $m_x\{\lambda_j(U(x(t),t))\}$ and $M_x\{\lambda_j(U(x(t),t))\}$ denote the essential minimum and the essential maximum of $\lambda_j(U(\cdot,t))$ at the point $x(t)$. As it was proved by Filippov [11], among all solutions of (4.5) passing through a point (x_0,t_0), there is an upper solution $\bar{x}(t)$ and a lower solution $\underline{x}(t)$, that is, the solutions of (4.5) such that any other solution $x(t)$ of (4.5) satisfies the inequality $\underline{x}(t) \le x(t) \le \bar{x}(t)$. The lower and upper solutions, for $t > t_0$, are called the minimal and maximal forward j-characteristics, respectively. An important feature about solutions in $L^\infty \cap BV_{loc}$ is that, given any generalized i-characteristic $y(t)$, it must propagate either with shock speed or with characteristic speed (*cf.* [7]). This allows one to treat $(y(t),t)$ simply as a shock curve of $U(x,t)$ in the (x,t)-plane.

Lemma 4.1 (DiPerna [9]). *Let* (1.7) *be an $m \times m$ strictly hyperbolic system endowed with a strictly convex entropy pair, whose characteristic fields are either genuinely nonlinear or linearly degenerate. Suppose $U(x,t) \in L^\infty \cap BV_{loc}(\Pi_T)$ is an entropy solution of* (1.7)-(1.8) *in Π_T. Let $x^m_{max}(t)$ denote the maximal forward m-characteristic through* $(0,0)$. *Let $x^1_{min}(t)$ denote the minimal forward 1-characteristic passing through* $(0,0)$. *Then $U(x,t) = U_L$, for a.e. (x,t) with $x < x^1_{min}(t)$, $0 \le t < T$, and $U(x,t) = U_R$, for a.e. (x,t) with $x > x^m_{max}(t)$, $0 \le t < T$.*

We now state and prove the uniqueness result for the general case.

Theorem 4.1. *Let $U(x,t) = (u(x,t), v(x,t), E(x,t)) \in BV_{loc}(\Pi_T;\mathcal{V})$, $\mathcal{V} \subset \{(u,v,S)\,|\,v > 0\} \subset \mathbb{R}^3$, be an entropy solution of* (1.1) *and* (1.8) *in Π_T. Assume* (1.14)-(1.15) *and* (1.21) *are satisfied. Then $U(x,t) = R(x/t)$, a.e. in Π_T, provided $\mathcal{V}$ is small.*

Proof. Following an idea in [9], we consider the auxiliary function in Π_T:

$$\widetilde{U}(x,t) = \begin{cases} U_L, & x < x(t), \\ U_M, & x(t) < x < \max\{x(t), \sigma_1 t\}, \\ R(x/t), & x > \max\{x(t), \sigma_1 t\}, \end{cases}$$

where $x(t)$ is the minimal 1-characteristic of $U(x,t)$, and $x = \sigma t$ is the line of 1-shock discontinuity in $R(x/t)$. We then consider the measures

$$\tilde{\theta} = \partial_t \eta_*(W(x,t)) + \partial_x q_*(W(x,t)) - \partial_S \eta_*(\widetilde{W}(x,t))\partial_t S, \quad (x,t) \in \Pi_T - \{\ell_T \cup L_T\},$$

$$\tilde{\gamma} = \partial_t \alpha(W(x,t), \widetilde{W}(x,t)) + \partial_x \beta(W(x,t), \widetilde{W}(x,t)), \quad (x,t) \in \Pi_T,$$

where $\widetilde{W}$ equals $\widetilde{U}$ written in the coordinates (u,v,S), $L_t = \{(x(s),s)\,|\,0 < s < t\}$, and $\alpha(W,\widetilde{W})$ and $\beta(W,\widetilde{W})$ are defined by (3.4)-(3.5). Our problem essentially reduces to analyzing the measure $\tilde{\gamma}$ over the region where the Riemann solution is a rarefaction wave and over the curve $(x(t),t)$, which for simplicity may be taken as the jump set of $\widetilde{W}(x,t)$.

Again, using the Gauss-Green formula for BV functions and the finiteness of propagation speeds of the solutions, we have

$$\tilde{\gamma}\{\Pi_t\} = \int_{-\infty}^{\infty} \alpha(W(x,t),\widetilde{W}(x,t))\,dx. \tag{4.6}$$

On the other hand, since $\tilde{\gamma}$ reduces to the measure $\tilde{\theta}$ on the open sets where $\widetilde{W}$ is a constant, and $\widetilde{W}(x,t)=\overline{W}(x,t)$ over $\bar{\Omega}_3$,

$$\tilde{\gamma}\{\Pi_t\} = \tilde{\gamma}\{L_t\} + \tilde{\gamma}\{\bar{\Omega}_3(t)\} + \tilde{\theta}\{\Pi_t - \big(L_t \cup \ell_t \cup \bar{\Omega}_3(t)\big)\}, \tag{4.7}$$

where we have used the fact that $\tilde{\gamma}\{\ell_t\}=0$. Hence, if one shows that

$$\tilde{\gamma}\{L_t\} \le 0, \tag{4.8}$$

the problem will reduce to the same verification as in the shock-free case. Thus, we consider the functional

$$D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) = \sigma[\alpha(W,\widetilde{W})] - [\beta(W,\widetilde{W})].$$

We will prove that

$$D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) \le 0, \tag{4.9}$$

if W_-, W_+ are connected by a 1-shock of speed $\sigma = x'(t)$, and $\widetilde{W}_-$, $\widetilde{W}_+$ are connected by a 1-shock of speed $\bar{\sigma}$, and also $W_- = \widetilde{W}_-$. Using Lemma 4.1, it is then clear that (4.9) immediately implies (4.8). Thus, when $W_- = \widetilde{W}_-$, an easy calculation shows that

$$\begin{aligned} D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) =& d(\sigma, W_-, W_+) - d(\bar{\sigma}, W_-, \widetilde{W}_+) - (\sigma - \bar{\sigma})\alpha(W_-, \widetilde{W}_+) \\ & - \partial_S \eta(\widetilde{W}_+)\left(\sigma(S_- - S_+) - \bar{\sigma}(S_- - \widetilde{S}_+)\right), \end{aligned} \tag{4.10}$$

where $d(\sigma, W_-, W_+) = \sigma[\eta(W)] - [q(W)]$, and $(\eta, q) = (\eta_*, q_*)$ is the entropy pair in (1.12)-(1.13). From the Rankine-Hugoniot relations for (1.1), we may view the state $W_+ = (u_+, v_+, S_+)$ connected on the right by a 1-shock to a state $W_- = (u_-, v_-, S_-)$ as parameterized by the shock speed σ, with $\sigma \le \lambda_1(W_-) < 0$. We recall that, through this parameterization, $S(\sigma)$ satisfies (see [21])

$$S(\sigma) = S(\lambda_1(W_-)) - \frac{\dddot{S}(\lambda_1(W_-))}{6}(\lambda_1(W_-) - \sigma)^3 + O((\lambda_1(W_-) - \sigma)^4), \tag{4.11}$$

and

$$\dddot{S}(\lambda_1(W_-)) = -p_{vv}(\dot{v})^3/2\theta < 0. \tag{4.12}$$

According to the parameterization, we set $W_+ = W_+(\sigma)$ and $\widetilde{W}_+ = W_+(\bar{\sigma})$ in (4.10). For concreteness, we assume $\bar{\sigma} > \sigma$. Now, we have

$$\begin{aligned} & - \partial_S \eta(\widetilde{W}_+)\,(\sigma(S_- - S_+(\sigma)) - \bar{\sigma}(S_- - S_+(\bar{\sigma}))) \\ & = -\partial_S \eta(\widetilde{W}_+)(\sigma - \bar{\sigma})\left(S_- - S_+(\bar{\sigma}) - \sigma\frac{S_+(\bar{\sigma}) - S_+(\sigma)}{\bar{\sigma} - \sigma}\right) \\ & = -\partial_S \eta(\widetilde{W}_+)(\sigma - \bar{\sigma})\left(S_- - S_+(\bar{\sigma}) - \sigma\dot{S}(\hat{\sigma})\right), \end{aligned}$$

where $\hat{\sigma}$ satisfies $\sigma \le \hat{\sigma} \le \bar{\sigma} \le \lambda_1(\bar{W}_-)$. Define

$$h(\sigma) \equiv d(\sigma, W_-, W_+(\sigma)) = \sigma[\eta(W)] - [q(W)].$$

One easily verifies that

$$\dot h(\bar\sigma) = \alpha(W_-, W_+(\bar\sigma)) + \partial_S\eta(\widetilde{W}_+)\left(S_- - S_+(\bar\sigma) - \bar\sigma\dot S_+(\bar\sigma)\right).$$

Now, from $0 > \lambda_1(W_-) \geq \bar\sigma \geq \sigma$ and $0 \geq \dot S(\bar\sigma) \geq \dot S(\hat\sigma)$ due to (4.11) and (4.12), it follows that

$$\partial_S\eta(\widetilde{W}_+)\bar\sigma\dot S_+(\bar\sigma) \leq \partial_S\eta(\widetilde{W}_+)\sigma\dot S(\hat\sigma),$$

and hence

$$\dot h(\bar\sigma) \geq \alpha(W_-, \widetilde{W}_+) + \partial_S\eta(\widetilde{W}_+)\left(S_- - S_+ - \sigma\dot S(\hat\sigma)\right).$$

Therefore, we have

$$D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) \leq h(\sigma) - h(\bar\sigma) - \dot h(\bar\sigma)(\sigma - \bar\sigma).$$

Observe that the above inequality is also true in the case where $\sigma > \bar\sigma$. Now, we will show that, for $|\lambda - \lambda_1(W_-)|$ sufficiently small,

$$\ddot h(\lambda) \leq 0. \tag{4.13}$$

Indeed, it is easy to verify

$$\frac{d}{d\lambda}\alpha(W_-, W_+(\lambda)) = -\nabla^2\eta(W_+(\lambda))\left(\dot W_+(\lambda), W_- - W_+(\lambda)\right).$$

Then

$$\begin{aligned}
\ddot h(\lambda) &= \frac{d}{d\lambda}\alpha(W_-, W_+(\lambda)) + \partial_S\nabla\eta(W_+(\lambda))\dot W_+(\lambda)(S_- - S_+(\lambda) - \lambda\dot S_+(\lambda)) \\
&\quad - \partial_S\eta(W_+(\lambda))(2\dot S_+(\lambda) + \lambda\ddot S_+(\lambda)) \\
&= -\nabla^2\eta(W_+(\lambda))\left(\dot W_+(\lambda), W_- - W_+(\lambda)\right) - \partial_S\eta(W_+(\lambda))\lambda\ddot S_+(\lambda) \\
&\quad + O((\lambda_1(W_-) - \lambda)^2) \\
&= \Big\{-\nabla^2\eta(W_-)(\dot W_+(\lambda_1(W_-)), \dot W_+(\lambda_1(W_-))) + \partial_S\eta(W_-)\ddot S(\lambda_1(W_-))\lambda_1(W_-)\Big\} \\
&\quad \times (\lambda_1(W_-) - \lambda) + O((\lambda_1(W_-) - \lambda)^2).
\end{aligned}$$

Thus we only need to show that

$$\nabla^2\eta(W_-)(\dot W_+(\lambda_1(W_-)), \dot W_+(\lambda_1(W_-))) - \partial_S\eta(W_-)\ddot S(\lambda_1(W_-))\lambda_1(W_-) > 0.$$

Hence, the left-hand side of the inequality above equals to

$$(\dot u(\lambda_1(W_-)))^2 - p_v(\dot v(\lambda_1(W_-)))^2 - \sqrt{-p_v}\frac{p_{vv}}{2\theta}(\dot v(\lambda_1(W_-)))^3\partial_S\eta. \tag{4.14}$$

Now, from the normalization $\nabla\lambda_1(W)\cdot\tilde r_1(W) = 1$, one obtains

$$\dot u(\lambda_1(W_-)) = -\frac{4p_v}{p_{vv}}, \qquad \dot v(\lambda_1(W_-)) = \frac{4\sqrt{-p_v}}{p_{vv}},$$

and so (4.14) equals to

$$(\dot u(\lambda_1(W_-)))^2 - p_v(v_-, S_-)(\dot v(\lambda_1(W_-)))^2(1 - \frac{2\partial_S\eta(W_-)}{\theta(v_-, S_-)}).$$

Therefore, hypothesis (1.19) guarantees the positiveness of (4.14), which implies (4.13) and in turn gives (4.9). Now, by (4.6), we get that $W(x,t) = \widetilde{W}(x,t)$, a.e. in Π_T. In particular, $\widetilde{W}$ is an entropy solution of (1.1) and (1.8), and then Rankine-Hugoniot relations imply that $\widetilde{W}$ must coincide with the classical Riemann solution $\overline{W}$. This concludes the proof. □

Combining (4.1) with Proposition 2.1, we arrive at the following conclusion.

Theorem 4.2. *Let* $U(x,t) = (u(x,t), v(x,t), E(x,t)) \in BV_{loc}(\mathbb{R}^2_+; \mathcal{V})$ *be an entropy solution of* (1.1) *and* (1.10)*, satisfying* (1.23)*. Assume* (1.14)-(1.15) *and* (1.21) *are satisfied. Then* $U(x,t)$ *asymptotically tends to the corresponding Riemann solution* $R(x/t)$ *in the sense of* (1.22)*, provided* $\mathcal{V}$ *is small.*

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References

[1] Bressan, A., Crasta, G., Piccoli, B., *Well posedness of the Cauchy problem for $n \times n$ systems of conservation laws*, Memoirs Amer. Math. Soc. (1999).

[2] Bressan, A. and LeFloch, P., *Uniqueness of weak solutions to systems of conservation laws,* Arch. Rational Mech. Anal. 140 (1997), 301-317.

[3] Bressan, A., Liu, T.-P., Yang, T., *L^1 stability estimates for $n \times n$ conservation laws*, Arch. Rational Mech. Anal. (1999).

[4] Chen, G.-Q. and Frid, H., *Large-time behavior of entropy solutions of conservation laws*, J. Diff. Eqs. **152** (1999), 308–357.

[5] Chen, G.-Q. and Frid, H., *Large-time behavior of entropy solutions in L^∞ for multidimensional conservation laws,* In: Advances in Nonlinear Partial Differential Equations and Related Areas, pp. 28-44, G.-Q. Chen et al (eds.), World Scientific: Singapore, 1998.

[6] Chen, G.-Q. and Frid, H., *Divergence-measure fields and conservation laws*, Arch. Rational Mech. Anal. 1999.

[7] Dafermos, C. M., *Generalized characteristics in hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal. **107** (1989), 127-155.

[8] Dafermos, C. M., *Entropy and the stability of classical solutions of hyperbolic systems of conservation laws*, In: *Recent Mathematical Methods in Nonlinear Wave Propagation* (Montecatini Terme, 1994), 48–69, Lecture Notes in Math. 1640, Springer, Berlin, 1996.

[9] DiPerna, R., *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J. **28** (1979), 137–188.

[10] Evans, L. C. and Gariepy, R. F., Lecture Notes on Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, Florida, 1992.

[11] Filippov, A. F., *Differential equations with discontinuous right-hand side*, Mat. Sb. (N.S.), **51** (1960), 99-128. English translation: Amer. Math. Soc. Transl. Ser. 2, **42** (1960), 199-231.

[12] Glimm, J., *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. **18** (1965), 95–105.

[13] Glimm, J. and Lax, P. D., *Decay of solutions of nonlinear hyperbolic conservation laws*, Memoirs Amer. Math. Soc. **101** (1970).

[14] Lax, P. D., *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. **10** (1957), 537-566.

[15] Lax, P. D., Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, CBMS. **11**, SIAM, Philadelphia, 1973.

[16] Lax, P. D., *Shock waves and entropy*, In: Contributions to Functional Analysis, ed. E. A. Zarantonello, Academic Press, New York, 1971, pp. 603–634.

[17] Liu, T.-P., *Initial-boundary value problems for gas dynamics*, Arch. Rational Mech. Anal. **64** (1977), 137–168.

[18] Liu, T.-P., *Linear and nonlinear large time behavior of solutions of general systems of conservation laws*, Comm. Pure Appl. Math. **30** (1977), 767-796.

[19] Liu, T.-P. and Yang, T., *Well-posedness theory for hyperbolic conservation laws*, Preprint, 1999.

[20] Peng, Y.-J., *Solutions faibles globales pour l'equation d'Euler d'un fluide compressible avec de grandes donnes initiales*, Comm. Partial Diff. Eqs. **17** (1992), 161–187.

[21] Smoller, J., Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1983.

[22] Temple, B., *Solutions in the large for the nonlinear hyperbolic conservation laws of gas dynamics*, J. Diff. Eqs. **41** (1981), 96–161.
[23] Temple, J.B., Young, R., *Large time stability of sound waves*, Comm. Math. Phys. **179** (1996), 417–466.
[24] Volpert, A. I., *The space BV and quasilinear equations*, Mat. Sb. (N.S.) **73** (1967), 255–302, Math. USSR Sbornik **2** (1967), 225–267 (in English).
[25] Wagner, D. H., *Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions*, J. Diff. Eqs. 68 (1987), 118-136.

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