UNIQUENESS AND STABILITY OF RIEMANN SOLUTIONS WITH LARGE OSCILLATION IN GAS DYNAMICS

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ABSTRACT. We prove the uniqueness of Riemann solutions in the class of entropy solutions in $L^\infty \cap BV_{loc}$ with arbitrarily large oscillation for the 3×3 system of Euler equations in gas dynamics. The proof for solutions with *large* oscillation is based on a detailed analysis of the global behavior of shock curves in the phase space and the singularity of centered rarefaction waves near the center in the physical plane. The uniqueness of Riemann solutions yields their inviscid large-time stability under *arbitrarily large* $L^1 \cap L^\infty \cap BV_{loc}$ perturbation of the Riemann initial data, as long as the corresponding solutions are in L^∞ and have local bounded total variation satisfying a natural condition on its growth with time. No specific reference to any particular method for constructing the entropy solutions is needed. The uniqueness result for Riemann solutions can easily be extended to entropy solutions $U(x,t)$, piecewise Lipschitz in x, for any $t > 0$, with arbitrarily large oscillation.

1. INTRODUCTION

We are concerned with the large-time behavior of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$, $\mathbb{R}^2_+ := \mathbb{R} \times [0, \infty)$, with arbitrarily large oscillation for the 3×3 system of Euler equations in gas dynamics, whose initial data are a large $L^1 \cap L^\infty \cap BV_{loc}$ perturbation of Riemann initial data. More specifically, for any entropy solution $U \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ which represents the evolution through the Euler equations with large initial data that are an $L^1 \cap L^\infty \cap BV_{loc}$ perturbation of the initial data of the Riemann solution $R(x/t)$, the problem is whether $U(\xi t, t) \to R(\xi)$ in $L^1_{loc}(\mathbb{R})$ as $t \to \infty$. In this paper we establish the uniqueness and stability of Riemann solutions in the class of entropy solutions with *arbitrarily large oscillation* in the physical region for polytropic gas dynamics.

The Euler system for gas dynamics in Lagrangian coordinates reads

$$(1.1) \qquad \begin{cases} \partial_t u + \partial_x p = 0, \\ \partial_t v - \partial_x u = 0, \\ \partial_t (e + \frac{u^2}{2}) + \partial_x (pu) = 0, \end{cases}$$

where u, p, v, and e represent the velocity, the pressure, the specific volume ($v = 1/\rho$, ρ the density), and the internal energy of the fluids, respectively. Other important physical variables are the temperature θ and the entropy S.

1991 *Mathematics Subject Classification.* Primary: 35B40, 35A05, 76N15; Secondary: 35B35, 35L65.

Key words and phrases. compressible Euler equations, gas dynamics, discontinuous entropy solutions, Riemann solutions, uniqueness, large-time stability, scaling sequence, compactness.

For polytropic gases, $pv = R\theta$, $e = c_v\theta$, where R and c_v are positive constants, and the choice of v and S as the independent variables leads to

$$p(v,S) = \kappa e^{S/c_v} v^{-\gamma}, \quad \gamma = 1 + \frac{R}{c_v} > 1. \tag{1.2}$$

Then $p(v,S)$ satisfies

$$p_v(v,S) < 0, \qquad p_{vv}(v,S) > 0, \qquad \nabla^2_{v,S} p(v,S) > 0, \tag{1.3}$$

for bounded $(v,S), v > 0$.

Any smooth solution of (1.1) is also a smooth solution of the system

$$\begin{cases} \partial_t u + \partial_x p = 0, \\ \partial_t v - \partial_x u = 0, \\ \partial_t S = 0. \end{cases} \tag{1.4}$$

However, for discontinuous solutions of (1.1), the last equation of (1.4) no longer holds, even in the weak sense, and it must be replaced by the so-called Clausius inequality

$$\partial_t a(S) \ge 0 \tag{1.5}$$

in the sense of distributions for any C^1 function $a(S)$ with $a'(S) \ge 0$.

System (1.1) can be written into the general form of conservation laws:

$$\partial_t U + \partial_x F(U) = 0, \tag{1.6}$$

by setting $U = (u, v, E)$ with $E = \frac{1}{2}u^2 + e$, and $F(U) = (p, -u, pu)$ with p as a function of (u, v, E). For a general exposition of the theory of hyperbolic systems of conservation laws, we refer to [8], [18], [27], and [28]. To avoid ambiguity, we henceforth denote $W = (u, v, S)$ to distinguish from $U = (u, v, E)$.

The Riemann problem for system (1.6) is a special Cauchy problem with initial data:

$$U|_{t=0} \equiv R_0(x) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0, \end{cases} \tag{1.7}$$

where U_L and U_R are constant states.

We are interested in the large-time behavior of solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of the Cauchy problem for (1.1) with initial data:

$$U|_{t=0} = U_0(x) := R_0(x) + P_0(x), \qquad \text{with } P_0 \in L^1 \cap L^\infty \cap BV_{loc}(\mathbb{R}), \tag{1.8}$$

which is then an $L^1 \cap L^\infty \cap BV_{loc}$ perturbation of the Riemann initial data. More precisely, we are interested in verifying the asymptotic stability of the classical Riemann solution $R(x/t)$, for the Riemann problem (1.6) and (1.7), in the sense that

$$ess \lim_{t\to\infty} \int_{-L}^{L} |U(\xi t, t) - R(\xi)| d\xi = 0, \qquad \text{for any } L > 0. \tag{1.9}$$

By the framework (see Theorem 5.1 in §5), established in Chen-Frid [5, 6], for any entropy solution of (1.6) and (1.8), $U \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$, the asymptotic stability problem can be reduced to the problem of the uniqueness of the Riemann solution in the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$, provided the local total variation of $U(x,t)$ satisfies the natural growth condition:

$$\begin{cases} \text{There exists } c_0 > 0 \text{ such that, for all } c \geq c_0, \text{ there is } C > 0, \\ \text{depending only on } c, \text{ such that } TV(U|\mathcal{K}_{c,T}) \leq CT \quad \text{for any } T > 0, \end{cases} \tag{1.10}$$

where

$$\mathcal{K}_{c,T} = \{(x,t) \in \mathbb{R}^2_+ \ : \ |x| < ct, \ t \in (0,T)\}. \tag{1.11}$$

We recall that the growth condition (1.10), which is required only for the study of the asymptotic behavior, is natural since any solution obtained by the Glimm method or related methods satisfies (1.10). In particular, (1.10) is verified for BV solutions satisfying the Glimm-Lax condition [16]:

$$TV_x(U(\cdot,t)|(-L,L)) \leq C_0 \frac{L}{t}, \tag{1.12}$$

for any $t > 0$ and $L > c_0 t$, for some fixed $C_0, c_0 > 0$, where TV_x denotes the total variation in x, for fixed $t > 0$. A crucial observation about condition (1.10) for the framework of [5, 6] is that, if $U(x,t)$ satisfies (1.10), then $U^T(x,t) := U(Tx,Tt)$ also does, for any $T > 0$. In particular, from the well-known compact embedding of BV in L^1, it follows that $U^T(x,t)$ is precompact in $L^1_{loc}(\mathbb{R}^2_+)$.

Our uniqueness result for entropy solutions for the 3×3 system with *large oscillation* does not require condition (1.10). The proof for the uniqueness is based on our new detailed analysis of the *global behavior* of shock curves in the phase space and on the feature of singularity of centered rarefaction waves near the center in the physical plane. This uniqueness result then yields the large-time stability of the Riemann solution under arbitrarily large $L^1 \cap L^\infty \cap BV_{loc}$ initial perturbation, in the sense of (1.9), for any entropy solution $U \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of (1.1) and (1.8), whose local total variation satisfies the natural growth condition (1.10).

We remark that our uniqueness result for Riemann solutions can easily be extended to the uniqueness of entropy solutions of (1.1) which are piecewise Lipschitz in x, for any $t > 0$, in the same spirit as DiPerna's theorem in [12] for the 2×2 case. We also remark that all results in this paper have a straightforward equivalent version for system (1.1) written in Eulerian coordinates, namely,

$$\begin{cases} \partial_\tau \rho + \partial_y(\rho u) = 0, \\ \partial_\tau(\rho u) + \partial_y(p + \rho u^2) = 0, \\ \partial_\tau(\rho(\frac{1}{2}u^2 + e)) + \partial_y(\rho u(\frac{1}{2}u^2 + e) + pu) = 0, \end{cases} \tag{1.13}$$

where $\rho = 1/v$ is the density and $\tau = t, y = \int_0^x v(s,t)\,ds + \int_0^t u(0,\sigma)\,d\sigma$. In order to avoid repetitions, we will not state the corresponding results for (1.13) which are obtained by using the well-known equivalence between (1.1) and (1.13) (see, *e.g.*, [32]).

Finally, we recall some correlated results. First, we mention the important results for $m \times m$ systems in Bressan-Crasta-Piccoli [1], with a simplified approach in Bressan-Liu-Yang [2] and Liu-Yang [24] (also see Hu-LeFloch [17]) on the L^1-stability of entropy solutions in $L^\infty \cap BV$ obtained by either the Glimm scheme [15] or the wave front-tracking method, or more generally satisfying an additional regularity property, with small total variation in x uniformly for all $t > 0$. Also see Lewicka-Trivisa [20]. In DiPerna [12], a uniqueness theorem of Riemann solutions was established for 2×2 systems in the class of entropy solutions in $L^\infty \cap BV_{loc}$ with small oscillation. In Chen-Frid [6], it is shown the uniqueness and stability of Riemann solutions for the 3×3 system of Euler equations, with general state equation for pressure, but with shocks of small strength, in the class of entropy solutions in $L^\infty \cap BV_{loc}$ with small oscillation. The uniqueness and large-time stability results in this paper impose neither smallness on the oscillation nor additional regularity of the solutions, as well as do not require specific reference to any particular method for constructing the entropy solutions. In this connection, we also recall that, for system (1.1) for polytropic gases, there are several existence results of solutions with large oscillation in $L^\infty \cap BV_{loc}$ via the Glimm scheme [15], especially when the adiabatic exponent $\gamma > 1$ is close to one (see, *e.g.*, [23, 25, 29, 30]). We also refer the reader to Dafermos [10] for the stability of Lipschitz solutions for hyperbolic systems of conservation laws.

This paper is organized as follows. In Section 2, we discuss some basic properties of system (1.1), its entropy functions and entropy solutions in $L^\infty \cap BV_{loc}$. In Section 3, we carefully analyze the global behavior of shock curves in the phase space and the singularity of centered rarefaction waves near the center in the physical space, and discuss the behavior of Riemann solutions of (1.1). Then, in Section 4, we prove the uniqueness of Riemann solutions in the class of entropy solutions in $L^\infty \cap BV_{loc}$ with arbitrarily large oscillation in the physical region. Finally, we show that the uniqueness result yields the large-time stability of Riemann solutions under $L^1 \cap L^\infty \cap BV_{loc}$ initial perturbation in Section 5.

2. Euler Equations and Entropy Solutions

System (1.1) is strictly hyperbolic in $v > 0$ and has the eigenvalues

$$\lambda_1 = -\sqrt{-p_v}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{-p_v},$$

and the corresponding right-eigenvectors r_1, r_2, r_3. The first and third families of (1.1) are genuinely nonlinear, *i.e.*, $\nabla\lambda_j \cdot r_j \neq 0$, $j = 1, 3$, and the second family is linearly degenerate, *i.e.*, $\nabla\lambda_2 \cdot r_2 = 0$. For the eigenvectors, we have

$$r_j = \nabla g\, \tilde{r}_j, \qquad j = 1, 2, 3,$$

where $U = g(W)$ is the transformation which transforms (1.1) into (1.4), and

$$\tilde{r}_1 = a(W)(-\lambda_1, 1, 0)^\top, \quad \tilde{r}_2 = (0, p_S, -p_v)^\top, \quad \tilde{r}_3 = a(W)(-\lambda_1, -1, 0)^\top.$$

Here $a(W) = 2\sqrt{-p_v}/p_{vv}$ is a normalization factor such that $\nabla_W \lambda_j \cdot \tilde{r}_j = 1$, which is equivalent to $\nabla_U \lambda_j \cdot r_j = 1$, $j = 1, 3$. We recall that an entropy-entropy flux pair for (1.6) is a pair $(\eta, q)(U)$ of C^1 functions satisfying

$$\nabla\eta(U)\nabla F(U) = \nabla q(U). \tag{2.1}$$

Clearly, $(a(S), 0), a(S) \in C^1$, is an entropy-entropy flux pair for (1.1). Throughout this paper, the mechanical energy-energy flux pair $(\eta_*, q_*)(W)$ of (1.1), a special entropy-entropy flux pair, plays an important role:

$$\eta_*(W) = \frac{u^2}{2} + e(v, S), \qquad q_*(W) = up(v, S). \tag{2.2}$$

Then $\eta_*(W)$ is strictly convex in W in any bounded domain $\mathcal{V} \subset \{v > 0\} \subset \mathbb{R}^3$, and

$$\partial_S \eta_*(W) = \theta(v, S).$$

Notice that

$$\nabla^2\eta(W)(\tilde r_i(W),\tilde r_j(W))=0, \qquad i\neq j, \tag{2.3}$$

for any entropy η. This fact implies that

$$\tilde l_j(W)=\tilde r_j(W)^\top\nabla^2\eta_*(W) \tag{2.4}$$

is a left-eigenvector of $\nabla f(W)$, $f(W)=(p(v,S),-u,0)^\top$, corresponding to the eigenvalue $\lambda_j(W)$, $j=1,3$. It is easy to check that $\tilde l_j(W)$ is a positive multiple of

$$\Big(1,(-1)^{\frac{j-1}{2}}\sqrt{-p_v(v,S)},(-1)^{\frac{j+1}{2}}p_S(v,S)/\sqrt{-p_v(v,S)}\Big). \tag{2.5}$$

Definition 2.1. A bounded measurable function $U(x,t)=(u,v,E)(x,t)$ is an entropy solution of (1.1) and (1.8) in $\Pi_T:=\mathbb{R}\times[0,T)$ if $U(x,t)$ is in the physical domain $\{(u,v,E)\,:\,v>0\}$ and satisfies the following.

(i). Equations in (1.1) hold in the weak sense in Π_T, *i.e.*, for all $\phi\in C_0^1(\Pi_T)$,

$$\int_{\Pi_T}\{U\partial_t\phi+F(U)\partial_x\phi\}\,dx\,dt+\int_{-\infty}^{\infty}U_0(x)\phi(x,0)dx=0, \tag{2.6}$$

with $U=(u,v,E)$ and $F(U)=(p,-u,pu)$.

(ii). The Clausius inequality holds in the sense of distributions in Π_T, *i.e.*, for all nonnegative $\phi\in C_0^1(\Pi_T)$,

$$\int_{\Pi_T}a(S)\partial_t\phi\,dx\,dt+\int_{-\infty}^{\infty}a(S_0(x))\phi(x,0)dx\le 0, \tag{2.7}$$

for any $a(S)\in C^1$ and $a'(S)\ge 0$.

Observe that (2.6) implies that any entropy solution $W(x,t)$ satisfies

$$\int_{\Pi_T}\{\eta_*(W)\partial_t\psi+q_*(W)\partial_x\psi\}\,dx\,dt+\int_{-\infty}^{\infty}\eta_*(W_0(x))\psi(x,0)\,dx=0, \tag{2.8}$$

for any $\psi\in C_0^1(\Pi_T)$, where (η_*,q_*) is the energy-energy flux pair in (2.2).

The Rankine-Hugoniot condition for any discontinuity with left state U_- and right state U_+ in the weak solutions defined by (2.6) are

$$\sigma(U_+-U_-)=F(U_+)-F(U_-), \tag{2.9}$$

where σ is the speed of the discontinuity.

The Clausius inequality (2.7) is equivalent to the Lax entropy inequalities

$$\lambda_{j-1}(U_-)<\sigma<\lambda_j(U_-), \qquad \lambda_j(U_+)<\sigma<\lambda_{j+1}(U_+),$$

for a j-shock wave, $j=1,3$, with left state U_- and right state U_+ that corresponds to the j-family of characteristic fields. Then, for a 1-shock wave with speed σ, left state U_-, and right state U_+,

$$\lambda_1(U_+)<\sigma<\lambda_1(U_-)<0,$$

and, for a 3-shock wave with speed σ, left state U_-, and right state U_+,

$$0<\lambda_3(U_+)<\sigma<\lambda_3(U_-).$$

Concerning the existence of entropy solutions for the Cauchy problem (1.1) and (1.8), we recall the following result of Liu [23] via the Glimm scheme [15] (also see [29]).

Theorem 2.1 (Liu [23]). *Let $\mathcal{K} \subset \{(u,v,S)\,:\, v>0\} \subset \mathbb{R}^3$ be a compact set, and let $N>0$ be any positive constant. Then there exists a constant $C_0 = C_0(\mathcal{K}, N) > 0$ independent of $\gamma \in (1, 5/3)$ such that, for $(u_0, v_0, S_0)(x) \in \mathcal{K}$ for all $x \in \mathbb{R}$ with $TV(u_0, v_0, S_0) \le N$, when*

$$(\gamma-1)TV(u_0,v_0,S_0) \le C_0 < \infty,$$

then there exists a global BV entropy solution $(u,v,E)(x,t)$ for the Cauchy problem (1.1) and (1.8) with initial data determined by $(u_0,v_0,S_0)(x)$.

The above theorem allows large oscillation of the initial data as γ is close to 1; also see Temple [29] and Peng [25] for some generalizations. For the isentropic case, whose analytical treatment was launched by DiPerna [13], the existence of L^∞ solutions even with arbitrarily L^∞ initial data can be found in Chen [4], Ding-Chen-Luo [11], Lions-Perthame-Tadmor [22], and Lions-Perthame-Souganidis [21] for polytropic gases, and Chen and LeFloch [7] for more general pressure laws.

3. Riemann Solutions and Behavior of Nonlinear Waves

In this section, we analyze the global behavior of shock curves in the phase space and recall the singularity of centered rarefaction waves in the physical plane, which are essential to determine the uniqueness of Riemann solutions with arbitrarily large oscillation in Section 4.

3.1. Shock Curves.

Given a state U_-, we consider possible states U that can be connected to state U_- on the right by a shock wave or contact discontinuity. The Rankine-Hugoniot condition for discontinuities with speed σ in the weak solutions for (1.1) is

$$\sigma[u] = [p], \tag{3.1}$$

$$\sigma[v] = -[u], \tag{3.2}$$

$$\sigma[e + \frac{1}{2}u^2] = [pu]. \tag{3.3}$$

Here and in what follows we use the notation $[H] := H_+ - H_-$, where H_- and H_+ are the values of function H on the left-hand side and the right-hand side of the discontinuity curve, respectively.

If $\sigma = 0$, then the discontinuity is a contact discontinuity which corresponds to the second family of characteristic fields.

If $\sigma \neq 0$, then the discontinuity is a shock wave, which corresponds to either the first or third family of characteristic fields.

The Lax entropy inequalities and the Rankine-Hugoniot condition (3.1)–(3.3) imply that, on a 1-shock wave,

$$[p] > 0, \qquad [v] < 0, \qquad [u] < 0,$$

and, on a 3-shock wave,

$$[p] < 0, \qquad [v] > 0, \qquad [u] < 0.$$

From (3.1)–(3.3), we have

$$e - e_- + \frac{1}{2}(p + p_-)(v - v_-) = 0. \tag{3.4}$$

Denote $\pi = \frac{p}{p_-}$ and $\tau = \frac{v}{v_-}$. Then

$$pv = p_- v_- \left(1 - \frac{\gamma - 1}{2}(\pi + 1)(\tau - 1)\right),$$

which implies

$$\tau = \frac{\pi + \beta}{\beta\pi + 1}, \quad \text{with} \quad \beta = \frac{\gamma + 1}{\gamma - 1}. \tag{3.5}$$

Note that

$$[u] = -\sigma[v] = (-1)^{\frac{j-1}{2}} \sqrt{-\frac{[p]}{[v]}}[v] = -\sqrt{-[p][v]}.$$

Then, denoting c the sound speed, i.e., $c = \sqrt{\gamma p v}$, one has

$$u - u_- = -\frac{c_-}{\sqrt{\gamma}} \sqrt{(\pi - 1)(1 - \tau)} = (-1)^{\frac{j-1}{2}} c_- \sqrt{\frac{2}{\gamma(\gamma - 1)}} \frac{1 - \pi}{\sqrt{\beta\pi + 1}}. \tag{3.6}$$

Let $\pi = e^{-x}$. From (3.5) and (3.6), the j-shock curve is determined by

$$\frac{p}{p_-} = e^{-x}, \qquad (-1)^{\frac{j-1}{2}} x \leq 0, \tag{3.7}$$

$$\frac{v}{v_-} = \frac{1 + \beta e^x}{\beta + e^x}, \tag{3.8}$$

$$\frac{u - u_-}{c_-} = (-1)^{\frac{j-1}{2}} \sqrt{\frac{2}{\gamma(\gamma - 1)}} \frac{1 - e^{-x}}{\sqrt{1 + \beta e^{-x}}}, \tag{3.9}$$

with speed

$$\sigma = (-1)^{\frac{j+1}{2}} \frac{c_-}{v_-} \sqrt{\frac{1 + \beta e^{-x}}{\beta + 1}}. \tag{3.10}$$

Now we choose the speed σ as a parameter for the shock curve (x is a function of σ: $x = x(\sigma)$), and compute the derivatives of $x(\sigma)$ in $\sigma < 0$ (1-shock) and in $\sigma > 0$ (3-shock).

Denote $' = \frac{d}{dx}$ and $\dot{} = \frac{d}{d\sigma}$. Since

$$\sigma^2 = \frac{c_-^2}{v_-^2} \frac{1 + \beta e^{-x(\sigma)}}{\beta + 1}, \tag{3.11}$$

we take derivative both sides of (3.11) in σ to have

$$\dot{x}(\sigma) = -2 \frac{\beta + 1}{\beta} \frac{v_-^2}{c_-^2} e^{x(\sigma)} \sigma,$$

which implies

$$\dot{x}(\sigma) = (-1)^{\frac{j-1}{2}} 2 \frac{\beta + 1}{\beta} \frac{v_-}{c_-} e^{x(\sigma)} \sqrt{\frac{1 + \beta e^{-x(\sigma)}}{\beta + 1}}. \tag{3.12}$$

Taking the second-order derivative both sides of (3.11) in σ yields

$$\dot{x}(\sigma)^2 - \ddot{x}(\sigma) = 2 \frac{\beta + 1}{\beta} \frac{v_-^2}{c_-^2} e^{x(\sigma)} = \frac{\beta}{2(\beta + e^{x(\sigma)})} \dot{x}(\sigma)^2 > 0. \tag{3.13}$$

Then

$$\ddot{x}(\sigma) = 2\frac{\beta+1}{\beta}\frac{v_-^2}{c_-^2}e^{x(\sigma)}(1+\frac{2}{\beta}e^{x(\sigma)}) = \frac{e^{x(\sigma)}+\beta/2}{e^{x(\sigma)}+\beta}\dot{x}(\sigma)^2 > 0. \tag{3.14}$$

Taking the third-order derivative both sides of (3.11) in σ yields

$$\dddot{x}(\sigma) - 3\dot{x}(\sigma)\ddot{x}(\sigma) + \dot{x}(\sigma)^3 = 0. \tag{3.15}$$

On the other hand, from (3.7), we have

$$p' = -p, \;\; p'' = p, \;\; p''' = -p,$$

and then

$$\dot{p} = -p\dot{x}, \quad \ddot{p} = p((\dot{x})^2 - \ddot{x}) > 0, \quad \dddot{p} = p(-(\dot{x})^3 + 3\dot{x}\ddot{x} - \dddot{x}) = 0. \tag{3.16}$$

From (3.8), we similarly have

$$\dot{v} = v'\dot{x}, \quad \ddot{v} = 3((\dot{x})^2 - \ddot{x})v', \quad \dddot{v} = \frac{6\beta v'\dot{x}}{\beta + e^x}((\dot{x})^2 - \ddot{x}). \tag{3.17}$$

Note that $\frac{S}{c_v} = \ln(\frac{1}{\kappa}pv^\gamma)$. Then

$$\frac{\dot{S}}{c_v} = \frac{\dot{p}}{p} + \gamma\frac{\dot{v}}{v} = -\frac{\beta(e^x-1)^2\dot{x}}{(\beta+e^x)(1+\beta e^x)}, \tag{3.18}$$

$$\frac{\ddot{S}}{c_v} = \frac{\ddot{p}}{p} - \frac{\dot{p}^2}{p^2} + \gamma\frac{\ddot{v}}{v} - \gamma\frac{\dot{v}^2}{v^2} = \frac{(\dot{x})^2}{(\beta+e^x)^2(1+\beta e^x)^2}P(e^x), \tag{3.19}$$

where

$$P(y) = \beta(y-1)\left(-\beta y^3 - (\frac{3}{2}\beta^2+\beta+2)y^2 - \frac{1}{2}(\beta^2+5\beta)y + \frac{\beta}{2}\right), \quad y > 0.$$

Lemma 3.1. *Along any shock curve, $S = S(\sigma)$ satisfies*

$$2\dot{S}(\sigma) + \sigma\ddot{S}(\sigma) \le 0.$$

Proof. This can be seen via a direct calculation, which yields

$$2\dot{S}(\sigma) + \sigma\ddot{S}(\sigma) = \frac{c_v\dot{x}(\sigma)(1-e^{x(\sigma)})}{(\beta+e^{x(\sigma)})(1+\beta e^{x(\sigma)})^2}Q(e^{x(\sigma)}),$$

while

$$\begin{aligned} Q(y) =& 2\beta(y-1)(1+\beta y) + 2\left(-\beta y^3 - (\frac{3}{2}\beta^2+\beta+2)y^2 - \frac{1}{2}(\beta^2+5\beta)y + \frac{\beta}{2}\right) \\ =& -2\beta y^3 - (\beta^2+2\beta+4)y^2 - 3\beta(\beta+1)y - \beta < 0. \end{aligned}$$

Since $\dot{x}(\sigma)(1-e^{x(\sigma)})$ is always nonnegative, the result follows. □

3.2. **Rarefaction Waves.** Given a state $U_- = (u_-, v_-, E_-)$, we consider possible states $U = (u, v, E)$ that can be connected to state U_- on the right by a centered rarefaction wave in the j-families, $j = 1, 3$. Consider the self-similar solutions $(u, v, E)(\xi), \xi = x/t$, of the Riemann problem (1.1) and (1.7). Then we have

$$\begin{cases} \xi = \lambda_j(u,v,E)(\xi), \quad j = 1,3, \\ \frac{du}{d\xi} + \xi\frac{dv}{d\xi} = 0, \\ \frac{dE}{d\xi} + (\xi u + p)\frac{dv}{d\xi} = 0, \end{cases}$$

with boundary condition:

$$(u, v, E)(\lambda_j(U_-)) = U_-,$$

and, on the j-family centered rarefaction waves,

$$\frac{\partial U}{\partial x} = \frac{1}{t}\frac{dU}{d\xi} = \frac{1}{t} r_j(U(\frac{x}{t})), \quad j = 1, 3. \tag{3.20}$$

In particular,

$$\frac{\partial W}{\partial x} = \frac{1}{t}\tilde{r}_j(W(\frac{x}{t})), \quad j = 1, 3. \tag{3.21}$$

For rarefaction waves $R(x/t)$ with right state U_+, denoting

$$w_j = u + (-1)^{\frac{j-1}{2}} \int_v^\infty \sqrt{-p_v(s, S_\pm)} ds, \qquad j = 1, 3,$$

with $w_1(U_-) - w_3(U_+) > 0$, one has

$$\begin{cases} w_1(U_-) \le w_1(R(x/t)) \le w_1(U_+), \ \ w_3(U_-) \le w_3(R(x/t)) \le w_3(U_+), \\ w_1(R(x/t)) - w_3(R(x/t)) > 0, \qquad S(x/t) = S_\pm. \end{cases}$$

These rarefaction waves are identical to those for the isentropic case with the 2-family in the isentropic case corresponding to the 3-family in the non-isentropic case.

3.3. **Solvability.** For the Riemann problem (1.1), (1.2), and (1.7), we have

Lemma 3.2. *Given the states $W_L = (u_L, v_L, S_L)$ and $W_R = (u_R, v_R, S_R)$, there exists a unique global Riemann solution in the class of the self-similar piecewise smooth solutions consisting of shocks, rarefaction waves, and contact discontinuities, provided that the Riemann data satisfy*

$$u_R - u_L < \frac{2}{\gamma - 1}(c(v_L, S_L) + c(v_R, S_R)),$$

where $c(v, S) = \sqrt{\gamma p(v, S) v}$.

The proof of Lemma 3.2 can be found in [26, 28, 3].

4. Uniqueness of Riemann Solutions in the Class of Entropy Solutions in $L^\infty \cap BV_{loc}$

In this section we prove the uniqueness of entropy solutions of the Riemann problem (1.1) and (1.7). Without loss of generality, we assume that the Riemann solution has the following generic form

$$R(x/t) = \begin{cases} U_L, & x/t < \sigma_1, \\ U_M, & \sigma_1 < x/t < 0, \\ U_N, & 0 < x/t \le \lambda_3(U_N), \\ R_3(x/t), & \lambda_3(U_N) < x/t < \lambda_3(U_R), \\ U_R, & x/t \ge \lambda_3(U_R), \end{cases} \tag{4.1}$$

where $\sigma_1 = \sigma_1(U_L, U_M)$ is the shock speed, determined by (3.10), and $R_3(\xi)$ is the solution of the boundary value problem

$$\begin{cases} \frac{dR_3(\xi)}{d\xi} = r_3(R_3(\xi)), \qquad \xi < \lambda_3(U_R), \\ R_3(\lambda_3(U_R)) = U_R. \end{cases} \tag{4.2}$$

The 1-shock wave connecting U_L and U_M satisfies the Lax entropy condition: $\lambda_1(U_M) < \sigma_1 < \lambda_1(U_L) < 0$. The states U_M and U_N are also completely determined by the shock curve formula (3.7)–(3.9) and (4.2). The best way to see this fact is first to recall that S is increasing across 1-shock waves and is constant over rarefaction curves, since S is a Riemann invariant of the first and third families (see [26, 28]). Similarly, u and p are both constant over the wave curves of the second (linearly degenerate) family. Hence, in the space (u,p,S), we can project the curves S_1 and R_3 on the plane (u,p), find the intersection (u_M,p_M) of these projected curves, and immediately obtain the two intersection points $(u_M,p_M,S_M), (u_M,p_M,S_N)$, of the line $\{(u,p,S) : u = u_M, p = p_M\}$ with the 1-shock curve S_1 and the 3-rarefaction curve R_3 in the phase space.

To handle shock waves, we use the concept of generalized characteristics introduced by Dafermos (cf. [9]). A generalized j-characteristic associated with a solution $U \in L^\infty \cap BV_{loc}$ of (1.1) is defined as a trajectory of the equation

$$\dot{x}(t) = \lambda_j(U(x(t),t)), \tag{4.3}$$

where (4.3) is interpreted in the sense of Filippov [14]. Thus, a (generalized) j-characteristic is a Lipschitz continuous curve $(x(t),t)$ whose speed of propagation $\dot{x}(t)$ satisfies

$$\dot{x}(t) \in \big[m_x\{\lambda_j(U(x(t),t))\}, M_x\{\lambda_j(U(x(t),t))\}\big], \tag{4.4}$$

where $m_x\{\lambda_j(U(x(t),t))\}$ and $M_x\{\lambda_j(U(x(t),t))\}$ denote the essential minimum and the essential maximum of $\lambda_j(U(\cdot,t))$ at the point $x(t)$, respectively. As it was proved by Filippov [14], among all solutions of (4.3) passing through a point (x_0,t_0), there is an upper solution $\bar{x}(t)$ and a lower solution $\underline{x}(t)$, that is, the solutions of (4.3) such that any other solution $x(t)$ of (4.3) satisfies the inequality $\underline{x}(t) \le x(t) \le \bar{x}(t)$. The lower and upper solutions, for $t > t_0$, are called the minimal and maximal forward j-characteristics, respectively. An important feature about solutions in $L^\infty \cap BV_{loc}$ is that, given any generalized i-characteristic $y(t)$, it must propagate either with shock speed or with characteristic speed (*cf.* [9]). This allows one to treat $(y(t),t)$ simply as a shock curve of $U(x,t)$ in the (x,t)-plane.

Lemma 4.1 (DiPerna [12])**.** *Let* (1.6) *be an $m \times m$ strictly hyperbolic system endowed with a strictly convex entropy. Suppose that $U \in L^\infty \cap BV_{loc}(\Pi_T)$ is an entropy solution of* (1.6)–(1.7) *in Π_T. Let $x^m_{max}(t)$ denote the maximal forward m-characteristic through* $(0,0)$*. Let $x^1_{min}(t)$ denote the minimal forward 1-characteristic passing through* $(0,0)$*. Then $U(x,t) = U_L$, for a.e. (x,t) with $x < x^1_{min}(t)$, $0 \le t < T$, and $U(x,t) = U_R$, for a.e. (x,t) with $x > x^m_{max}(t)$, $0 \le t < T$.*

We now state and prove our uniqueness result.

Theorem 4.1. *Let $U = (u,v,E) \in BV_{loc}(\Pi_T;\mathcal{V})$, $\mathcal{V} \subset \{(u,v,E) : v > 0\} \subset \mathbb{R}^3$, be an entropy solution of* (1.1) *and* (1.7) *in Π_T. Then $U(x,t) = R(x/t)$, a.e. in Π_T.*

Proof. 1. Motivated by [12], we consider the auxiliary function in Π_T:

$$\widetilde{U}(x,t) = \begin{cases} U_L, & x < x(t), \\ U_M, & x(t) < x < \max\{x(t), \sigma_1 t\}, \\ R(x/t), & x > \max\{x(t), \sigma_1 t\}, \end{cases}$$

where $x(t)$ is the minimal 1-characteristic of $U(x,t)$, and $x = \sigma_1 t$ is the line of 1-shock discontinuity in $R(x/t)$. One of the main ingredients in the proof is to use the state variables $W = (u,v,S)$ as the basic variables, rather than the conserved variables $U = (u,v,E)$, and we let $\widetilde{W}(x,t)$ denote $R(x/t)$ in these state variables. Motivated by a procedure introduced by Dafermos (*cf.* [10, 12]), we identify a Lyapunov functional through the following quadratic entropy-entropy flux pairs obtained from (η_*, q_*):

$$\alpha(W,\widetilde{W}) := \eta_*(W) - \eta_*(\widetilde{W}) - \nabla\eta_*(\widetilde{W})\cdot(W-\widetilde{W}), \tag{4.5}$$

$$\beta(W,\widetilde{W}) := q_*(W) - q_*(\widetilde{W}) - \nabla\eta_*(\widetilde{W})\cdot(f(W)-f(\widetilde{W})). \tag{4.6}$$

Consider the measures

$$\mu := \partial_t\alpha(W(x,t),\widetilde{W}(x,t)) + \partial_x\beta(W(x,t),\widetilde{W}(x,t)), \quad (x,t)\in\Pi_T,$$

$$\nu := \partial_t\eta_*(W(x,t)) + \partial_x q_*(W(x,t)) - \partial_S\eta_*(\widetilde{W}(x,t))\partial_t S(x,t), \;\; (x,t)\in\Pi_T - \{\ell_T\cup L_T\},$$

where $\ell_t = \{(0,s) : 0\le s<t\}$, and $L_t = \{(x(s),s) : 0\le s<t\}$.

Our problem essentially reduces to analyzing the measure μ over the region, where the Riemann solution is a rarefaction wave, and over the curve $(x(t),t)$, which for simplicity may be taken as the jump set of $\widetilde{W}(x,t)$.

2. The first important fact is that $\mu\{\ell_T\} = 0$, since $\mu\{\ell_T\} = \int_{\ell_T}[\beta(W,\widetilde{W})]\,d\mathcal{H}^1$ and $[\beta(W,\widetilde{W})] = 0$, $\mathcal{H}^1$-a.e. over ℓ_T. The latter follows from $\beta(W,\widetilde{W}) = (u-\bar{u})(p-\bar{p})$ and the fact that u, p, $\bar{u}$, $\bar{p}$ cannot change across the jump discontinuities of W and $\widetilde{W}$ over ℓ_T, because of the Rankine-Hugoniot relation (3.1)–(3.3).

3. Set

$$\Omega_3 = \{\,(x,t) : \lambda_3(U_N) < x/t < \lambda_3(U_R),\ t>0\,\},$$

the rarefaction wave region of the classical Riemann solution. Over this region, $\widetilde{W} = \overline{W}$, and μ satisfies

$$\begin{aligned}\mu =&\partial_t\alpha(W,\overline{W}) + \partial_x\beta(W,\overline{W})\\ =&\partial_t\eta_*(W) + \partial_x q_*(W) - \partial_S\eta_*(\overline{W})\partial_t S\\ &- \nabla^2\eta_*(\overline{W})(\partial_t\overline{W}, W-\overline{W}) + \nabla^2\eta_*(\overline{W})(\partial_x\overline{W}, f(W)-f(\overline{W}))\\ =&\nu - \nabla^2\eta_*(\overline{W})(\partial_x\overline{W}, Qf(W,\overline{W})),\end{aligned} \tag{4.7}$$

where we used the fact that $\nabla^2\eta_*\nabla f$ is symmetric, and $Qf(W,\overline{W}) = f(W)-f(\overline{W})-\nabla f(\overline{W})(W-\overline{W})$ is the quadratic part of f at $\overline{W}$. Since $\tilde{l}_j(\overline{W}) = \tilde{r}_j(\overline{W})^\top\nabla^2\eta_*(\overline{W})$ is a left-eigenvector of $\nabla f(\overline{W})$ corresponding to the eigenvalue $\lambda_j(\overline{W})$, $j=1,3$, and, for $(x,t)\in\Omega_j$,

$$\frac{\partial\overline{W}(x,t)}{\partial x} = \frac{1}{t}\tilde{r}_j(\overline{W}(x,t)), \qquad j=1,3.$$

Then, by (2.4) and (4.7), for any Borel set $E\subset\Omega_3$, we have

$$\mu(E) = \nu(E) - \int_E \frac{1}{t}\tilde{l}_3(\overline{W})Qf(W,\overline{W})\,dxdt.$$

Since $\tilde{l}_3(\overline{W})$ is a positive multiple of

$$\left(1, -\sqrt{-p_v(\overline{v},\overline{S})}, -p_S(\overline{v},\overline{S})/\sqrt{-p_v(\overline{v},\overline{S})}\right),$$

and

$$Qf(W,\overline{W}) := (p(v,S) - p(\overline{v},\overline{S}) - p_v(\overline{v},\overline{S})(v-\overline{v}) - p_S(\overline{v},\overline{S})(S-\overline{S}),0,0)^\top,$$

we use (1.3) to have $\tilde{l}_3(\overline{W})Qf(W,\overline{W}) \geq 0$, which implies that

$$\mu(\Omega_3) \leq 0.$$

4. Using the Gauss-Green formula for BV functions and the finiteness of propagation speeds of the solutions, we have

$$\mu\{\Pi_t\} = \int_{-\infty}^{\infty} \alpha(W(x,t),\widetilde{W}(x,t))\,dx. \tag{4.8}$$

On the other hand, since $\tilde{\gamma}$ reduces to the measure $\tilde{\theta}$ on the open sets where $\widetilde{W}$ is a constant, and $\widetilde{W}(x,t) = \overline{W}(x,t)$ over $\bar{\Omega}_3$,

$$\mu\{\Pi_t\} = \mu\{L_t\} + \mu\{\bar{\Omega}_3(t)\} + \nu\big\{\Pi_t - \big(L_t \cup \ell_t \cup \bar{\Omega}_3(t)\big)\big\}, \tag{4.9}$$

where we have used the fact that $\mu\{\ell_t\} = 0$.

5. Hence, it suffices to show

$$\mu\{L_t\} \leq 0. \tag{4.10}$$

Thus, we consider the functional

$$D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) := \sigma[\alpha(W,\widetilde{W})] - [\beta(W,\widetilde{W})].$$

We will prove that

$$D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) \leq 0, \tag{4.11}$$

if W_-, W_+ are connected by a 1-shock of speed $\sigma = x'(t)$, and $\widetilde{W}_-$, $\widetilde{W}_+$ are connected by a 1-shock of speed $\bar{\sigma}$, and also $W_- = \widetilde{W}_-$. Using Lemma 4.1, it is then clear that (4.11) immediately implies (4.10). Thus, when $W_- = \widetilde{W}_-$, an easy calculation shows that

$$\begin{aligned} D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) &= d(\sigma, W_-, W_+) - d(\bar{\sigma}, W_-, \widetilde{W}_+) - (\sigma - \bar{\sigma})\alpha(W_-, \widetilde{W}_+) \\ &\quad - \partial_S\eta(\widetilde{W}_+)\left(\sigma(S_- - S_+) - \bar{\sigma}(S_- - \widetilde{S}_+)\right), \end{aligned} \tag{4.12}$$

where $d(\sigma, W_-, W_+) := \sigma[\eta(W)] - [q(W)]$, and $(\eta, q) = (\eta_*, q_*)$ is the entropy pair in (2.2). From the Rankine-Hugoniot relation (3.1)–(3.3), we may view the state $W_+ = (u_+, v_+, S_+)$ connected on the right by a 1-shock to a state $W_- = (u_-, v_-, S_-)$ as parametrized by the shock speed σ, with $\sigma \leq \lambda_1(W_-) < 0$.

6. According to the parametrization, we set $W_+ = W_+(\sigma)$ and $\widetilde{W}_+ = W_+(\bar{\sigma})$ in (4.12). For concreteness, we assume $\bar{\sigma} > \sigma$. Then

$$\begin{aligned} &- \partial_S\eta(\widetilde{W}_+)\left(\sigma(S_- - S_+(\sigma)) - \bar{\sigma}(S_- - S_+(\bar{\sigma}))\right) \\ &= -\partial_S\eta(\widetilde{W}_+)(\sigma - \bar{\sigma})\left(S_- - S_+(\bar{\sigma}) - \sigma\frac{S_+(\bar{\sigma}) - S_+(\sigma)}{\bar{\sigma} - \sigma}\right) \\ &= -\partial_S\eta(\widetilde{W}_+)(\sigma - \bar{\sigma})\left(S_- - S_+(\bar{\sigma}) - \sigma\dot{S}(\hat{\sigma})\right), \end{aligned}$$

where $\hat{\sigma}$ satisfies $\sigma \leq \hat{\sigma} \leq \bar{\sigma} \leq \lambda_1(\bar{W}_-)$.

Observe that the Rankine-Hugoniot relation (3.1)–(3.3) for (1.1) implies

$$d(\sigma, W_-, W_+(\sigma)) = d(\bar{\sigma}, W_-, W_+(\bar{\sigma})) = 0,$$

especially,

$$\sigma(\eta(W_+(\sigma)) - \eta(W_-)) - (q(W_+(\sigma)) - q(W_-)) = 0, \tag{4.13}$$

for all σ. Taking derivative both sides of identity (4.13) yields

$$\alpha(W_-, W_+(\bar{\sigma})) + \partial_S \eta(\widetilde{W}_+)\left(S_- - S_+(\bar{\sigma}) - \bar{\sigma}\dot{S}_+(\bar{\sigma})\right) = 0.$$

Now, from $0 > \lambda_1(W_-) \geq \bar{\sigma} \geq \sigma$, we have

$$\alpha(W_-, \widetilde{W}_+) + \partial_S \eta(\widetilde{W}_+)\left(S_- - S_+(\bar{\sigma}) - \sigma\dot{S}(\hat{\sigma})\right) \leq 0. \tag{4.14}$$

The fact (4.14) can be seen as follows. First, define

$$G(\sigma) := (\bar{\sigma} - \sigma)\bar{\sigma}\dot{S}(\bar{\sigma}) - \sigma S(\bar{\sigma}) + \sigma S(\sigma).$$

Then $G(\bar{\sigma}) = G'(\bar{\sigma}) = 0$. Using Lemma 3.1 yields

$$G''(\sigma) = 2\dot{S}(\sigma) + \sigma\ddot{S}(\sigma) \leq 0.$$

Then

$$G'(\sigma) \geq G'(\bar{\sigma}) = 0,$$

and hence

$$G(\sigma) \leq G(\bar{\sigma}) = 0,$$

which implies

$$(\bar{\sigma} - \sigma)\bar{\sigma}\dot{S}(\bar{\sigma}) \leq \sigma S(\bar{\sigma}) - \sigma S(\sigma).$$

Since $\sigma \leq \bar{\sigma} \leq \lambda_1(W_-) < 0$,

$$\bar{\sigma}\dot{S}(\bar{\sigma}) \leq \sigma \frac{S(\bar{\sigma}) - S(\sigma)}{\bar{\sigma} - \sigma},$$

which implies (4.14).

Therefore, we have

$$D(\sigma, W_-, W_+, \widetilde{W}_-, \widetilde{W}_+) \leq 0,$$

for the case $\sigma < \bar{\sigma}$. Observe that the above inequality is also true in the case where $\sigma > \bar{\sigma}$. This arrives at (4.11).

7. Now, by (4.8), we conclude that $W(x,t) = \widetilde{W}(x,t)$, a.e. in Π_T. In particular, $\widetilde{W}(x,t)$ is an entropy solution of (1.1) and (1.7), and then the Rankine-Hugoniot condition (3.1)-(3.3) implies that $\widetilde{W}(x,t)$ must coincide with the classical Riemann solution $\overline{W}(x,t)$. This concludes the proof. □

5. Large-Time Stability of Riemann Solutions with Large Oscillation

In this section we show that the uniqueness of the Riemann solution $R(x/t)$, corresponding to the Riemann data (1.7), implies the large-time stability of the Riemann solution in the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of (1.1) and (1.8), whose local total variation satisfies the growth condition (1.10).

For the sake of completeness, we first present the framework in Theorem 5.1, established in [5], and describe the main ideas for the proof of Theorem 5.1. This framework exhibits the relationship among uniqueness of Riemann solutions, compactness of the scaling sequence $U^T(x,t) := U(Tx, Tt)$ of any perturbing entropy solution $U(x,t)$, and asymptotic stability in the sense of (1.9).

Theorem 5.1 (Chen-Frid [5]). *Let $\mathcal{S}(\mathbb{R}^2_+)$ denote a class of functions defined on $\mathbb{R}^2_+$. Assume that the Cauchy problem* (1.6) *and* (1.8) *satisfies the following:*
(i) *System* (1.6) *has a strictly convex entropy;*
(ii) *The Riemann solution is unique in the class $\mathcal{S}(\mathbb{R}^2_+)$;*
(iii) *Given an entropy solution of* (1.6) *and* (1.8), *$U \in \mathcal{S}(\mathbb{R}^2_+)$, the sequence $U^T(x,t)$ is precompact in $L^1_{loc}(\mathbb{R}^2_+)$, and any limit function of its subsequence is still in $\mathcal{S}(\mathbb{R}^2_+)$.*
Then the Riemann solution $R(x/t)$ is asymptotically stable in $\mathcal{S}(\mathbb{R}^2_+)$, with respect to the corresponding initial perturbation $P_0(x)$, in the sense of (1.9).

This can be seen as follows. First, given any sequence $\{U^{T_k}(x,t)\}_{k=1}^\infty$, with $T_k \to \infty$, condition (iii) implies that there exists a subsequence converging in L^1_{loc} to $\tilde{U} \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ satisfying the same initial data as $R(x/t)$. Condition (ii) then ensures that $\tilde{U}(x,t) = R(x/t)$ a.e.. This shows that the whole family $\{U^T(x,t)\}_{T>0}$ converges to the Riemann solution $R(x/t)$ in $L^1_{loc}(\mathbb{R}^2_+)$, as $T \to \infty$. Hence, for any $0 < r < \infty$, we have

$$\frac{1}{T^2}\int_0^T \int_{|\xi|\le r} |U(\xi t,t) - R(\xi)| t\, d\xi dt = \frac{1}{T^2}\int_0^T \int_{|x|\le rt} |U(x,t) - R(x/t)| dxdt$$
$$= \int_0^1 \int_{|x|\le rt} |U^T(x,t) - R(x/t)| dxdt \to 0, \qquad \text{when} \quad T\to\infty,$$

which implies

$$\frac{1}{T}\int_0^T \int_{|\xi|\le r} |U(\xi t,t) - R(\xi)|\, d\xi dt \to 0, \qquad \text{when} \quad T\to\infty. \tag{5.1}$$

In view of condition (i) and the piecewise Lipschitz continuity of $R(\xi)$ given by condition (ii), we can use Theorem 2.1 of Chen-Frid [5] to conclude that the Riemann solution is asymptotically stable in the sense of (1.9) with respect to the initial perturbation $P_0(x)$. We briefly recall the main points of the argument in the proof.

Let $\eta(U)$ be a strictly convex entropy of (1.6) with associated entropy flux $q(U)$. We consider the Dafermos quadratic entropy pair:

$$\alpha(U,V) := \eta(U) - \eta(V) - \nabla\eta(V)\cdot(U-V),$$
$$\beta(U,V) := q(U) - q(V) - \nabla\eta(V)\cdot(F(U)-F(V)).$$

Using the entropy inequality

$$\partial_t \eta(U) + \partial_x q(U) \le 0,$$

and the equations

$$\partial_t R + \partial_x F(R) = 0,$$

in the sense of distributions, and

$$\partial_t \eta(R) + \partial_x q(R) = 0,$$

which holds in any wedge $\xi_1 < x/t < \xi_2$ where $R(\xi)$ is Lipschitz, we conclude

$$\frac{d}{dt}Y(t) \le \frac{C}{t}, \qquad Y(t) := \int_{\xi_1}^{\xi_2} \alpha(U(\xi t,t), R(\xi))\, d\xi, \tag{5.2}$$

in the sense of distributions for a certain constant $C > 0$. Now, (5.1) implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Y(t)\,dt = 0. \tag{5.3}$$

Then it can be proved that (5.2) and (5.3) imply $Y(t) \to 0$ as $t \to \infty$ (see [5] for the details). Since $R(\xi)$ is piecewise Lipschitz, (1.9) follows.

Now we use Theorem 5.1 to show the large-time stability of Riemann solutions in the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$.

First, we observe that system (1.1) has a strictly convex entropy $S(u, v, E)$ for $(u, v, E) \in \mathcal{V}$, and then condition (i) of Theorem 5.1 is verified. We choose $\mathcal{S}(\mathbb{R}^2_+)$ as the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ satisfying (1.10). As remarked in §1, if $U(x,t)$ satisfies (1.10), $U^T(x,t)$ also does with the same constant C, depending only on c. Then, the well-known compact embedding of BV in L^1 implies that the sequence $U^T(x,t)$ is compact in $L^1_{\rm loc}(\mathbb{R}^2_+)$. Hence, condition (iii) also holds.

Therefore, the uniqueness result established in §4 yields the large-time stability of entropy solutions satisfying (1.10).

Theorem 5.2. *Any Riemann solution of system* (1.1) *with arbitrarily large Riemann initial data* (1.7) *is large-time asymptotically stable in the sense of* (1.9) *in the class of entropy solutions of* (1.1) *with arbitrarily large initial perturbation* (1.8) *and satisfying* (1.10).

Acknowledgments. Gui-Qiang Chen's research was supported in part by the National Science Foundation grants DMS-9971793, INT-9726215, and INT-9987378. Hermano Frid's research was supported in part by CNPq-Brazil, proc. 352871/96-2 and FAPERJ, proc. E-26/151.190/2000.

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