CONCENTRATION AND CAVITATION IN THE VANISHING PRESSURE LIMIT OF SOLUTIONS TO THE EULER EQUATIONS FOR NONISENTROPIC FLUIDS

GUI-QIANG CHEN HAILIANG LIU

ABSTRACT. Numerical simulations in [5, 6] for the Euler equations for gas dynamics in the regime of small pressure showed that, for one case, the particles seem to be more sticky and tend to concentrate near some shock locations; and for the other case, in the region of rarefaction waves, the particles seem to be far apart and tend to form cavitation in the region. In this paper we identify and analyze the phenomena of concentration and cavitation by studying vanishing pressure limit of solutions of the full Euler equations for nonisentropic compressible fluids with a scaled pressure. It is rigorously shown that any Riemann solution containing two shocks and possibly one contact discontinuity to the Euler equations for nonisentropic fluids tends to a δ-shock solution to the corresponding transport equations, and the intermediate densities between the two shocks tend to a weighted δ-measure that, along with the two shocks and possibly contact discontinuity, forms the δ-shock as the pressure vanishes. By contrast, it is also shown that any Riemann solution containing two rarefaction waves and possibly one contact discontinuity to the Euler equations for nonisentropic fluids tends to a two-contact-discontinuity solution to the transport equations, and the nonvacuum intermediate states between the two rarefaction waves tend to a vacuum state as the pressure vanishes. Some numerical results exhibiting the processes of concentration and cavitation are presented as the pressure decreases.

1. INTRODUCTION

Fluids are substances whose molecular structure offers no resistance to external shear forces: even the smallest force causes deformation of a fluid particle. In most cases of interest, a fluid can be regarded as continuum, thus satisfies the balance laws of density, momentum, and energy. Fluids with large Mach number and far from solid surfaces can be described as the inviscid Euler equations. Pressure as a thermodynamic variable depends on other fluid variables such as density and internal energy. Such dependence is possible to be estimated from statistic mechanics or kinetic theory and is usually obtained by laboratory measurement. If the gas is ideal, then the pressure may become small in some physical regimes, for example, when the density or internal energy is small.

Date: August 15, 2003.

1991 *Mathematics Subject Classification.* Primary: 35L65, 35B30, 76E19, 35Q35, 35L67; Secondary: 35B25,65M06.

Key words and phrases. Concentration, cavitation, δ-shocks, vacuum states, Euler equations, vanishing pressure limit, transport equations, measure solutions, nonisentropic fluids, pressureless fluids, numerical simulations.

It is well-known, in the compressible fluid flow, if the speed is larger than the sound speed, shocks may form when particles collide. However, as observed numerically in Chang-Chen-Yang [5, 6] for gas dynamics in the regime of small pressure: For one case, the particles seem to be more sticky and tend to concentrate at some shock locations which move with the associated shock speeds; and for the other case, in the region of rarefaction waves, the particles seem to be far apart and tend to form cavitation in the region. Such phenomena may be regarded as a tendency towards the concentration and cavitation in terms of the density. We have rigorously justified the phenomena of concentration and cavitation for isentropic fluids by looking at the vanishing pressure limit in [11]. One of the main objectives of this paper is to show rigorously that the phenomena of concentration and cavitation in the solutions are fundamental in inviscid nonisentropic flow, which occur not only in the multidimensional situations, but also even in the one-dimensional case.

Consider the nonisentropic compressible fluids modeled by the full Euler equations in Eulerian coordinates

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.1}$$

$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + P\right) = 0, \tag{1.2}$$

$$\partial_t(\rho E) + \partial_x\left((\rho E + P)\, v\right) = 0, \tag{1.3}$$

where ρ represents the density, $m = \rho v$ the momentum, ρE the total energy, P the scalar pressure, and $\gamma > 1$ the adiabatic exponent, respectively; and ρ and m are in a physical region $\{(\rho, m) :\ \rho \geq 0, |m| \leq V_0 \rho\}$ for some $V_0 > 0$. For $\rho > 0$, $v = m/\rho$ is the velocity with $|v| \leq V_0$. In order to analyze the vanishing pressure limit of solutions, we assume that the scalar pressure P is a scaled function of the conserved quantities of the density ρ, momentum m, and total energy ρE for polytropic gases:

$$P = \epsilon\,(\gamma - 1)(\rho E - \frac{m^2}{2\rho}) \tag{1.4}$$

with a small scaling parameter $\epsilon > 0$ modeling the strength of pressure P. Experimental and numerical studies of flows are often carried out on models, and the results are displayed in dimensionless form, thus allowing scaling to real flow conditions. The scaling (1.4) may not be claimed as a faithful description of the whole fluid flow, but it does in the regime of small pressure in the flow. Although the parameter ϵ can be considered very small and reflects the strength of the underlying pressure, it does not vanish in general. We propose to include this parameter in hopes of understanding the process of the formation of concentration and cavitation in nonisentropic compressible fluid flow.

System (1.1)–(1.4) is an archetype of hyperbolic systems of conservation laws of the form

$$\partial_t u + \partial_x f(u, \epsilon) = 0, \tag{1.5}$$

with $u = (\rho, \rho v, \rho E)^\top$ and $f(u, \epsilon) = (\rho v, \rho v^2 + \epsilon p, (\rho E + \epsilon p)v)^\top$ with $p = (\gamma - 1)(\rho E - \frac{m^2}{2\rho})$.

As $\epsilon \to 0$, the limit system formally becomes the following transport equations

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.6}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2) = 0, \tag{1.7}$$

with an additional conservation law

$$\partial_t(\rho E) + \partial_x(\rho v E) = 0. \tag{1.8}$$

In this paper, we rigorously study the limit process and prove that the additional conservation law (1.8) actually yields the entropy inequality

$$(\rho v^2)_t + (\rho v^3)_x \leq 0 \tag{1.9}$$

in the sense of distributions for the Riemann solutions to (1.6)–(1.7). The closed system (1.6)–(1.7), with the entropy inequality (1.9), is also called the one-dimensional system of pressureless Euler equations, modeling the motion of free particles which stick under collision and other related motions (see [4, 13, 34]).

The transport equations (1.6)–(1.7) have been analyzed extensively since 1994; for example, see [1, 2, 3, 4, 14, 13, 15, 20, 21, 22, 24, 31, 33] and the references cited therein. Also see [16, 17, 18, 25, 29, 30] for related equations and results. It has been shown that, for the transport equations, δ-shocks and vacuum states do occur in the Riemann solutions satisfying the entropy inequality (1.9). The occurrence of δ-shocks and vacuum states can be regarded as a result of resonance between the two characteristic fields of the transport equations (1.6)–(1.7).

In this paper, we rigorously analyze the phenomena of concentration and cavitation in the vanishing pressure limit of inviscid nonisentropic fluid flow. This limit can be regarded as a singular flux-function limit of entropy solutions to hyperbolic conservation laws (1.5). We prove that such phenomena occur even naturally in the one-dimensional case: any Riemann solution containing two shocks and possibly one contact discontinuity to the Euler equations (1.1)–(1.4) tends to a δ-shock solution to the transport equations (1.6)–(1.7), and the intermediate densities between the two shocks tend to a weighted δ-measure that, along with the two shocks and possibly one contact discontinuity, forms a δ-shock; by contrast, any Riemann solution with two rarefaction waves and possibly one contact discontinuity for (1.1)–(1.4) tends to a two-contact-discontinuity solution to (1.6)–(1.7), the nonvacuum intermediate states between the two rarefaction waves tend to a vacuum state; and the conservation law of energy (1.8) yields the entropy inequality (1.9) naturally imposed on the δ-shocks. This shows that a δ-shock for the transport equations is a result of concentration of the density, while a vacuum state is a result of cavitation in the vanishing pressure limit; both are fundamental and physical in fluid dynamics.

Since strict hyperbolicity of the limiting system fails, the phenomena of concentration and cavitation as the pressure decreases can be regarded as a process of resonance formation between the two characteristic fields from the point of view of hyperbolic conservation laws. These phenomena show that the flux-function limit can be very singular: the limit functions of solutions are no longer in the spaces of functions, BV or L^∞; and the space of Radon measures, for which the divergences of certain entropy and entropy flux fields are Radon measures, is a natural space for dealing with such a limit in general. See [8, 9, 10] for a theory of divergence-measure fields.

In Section 2, we discuss δ-shocks and vacuum states for the transport equations (1.6)–(1.7) and examine the dependence of the Riemann solutions on the parameter $\epsilon > 0$ for the Euler equations (1.1)–(1.4). In Section 3, we analyze the phenomenon of concentration in the vanishing pressure limit of the Riemann solutions to the Euler equations (1.1)–(1.4). In Section 4, we analyze the phenomenon of cavitation in the vanishing pressure limit of the Riemann solutions to (1.1)–(1.4). In Section 5, we present some representative numerical results, obtained by using the localized central scheme (see [19, 26]), to examine the processes of concentration and cavitation in the Riemann solutions to (1.1)–(1.4) as the pressure decreases.

2. δ-Shocks, Vacuum States, and Riemann Solutions

In this section, we first briefly discuss δ-shocks and vacuum states in the Riemann solutions to the transport equations (1.6)–(1.7), and then we examine the dependence of the Riemann solutions on the parameter $\epsilon > 0$ for the Euler equations (1.1)–(1.4).

2.1. **δ-Shocks and Vacuum States for the Transport Equations.** Consider the Riemann problem for the transport equations (1.6)–(1.7) with Riemann initial data:

$$(\rho, v)(x, 0) = (\rho_\pm, v_\pm), \qquad \pm x > 0, \tag{2.1}$$

for $\rho_\pm > 0$. Since the equations and the Riemann data are invariant under uniform stretching of coordinates:

$$(x, t) \to (\alpha x, \alpha t),$$

we look for the self-similar solutions of (1.6)–(1.7) and (2.1):

$$(\rho, v)(x, t) = (\rho, v)(\xi), \qquad \xi = x/t,$$

for which the Riemann problem is reduced to the boundary value problem of the ordinary differential equations:

$$\begin{aligned} &-\xi\rho_\xi + (\rho v)_\xi = 0,\\ &-\xi(\rho v)_\xi + (\rho v^2)_\xi = 0,\\ &(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \end{aligned}$$

As in [31], in the case $v_- < v_+$, we can obtain the following solution that consists of two contact discontinuities and a vacuum state determined by the Riemann data $(\rho_\pm, v_\pm)$:

$$(\rho, v)(\xi) = \begin{cases} (\rho_-, v_-), & -\infty < \xi \le v_-,\\ (0, \xi), & v_- \le \xi \le v_+,\\ (\rho_+, v_+), & v_+ \le \xi < \infty. \end{cases}$$

In the case $v_- > v_+$, a key observation is that the singularity cannot be a jump with finite amplitude, that is, there is no solution which is piecewise smooth and bounded; hence a solution containing a weighted δ-measure (i.e. δ-shock), supported on a line, was constructed in order to establish the existence in a space of measures from the mathematical point of view (also see [29, 30]).

To define the measure solutions, the weighted δ-measure $w(t)\delta_S$ with weight $w \in C[0, \infty)$, supported on a smooth curve $S = \{(x(s), t(s)) : a < s < b\}$ can be defined by

$$\langle w(\cdot)\delta_S, \psi(\cdot,\cdot)\rangle = \int_a^b w(t(s))\psi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2}ds$$

for any $\psi \in C_0^\infty(\mathsf{IR} \times \mathsf{IR}_+)$, $\mathsf{IR} = (-\infty, \infty)$ and $\mathsf{IR}_+ = (0, \infty)$.

With this definition, one can construct a family of solutions in the case $v_- > v_+$. A δ-measure solution with parameter σ can be obtained as

$$\rho(x,t) = \rho_0(x,t) + w_0(t)\delta_S, \qquad v(x,t) = v_0(x,t).$$

Here $S = \{(x,t) : x = \sigma t,\ 0 \le t < \infty\}$,

$$\rho_0(x,t) = \rho_- + [\rho]\, H(x-\sigma t), \qquad v_0(x,t) = v_- + [v]\, H(x-\sigma t),$$

$$w_0(t) = \frac{t}{1+\sigma^2}(\sigma[\rho] - [\rho v]),$$

where $[g] := g_+ - g_-$ denotes the jump of function g across the discontinuity from the left state g_- to the right state g_+, and $H(x)$ is the Heaviside function that is 0 when $x < 0$ and 1 when $x > 0$.

Then one can conclude that the δ-measure solution satisfies

$$\langle \rho, \phi_t\rangle + \langle \rho v, \phi_x\rangle = 0, \tag{2.2}$$

$$\langle \rho v, \phi_t\rangle + \langle \rho v^2, \phi_x\rangle = 0, \tag{2.3}$$

for $\phi \in C_0^\infty(\mathrm{IR} \times \mathrm{IR}_+)$, where

$$\langle \rho v^k, \phi\rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 v_0^k \phi dxdt \, + \langle w_k\, \delta_S, \phi\rangle, \quad k = 0, 1, 2,$$

and

$$w_k(t) = \frac{t}{1+\sigma^2}([\rho v^{k+1}] - \sigma[\rho v^k]). \tag{2.4}$$

A unique solution can be singled out by the δ-Rankine-Hugoniot condition:

$$\sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \tag{2.5}$$

that satisfies the δ-entropy condition:

$$v_+ < \sigma < v_-, \tag{2.6}$$

which is consistent with (1.9). The entropy condition means that, in the (x,t)-plane, all the characteristic lines on either side of a δ-shock run into the line of δ-shock, which implies that a δ-shock is an overcompressive shock.

2.2. **Riemann Solutions to the Euler Equations for Nonisentropic Fluids.** The Euler equations (1.1)–(1.4) for smooth solutions with $\rho(x,t) > 0$ can be written as

$$\partial_t u + A(u)\partial_x u = 0, \quad u = (\rho, \rho v, \rho E)^\top,$$

where

$$A(u) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{(\gamma-1)\epsilon-2}{2}v^2 & (2-\epsilon(\gamma-1))v & \epsilon(\gamma-1) \\ -\frac{1}{\epsilon(\gamma-1)}(c^\epsilon)^2 v + \frac{\epsilon(\gamma-1)-1}{2}v^3 & \frac{1}{\epsilon(\gamma-1)}(c^\epsilon)^2 + \frac{1-2\epsilon(\gamma-1)}{2}v^2 & (1+\epsilon(\gamma-1))v \end{pmatrix}.$$

Then the eigenvalues of system (1.1)–(1.4) are

$$\lambda_1 = v - c^\epsilon, \quad \lambda_2 = v, \quad \lambda_3 = v + c^\epsilon \qquad \text{for } \rho > 0,$$

with associated right eigenvectors

$$r_j = \begin{pmatrix} 1 \\ v + \theta_j c^\epsilon \\ \frac{1}{2}v^2 + \theta_j c^\epsilon v + \frac{\theta_j^2}{\epsilon(\gamma-1)}(c^\epsilon)^2 \end{pmatrix}$$

and left eigenvectors

$$l_j = \left(\frac{1}{2}v^2 - \frac{\theta_j}{\epsilon(\gamma-1)}c^\epsilon v + \frac{(\theta_j^2-1)}{\epsilon(\gamma-1)}(c^\epsilon)^2, -v + \frac{\theta_j}{\epsilon(\gamma-1)}c^\epsilon, 1 \right),$$

where $\theta_j = sign(j-2)$ and

$$c^\epsilon = c(e,\epsilon) := \sqrt{\epsilon\gamma_\epsilon(\gamma-1)e}, \qquad \gamma_\epsilon = 1 + \epsilon(\gamma-1).$$

The Riemann invariants along the characteristic fields are

$$\left\{ S^\epsilon, v + \frac{2}{\epsilon(\gamma-1)}c^\epsilon \right\}, \quad \{v, p\}, \quad \left\{ S^\epsilon, v - \frac{2}{\epsilon(\gamma-1)}c^\epsilon \right\}, \tag{2.7}$$

respectively, where

$$S^\epsilon = ln(p\rho^{-\gamma_\epsilon}/\kappa) \text{ with } \quad \kappa > 0 \tag{2.8}$$

and

$$p = (\gamma-1)\rho e. \tag{2.9}$$

The Riemann solutions, which are the functions of $\xi = x/t$, are solutions of

$$\begin{aligned} &-\xi\rho_\xi + (\rho v)_\xi = 0, \\ &-\xi(\rho v)_\xi + (\rho v^2 + \epsilon p)_\xi = 0, \\ &-\xi(\rho E)_\xi + ((\rho E + \epsilon p)v)_\xi = 0, \\ &(\rho, v, E)(\pm\infty) = (\rho_\pm, v_\pm, E_\pm). \end{aligned}$$

Then the Rankine-Hugoniot conditions for discontinuous solutions to (1.1)–(1.4) on a discontinuity are

$$\begin{aligned} \sigma[\rho] &= [\rho v], \\ \sigma[\rho v] &= [\rho v^2 + \epsilon p], \\ \sigma[\rho E] &= [(\rho E + \epsilon p)v], \end{aligned}$$

which can be rewritten in terms of $w := v - \sigma$ and $J := \rho w = \rho v - \sigma\rho$ as

$$\begin{aligned} [J] &= 0, \\ [Jw + \epsilon p] &= 0, \\ J\left[\frac{2}{\epsilon(\gamma-1)}(c^\epsilon)^2 + w^2 \right] &= 0. \end{aligned}$$

Hereafter, we use the usual notation $[g] = g_r - g_l$, where g_l and g_r are the values of function g on the left-hand and the right-hand sides of the discontinuity, respectively.

Contact Discontinuities. If $\sigma = v_l = v_r$, the discontinuity is a contact discontinuity S_c, which corresponds to the second family of characteristic fields. Then

$$[v] = 0, \qquad [p] = 0.$$

Shock Curves. If $\sigma \neq v_l, v_r$, then the discontinuity is a shock wave, which corresponds to either the first or third family of characteristic fields. Then the Lax entropy inequalities and the Rankine-Hugoniot conditions imply that, on a 1-shock S_1,

$$[p] > 0, \quad [\rho] > 0, \quad [v] < 0, \tag{2.10}$$

and, on a 3-shock S_3,

$$[p] < 0, \quad [\rho] < 0, \quad [v] < 0, \tag{2.11}$$

which means that the shocks are compressive.

We now explicitly calculate the two one-parameter families of shock curves to examine their dependence on ϵ.

Define

$$\pi = \frac{p_r}{p_l}, \quad z = \frac{\rho_r}{\rho_l}, \quad \alpha_\epsilon = \frac{\gamma-1}{2\gamma_\epsilon}, \quad \beta_\epsilon = \frac{2}{\epsilon(\gamma-1)} + 1.$$

The entropy inequalities (2.10) and (2.11) show that $\pi > 1$ and $z > 1$ for 1-shocks and $\pi < 1$ and $z < 1$ for 3-shocks. The relation $(c^\epsilon)^2 = \epsilon\gamma_\epsilon p/\rho$ gives

$$\frac{c_r^\epsilon}{c_l^\epsilon} = \sqrt{\frac{\pi}{z}}.$$

Similarly, $[J] = 0$, i.e., $\rho_l w_l = \rho_r w_r$, yields

$$\frac{w_r}{w_l} = \frac{\rho_l}{\rho_r} = \frac{1}{z}.$$

Substituting the above relations into $J[\frac{2}{\epsilon(\gamma-1)}(c^\epsilon)^2 + w^2] = 0$ leads to

$$\left(\frac{w_l}{c_l^\epsilon}\right)^2 = \frac{(\beta_\epsilon - 1)z(z-\pi)}{1-z^2}. \tag{2.12}$$

Also, using $[\epsilon p + Jw] = 0$, we find

$$\left(\frac{w_l}{c_l^\epsilon}\right)^2 = \frac{z(1-\pi)}{\gamma_\epsilon(1-z)}. \tag{2.13}$$

A combination of the above two relations yields

$$z = \frac{1+\pi\beta_\epsilon}{\pi+\beta_\epsilon} < \beta_\epsilon.$$

Since $\beta_\epsilon - z = \frac{1-z^2}{z-\pi}$, it follows from (2.12) that

$$\frac{w_l}{c_l^\epsilon} = (-1)^{\frac{j-1}{2}}\sqrt{\frac{(\beta_\epsilon-1)z}{\beta_\epsilon - z}}$$

for j-shocks, $j = 1, 3$. The relation $w = v - \sigma$ gives the shock speed:

$$\sigma = v_l + (-1)^{\frac{j+1}{2}} c_l^\epsilon \sqrt{\frac{(\beta_\epsilon-1)z}{\beta_\epsilon - z}} = v_l + (-1)^{\frac{j+1}{2}} c_l^\epsilon \sqrt{\frac{1+\pi\beta_\epsilon}{1+\beta_\epsilon}} \tag{2.14}$$

for j-shocks, $j = 1, 3$. Noting that $w_l = z w_r$ and $w = v - \sigma$, we then have

$$v_r - v_l = \frac{z-1}{z}(\sigma - v_l) = (-1)^{\frac{j-1}{2}} c_l^\epsilon \sqrt{\frac{\beta_\epsilon - 1}{\gamma_\epsilon}}\frac{\pi - 1}{\sqrt{1+\pi\beta_\epsilon}}.$$

Following [27], one introduces $\pi = e^{-s} \geq 1$ for $s \leq 0$. In terms of this parametrization, we obtain the following formulas for the j-shock curves, $j = 1, 3$, respectively:

1-Shock Curve: For $s \leq 0$,

$$\frac{p_r}{p_l} = e^{-s}, \quad \frac{\rho_r}{\rho_l} = \frac{\beta_\epsilon + e^s}{1+\beta_\epsilon e^s}, \quad \frac{v_r - v_l}{c_l^\epsilon} = \frac{2\sqrt{\alpha_\epsilon/\epsilon}}{\gamma - 1}\frac{1-e^{-s}}{\sqrt{1+\beta_\epsilon e^{-s}}}.$$

3-Shock Curve: For $s \le 0$,

$$\frac{p_r}{p_l} = e^s, \quad \frac{\rho_r}{\rho_l} = \frac{\beta_\epsilon + e^{-s}}{1 + \beta_\epsilon e^{-s}}, \quad \frac{v_r - v_l}{c_l^\epsilon} = \frac{2\sqrt{\alpha_\epsilon/\epsilon}}{\gamma - 1} \frac{e^s - 1}{\sqrt{1 + \beta_\epsilon e^s}}.$$

Rarefaction Wave Curves. Since the 1-Riemann invariants are constant in any 1-rarefaction wave R_1, we have

$$S_r^\epsilon = S_l^\epsilon \tag{2.15}$$

and

$$v_r + (\beta_\epsilon - 1)c_r^\epsilon = v_l + (\beta_\epsilon - 1)c_l^\epsilon. \tag{2.16}$$

Then (2.15) and the relation $p = \kappa e^{S^\epsilon} \rho^{\gamma_\epsilon}$ yield

$$\frac{p_r}{p_l} = \left(\frac{\rho_r}{\rho_l}\right)^{\gamma_\epsilon}.$$

From $(c^\epsilon)^2 = \epsilon\gamma_\epsilon p/\rho$, we have

$$\frac{p_r}{p_l} = \left(\frac{c_r^\epsilon}{c_l^\epsilon}\right)^{(\beta_\epsilon - 1)\gamma_\epsilon}. \tag{2.17}$$

Using (2.16), we have

$$\frac{v_r - v_l}{c_l^\epsilon} = (\beta_\epsilon - 1)\left(1 - \frac{c_r^\epsilon}{c_l^\epsilon}\right). \tag{2.18}$$

The fact that $\lambda_1 = v - c^\epsilon$ must increase in a 1-rarefaction wave R_1 implies $v_r - v_l > c_r^\epsilon - c_l^\epsilon$. Combining this with (2.17) and (2.18) yields $0 < c_r^\epsilon/c_l^\epsilon < 1$ and $0 < p_r/p_l < 1$. Thus, we can introduce a parameter s by

$$s = -log\left(\frac{p_r}{p_l}\right) \ge 0.$$

Then we have

1-Rarefaction Wave Curve: For $s \ge 0$,

$$\frac{p_r}{p_l} = e^{-s}, \quad \frac{\rho_r}{\rho_l} = e^{-s/\gamma_\epsilon}, \quad \frac{v_r - v_l}{c_l^\epsilon} = (\beta_\epsilon - 1)\left(1 - e^{-\epsilon\alpha_\epsilon s}\right);$$

3-Rarefaction Wave Curve: For $s \ge 0$,

$$\frac{p_r}{p_l} = e^{s}, \quad \frac{\rho_r}{\rho_l} = e^{s/\gamma_\epsilon}, \quad \frac{v_r - v_l}{c_l^\epsilon} = (\beta_\epsilon - 1)\left(e^{\epsilon\alpha_\epsilon s} - 1\right).$$

Therefore, we have

1-Family Curves: For $s \in \mathrm{IR}$,

$$\begin{aligned} \frac{p_r}{p_l} &= e^{-s}, \\ \frac{\rho_r}{\rho_l} &= f_1^\epsilon(s) := \begin{cases} e^{-s/\gamma_\epsilon}, & s \ge 0, \\ \frac{\beta_\epsilon + e^s}{1 + \beta_\epsilon e^s}, & s \le 0, \end{cases} \\ \frac{v_r - v_l}{c_l^\epsilon} &= h_1^\epsilon(s) := \begin{cases} (\beta_\epsilon - 1)\left(1 - e^{-\epsilon\alpha_\epsilon s}\right), & s \ge 0, \\ \frac{2\sqrt{\alpha_\epsilon/\epsilon}}{\gamma - 1} \frac{1 - e^{-s}}{\sqrt{1 + \beta_\epsilon e^{-s}}}, & s \le 0; \end{cases} \end{aligned}$$

2-Family Curves: For $s \in \mathrm{IR}$,

$$\frac{p_r}{p_l} = 1, \quad \frac{\rho_r}{\rho_l} = e^s, \quad v_r = v_l;$$

3-Family Curves: For $s \in \mathrm{IR}$,

$$\frac{p_r}{p_l} = e^s,$$

$$\frac{\rho_r}{\rho_l} = f_3^\epsilon(s) = \frac{1}{f_1^\epsilon(s)},$$

$$\frac{v_r - v_l}{c_l^\epsilon} = h_3^\epsilon(s) := \begin{cases} (\beta_\epsilon - 1)\left(e^{\epsilon\alpha_\epsilon s} - 1\right), & s \geq 0, \\ \frac{2\sqrt{\alpha_\epsilon/\epsilon}}{\gamma-1}\frac{e^s-1}{\sqrt{1+\beta_\epsilon e^s}}, & s \leq 0. \end{cases}$$

Riemann Solutions. Consider the Riemann problem for (1.1)–(1.4):

$$u(x,0) = u_\pm, \qquad \pm x > 0. \tag{2.19}$$

With the above explicit formulas, it suffices for solving the Riemann problem to find $s_1^\epsilon, s_2^\epsilon$, and s_3^ϵ determined by

$$\rho_+ = f_1^\epsilon(s_1^\epsilon)e^{s_2^\epsilon}f_3^\epsilon(s_3^\epsilon)\rho_-, \tag{2.20}$$

$$p_+ = e^{s_3^\epsilon - s_1^\epsilon}p_-, \tag{2.21}$$

$$v_+ = v_- + c_-^\epsilon\left(h_1^\epsilon(s_1^\epsilon) + \frac{e^{-(s_1^\epsilon+s_2^\epsilon)/2}}{\sqrt{f_1^\epsilon(s_1^\epsilon)}}h_3^\epsilon(s_3^\epsilon)\right). \tag{2.22}$$

Define

$$A = \rho_+/\rho_-, \quad B = p_+/p_-, \quad C^\epsilon = (v_+ - v_-)/c_-^\epsilon.$$

Then one can find $s_i^\epsilon, i = 1, 2, 3$, successively as follows:

$$h_1^\epsilon(s_1^\epsilon) + \sqrt{\frac{B}{A}}h_1^\epsilon(s_1^\epsilon + logB) = C^\epsilon, \tag{2.23}$$

$$s_3^\epsilon = s_1^\epsilon + logB, \tag{2.24}$$

$$s_2^\epsilon = log\left(\frac{A}{f_1^\epsilon(s_1^\epsilon)f_3^\epsilon(s_3^\epsilon)}\right), \tag{2.25}$$

where we used the fact that $f_1^\epsilon(s)f_3^\epsilon(s) = 1$ and $h_3^\epsilon(s) = e^{s/2}h_1^\epsilon(s)\sqrt{f_1^\epsilon(s)}$.

Combining the above relations with the fact that $(h_1^\epsilon)'(s_1^\epsilon) > 0$ and $h_1^\epsilon(\mathrm{IR}) = (-\infty, \beta_\epsilon - 1]$, we have

Theorem 2.1. *For any fixed $\epsilon > 0$, there exists a unique Riemann solution $R(x/t)$, staying away from the vacuum, of the Riemann problem* (1.1)–(1.4) *with Riemann data* (2.19) *if and only if*

$$v_+ - v_- < \frac{2}{\gamma - 1}\frac{c_-^\epsilon + c_+^\epsilon}{\epsilon}.$$

Otherwise, the vacuum is present in the Riemann solution.

This means that, given the Riemann data staying away from the vacuum, there exists always a unique Riemann solution staying away from the vacuum when ϵ is sufficiently small, since

$$\frac{c_-^\epsilon + c_+^\epsilon}{\epsilon} \to \infty \qquad \text{as } \epsilon \to 0,$$

which is different from the situation for isentropic flow [11]. On the other hand, we will see below that, although the intermediate densities in the Riemann solution stay away from zero for any fixed $\epsilon > 0$, they become closer and closer to zero as $\epsilon \to 0$, a phenomenon of cavitation.

When $u_+ \in S_2 S_c S_1(u_-)$, $R(x/t)$ may contain a 1-shock, a 2-contact discontinuity, a 3-shock, and two nonvacuum intermediate constant states between the shocks and the contact discontinuity; and, when $u_+ \in R_2 S_c R_1(u_-)$, $R(x/t)$ may contain a 1-rarefaction wave, a 2-contact discontinuity, a 3-rarefaction wave, and two nonvacuum intermediate constant states between the rarefaction waves and the contact discontinuity. Since the other two regions $S_2 S_c R_1(u_-)$ and $R_2 S_c S_1(u_-)$ in $\mathbb{R}^3$ have empty interior when ϵ tends to zero, it suffices to analyze the limit process for the two cases $u_+ \in S_2 S_c S_1(u_-)$ (in Section 3) and $u_+ \in R_2 S_c R_1(u_-)$ (in Section 4). For more details about Riemann solutions, see [27, 28].

3. Concentration in the Vanishing Pressure Limit

In this section, we study the phenomenon of concentration and the formation of δ-shocks in the vanishing pressure limit of Riemann solutions to the Euler equations for nonisentropic fluids in the case $u_+ \in S_2 S_c S_1(u_-)$ with $v_- > v_+$.

3.1. Limiting Behavior of the Riemann Solutions as $\epsilon \to 0$.

Let $u_i^\epsilon := (\rho_i^\epsilon, \rho_i^\epsilon v_*^\epsilon, \rho_i^\epsilon E_i^\epsilon), i = 1, 2$, with $p_1^\epsilon = p_2^\epsilon = p_*^\epsilon$, be the intermediate states in the sense that (p_-, ρ_-, v_-) and $(p_*^\epsilon, \rho_1^\epsilon, v_*^\epsilon)$ determine a 1-shock S_1 with speed σ_1^ϵ and that $(p_*^\epsilon, \rho_2^\epsilon, v_*^\epsilon)$ and (p_+, ρ_+, v_+) determine a 3-shock S_3 with speed σ_3^ϵ. The difference $\rho_2^\epsilon - \rho_1^\epsilon$ is the jump of density which determines a 2-contact discontinuity S_c.

Based on these facts, we can obtain the limits of $s_i^\epsilon, i = 1, 2, 3$, as $\epsilon \to 0$, respectively.

Lemma 3.1. *If $v_- > v_+$, then*

$$\lim_{\epsilon \to 0} \epsilon\, e^{-s_1^\epsilon} = q_1 := \frac{\rho_+ \rho_- (v_+ - v_-)^2}{p_-(\sqrt{\rho_+} + \sqrt{\rho_-})^2},$$
$$\lim_{\epsilon \to 0} \epsilon\, e^{-s_3^\epsilon} = q_3 := \frac{\rho_+ \rho_- (v_+ - v_-)^2}{p_+(\sqrt{\rho_+} + \sqrt{\rho_-})^2},$$
$$\lim_{\epsilon \to 0} e^{s_2^\epsilon} = q_2 := \frac{2 + (\gamma - 1) q_3}{2 + (\gamma - 1) q_1}.$$

Proof. From (2.23), it follows that

$$\frac{1 - e^{-s_1^\epsilon}}{\sqrt{1 + \beta_\epsilon e^{-s_1^\epsilon}}} + \sqrt{\frac{B}{A}} \frac{1 - e^{-s_1^\epsilon}/B}{\sqrt{1 + \beta_\epsilon e^{-s_1^\epsilon}/B}} = \frac{(\gamma - 1) C^\epsilon}{2\sqrt{\alpha_\epsilon/\epsilon}} = \frac{v_+ - v_-}{\sqrt{2 e_-}}. \tag{3.1}$$

Note that $c_-^\epsilon = \sqrt{\epsilon e_- \gamma_\epsilon(\gamma-1)}$ and $\alpha_\epsilon = (\gamma-1)/(2\gamma_\epsilon)$. Passing to the limit $\epsilon \to 0$ yields

$$
\begin{aligned}
(3.2) \qquad & \frac{1-e^{-s_1^\epsilon}}{\sqrt{1+\beta_\epsilon e^{-s_1^\epsilon}}} + \sqrt{\frac{B}{A}}\frac{1-e^{-s_1^\epsilon}/B}{\sqrt{1+\beta_\epsilon e^{-s_1^\epsilon}/B}} \\
& \to -\frac{q_1}{\sqrt{2q_1/(\gamma-1)}} - \sqrt{\frac{B}{A}}\frac{q_1/B}{\sqrt{2q_1/(B(\gamma-1))}} \\
& = -\sqrt{q_1(\gamma-1)}\frac{1+\sqrt{A}}{\sqrt{2A}},
\end{aligned}
$$

where $q_1 = \lim_{\epsilon\to 0} \epsilon e^{-s_1^\epsilon}$. Combining (3.1) with (3.2), we obtain q_1 as asserted.

The second limit q_3 follows from the above limit and the relation

$$\epsilon e^{-s_3^\epsilon} = \epsilon e^{-s_1^\epsilon}/B.$$

Note that

$$s_2^\epsilon = log\left(\frac{A}{f_1^\epsilon(s_1^\epsilon)f_3^\epsilon(s_3^\epsilon)}\right)$$

and

$$f_1^\epsilon(s_1^\epsilon)f_3^\epsilon(s_3^\epsilon) = \frac{(\beta_\epsilon + e^{s_1^\epsilon})(1+\beta_\epsilon e^{s_3^\epsilon})}{(\beta_\epsilon + e^{s_3^\epsilon})(1+\beta_\epsilon e^{s_1^\epsilon})} \longrightarrow \frac{q_3(\gamma-1)+2}{Bq_3(\gamma-1)+2} \qquad \text{as } \epsilon \to 0.$$

We thus obtain the limit of $e^{s_2^\epsilon}$ as $\epsilon \to 0$.

Lemma 3.2. *As $\epsilon \to 0$, the intermediate densities ρ_1^ϵ and ρ_2^ϵ become unbounded, i.e.,*

$$\lim_{\epsilon\to 0} \epsilon\rho_1^\epsilon = \frac{2\rho_- q_1}{2+(\gamma-1)q_1}, \qquad \lim_{\epsilon\to 0} \epsilon\rho_2^\epsilon = \frac{2\rho_+ q_3}{2+(\gamma-1)q_3}.$$

Proof. Note that $\rho_1^\epsilon = \rho_- f_1^\epsilon(s_1^\epsilon) = \rho_- \frac{\beta_\epsilon + e^{s_1^\epsilon}}{1+\beta_\epsilon e^{s_1^\epsilon}}$, where $\beta_\epsilon e^{s_1^\epsilon} \to \frac{2}{(\gamma-1)q_1}$ as $\epsilon \to 0$. Thus,

$$\lim_{\epsilon\to 0} \epsilon\rho_1^\epsilon = \rho_- \lim_{\epsilon\to 0} \frac{2/(\gamma-1) + \epsilon(1+e^{s_1^\epsilon})}{1+\beta_\epsilon e^{s_1^\epsilon}} = \frac{2\rho_- q_1}{2+(\gamma-1)q_1}.$$

The limit of ρ_2^ϵ follows from the relation

$$\rho_2^\epsilon = \rho_+ f_1^\epsilon(s_3^\epsilon)$$

and Lemma 3.1.

We now consider the intermediate velocity v_*^ϵ and pressure p_*^ϵ between the two shocks.

Lemma 3.3. *The intermediate velocity v_*^ϵ and pressure p_*^ϵ satisfy*

$$\lim_{\epsilon\to 0} v_*^\epsilon = \sigma := \frac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_+}+\sqrt{\rho_-}} \tag{3.3}$$

and

$$\lim_{\epsilon\to 0} \epsilon p_*^\epsilon = \frac{\rho_+\rho_-(v_+-v_-)^2}{(\sqrt{\rho_+}+\sqrt{\rho_-})^2}. \tag{3.4}$$

Proof. Using the formula for the 1-shock curve, we have

$$v_*^\epsilon = v_- + \frac{2}{\gamma-1} c_-^\epsilon \sqrt{\alpha_\epsilon/\epsilon}\, \frac{1-e^{-s_1^\epsilon}}{\sqrt{1+\beta_\epsilon e^{-s_1^\epsilon}}}.$$

Noting that $c_-^\epsilon = \sqrt{\epsilon e_- \gamma_\epsilon(\gamma-1)}$ and $\alpha_\epsilon = (\gamma-1)/(2\gamma_\epsilon)$, we find

$$\lim_{\epsilon\to 0} v_*^\epsilon = v_- + \sqrt{2e_-}\lim_{\epsilon\to 0}\frac{1-e^{-s_1^\epsilon}}{\sqrt{1+\beta_\epsilon e^{-s_1^\epsilon}}} = v_- - \sqrt{(\gamma-1)e_- q_1}\,,$$

which yields (3.3) by using the expressions for q_1.

For the intermediate pressure, one has

$$p_*^\epsilon = p_- e^{-s_1^\epsilon}, \qquad p_+ = p_*^\epsilon e^{s_3^\epsilon}.$$

Combining this with the estimates for $s_i^\epsilon, i = 1, 3$, yields (3.4).

Lemma 3.4.

$$\lim_{\epsilon\to 0}\sigma_1^\epsilon = \lim_{\epsilon\to 0}\sigma_2^\epsilon = \lim_{\epsilon\to 0}\sigma_3^\epsilon = \frac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_+}+\sqrt{\rho_-}} = \sigma \in (v_+, v_-).$$

Proof. The limit for $\sigma_2^\epsilon = v_*^\epsilon$ directly follows from the known limit for v_*^ϵ above. By (2.14) with $\pi = e^{-s_1^\epsilon}$, we have

$$\lim_{\epsilon\to 0}\sigma_1^\epsilon = v_- - \lim_{\epsilon\to 0} c_-^\epsilon \sqrt{\frac{1+\beta_\epsilon e^{-s_1^\epsilon}}{1+\beta_\epsilon}} = v_- - \sqrt{(\gamma-1)e_- q_1}\,.$$

Combining this with the expression of q_1 yields the desired limit for σ_1^ϵ. The limit for σ_3^ϵ can similarly be proved.

Lemma 3.5.

$$\lim_{\epsilon\to 0}\rho_1^\epsilon(\sigma_1^\epsilon - \sigma_2^\epsilon) = \frac{\rho_-\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(v_+ - v_-), \tag{3.5}$$

and

$$\lim_{\epsilon\to 0}\rho_2^\epsilon(\sigma_2^\epsilon - \sigma_3^\epsilon) = \frac{\rho_+\sqrt{\rho_-}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(v_+ - v_-). \tag{3.6}$$

Proof. Note that $\sigma_1^\epsilon = v_- - c_-^\epsilon\sqrt{\frac{1+\beta_\epsilon e^{-s_1^\epsilon}}{1+\beta_\epsilon}}$ and

$$\sigma_2^\epsilon = v_*^\epsilon = v_- + \frac{2c_-^\epsilon\sqrt{\alpha_\epsilon/\epsilon}}{\gamma-1}\frac{1-e^{-s_1^\epsilon}}{\sqrt{1+\beta_\epsilon e^{-s_1^\epsilon}}}.$$

Thus, we have

$$\begin{aligned}
\rho_1^\epsilon(\sigma_2^\epsilon - \sigma_1^\epsilon) &= \rho_1^\epsilon c_-^\epsilon\left(\sqrt{\frac{2}{\gamma_\epsilon(\gamma-1)}}\frac{1-e^{-s_1^\epsilon}}{\sqrt{1+\beta_\epsilon e^{-s_1^\epsilon}}} + \sqrt{\frac{1+\beta_\epsilon e^{-s_1^\epsilon}}{1+\beta_\epsilon}}\right)\\
&= \frac{\rho_1^\epsilon c_-^\epsilon}{\sqrt{1+\beta_\epsilon e^{-s_1^\epsilon}}}\sqrt{\frac{2}{\gamma_\epsilon}}\left(\frac{1}{\sqrt{\gamma-1}}(1-e^{-s_1^\epsilon}) + \frac{\epsilon\sqrt{\gamma-1}}{2}(1+\beta_\epsilon e^{-s_1^\epsilon})\right)\\
&= \sqrt{\frac{2p_-}{\rho_-}}\frac{\epsilon\rho_1^\epsilon}{\sqrt{\epsilon^2+\epsilon\beta_\epsilon\cdot\epsilon e^{-s_1^\epsilon}}}\left(\frac{1}{\sqrt{\gamma-1}} + \frac{\epsilon\sqrt{\gamma-1}}{2}(1+e^{-s_1^\epsilon})\right)\\
&\to \frac{2q_1\sqrt{2\rho_- p_-}}{(2+(\gamma-1)q_1)\sqrt{2q_1/(\gamma-1)}}\frac{2+(\gamma-1)q_1}{2\sqrt{\gamma-1}} = \sqrt{p_-\rho_- q_1}\,,
\end{aligned}$$

which gives (3.5). To prove (3.6), we use

$$\sigma_3^\epsilon = v_+ + c_+^\epsilon \sqrt{\frac{1+\beta_\epsilon e^{-s_3^\epsilon}}{1+\beta_\epsilon}}$$

and

$$\sigma_2^\epsilon = v_*^\epsilon = v_+ - c_2^\epsilon h_3^\epsilon(s_3^\epsilon),$$

where

$$c_2^\epsilon = \frac{\epsilon\gamma_\epsilon p_*^\epsilon}{\rho_2^\epsilon} = \epsilon\gamma_\epsilon \frac{p_+}{\rho_+} f_3^\epsilon(s_3^\epsilon) e^{-s_3^\epsilon}.$$

Therefore, we have

$$\sigma_3^\epsilon - \sigma_2^\epsilon = c_+^\epsilon \left(\sqrt{\frac{\epsilon(\gamma-1)}{2\gamma_\epsilon}} \sqrt{1+\beta_\epsilon e^{-s_3^\epsilon}} + \frac{2\sqrt{\alpha_\epsilon/\epsilon}}{\gamma-1} \frac{1-e^{-s_3^\epsilon}}{\sqrt{1+\beta_\epsilon e^{-s_3^\epsilon}}} \right).$$

Performing similar analysis to that for the first limit, we have

$$\begin{aligned} \rho_2^\epsilon(\sigma_3^\epsilon - \sigma_2^\epsilon) &= \frac{\rho_2^\epsilon c_+^\epsilon}{\sqrt{1+\beta_\epsilon e^{-s_3^\epsilon}}} \sqrt{\frac{2}{\gamma_\epsilon}} \left(\frac{1}{\sqrt{\gamma-1}} + \frac{\epsilon\sqrt{\gamma-1}}{2}(1+e^{-s_3^\epsilon}) \right) \\ &\to \sqrt{\rho_+ p_+ q_3} \qquad \text{as } \epsilon \to 0, \end{aligned}$$

which arrives at (3.6).

Remark 3.1. The quantity σ that is the limit of v_*^ϵ and $\sigma_j^\epsilon, j = 1,2,3$, in Lemma 3.4 uniquely determines the δ-shock solution as the limit of the Riemann solutions when $\epsilon \to 0$ and is consistent with the δ-Rankine-Hugoniot condition (2.5) and the δ-entropy condition (2.6), as proposed for the Riemann solutions for the pressureless Euler equations.

3.2. **Concentration and δ-Shocks.** We now show the following theorem characterizing the vanishing pressure limit for the case $v_- > v_+$.

Theorem 3.1. *Let $(\rho^\epsilon, \rho^\epsilon v^\epsilon, \rho^\epsilon E^\epsilon)$ be the Riemann solutions to* (1.1)–(1.4) *and* (2.19)*, which contain two shocks and possibly one contact discontinuity. Then ρ^ϵ, $\rho^\epsilon v^\epsilon$, and $\rho^\epsilon E^\epsilon$ converge in the sense of distributions, respectively, and the limit functions are all sums of a step function and a δ-measure with weights*

$$\frac{t}{\sqrt{1+\sigma^2}}([\rho v] - \sigma[v]), \quad \frac{t}{\sqrt{1+\sigma^2}}([\rho v^2] - \sigma[\rho v]), \quad \frac{t}{\sqrt{1+\sigma^2}}([\rho v E] - \sigma[\rho E]),$$

respectively, where $\sigma = (\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+)/(\sqrt{\rho_+} + \sqrt{\rho_-})$.

Proof. 1. For $\xi = x/t$, the Riemann solutions are determined by

$$\rho^\epsilon(\xi) = \begin{cases} \rho_-, & \xi < \sigma_1^\epsilon, \\ \rho_1^\epsilon, & \sigma_1^\epsilon < \xi < \sigma_2^\epsilon, \\ \rho_2^\epsilon, & \sigma_2^\epsilon < \xi < \sigma_3^\epsilon, \\ \rho_+, & \xi > \sigma_3^\epsilon, \end{cases}$$

and

$$v^\epsilon(\xi) = \begin{cases} v_-, & \xi < \sigma_1^\epsilon, \\ v_*^\epsilon, & \sigma_1^\epsilon < \xi < \sigma_3^\epsilon, \\ v_+, & \xi > \sigma_3^\epsilon, \end{cases}$$

which satisfy the following weak formulations:

(3.7)
$$-\int_{-\infty}^{\infty}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)\psi'(\xi)d\xi+\int_{-\infty}^{\infty}\rho^\epsilon(\xi)\psi(\xi)d\xi=0,$$

(3.8)
$$-\int_{-\infty}^{\infty}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)v^\epsilon(\xi)\psi'(\xi)d\xi+\int_{-\infty}^{\infty}\rho^\epsilon(\xi)v^\epsilon(\xi)\psi(\xi)d\xi=\epsilon\int_{-\infty}^{\infty}p^\epsilon(\xi)\psi'(\xi)d\xi,$$

(3.9)
$$\begin{aligned}-\int_{-\infty}^{\infty}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)E^\epsilon(\xi)\psi'(\xi)d\xi+\int_{-\infty}^{\infty}\rho^\epsilon(\xi)E^\epsilon(\xi)\psi(\xi)d\xi\\ =\epsilon\int_{-\infty}^{\infty}p^\epsilon(\xi)v^\epsilon(\xi)\psi'(\xi)d\xi\end{aligned}$$

for any $\psi\in C_0^\infty(\mathbb{R})$.

2. The first integral in (3.7) can be decomposed into:
$$-\left\{\int_{-\infty}^{\sigma_1^\epsilon}+\int_{\sigma_1^\epsilon}^{\sigma_2^\epsilon}+\int_{\sigma_2^\epsilon}^{\sigma_3^\epsilon}+\int_{\sigma_3^\epsilon}^{\infty}\right\}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)\psi'(\xi)d\xi.$$

The sum of the first and last terms is
$$\begin{aligned}&-\int_{-\infty}^{\sigma_1^\epsilon}(v_--\xi)\rho_-\psi'(\xi)d\xi-\int_{\sigma_3^\epsilon}^{\infty}(v_+-\xi)\rho_+\psi'(\xi)d\xi\\ &=-v_-\rho_-\psi(\sigma_1^\epsilon)+v_+\rho_+\psi(\sigma_3^\epsilon)+\rho_-\sigma_1^\epsilon\psi(\sigma_1^\epsilon)\\ &\quad-\int_{-\infty}^{\sigma_1^\epsilon}\rho_-\psi(\xi)d\xi-\rho_+\sigma_3^\epsilon\psi(\sigma_3^\epsilon)-\int_{\sigma_3^\epsilon}^{\infty}\rho_+\psi(\xi)d\xi,\end{aligned}$$

which converges, as $\epsilon\to 0$, to
$$([\rho v]-\sigma[\rho])\psi(\sigma)-\int_{-\infty}^{\sigma}\rho_-\psi(\xi)d\xi-\int_{\sigma}^{\infty}\rho_+\psi(\xi)d\xi=([\rho v]-\sigma[\rho])\psi(\sigma)-\int_{-\infty}^{\infty}\rho_0(\xi-\sigma)\psi(\xi)d\xi$$

with
$$\rho_0(\xi)=\rho_-+[\rho]H(\xi).$$

Furthermore, we have
$$\begin{aligned}&\int_{\sigma_1^\epsilon}^{\sigma_2^\epsilon}(v_*^\epsilon-\xi)\rho_1^\epsilon\psi'(\xi)d\xi\\ &=\rho_1^\epsilon(\sigma_2^\epsilon-\sigma_1^\epsilon)\left(v_*^\epsilon\frac{\psi(\sigma_2^\epsilon)-\psi(\sigma_1^\epsilon)}{\sigma_2^\epsilon-\sigma_1^\epsilon}-\frac{\sigma_2^\epsilon\psi(\sigma_2^\epsilon)-\sigma_1^\epsilon\psi(\sigma_1^\epsilon)}{\sigma_2^\epsilon-\sigma_1^\epsilon}+\frac{1}{\sigma_2^\epsilon-\sigma_1^\epsilon}\int_{\sigma_1^\epsilon}^{\sigma_2^\epsilon}\psi(\xi)d\xi\right),\end{aligned}$$

which, in virtue of the smoothness of the test function $\psi(\xi)$, converges to
$$(v_+-v_-)\frac{\rho_-\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}\left(-\sigma\psi'(\sigma)+\sigma\psi'(\sigma)+\psi(\sigma)-\psi(\sigma)\right)=0$$

as $\epsilon\to 0$, where we used the facts that $\lim_{\epsilon\to 0}v_*^\epsilon=\sigma$ and $\lim_{\epsilon\to 0}\sigma_i^\epsilon=\sigma$ for $i=1,2$, and (3.5). Similarly, we have
$$\lim_{\epsilon\to 0}\int_{\sigma_2^\epsilon}^{\sigma_3^\epsilon}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)\psi'(\xi)d\xi=0.$$

Returning to (3.7), one has

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(\rho^\epsilon(\xi)-\rho_0(\xi-\sigma))\psi(\xi)d\xi=([\rho v]-\sigma[\rho])\psi(\sigma)$$

for all function $\psi\in C_0^\infty(\mathbb{R})$.

3. We now justify the limit of the momentum $m^\epsilon=\rho^\epsilon v^\epsilon$ by using the weak formulation of the momentum equation (3.8). As done previously, we can obtain the limit for the first term on the left hand side of (3.8) as

$$\begin{aligned}&-\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)v^\epsilon(\xi)\psi'(\xi)d\xi\\&=\psi(\sigma)([\rho v^2]-\sigma[\rho v])-\int_{-\infty}^{\sigma}\rho_-v_-\psi(\xi)d\xi-\int_{\sigma}^{\infty}\rho_+v_+\psi(\xi)d\xi\\&=\psi(\sigma)([\rho v^2]-\sigma[\rho v])-\int_{\mathbb{R}}m_0(\xi-\sigma)\psi(\xi)d\xi\end{aligned}$$

with

$$m_0(\xi)=\rho_-v_-+[\rho v]H(\xi).$$

The term on the right hand side of (3.8) equals to

$$\begin{aligned}\epsilon\int_{-\infty}^{\infty}p^\epsilon(\xi)\psi'(\xi)d\xi&=\epsilon\left(\int_{-\infty}^{\sigma_1^\epsilon}p_-+\int_{\sigma_1^\epsilon}^{\sigma_3^\epsilon}p_*^\epsilon+\int_{\sigma_3^\epsilon}^{\infty}p_+\right)\psi'(\xi)d\xi\\&=\epsilon p_-\psi(\sigma_1^\epsilon)+\epsilon p_*^\epsilon(\psi(\sigma_3^\epsilon)-\psi(\sigma_1^\epsilon))-\epsilon p_+\psi(\sigma_3^\epsilon)\\&=o(\epsilon)+O(1)(\psi(\sigma_3^\epsilon)-\psi(\sigma_1^\epsilon))\to 0\qquad\text{as}\quad\epsilon\to 0,\end{aligned}$$

where we used the fact that ϵp_*^ϵ is bounded and $\lim_{\epsilon\to 0}\sigma_i^\epsilon=\sigma$ for $i=1,3$.

Returning to the weak formulation (3.8), one has

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(\rho^\epsilon(\xi)v^\epsilon(\xi)-m_0(\xi-\sigma))\psi(\xi)d\xi=\psi(\sigma)(\sigma[\rho v]-[\rho v^2]).$$

4. Next we consider the limit of the total energy E^ϵ by using the weak formulation of the energy equation (3.9). Performing the similar analysis as above and using the identity

$$\rho E=\frac{1}{2}\rho v^2+\frac{p}{\gamma-1},$$

we have

$$\begin{aligned}&-\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(v^\epsilon(\xi)-\xi)\rho^\epsilon(\xi)E^\epsilon(\xi)\psi'(\xi)d\xi\\&=\psi(\sigma)([\rho vE]-\sigma[\rho E])-\int_{-\infty}^{\sigma}\rho_-E_-\psi(\xi)d\xi-\int_{\sigma}^{\infty}\rho_+E_+\psi(\xi)d\xi.\end{aligned}$$

The term on the right hand side of (3.9) tends to zero as $\epsilon\to 0$. Therefore, we have

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(\rho^\epsilon E^\epsilon(\xi)-\mathcal{E}_0(\xi-\sigma))\psi(\xi)d\xi=\psi(\sigma)(\sigma[\rho E]-[\rho vE])$$

with

$$\mathcal{E}_0(\xi)=\rho_-E_-+[\rho E]H(\xi).$$

5. We are now in the position to study the limits of density, momentum, and energy by tracing the time-dependence of weights of the δ-measures.

Let $\psi \in C_0^\infty(\mathrm{IR} \times \mathrm{IR}_+)$ and set $\tilde{\psi}(\xi, t) := \psi(\xi t, t)$. Then we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \rho^\epsilon(x,t)\psi(x,t)dxdt &= \lim_{\epsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \rho^\epsilon(\frac{x}{t})\psi(x,t)dxdt\\ &= \lim_{\epsilon\to 0}\int_0^\infty t\left(\int_{-\infty}^\infty \rho^\epsilon(\xi)\tilde{\psi}(\xi,t)d\xi\right)dt,\end{aligned}$$

since ρ^ϵ is a self-similar solution depending only on $\xi = x/t$. Therefore, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{-\infty}^\infty \rho^\epsilon(\xi)\tilde{\psi}(\xi,t)d\xi &= \int_{-\infty}^\infty \rho_0(\xi-\sigma)\tilde{\psi}(\xi,t)d\xi + ([\rho v]-\sigma[\rho])\tilde{\psi}(\sigma,t)\\ &= t^{-1}\int_{-\infty}^\infty \rho_0(x-\sigma t)\psi(x,t)dx + ([\rho v]-\sigma[\rho])\psi(\sigma t,t).\end{aligned}$$

Combining the above two relations, we have

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^\infty\int_{\infty}^\infty \rho^\epsilon(x,t)\psi(x,t)dxdt\\ &= \int_0^\infty\int_{-\infty}^\infty \rho_0(x-\sigma t)\psi(x,t)dxdt + \int_0^\infty t([\rho v]-\sigma[\rho])\psi(\sigma t,t)dt.\end{aligned}$$

The last term, by definition, equals to

$$\langle w_0(t)\delta_S, \psi(x,t)\rangle$$

with $w_0(t) = \frac{t}{\sqrt{1+\sigma^2}}([\rho v]-\sigma[\rho])$ as in (2.4).

Similarly, we can show that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^\infty\int_{-\infty}^\infty (\rho^\epsilon v^\epsilon)(x,t)\psi(x,t)dxdt\\ &= \int_0^\infty\int_{-\infty}^\infty m_0(x-\sigma t)\psi(x,t)dxdt + \langle w_1(t)\delta_S, \psi(x,t)\rangle\end{aligned}$$

with $w_1(t) = \frac{t}{\sqrt{1+\sigma^2}}([\rho v^2]-\sigma[\rho v])$, and

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \rho^\epsilon(x,t)E^\epsilon(x,t)\psi(x,t)dxdt\\ &= \int_0^\infty\int_{-\infty}^\infty \mathcal{E}_0(x-\sigma t)\psi(x,t)dxdt + \langle \tilde{w}(t)\delta_S, \psi(x,t)\rangle\end{aligned}$$

with $\tilde{w}(t) = \frac{t}{\sqrt{1+\sigma^2}}([\rho vE]-\sigma[\rho E])$ as defined in (2.4).

Finally, we conclude the entropy consistency by proving that the conservation law of energy (1.3) actually yields the correct entropy inequality (1.9) for the transport equations (1.6) and (1.7).

Theorem 3.2. *The limit functions (ρ, v) are a measure solution of the transport equation* (1.6)–(1.7) *satisfying*

$$\partial_t(\rho v^2) + \partial_x(\rho v^3) \le 0$$

in the sense of distributions.

Proof. Since $\partial_t(\rho^\epsilon E^\epsilon) + \partial_x(\rho^\epsilon v^\epsilon E^\epsilon) = 0$ in the sense of distributions and $\rho E = \frac{p}{\gamma-1} + \frac{1}{2}\rho v^2$, then

$$\partial_t(\rho^\epsilon (v^\epsilon)^2) + \partial_x(\rho^\epsilon (v^\epsilon)^3) = -\frac{2}{\gamma-1}(\partial_t p^\epsilon + \partial_x(p^\epsilon v^\epsilon)).$$

That is, for any nonnegative test function $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$,

$$-\left(\langle \rho^\epsilon (v^\epsilon)^2, \phi_t\rangle + \langle \rho^\epsilon (v^\epsilon)^3, \phi_x\rangle\right) = \frac{2}{\gamma - 1}\left(\langle p^\epsilon, \phi_t\rangle + \langle p^\epsilon v^\epsilon, \phi_x\rangle\right).$$

Note that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^\infty \int_{-\infty}^\infty (p^\epsilon \phi_t + p^\epsilon v^\epsilon \phi_x) dx dt \\
&= \lim_{\epsilon\to 0}\int_0^\infty \left(\sigma_1^\epsilon (p_*^\epsilon - p_-) - (p_*^\epsilon v_*^\epsilon - p_- v_-)\right)\phi(\sigma_1^\epsilon t, t) dt \\
&\quad + \lim_{\epsilon\to 0}\int_0^\infty \sigma_2^\epsilon\left((p_+ - p_*^\epsilon) - (p_+ v_+ - p_*^\epsilon v_*^\epsilon)\right)\phi(\sigma_2^\epsilon t, t) dt \\
&= \int_0^\infty \left(\sigma(p_+ - p_-) - (p_+ v_+ - p_- v_-)\right)\phi(\sigma t, t) dt \\
&= \int_0^\infty (v_- - v_+)(\alpha p_+ + (1-\alpha) p_-)\phi(\sigma t, t) dt < 0,
\end{aligned}$$

since, for $\alpha = \frac{\sqrt{\rho_-}}{\sqrt{\rho_+}+\sqrt{\rho_-}} \in (0,1)$,

$$\sigma = \frac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_+} + \sqrt{\rho_-}} = \alpha v_- + (1-\alpha) v_+.$$

This verifies the entropy consistency as claimed.

4. Cavitation in the Vanishing Pressure Limit

In this section, we show the phenomenon of cavitation in the vanishing pressure limit of the Riemann solutions to (1.1)–(1.4) in the case $u_+ \in R_2 S_c R_1(u_-)$ with $v_- < v_+$ and $\rho_\pm > 0$.

Let $u_*^\epsilon = (\rho_i^\epsilon, \rho_i^\epsilon v_*^\epsilon, E_i^\epsilon)$, $i = 1, 2$, with $p_*^\epsilon = p_1^\epsilon = p_2^\epsilon$ be the intermediate states in the sense that (p_-, ρ_-, v_-) and $(p_*^\epsilon, \rho_1^\epsilon, v_*^\epsilon)$ determine a 1-rarefaction wave R_1, and that $(p_*^\epsilon, \rho_2^\epsilon, v_*^\epsilon)$ and (p_+, ρ_+, v_+) determine a 3-rarefaction wave R_3. The difference $\rho_2^\epsilon - \rho_1^\epsilon$ is the jump of density across the contact discontinuity S_c.

The limits of the intermediate states of the Riemann solutions are determined by the limits of $s_i^\epsilon, i = 1, 2, 3$.

Lemma 4.1. *If $v_- < v_+$, then*

$$\lim_{\epsilon\to 0}\sqrt{\epsilon}s_1^\epsilon = \lim_{\epsilon\to 0}\sqrt{\epsilon}s_3^\epsilon = \frac{(\gamma-1)\sqrt{\rho_+\rho_-}}{2(\sqrt{\rho_+ p_-} + \sqrt{\rho_- p_+})}(v_+ - v_-),$$

$$\lim_{\epsilon\to 0} s_2^\epsilon = log\left(\frac{A}{B}\right).$$

Proof. The fact that $\lim_{\epsilon\to 0}\sqrt{\epsilon}s_1^\epsilon = \lim_{\epsilon\to 0}\sqrt{\epsilon}s_3^\epsilon$ follows directly from the relation $s_1^\epsilon = s_3^\epsilon + logB$. It suffices to show the limit for $\sqrt{\epsilon}s_1^\epsilon$, for which we need the relation:

$$h_1^\epsilon(s_1^\epsilon) + \sqrt{\frac{B}{A}}h_1^\epsilon(s_1^\epsilon + logB) = C^\epsilon.$$

Using $h_1^\epsilon(s_1^\epsilon) = \frac{2}{\epsilon(\gamma-1)}\left(1-e^{-\epsilon\alpha_\epsilon s_1^\epsilon}\right)$ and $C^\epsilon = (v_+ - v_-)/c_-^\epsilon$, we can show the following limit:

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\left(1-e^{-\epsilon\alpha_\epsilon s_1^\epsilon}\right) = \frac{1}{2}\sqrt{\frac{\gamma-1}{e_-}}\frac{\sqrt{A}}{\sqrt{A}+\sqrt{B}}(v_+ - v_-).$$

It follows from the above limit that

$$\lim_{\epsilon\to 0}\sqrt{\epsilon}s_1^\epsilon = \frac{(\gamma-1)\sqrt{\rho_+\rho_-}}{2(\sqrt{\rho_+ p_-}+\sqrt{\rho_- p_+})}(v_+ - v_-)$$

as asserted.

For the limit of s_2^ϵ, we use

$$s_2^\epsilon = log\left(\frac{A}{f_1^\epsilon(s_1^\epsilon)f_3^\epsilon(s_3^\epsilon)}\right) = log\left(\frac{A}{e^{(s_3^\epsilon - s_1^\epsilon)/\gamma_\epsilon}}\right)$$

and $s_3^\epsilon = s_1^\epsilon + logB$ to yield the desired limit.

With the above lemma, we are able to conclude the following theorem.

Theorem 4.1. *When $v_- < v_+$, the cavitation occurs as $\epsilon \to 0$. More precisely,*

$$\lim_{\epsilon\to 0}\rho_1^\epsilon = \lim_{\epsilon\to 0}\rho_2^\epsilon = 0, \qquad \lim_{\epsilon\to 0}\frac{\rho_2^\epsilon}{\rho_1^\epsilon} = \frac{A}{B}.$$

These limits follow directly from the relations

$$\rho_1^\epsilon = \rho_- e^{-s_1^\epsilon/\gamma_\epsilon}, \qquad \rho_2^\epsilon = \rho_+ e^{-s_3^\epsilon/\gamma_\epsilon},$$

and

$$\rho_2^\epsilon = \rho_1^\epsilon e^{s_2^\epsilon}.$$

5. Processes of Concentration and Cavitation: Numerical Simulations

In order to understand the processes of concentration and cavitation in the Riemann solutions to the Euler equations (1.1)–(1.4) when the pressure vanishes, we present a selected group of representative numerical results. We have performed many more numerical tests to make sure what we present are not numerical artifacts.

To discretize the conservation laws (1.5), we use the central scheme of the form

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n,\epsilon) - f(u_{j-1}^n,\epsilon)\right) \\ &\quad + \frac{\lambda}{2}\left(a_{j+1/2,\epsilon}^n(u_{j+1}^n - u_j^n) - a_{j-1/2,\epsilon}^n(u_j^n - u_{j-1}^n)\right), \end{aligned} \tag{5.1}$$

where $a_{j\pm 1/2,\epsilon}^n$ are the maximal local speeds and $\lambda = \Delta t/\Delta x$. This scheme coincides with the so-called local Lax-Friedrichs scheme in [26]; for its higher-order version, see [19]. The main feature of this type of central schemes is to use more precise information about the local speeds of wave propagation, in comparison with the original Lax-Friedrichs scheme as well as its higher-order extension–the NT scheme [23].

To illustrate the process of concentration, we solve the Riemann problem for (1.1)–(1.4) for $\gamma = 1.4$ for an ideal gas with Riemann data determined by

$$(\rho, v, E)(x,0) = \begin{cases} (1.0, 1.5, 2.5) & \text{if } x < 0, \\ (0.2, 0.0, 1.25) & \text{if } x > 0. \end{cases}$$

We calculate by the first-order scheme (5.1) (see e.g. [19, 26]) up to $t = 0.1, 0.3$ with mesh 200. The numerical simulations for different choices of ϵ are presented in Figs. 5.1–5.3 for $t = 0.1$ and in Figs. 5.4–5.6 for $t = 0.3$.

These figures show the process of concentration in the vanishing pressure limit of the Riemann solutions containing two shocks and one contact discontinuity in nonisentropic Euler flow. We start with $\epsilon = 0.085$, then $\epsilon = 0.055$, and finally $\epsilon = 0.025$. Figs. 5.1a–5.6a show the concentration of density yielding a weighted δ-measure in the limit, in which the horizontal axis is for the space variable x and the vertical axis is for the density. Figs. 5.1b–5.6b show the change of the velocity as ϵ decreases yielding a step function in the limit, in which the horizontal axis is for the space variable x and the vertical axis is for the velocity.

We can clearly see from these numerical results that, when ϵ decreases, the locations of the two shocks become closer to the contact discontinuity, and the densities of the intermediate states increase dramatically, while the velocity is closer to a step function; in the vanishing pressure limit, the two shocks and the contact discontinuity coincide to form, along with the intermediate states, a δ-shock of the transport equations (1.6) and (1.7), while the velocity is a step function.

The process of cavitation is simulated for the Riemann problem (1.1)–(1.4) with the initial data determined by

$$(\rho, v, E)(x, 0) = \begin{cases} (1.0, 0, 2.5) & \text{if} \quad x < 0, \\ (0.2, 1.5, 1.25) & \text{if} \quad x > 0. \end{cases}$$

In the rarefaction wave cases, we employ the first-order central scheme (3.7) to compute the solution up to $t = 0.1, 0.3$, respectively. Numerical simulations are presented in Figs. 5.7–5.9 for $t = 0.1$ and in Figs. 5.10–5.12 for $t = 0.3$. These figures show the process of cavitation in the vanishing pressure limit of the Riemann solutions containing two rarefaction waves and one contact discontinuity, starting away from the vacuum, in nonisentropic Euler flow. We start with $\epsilon = 0.085$, then $\epsilon = 0.055$, and finally $\epsilon = 0.025$. Figs. 5.7a–5.12a show the cavitation of the density yielding a vacuum state between the left boundary of the 1-rarefaction wave and the right boundary of the 2-rarefaction wave in the limit, in which the horizontal axis is for the space variable x and the vertical axis is for the density. Figs. 5.7b–5.12b show the change of the momentum as ϵ decreases yielding a linear function between the left boundary of the 1-rarefaction wave and the right boundary of the 2-rarefaction wave in the limit, in which the horizontal axis is for the space variable x and the vertical axis is for the momentum.

We can clearly see from these numerical results that, when ϵ decreases, the left boundary of the 1-rarefaction wave and the right boundary of the 2-rarefaction wave are fixed; the right boundary of the 1-rarefaction wave and the left boundary of the 2-rarefaction wave become closer and closer, while the states between the left boundary of the 1-rarefaction wave and the right boundary of the 2-rarefaction wave in the Riemann solution tends to a vacuum state; and, in the limit, the left boundary of the 1-rarefaction wave and the right boundary of the 2-rarefaction wave become two contact discontinuities of the transport equation (1.6)–(1.7).

Acknowledgments. Gui-Qiang Chen's research was supported in part by the National Science Foundation under Grants DMS-0204225, DMS-0204455, and INT-9987378. Hailiang Liu's research was supported in part by the National Science Foundation under Grant DMS01-07917.

References

[1] F. Bouchut, On zero pressure gas dynamics, In: Advances in Kinetic Theory and Computing, pp. 171–190, Ser. Adv. Math. Appl. Sci. **22**, World Sci. Publishing: River Edge, NJ, 1994.

[2] F. Bouchut and F. James, Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness, *Commun. Partial Diff. Eqs.* **24** (1999), 2173–2189.

[3] F. Bouchut, S. Jin, and X. Li, Numerical approximation of pressureless and isothermal gas dynamics, SIAM J. Numer. Anal. **41** (2003), 135–158.

[4] Y. Brenier and E. Grenier, Sticky particles and scalar conservation laws, *SIAM J. Numer. Anal.* **35** (1998), 2317–2328.

[5] T. Chang, G.-Q. Chen, and S. Yang, 2-D Riemann problem in gas dynamics and formation of spiral, In: *Nonlinear Problems in Engineering and Science–Numerical and Analytical Approach (Beijing, 1991)*, pp. 167–179, Science Press: Beijing, 1992.

[6] T. Chang, G.-Q. Chen, and S. Yang, On the Riemann problem for two-dimensional Euler equations, I: Interaction of shocks and rarefaction waves, Discrete Contin. Dynam. Systems, **1** (1995), 555–584; II: Interaction of contact discontinuities, Discrete Contin. Dynam. Systems, **6** (2000), 419–430.

[7] T. Chang and L. Hsiao, The Riemann Problem and Interaction of Waves in Gas Dynamics, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc.: New York, 1989.

[8] G.-Q. Chen and H. Frid, Divergence-measure fields and hyperbolic conservation laws, Arch. Rational Mech. Anal. **147** (1999), 89–118.

[9] G.-Q. Chen and H. Frid, Extended divergence-measure fields and the Euler equations of gas dynamics, Commun. Math. Phys. **228** (2002), 201–217.

[10] G.-Q. Chen and H. Frid, On the theory of divergence-measure fields and its applications, Bol. Soc. Brasil. Mat. (N.S.) **32** (2001), 401–433.

[11] G.-Q. Chen and H. Liu, Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids, SIAM J. Math. Anal. **34** (2003), 925-938.

[12] G.-Q. Chen and D. Wang, The Cauchy problem for the Euler equations for compressible fluids, In: *Handbook of Mathematical Fluid Dynamics,* Vol. **1**, pp. 421-543, Elsevier: Amsterdam, The Netherlands, 2002.

[13] W. E, Yu. G. Rykov, and Ya. G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics, *Commun. Math. Phys.* **177** (1996), 349–380.

[14] E. Grenier, Existence globale pour le système des gaz sans pression, *C. R. Acad. Sci. Paris Sér. I. Math.* **321** (1995), 171–174.

[15] F. Huang and Z. Wang, Well posedness for pressureless flow, *Commun. Math. Phys.* **222** (2001), 117-146.

[16] K. T. Joseph, A Riemann problem whose viscosity solutions contain delta measure, *Asymptotic Analysis*, **7** (1993), 105-120.

[17] B. L. Keyfitz and H. Kranzer, Spaces of weighted measures for conservation laws with singular shock solutions, *J. Diff. Eqs.* **118** (1995), 420–451.

[18] D. J. Korchinski, Solutions of a Riemann problem for a 2×2 system of conservation laws possessing no classical solution, Adelphi University Thesis, 1977.

[19] A. Kurganov and E. Tadmor, New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations, *J. Compt. Phys.* **160**, (2000), 241–282.

[20] J. Li and H. Yang, Delta-shocks as limits of vanishing viscosity for multidimensional zero-pressure gas dynamics, *Quart. Appl. Math.* **59** (2001), 315–342.

[21] J. Li and T. Zhang, Generalized Rankine-Hugoniot relations of delta-shocks in solutions of transportation equations, In: *Advances in Nonlinear Partial Differential Equations and*

Related Areas (Beijing, 1997), 219–232, eds. G.-Q. Chen et al, World Sci. Publishing: River Edge, NJ, 1998.

[22] J. Li and T. Zhang, On the initial-value problem for zero-pressure gas dynamics, In:*Hyperbolic Problems: Theory, Numerics, Applications*, Vol. **2** (Zürich, 1998), 629–640, Birkhäuser: Basel, 1999.

[23] H. Nessyahu and E. Tadmor, Non-oscillatory central differencing for hyperbolic conservation laws, *J. Comput. Phys.* **87** (1990), 408-463.

[24] F. Poupaud and M. Rascle, Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients, *Commun. Partial Diff. Eqs.* **22** (1997), 337–358.

[25] M. Sever, A class of nonlinear, nonhyperbolic systems of conservation laws with well-posed initial value problem, J. Diff. Eqs. **180** (2002), 238–271.

[26] Chi-Wang Shu and S. Osher, Efficient implementation of essentially nonoscillatory shock-capturing schemes, *J. Comput. Phys.* **77** (1988), 439–471.

[27] R. Smith, The Riemann problem in gas dynamics, *Trans. Amer. Math. Soc.* **249** (1979), 1-50.

[28] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd edition, Springer-Verlag: New York, 1994.

[29] D. Tan and T. Zhang, Two dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws (I): four-J cases, J. Diff. Eqs. **111** (1994), 203-254; (II): Initial data consists of some rarefaction, J. Diff. Eqs. **111** (1994), 255-283.

[30] D. Tan, T. Zhang, and Y. Zheng, Delta-shock waves as limits of vanishing viscosity for hyperbolic system of conservation laws, J. Diff. Eqs. **112** (1994), 1-32.

[31] W. Sheng and T. Zhang, *The Riemann Problem for the Transport Equations in Gas Dynamics*, Mem. Amer. Soc. **654** (1999), AMS: Providence.

[32] Z. Wang and X. Ding, Uniqueness of generalized solutions for the Cauchy problem of transportation equations, *Acta Math. Sci.* (English Ed.), **17** (1997), 341-352.

[33] Z. Wang, F. Huang, and X. Ding, On the Cauchy problem of transportation equations, *Acta Math. Appl. Sinica (English Ser.)*, **13** (1997), 113–122.

[34] Ya. B. Zeldovich, Gravitational instability: an approximate theory for large density perturbations, Astron. Astrophys. **5** (1970), 84-89.

Gui-Qiang Chen
Department of Mathematics, Northwestern University
Evanston, IL 60208-2730, USA
gqchen@math.northwestern.edu
Homepage: http://www.math.northwestern.edu/~gqchen/preprints

Hailiang Liu
Department of Mathematics, Iowa State University
Ames, IA 50011, USA
hliu@iastate.edu
Homepage: http://www.math.iastate.edu/hliu

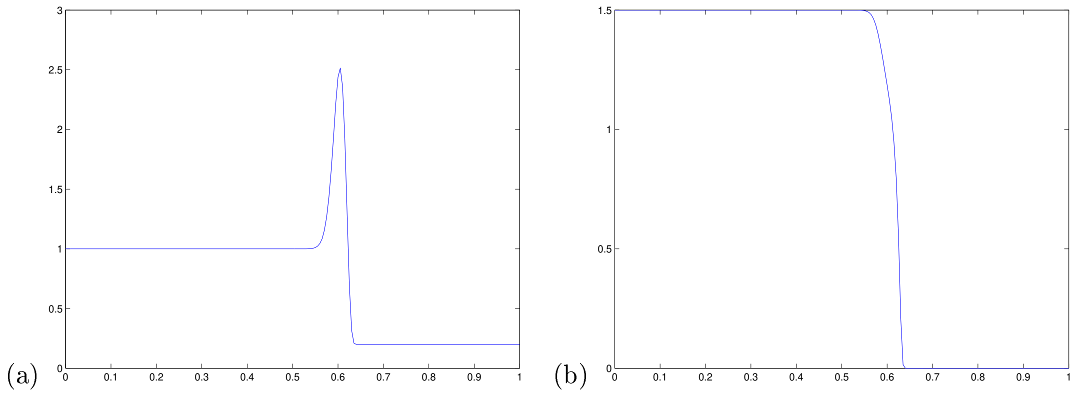


FIGURE 5.1. Density and velocity for $\epsilon = 0.085$ and $t = 0.1$

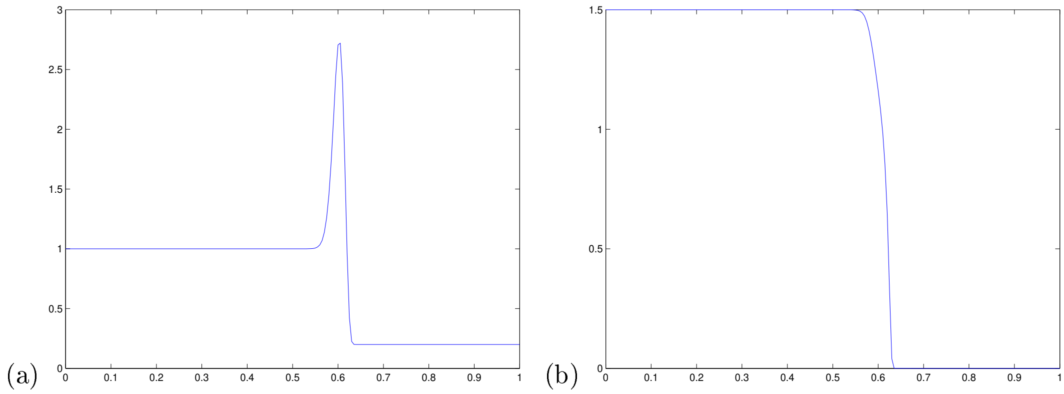


FIGURE 5.2. Density and velocity for $\epsilon = 0.055$ and $t = 0.1$

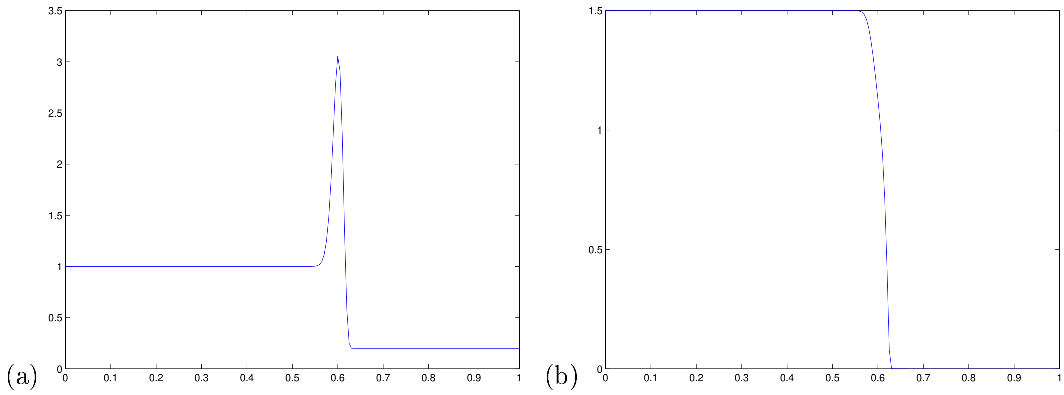


FIGURE 5.3. Density and velocity for $\epsilon = 0.025$ and $t = 0.1$

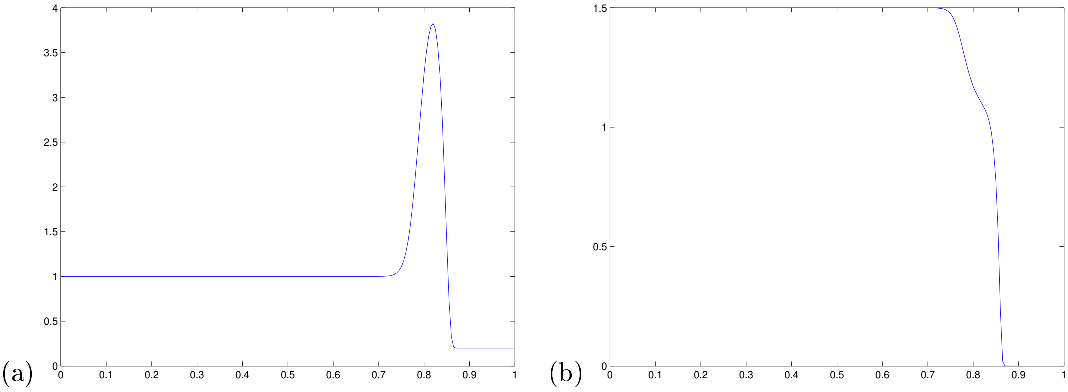


FIGURE 5.4. Density and velocity for $\epsilon = 0.085$ and $t = 0.3$

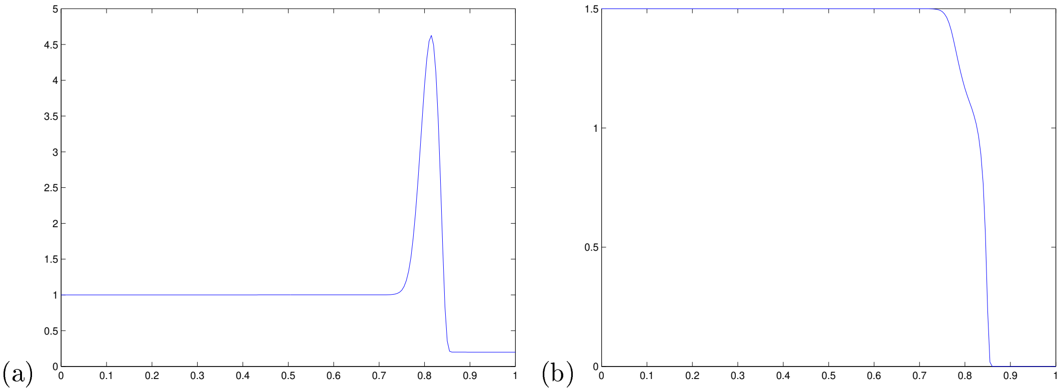


FIGURE 5.5. Density and velocity for $\epsilon = 0.055$ and $t = 0.3$

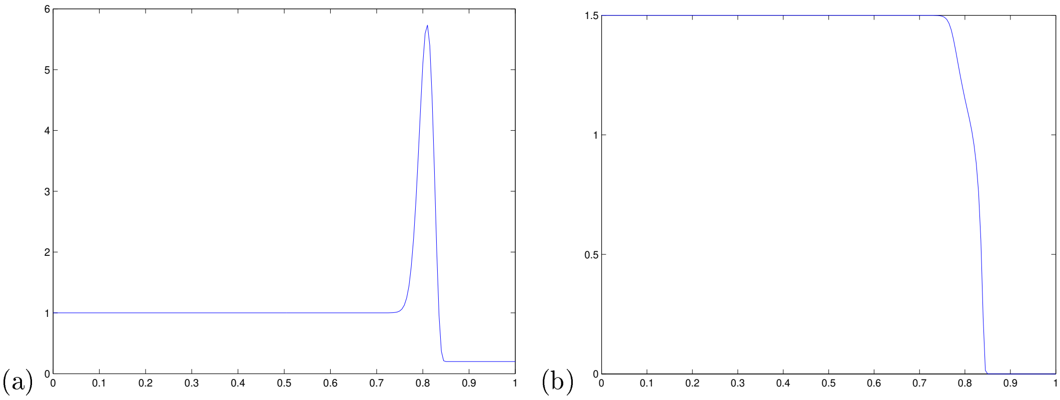


FIGURE 5.6. Density and velocity for $\epsilon = 0.025$ and $t = 0.3$

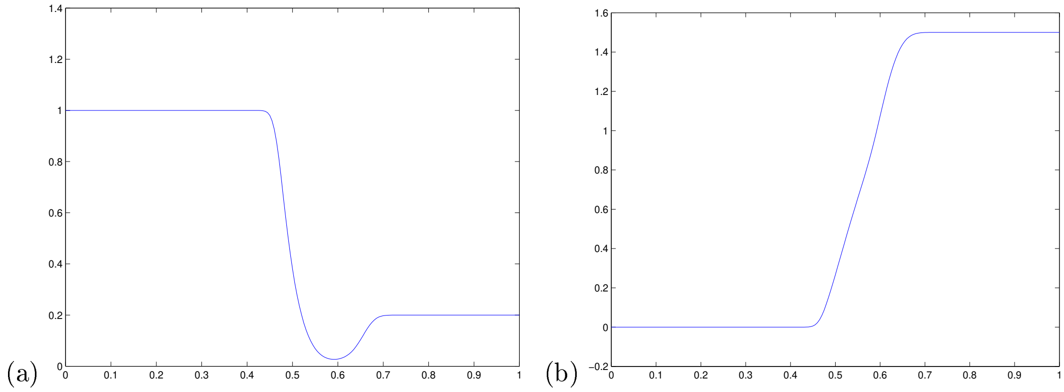


FIGURE 5.7. Density and momentum for $\epsilon = 0.085$ and $t = 0.1$

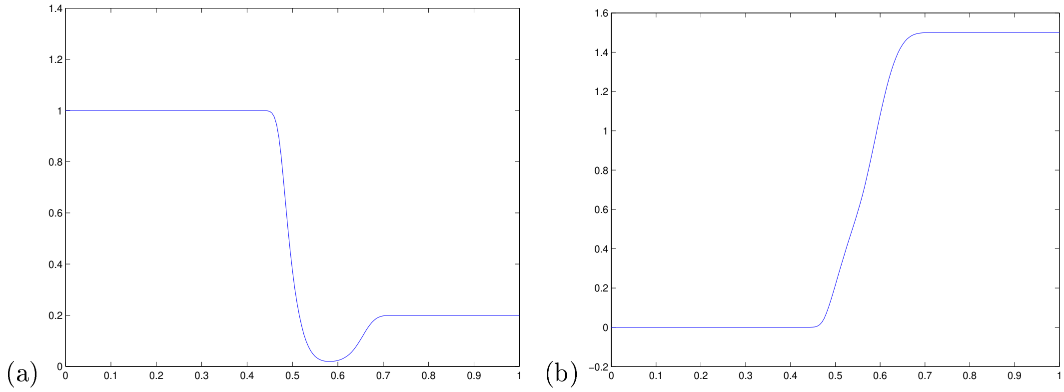


FIGURE 5.8. Density and momentum for $\epsilon = 0.055$ and $t = 0.1$

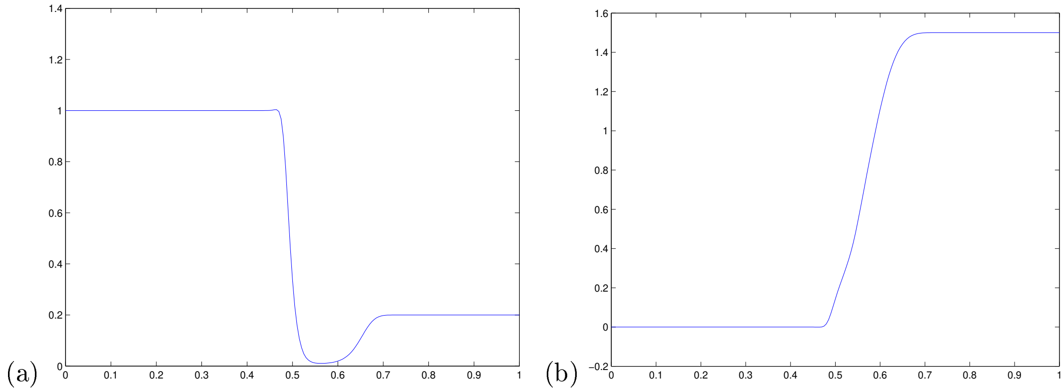
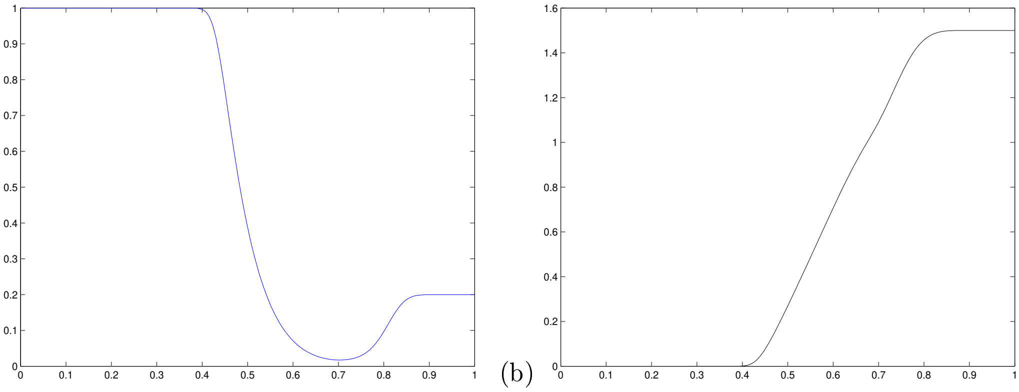
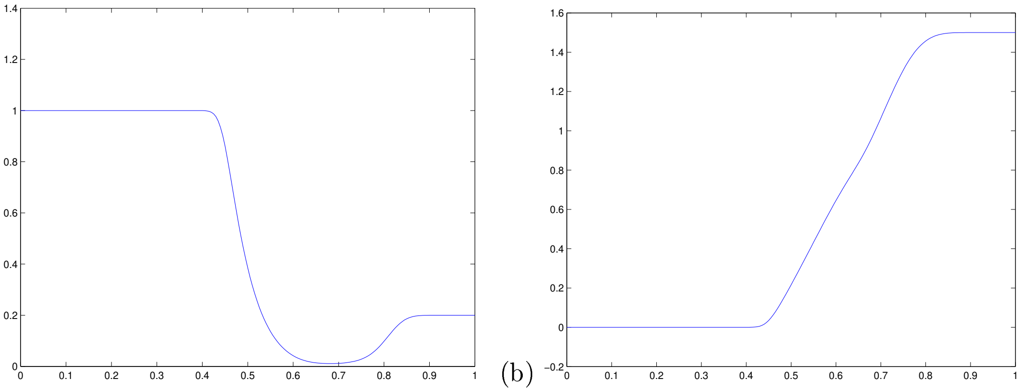


FIGURE 5.9. Density and momentum for $\epsilon = 0.025$ and $t = 0.1$



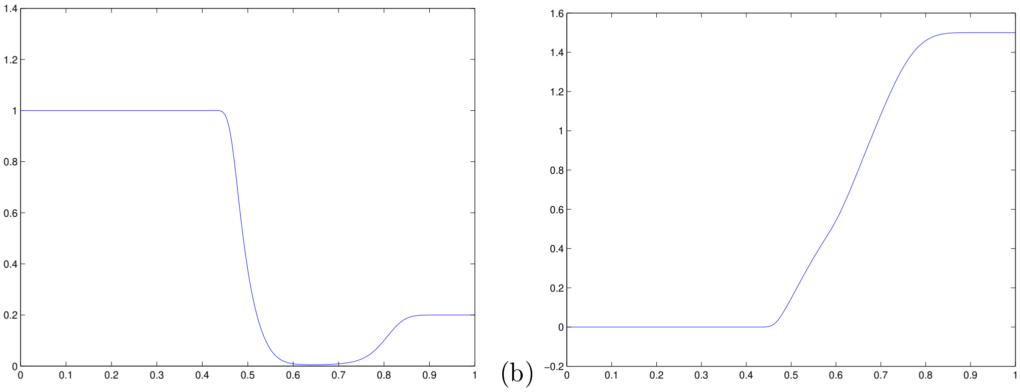
(a) (b)

FIGURE 5.10. Density and momentum for $\epsilon = 0.085$ and $t = 0.3$



(a) (b)

FIGURE 5.11. Density and momentum for $\epsilon = 0.055$ and $t = 0.3$



(a) (b)

FIGURE 5.12. Density and momentum for $\epsilon = 0.025$ and $t = 0.3$