Compactness and Asymptotic Behavior of Entropy Solutions without Locally Bounded Variation for Hyperbolic Conservation Laws

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Abstract. We discuss some recent developments and ideas in studying the compactness and asymptotic behavior of entropy solutions without locally bounded variation for nonlinear hyperbolic systems of conservation laws. Several classes of nonlinear hyperbolic systems with resonant or linear degeneracy are analyzed. The relation of the asymptotic problems to other topics such as scale-invariance, compactness of solutions, and singular limits is described.

1. Introduction

We are concerned with the behavior of entropy solutions without locally bounded variation for the Cauchy problem of hyperbolic conservation laws:

$$\partial_t u + \nabla_x \cdot f(u) = 0, \qquad x \in \mathbb{R}^n,\, u \in \mathbb{R}^m, \tag{1}$$

$$u|_{t=0} = u_0(x), \tag{2}$$

where $f : \mathbb{R}^m \to \mathbb{R}^{m\times n}$ is a nonlinear flux function. Such a system arises from many important areas since the conservation law is a fundamental law of nature. The archetypical example is the compressible Euler equations in fluid dynamics. One of the main difficulties in solving (1)-(2) is the development of shock waves, observed in nature, no matter how smooth the initial data are. Therefore, the Sobolev spaces $W^{k,p}, k \geq 1$, are not well-posed for the solutions, for which many powerful techniques are not directly applicable. One expects that the solutions are at most in the space of functions of bounded variations. The Glimm theory indicates that this is indeed the case for strictly hyperbolic systems with initial data of small total variation (see Glimm [21]). The well-posedness in BV is optimal in general since several recent examples indicate that, for the Cauchy problem of strictly hyperbolic systems, the total variation of the solutions blows up in a finite time for the initial data of large variation (e.g. see [25] for example). Another main difficulty is the resonances occurred among different characteristic fields so that the systems become nonstrictly hyperbolic. Such a feature causes extra difficulties even in the linear case. For the multidimensional case, such a situation is generic.

For example, in the 3-D case, if the number of equations $m \equiv 2 (mod\, 4)$, Lax [28] proved that such systems must be nonstrictly hyperbolic. This is also the case even for $m \equiv \pm 2, \pm 3, \pm 4\, (mod\, 8)$(see [20]). Many physical systems arising from different areas have such features. Recent existence theories for these systems with L^∞ large initial data indicate that the entropy solutions belong to the following class of functions:

$$u(x,t) \in L^\infty(\mathbb{R}^{n+1}_+), \quad \text{or } L^p(\mathbb{R}^{n+1}_+), \quad \text{for some } p \in [1,\infty), \tag{3}$$

and, for any entropy-entropy flux pair $(\eta, q) \in \mathbb{R} \times \mathbb{R}^n$ with convex $\eta(u)$,

$$\partial_t \eta(u) + \nabla \cdot q(u) \le 0 \tag{4}$$

in the sense of distributions.

In this article we discuss some recent developments and related ideas for studying the behavior of entropy solutions in the class of (3)-(4), especially the asymptotic problems. One of the main difficulties is that the solution class is much larger; and only information is the Lax entropy inequalities (4) in a weak topology: the distributional sense. Another main difficulty is the lack of analytical techniques in L^p, such as the method of generalized characteristics, to follow the characteristics. To explain the ideas more clearly, we will focus mainly on the L^∞ periodic entropy solutions in $x \in \mathbb{R}^n$ with period $P \equiv [0,1]^n$.

2. Asymptotic Decay via BV Estimates and Glimm-Lax Theory

It is well known that, for the linear case with nonzero propagation speeds and periodic initial data, the solution is periodic in time t. One can not expect the decay as $t \to \infty$. However, this is not true for the genuinely nonlinear case. The observation is that the genuine nonlinearity of equations forces the nonlinear waves of each characteristic family to interact vigorously and to cancel each other. The analysis of Glimm-Lax [22], for scalar equations and 2×2 systems, has indicated that the resulting mutual cancellation of interacting shock and rarefaction waves of the same family induces the decay of periodic solutions. In particular, for 2×2 genuinely nonlinear and strictly hyperbolic systems subject to the following condition: the interaction of two shocks of the same family always produces a shock of the same family plus a rarefaction wave of the opposite family, Glimm-Lax [22] established the decay theory: There exists a constant C_0 and some constant state $\bar{u}$ such that, when $osc(u_0 - \bar{u}) \le C_0$, there exists a global periodic entropy solution $u(x,t)$ in x satisfying $TV(u)([0,1] \times \{t\}) \le \frac{C}{t}$, with C independent of t. This implies that the solution decays like $1/t$ in L^∞. Recently, Dafermos [15] used the method of generalized characteristics to show that any periodic solution with small oscillation and local bounded variation of the 2×2 systems decays asymptotically, with a detailed structure picture as in Lax [27] for the scalar case. Also see Engquist-E [19] for the decay of periodic solutions with local bounded variation for the two-dimensional scalar conservation laws.

3. Asymptotic Decay via a New Approach

A further problem is whether the decay phenomenon holds for more general cases: (a) any L^∞ large periodic solutions without restrictions of either small oscillation or local bounded variation, and (b) more general nonlinear hyperbolic systems, especially degenerate systems and multidimensional scalar equations. The counterexample of Greenberg-Rascle [23] indicates that this is not always true if the flux functions are not so smooth; and the asymptotic behavior is very sensitive with respect to the smoothness of the flux functions. In this section, we discuss a new approach, developed in Chen-Frid [3, 4], to study the asymptotic behavior of periodic entropy solutions in a general framework.

3.1. Decay and Scale-Invariance

One of the main features of (1) is the scale-invariance in the sense that the self-similar scaling sequence $u^T(x,t) = u(Tx, Tt)$ also satisfies (1) for any $T > 0$.

Definition 3.1. The periodic solution $u(x,t)$ of the Cauchy problem (1)-(2) in $x \in \mathbb{R}^n$ asymptotically decays to $\bar{u}$, provided that

$$\|u^T - \bar{u}\|_{L^q_{loc}(\mathbb{R}^{n+1}_+)} \to 0, \quad \text{when } T \to \infty, \quad \text{for some } q \in [1, \infty). \tag{5}$$

Because of the self-similar structure of $u^T(x,t)$, the limit (5) can be translated equivalently in terms of the decay geometrically for $u(x,t) = U(x/t, t)$ along the rays $x/t = \xi$, $\xi \in \mathbb{R}^n$, passing the origin in the (x,t)-plane (also see [3]):

$$\frac{1}{T}\int_0^T |U(\xi,t) - \bar{u}|^q dt \to 0, \qquad \text{in} \quad L^1_{loc}(\mathbb{R}^n_\xi), \quad \text{when } T \to \infty. \tag{6}$$

The limits (5)-(6) for $u(x,t) \in L^\infty(\mathbb{R}^2_+)$ are also equivalent to

$$\frac{n+1}{T^{n+1}}\int_0^T |U(\xi,t) - \bar{u}|^q t^n dt \to 0, \qquad \text{in} \quad L^1_{loc}(\mathbb{R}^n_\xi), \quad \text{when} \quad T \to \infty.$$

Remark. Set $\mu^T(t) = \frac{1}{T}\chi_{[0,T]}(t)dt$. Then the limit (6) means

$$< \mu^T, |U - \bar{u}|^q > (\xi) \longrightarrow 0, \qquad \text{in} \quad L^1_{loc}(\mathbb{R}^n_\xi), \quad \text{when } T \to \infty. \tag{7}$$

Such averaging probability measures have been widely used to understand macroscopically the asymptotic behavior of physical quantities in statistical mechanics, kinetic theory, ergodic theory, and probability theory. One may extend these notions to more general settings. A periodic solution $u(x,t)$ asymptotically decays with respect to a family of probability measures $\{\mu^T(t)\}_{\{T>0\}}$ if $u(x,t) = U(x/t,t)$, locally integrable in $\mathbb{R}^{n+1}_+$, satisfies (7) and $\mu(t) = w - \lim \mu^T(t)$, in the space of Radon measures over some compactification for $[0, +\infty)$, satisfies supp $\mu(t) = \{+\infty\}$. It would be natural to develop such notions to understand the asymptotic behavior of stochastic solutions of nonlinear conservation laws.

3.2. Decay and Compactness

One of our main observations is that the compactness of L^∞ solution operator in L^1_{loc}, coupling with the weak convergence of periodic initial data to the mean, yields the decay of the L^∞ periodic solution. More precisely, we have

Theorem 3.1. [3, 4]. *Assume that $u(x,t) \in L^\infty(\mathbb{R}^{n+1}_+)$ is a periodic solution of (1)-(2), and its scaling sequence $\{u^T(x,t)\}$ is compact in $L^1_{loc}(\mathbb{R}^{n+1}_+)$. Then $u(x,t)$ asymptotically decays to $\bar{u} \equiv \int_P u_0(x)dx$ in the sense of (5) or (6).*

In Theorem 3.1, we assume that $\{u^T(x,t)\}$ is compact in $L^1_{loc}(\mathbb{R}^{n+1}_+)$, which is a corollary of the compactness of solution operator. Such a compactness can be achieved by the method of compensated compactness, the averaging method, and other analytical techniques. For example, we have

Theorem 3.2. [4, 7]. *Consider a system (1) with a strictly convex entropy pair (η_*, q_*). Assume that the uniformly bounded sequence $u^T(x,t) \in L^\infty(\mathbb{R}^{n+1}_+)$ satisfies (4) for any <u>convex</u> entropy pair $(\eta, q) \in \Lambda$, where Λ is a linear space of entropy pairs of (5) including (η_*, q_*). Then the measure sequence*

$$\partial_t \eta(u^T) + \nabla_x \cdot q(u^T) \qquad \text{is compact in } W^{-1,r}_{loc}(\mathbb{R}^{n+1}_+), \quad r \in (1, \infty), \tag{8}$$

for <u>any</u> entropy pair $(\eta, q) \in \Lambda$ with $|\nabla^2 \eta(u)| \le C_{\eta,K} \nabla^2 \eta_*(u)$ for $u \in K \Subset \mathbb{R}^m$.

Therefore, the compensated compactness method (e.g. [36, 34]) is one of the efficient methods to achieve the compactness with the aid of Theorem 3.2.

3.3. Decay and Entropy

The next question is whether the decay in Theorem 3.1 is achieved in a strong sense. Indeed, the entropy inequalities (4) provide such further information.

Theorem 3.3. [3, 4]. *Let the system (1) be endowed with a strictly convex entropy η_*. Then the asymptotic decay of an L^∞ periodic entropy solution $u(x,t)$ of (1)-(2) to the mean $\bar{u}$ of $u_0(x)$ over the period P in the sense of (5) implies its asymptotic decay in $L^q, 1 \le q < \infty$. That is, there exists a set $\mathcal{T} \subset (0, \infty)$, with meas$((0,\infty) - \mathcal{T}) = 0$, such that*

$$\int_P |u(x,t) - \bar{u}|^q dx \to 0, \qquad t \to \infty,\ t \in \mathcal{T}, \qquad \text{for any} \quad 1 \le q < \infty. \tag{9}$$

4. Applications

We now apply the approach established in §3 to analyze the decay of periodic solutions for various nonlinear conservation laws. The proofs of these theorems can be found in Chen-Frid [4, 7], Chen-LeFloch [10], and Chen-Dafermos [2]. In this section we discuss these results and raise some open problems.

4.1. Scalar conservation laws

For multidimensional scalar conservation laws with C^2 flux function $f(u)$ and initial data $u_0(x) \in L^\infty(\mathbb{R}^n)$ with period $P = [0,1]^n$, the existence of global entropy solutions of the Cauchy problem is due to Kruzkov [26]. Combining the approach in §3 with a theorem by Lions-Perthame-Tadmor [29], we conclude

Theorem 4.1. [4, 7]. *Assume that the flux function $f(u)$ satisfies*

$$\text{meas}\left\{ u \in \mathbb{R} \mid \tau + f'(u) \cdot \vec{k} = 0 \right\} = 0, \quad \textit{for any } (\tau, \vec{k}) \in S^n. \tag{10}$$

Let $u(x,t)$ be an entropy solution in $\mathbb{R}^{n+1}_+$ with periodic data $u_0(x)$. Then $u(x,t)$ asymptotically decays to $\bar{u} = \int_P u_0(x)dx$ in the sense of (9).

The condition (10) is implied by the generalized genuine nonlinearity condition: $\text{meas}\{ u \in \mathbb{R} \mid \vec{k} \cdot f''(u) = 0 \} = 0$, for all $\vec{k} \in S^{n-1}$. For the one-dimensional case, the assumption (10) can be further relaxed into the following nonlinearity assumption: There is no subinterval $(\alpha, \beta) \in [\min u_0(x), \max u_0(x)]$ on which $f(u)$ is affine. It would be interesting to generalize the condition (10) under which the decay phenomenon for the multidimensional case is still held.

4.2. 2×2 Strictly Hyperbolic Systems

Consider a 2×2 strictly hyperbolic and genuinely nonlinear system of conservation laws with periodic initial data $u_0(x) \in L^\infty(\mathbb{R}; \mathbb{R}^2)$.

Theorem 4.2. [4] *Let $u(x,t) \in L^\infty(\mathbb{R}^2_+)$ be a periodic entropy solution of the Cauchy problem of the 2×2 system. Then $u(x,t)$ asymptotically decays to $\bar{u} = \int_P u_0(x)dx$ in L^q, $1 \le q < \infty$, in the sense of (9).*

We assume neither the small oscillation and the local bounded variation of the periodic solution nor further conditions on the system in Theorem 4.2. The genuine nonlinearity of the system can be relaxed to allow reflection points. Such a typical example is the equations of elasticity:

$$\partial_t \tau - \partial_x v = 0, \quad \partial_t v - \partial_x \sigma(\tau) = 0, \qquad \sigma(\tau) \in C^2(\mathbb{R}_+), \, \sigma'(\tau) > 0. \tag{11}$$

The genuine nonlinearity is generally precluded in elasticity in general. One typical case is: $\text{sign}(\tau\sigma''(\tau)) > 0, \tau \neq 0$. A more general situation is:

- There exists an interval $(\alpha_0, 0)$ or $(0, \beta_0)$ in which $\sigma''(\tau) \neq 0$;
- There is no interval $(\alpha, \beta), \alpha > 0$ or $\beta < 0$, in which $\sigma(\tau)$ is affine.

For this case, combining a theorem in [24, 17] and the approach in §3, we have

Theorem 4.3. *Let the initial data $(\tau_0(x), v_0(x)) \in L^\infty(\mathbb{R})$ be periodic with period P. Then there exists a periodic entropy solution of (11), which asymptotically decays to $(\bar{\tau}, \bar{v}) = \int_P (\tau_0(x), v_0(x))dx$ in L^q, $1 \le q < \infty$, in the sense of (9).*

Remark. For (11) with $\sigma(\tau) \notin C^2$, Greenberg-Rascle [23] constructed an example that the periodic solution $(\tau(x,t), v(x,t))$ does not decay in L^q. This fact indicates that $\sigma(\tau) \in C^2$ is necessary to keep the decay phenomenon. It would be interesting to consider more general $\sigma(\tau) \in C^2$ so that the decay phenomenon still holds.

4.3. Isentropic Euler Equations

Consider the isentropic Euler equations for compressible fluids:

$$\partial_t \rho + \partial_x m = 0, \qquad \partial_t m + \partial_x (m^2/\rho + p(\rho)) = 0, \tag{12}$$

where ρ, m, and p are the density, the momentum, and the pressure, respectively. In the non-vacuum state $(\rho > 0)$, $v = m/\rho$ is the velocity. $p = p(\rho) \in C^4(0, \infty)$ is a given function of ρ depending on compressible fluids under consideration. Strict hyperbolicity and genuine nonlinearity away from the vacuum require that

$$p'(\rho) > 0, \quad \rho p''(\rho) + 2p'(\rho) > 0, \qquad \rho > 0. \tag{13}$$

Near the vacuum, $p(\rho)$ is only asymptotic to the γ-law pressure (as real gases):

$$|(p(\rho) - \kappa \rho^{\gamma})^{(k)}| \le C \rho^{\gamma + 1 - k}, \quad 0 \le k \le 4, \ \rho \ll 1. \tag{14}$$

Consider the Cauchy problem for (12) with the initial data

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)), \quad 0 \le \rho_0(x) \le C_0, \ |m_0(x)/\rho_0(x)| \le C_0 < \infty. \tag{15}$$

The main difficulty of this system is that strict hyperbolicity fails, and the flux function is only Lipschitz continuous at the vacuum state $\rho = 0$, which occurs in fluid mechanics. Nevertheless, a compactness theorem has been established by using only weak entropy pairs, a subspace of entropy pairs, consisting of those η vanishing on the vacuum $\rho = 0$ for any fixed $m/\rho \in (-\infty, \infty)$. For example, the mechanical energy-energy flux pair $\eta_* = \frac{1}{2}\frac{m^2}{\rho} + \rho \int_0^{\rho} \frac{p(r)}{r^2} dr$ and $q_* = \frac{m^3}{2\rho^2} + m \int_0^{\rho} \frac{p'(r)}{r} dr$ is a convex weak entropy pair. One can prove that, for $0 \le \rho \le C$, $|m/\rho| \le C$, $|\nabla \eta(u)| \le C_\eta, |\nabla^2 \eta(u)| \le C_\eta \nabla^2 \eta_*(u)$, for any weak entropy η, with C_η independent of u.

Theorem 4.4. [10]. *(a) There exists a global solution $(\rho(x,t), m(x,t))$ of the Cauchy problem (12)-(15), satisfying $0 \le \rho(x,t) \le C, |m(x,t)/\rho(x,t)| \le C$, for some C depending only on C_0 and γ, and $\partial_t \eta(\rho, m) + \partial_x q(\rho, m) \le 0$ in the sense of distributions for any convex weak entropy pairs (η, q).*
(b) The solution operator $(\rho, m)(\cdot, t) = S_t(\rho_0, m_0)(\cdot)$, determined by (a), is compact in $L^1_{loc}(\mathbb{R}^2_+)$ for $t > 0$.

For polytropic perfect gases, the similar results were proved by DiPerna [18] for the case $\gamma = 1 + 1/N, N \ge 5$ odd, for $L^2 \cap L^\infty(\mathbb{R})$ initial data, by Ding-Chen-Luo [16] for $\gamma = 3/2$ and Chen [1] for $1 < \gamma \le 5/3$ for usual gases with general L^∞ initial data. The results are also true for $\gamma \ge 3$ due to Lions-Perthame-Tadmor [30] and for $5/3 < \gamma < 3$ due to Lions-Perthame-Souganidis [31]. Using Theorem 4.4, we conclude

Theorem 4.5. *Let $(\rho(x,t), m(x,t)), 0 \le \rho(x,t) \le C, |m(x,t)/\rho(x,t)| \le C$, be a periodic entropy solution of (12)-(15) with period P. Then $(\rho(x,t), m(x,t))$ asymptotically decays to $\int_P (\rho_0(x), m_0(x)) dx$ in the sense of (9).*

4.4. 3×3 Euler Equations in Thermoelasticity

We now consider hyperbolic systems of conservation laws with linear degeneracy: there is at least one characteristic field is linearly degenerate. A typical example is the 3×3 system of Euler equations in Lagrangian coordinates:

$$\partial_t \tau - \partial_x v = 0, \quad \partial_t v + \partial_x p = 0, \quad \partial_t (e + v^2/2) + \partial_x (vp) = 0, \tag{16}$$

where τ, v, p, and e denote respectively the deformation gradient (specific volume for fluids, strain for solids), the velocity, the pressure, and the internal energy. Other relevant fields are the entropy s and the temperature θ. The system (16) is complemented by the Clausius inequality $\partial_t s \geq 0$. We consider the following class of constitutive relations for the new state vector (w, v, s) with the form

$$\tau = w + \alpha s, \quad p = h(w) > 0, \quad e = -\int_0^w h(\omega) d\omega + \beta s, \quad \theta = \alpha h(w) + \beta, \tag{17}$$

where α and β are positive constants, and $h(w) \in C^2(\mathbb{R})$ with $h'(w) < 0$ satisfying

$$sign(w - \hat{w}) \left((\alpha h(w) + \beta) h''(w) - 4\alpha h'(w)^2 \right) < 0, \quad w \neq \hat{w}. \tag{18}$$

Observe that the equations (17) are compatible with the first law of thermodynamics: $\theta\, ds = de + p\, dv$. The model (17) can be regarded as a “first-order correction” to the general constitutive equations (see [2] for the details).

Consider the Cauchy problem for (16)-(18) with periodic initial data

$$(w, v, s)|_{t=0} = (w_0(x), v_0(x), s_0(x)) \tag{19}$$

satisfying $(w_0(x), v_0(x)) \in \{(w, v) \,|\, |v \pm \int_{\hat{w}}^w \sqrt{-h'(\omega)} d\omega| \leq C_0\}, s_0(x) \in \mathcal{M}_{loc}(\mathbb{R})$.

Theorem 4.6. [2]. *(a) There exists a periodic distributional solution $(w(x,t), v(x,t), s(x,t)) \in L^\infty(\mathbb{R}^2_+; \mathbb{R}^2) \times \mathcal{M}_{loc}(\mathbb{R}^2_+; \mathbb{R})$ of (16)-(18), satisfying*

$$s_t(x,t) \in \mathcal{M}_{loc}(\mathbb{R}^2_+), \ \theta(w(x,t)) \geq 0, \ |s|\{\{|x| \leq cT_0\} \times [0, T_0]\} \leq CT_0^2, \tag{20}$$

for any $c, T_0 > 0$, with $C > 0$ independent of T_0. Moreover, $(w(x,t), v(x,t), s(x,t))$ satisfies the entropy condition:

$$\partial_t \eta(w, v) + \partial_x q(w, v) \leq 0, \qquad s_t \geq 0, \tag{21}$$

in the sense of distributions for any C^2 entropy pair $(\eta(v,w), q(v,w))$ of $\partial_t w - \partial_x v = 0, \partial_t v + \partial_x h(w) = 0$, for which the strong convexity condition holds:

$$\theta\eta_{ww} - \alpha h'(w)\eta_w \geq 0, \ \theta\eta_{vv} + \alpha\eta_w \geq 0, \ (\theta\eta_{ww} - \alpha h'(w)\eta_w)(\theta\eta_{vv} + \alpha\eta_w) - \eta_{ww}^2 \geq 0.$$

(b) Any sequence $(w^T(x,t), v^T(x,t))$, which is uniformly bounded and satisfies (21), is compact in $L^1_{loc}(\mathbb{R}^2_+)$ when $t > 0$.

Theorem 4.7. [4]. *Let $(\tau(x,t), v(x,t), s(x,t))$ be a periodic entropy solution of (16)-(18) with period P satisfying $(v(x,t), \tau(x,t) - \alpha s(x,t)) \in L^\infty(\mathbb{R}^2_+)$ and (20)-(21). Then the velocity $v(x,t)$ asymptotically decays to $\bar{v} = \int_P v_0(x) dx$ in L^q, $1 \leq q < \infty$. Moreover, the pressure $p(w(x,t))$ and the temperature $\theta(w(x,t))$ decay to $\tilde{p} = p(\Theta^{-1}(\int_P \Theta(w_0(x)) dx))$, and $\tilde{\theta} = \theta(\Theta^{-1}(\int_P \Theta(w_0(x)) dx))$, in $L^q, 1 \leq q < \infty$, respectively, where $\Theta(w) = \beta w + \alpha \int_0^w h(\omega) d\omega$.*

5. Final Remarks

1. Asymptotic Decay of L^p Entropy Solutions. The approach described here can be easily generalized to the decay problem for L^p entropy solutions for $1 < p < \infty$. For example, In Theorems 3.1–3.3, the assumption $u(x,t) \in L^\infty(\mathbb{R}^{n+1}_+)$ may be replaced by $u(x,t) \in L^p(\mathbb{R}^{n+1}_+), p > 2$. Then the decay of an L^p periodic entropy solution $u(x,t)$ of (1)-(2), with period P, in the sense of (5) or (6) in the L^2 norm implies (9) for any $q \in [2,p)$. It would be interesting to generalize the approach for the entropy solutions in L^p_W (weighted L^p spaces).

2. Hyperbolic Conservation Laws with Relaxation. The approach can be easily extended to the asymptotic problems for the entropy solutions of hyperbolic systems of conservation laws with relaxation. See [4] for the decay results for such systems discussed in [11, 12].

3. Asymptotic Behavior of Entropy Solutions with General Initial Data. The approach and ideas can be developed to study the asymptotic behavior of entropy solutions without locally bounded variation for general L^p initial data. See [5, 6] for such an approach.

4. Asymptotic Behavior of Entropy Solutions for Multidimensional Conservation Laws. In §4.1 we give the simplest multidimensional example: scalar conservation laws. It would be interesting to find some interesting classes of multidimensional systems of conservation laws so that this approach is applicable.

5. The Validity of Nonlinear Geometric Optics for Entropy Solutions. The ideas and observations discussed here have been applied to the validity of the approximation of weakly nonlinear geometric optics for weak entropy solutions without locally bounded variation for conservation laws (cf. [8]).

6. Generalized Characteristics in L^p. It would be interesting to explore some approaches to define generalized characteristics for the entropy solutions in L^p so that one could follow the characteristics to study the asymptotic problems. Again, the Lax entropy inequalities should play an important role in this effort.

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