EXISTENCE THEORY FOR THE ISENTROPIC EULER EQUATIONS

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ABSTRACT. We establish an existence theorem for entropy solutions to the Euler equations modeling isentropic compressible fluids. We develop a new approach for constructing mathematical entropies for the Euler equations, which are singular near the vacuum. In particular, we identify the *optimal assumption* required on the singular behavior on the pressure law at the vacuum in order to validate the *two-term* asymptotic expansion of the entropy kernel proposed earlier by the authors. For more general pressure laws, we introduce a new *multiple-term* expansion based on the Bessel functions with suitable exponents, and we also identify the optimal assumption to valid the multiple-term expansion and to establish the existence theory. Our results cover, as a special example, the density-pressure law $p(\rho) = \kappa_1\,\rho^{\gamma_1} + \kappa_2\,\rho^{\gamma_2}$ where $\gamma_1, \gamma_2 \in (1,3)$ and $\kappa_1, \kappa_2 > 0$ are arbitrary constants.

1. Introduction.

In this paper we establish an existence theory for the Euler equations modeling isentropic compressible fluids:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho\, v) = 0, \\ &\partial_t\big(\rho\, v\big) + \partial_x(\rho\, v^2 + p(\rho)) = 0, \end{aligned} \tag{1.1}$$

where $\rho \geq 0$ denotes the density, v the velocity, and $p(\rho) \geq 0$ the pressure of the fluid. Under the standard assumption

$$p'(\rho) > 0 \qquad \text{for} \ \ \rho > 0, \tag{1.2}$$

the system (1.1) is strictly hyperbolic away from the vacuum $\rho = 0$ at least. Denoting the sound speed by $c(\rho) := \sqrt{p'(\rho)}$, the two wave speeds of the system, $v - c(\rho)$ and $v + c(\rho)$, are real and distinct for $\rho > 0$. The partial differential equations (1.1) provide an example of particular interest in the mathematical theory of nonlinear hyperbolic systems of conservation laws. Recall that the main difficulties towards proving the existence of an (entropy) solution to (1.1) are as follows:

(1) Discontinuities may form in a later time even for initially smooth data and, therefore, (entropy) solutions must be understood in the sense of distributions.
(2) Since the equations come from continuum physics modeling, we are interested in solutions that are defined globally in time and start out from arbitrary large data at time $t = 0$, and should not impose any "smallness" condition on the solutions.

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(3) Additionally, the equations under consideration are "degenerate" near the vacuum, in the sense that strict hyperbolicity of (1.1) fails at the vacuum: at $\rho = 0$, the two wave speeds coincide if $p'(0) = 0$, which is satisfied by pressure laws arising in continuum physics and will therefore be assumed from now on.

The existence of entropy solutions for the Cauchy problem associated with (1.1) was established, in the case of polytropic perfect gases

$$p(\rho) = \kappa\,\rho^{\gamma}, \qquad \kappa > 0,\, \gamma > 1, \tag{1.3}$$

first by DiPerna [6], Ding, Chen, and Luo [5], and Chen [1] based on compensated compactness arguments, and then, motivated by a kinetic formulation of hyperbolic conservation laws, by Lions, Perthame, and Tadmor [8], and Lions, Perthame, and Souganidis [9]. General pressure laws $p(\rho)$ were covered first by Chen and LeFloch [2]; the existence of entropy flux-splittings was established by Chen and LeFloch [3].

The purpose of this paper is to deal with more general pressure functions, especially including the example

$$p(\rho) = \kappa_1\,\rho^{\gamma_1} + \kappa_2\,\rho^{\gamma_2}, \qquad \gamma_1, \gamma_2 > 1,\ \kappa_1, \kappa_2 > 0. \tag{1.4}$$

We refer to the statements (2.1) and (3.2) below for our precise assumption on the function $p(\rho)$. Recall that the results in [2] apply to (1.4) when $\gamma_2 > 1 + \gamma_1$. The "Green function approach" introduced here turns out to be more accurate than the "energy estimate approach" we developed earlier. It allows us to cover more general pressure functions and, in particular, to cover general exponents γ_1, γ_2 in (1.4).

It was pointed out by DiPerna [6] that, in the case (1.3) for polytropic perfect gases, the mathematical entropies associated with the Euler equations admit an explicit formula, based on a fundamental solution of the entropy equation. We refer to this fundamental solution as the *entropy kernel.* In [6, 5, 1, 8, 9], the explicit formula for the entropies associated with (1.3) enters in a central manner in establishing the reduction of the Young measures governed by Tartar's commutation equations. The approach developed in [2] for general pressure laws $p(\rho)$ requires less explicit calculations and solely certain properties of the entropy kernel (singularities and cancellation). Studying the existence of the entropy kernel and its regularity is delicate since the equation for entropies contains singular coefficients near the vacuum.

Our general strategy is as follows. We first apply the Fourier transform in the velocity variable so that the linear hyperbolic equation governing the entropy kernel is transformed into a family of second order differential equations with singular coefficients, in which the Fourier variable is as a parameter. To determine the regularity of the entropy kernel, then we determine the asymptotic behavior of the solutions in the Fourier variable, that is, to derive a suitable asymptotic expansion which is based on the Bessel functions.

Our main results are as follows. In Section 2, we determine the optimal assumption on $p(\rho)$ which is required to validate the two-term expansion derived in [2]. For the example (1.4), that condition is $\gamma_2 - 1 > 3(\gamma_1 - 1)/2$. However, the two-term expansion does not accurately describe the asymptotic properties of the entropy kernel. A multiple-term expansion is derived in Section 3 for a large class of pressure laws which, in particular, cover the example (1.4) for arbitrary exponents $\gamma_1, \gamma_2 > 1$. Finally, in Section 4, relying on the convergence framework established by the authors in [2], we arrive at the new existence theory for (1.1) with exponents in the physically relevant interval $(1, 3)$. Then, the compactness of the solution operator and the decay of periodic entropy solutions follow as in [2].

Note in passing that the genuine nonlinearity assumption

$$\rho\, p''(\rho) + 2\, p'(\rho) > 0 \qquad \text{for } \rho > 0 \tag{1.5}$$

is not needed for the proof of existence of the entropy kernel, but solely to establish that it remains non-negative. We refer to [4] for other applications of the techniques and ideas developed here.

2. A Two-Term Asymptotic Expansion.

To begin with, we treat a class of pressure laws that includes, for instance, the example (1.4) for large deviation between γ_1 and γ_2, specifically for $\gamma_2 > (3\gamma_1 - 2)/2$. In the present section only, we assume that there exists an exponent $\gamma \in (1, \infty)$, a smooth function $P = P(\rho)$, and some real $\epsilon > 0$ such that

$$\begin{aligned} &p(\rho) = \kappa\, \rho^\gamma \left(1 + \rho^{\theta(1+\epsilon)}\, P(\rho)\right), \\ &P(\rho) \ \text{and}\, \rho^3\, P'''(\rho) \ \text{are bounded as } \rho \to 0, \end{aligned} \tag{2.1}$$

where $\kappa := (\gamma - 1)^2/(4\gamma)$ after normalization. Of course, the function $P(\rho)$ may exhibit some singularities at $\rho = 0$. In fact, (2.1) implies solely that $\rho\, P'(\rho)$ and $\rho^2\, P''(\rho)$ remain bounded.

The following notation will be used:

$$k(\rho) := \int_0^\rho \frac{c(y)}{y}\, dy, \qquad \theta = \frac{\gamma - 1}{2}, \quad \lambda = \frac{3-\gamma}{2\,(\gamma - 1)}, \quad \nu = \lambda + \frac{1}{2} = \frac{1}{2\,\theta}.$$

Recall that the case $\gamma \in (1, 3)$ and $\epsilon \geq -1 + 1/\theta$ is treated in [2]. The entropy kernel solves the following Cauchy problem associated with a linear hyperbolic equation (see [2]):

$$\begin{aligned} &(a) \quad \chi_{\rho\rho} - k'(\rho)^2\, \chi_{vv} = 0, \\ &(b) \quad \chi(0, v) = 0, \\ &(c) \quad \chi_\rho(0, v) = \delta_{v=0}. \end{aligned} \tag{2.2}$$

The equation (2.2a) is a singular Euler-Poisson-Darboux equation, in which

$$k'(\rho)^2 = \theta^2\, \rho^{2\,(\theta-1)} \left(1 + o(\rho)\right) \quad \text{with } 2\,(\theta - 1) \in (-2, \infty).$$

Using the Fourier transform in the variable v and denoting by ξ the Fourier variable in (2.2) yield

$$\begin{aligned} &(a) \quad \hat\chi_{\rho\rho} = -k'(\rho)^2\, \xi^2\, \hat\chi, \\ &(b) \quad \hat\chi(0, \xi) = 0, \\ &(c) \quad \hat\chi_\rho(0, \xi) = 1. \end{aligned} \tag{2.3}$$

Thus, the problem reduces to a family of second order differential equations in the density variable ρ, depending on some parameter $\xi \in \boldsymbol{R}$.

In the particular case (1.3), it is not hard to check that the solution of (2.3) is closely related to the Bessel function $J_\nu = J_\nu(y)$ defined for all $y > 0$ by

$$J_\nu(y) = c_{0,\lambda}\, y^\nu \int_{-1}^{1} \cos(y\, x)\, \left(1 - x^2\right)^\lambda dx =: c_{0,\lambda}\, y^\nu\, \hat f_\lambda(y),$$

where the constant $c_{0,\lambda}$ is given by $c_{0,\lambda} = \left(\int_{-1}^1 \left(1 - x^2\right)^\lambda dx\right)^{-1}$. See for instance [7]. More precisely, using that J_ν is a solution of the second order differential equation

$$\frac{d^2 J_\nu}{dy^2} + \frac{1}{y}\,\frac{dJ_\nu}{dy} + \left(1 - \frac{\nu^2}{y^2}\right) J_\nu = 0, \tag{2.4}$$

one can check that the solution of (2.3) in the case $p^*(\rho) := \kappa\,\rho^\gamma$ is exactly

$$\hat\chi^*(\rho,\xi) := |\xi|^{-\nu}\,\rho^{1/2}\,J_\nu\big(|\xi|\,\rho^\theta\big).$$

Let us review first some properties of Bessel functions, which motivate the forthcoming discussion. First of all, we have the asymptotic expansions

$$J_\nu(y) = \begin{cases} y^\nu + O(y^{\nu+1}) & \text{when } y \to 0, \\ c_{1,\lambda}\, y^{-1/2}\, \cos\big(y - (\lambda+1)\,\pi/2\big) + O(y^{-3/2}) & \text{when } y \to \infty, \end{cases} \tag{2.5}$$

for some constant $c_{1,\lambda} > 0$. On the other hand, the second Bessel function $Y_\nu = Y_\nu(y)$ satisfies (after normalization)

$$Y_\nu(y) = \begin{cases} -y^{-\nu} + O(y^{-\nu+1}) & \text{when } y \to 0, \\ c_{2,\lambda}\, y^{-1/2}\, \sin\big(y - (\lambda+1)\,\pi/2\big) + O(y^{-3/2}) & \text{when } y \to \infty, \end{cases} \tag{2.6}$$

for some constant $c_{2,\lambda} > 0$. Similar formulas are available for the derivatives $J_\nu'(y)$ and $Y_\nu'(y)$.

To handle the above expansions, it is convenient to set, for $y \neq 0$,

$$Q_{\pm\nu}(y) := \begin{cases} |y|^{\pm\nu} & \text{for } |y| \le 1, \\ |y|^{-1/2} & \text{for } |y| \ge 1. \end{cases}$$

From the asymptotic formula (2.5) we see that, for some $C_1 > 0$,

$$\big|J_\nu(y)\big| \le C_1\, Q_\nu(y), \qquad y > 0. \tag{2.7}$$

Consider also the kernel K associated with the two Bessel functions ($\rho, s > 0$ and $\xi \in \boldsymbol{R}$):

$$K\big(\rho, s; \xi\big) = Y_\nu(\xi\, k(\rho))\, J_\nu(\xi\, k(s)) - J_\nu(\xi\, k(\rho))\, Y_\nu(\xi\, k(s)).$$

Based on (2.5) and (2.6), one easily checks that, for some $C_2 > 0$,

$$|K\big(\rho, s; \xi\big)| \le C_2\, Q_\nu\big(\xi\, k(\rho)\big)\, Q_\nu\big(\xi\, k(s)\big)^{-1}\, R(\xi\, k(s)), \qquad 0 \le s \le \rho,\ \xi \in \boldsymbol{R}, \tag{2.8}$$

where the function R is defined by

$$R(y) := \begin{cases} 1 & \text{if } |y| \le 1, \\ 1/|y| & \text{if } |y| \ge 1. \end{cases}$$

Finally, since J_ν and Y_ν are two independent solutions of (2.4) with the expansions (2.5) and (2.6) at 0, we see that their Wronskian is

$$\begin{aligned} w(\rho,\xi) :&= Y_\nu(\xi\, k(\rho))\, J_\nu'(\xi\, k(\rho)) - J_\nu'(\xi\, k(\rho))\, Y_\nu(\xi\, k(\rho)) \\ &= \frac{1}{\xi\,\theta k(\rho)} = \frac{2\,\nu}{\xi\, k(\rho)}. \end{aligned} \tag{2.10}$$

For the general problem (2.3), we now prove the following theorem.

Theorem 2.1. *Suppose that the function $p = p(\rho)$ satisfies the assumptions* (1.2) *(hyperbolicity) and* (2.1) *(behavior near the vacuum) for some $\gamma \in (1,\infty)$ and some arbitrary $\epsilon > 0$. Then there exists a solution of the problem* (2.3)*, $\hat\chi = \hat\chi(\rho,\xi)$, defined for $\rho \geq 0$ and $\xi \in \boldsymbol{R}$. It is smooth for $\rho > 0$, continuous when $\rho \to 0$, and is given by the expansion*

$$\hat\chi(\rho,\xi) = \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} \left(\frac{J_\nu\big(|\xi|\, k(\rho)\big)}{|\xi|^\nu} + N(\rho)\, \frac{J_{\nu+1}\big(|\xi|\, k(\rho)\big)}{|\xi|^{\nu+1}} + r(\rho,\xi)\right), \tag{2.11}$$

where

$$N(\rho) := -c_3 \int_0^\rho k^{-\lambda-1} k'^{-1/2} \big(k^{\lambda+1} k'^{-1/2}\big)''\, d\tau$$

for some constant $c_3 := (2\lambda+1)c_{0,\lambda}/\big(4(\lambda+1)c_{0,\lambda+1}\big) = (4\lambda^2-1)/(8\lambda(\lambda+1))$. The remainder satisfies

$$|r(\rho,\xi)| + \rho\, |r_\rho(\rho,\xi)| \leq C\, \frac{Q_\nu\big(\xi\, k(\rho)\big)}{|\xi|^{\min(2,1+\epsilon)}} \qquad \text{for some } C > 0. \tag{2.12}$$

Note that, in fact, $N = N(\rho)$ only depends on second order derivatives of the function k at most, since

$$N = -c_3\, k^{-\lambda-1} k'^{-1/2}\, \big(k^{\lambda+1} k'^{-1/2}\big)' + c_3 \int_0^\rho \big(k^{-\lambda-1} k'^{-1/2}\big)' \big(k^{\lambda+1} k'^{-1/2}\big)'\, d\tau.$$

Proof. To simplify the notation, we assume $\xi > 0$ throughout the proof. The functions

$$\hat\chi_1^\sharp(\rho,\xi) := \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} \frac{J_\nu\big(\xi\, k(\rho)\big)}{\xi^\nu}, \qquad \hat\chi_2^\sharp(\rho,\xi) := \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} \xi^\nu\, Y_\nu\big(\xi\, k(\rho)\big)$$

are two independent solutions of the "modified" equation (cf. (3.26) in [2]):

$$\hat\chi^\sharp_{\rho\rho} + \Big(k'(\rho)^2\, \xi^2 - \beta(\rho)\Big)\, \hat\chi^\sharp = 0, \tag{2.13}$$

where

$$\beta := k^{-\lambda-1}\, k'^{1/2}\, \Big(k^{\lambda+1}\, k'^{-1/2}\Big)''.$$

On the other hand, the remainder

$$\tilde r(\rho,\xi) := \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} r(\rho,\xi)$$

satisfies a non-homogeneous version of the same equation (cf. (3.30) in [2]). Indeed, we have

$$\tilde r_{\rho\rho} + \Big(k'(\rho)^2\, \xi^2 - \beta(\rho)\Big)\, \tilde r = H(\rho,\xi) - \beta(\rho)\, \tilde r \tag{2.14}$$

with (cf. Section 3, Step 1, in [2] for an explicit expression of $\alpha_\flat$, $\alpha_\sharp$, and A):

$$\begin{aligned} H(\rho,\xi) :&= -\Big(\alpha_\flat''(\rho) + \frac{2\lambda+3}{2(\lambda+1)}\, \alpha_\sharp''(\rho)\Big)\, \hat f_{\lambda+1}\big(\xi\, k(\rho)\big) =: A(\rho)\, \hat f_{\lambda+1}\big(\xi\, k(\rho)\big) \\ &= \frac{A(\rho)}{c_{0,\lambda+1}}\, \frac{J_{\nu+1}\big(\xi\, k(\rho)\big)}{|\xi k(\rho)|^{\nu+1}}. \end{aligned}$$

A tedious but straightforward calculation based on the assumption (2.1) shows that

$$
\begin{aligned}
g := -\beta \, \frac{k}{k'} &= -\lambda(\lambda+1)\, \frac{k'}{k} - \frac{3}{4}\, k\, k'^{-2}\, k''^2 + \frac{1}{2}\, k\, k'^{-2}\, k''' \\
&= O(\rho^{-1+\theta(1+\epsilon)}),
\end{aligned}
\tag{2.15}
$$

and

$$
B(\rho) := \left(\frac{\theta k(\rho)}{k'(\rho)}\right)^{1/2} \frac{A(\rho)}{c_{0,\lambda+1}\, k(\rho)^\nu} = O(\rho^{-1+\theta(1+\epsilon)}). \tag{2.16}
$$

Using that $\hat\chi_1^\sharp$ and $\hat\chi_2^\sharp$ are two independent solutions of the homogeneous equation (2.13) and relying on the method of variation of parameters, we easily arrive at the following integral formulation for the solution of (2.14):

$$
\tilde r(\rho,\xi) = \left(\frac{\theta k(\rho)}{k'(\rho)}\right)^{1/2} \int_0^\rho \frac{K(\rho,s;\xi)}{W(s,\xi)} \left(\frac{\theta k(s)}{k'(s)}\right)^{1/2} \Big(H(s,\xi) - \beta(s)\, \tilde r(s,\xi)\Big)\, ds, \tag{2.17}
$$

where the kernel K was defined earlier and the Wronskian W is

$$
\begin{aligned}
W(\rho,\xi) &:= \big(\hat\chi_1^\sharp\, \hat\chi_{2,\sharp}' - \hat\chi_{1,\sharp}'\, \hat\chi_2^\sharp\big)(\rho,\xi) \\
&= \xi\, \theta k(\rho)\, \Big(J_\nu(\xi k(\rho))\, Y_\nu'(\xi k(\rho)) - J_\nu'(\xi k(\rho))\, Y_\nu(\xi k(\rho))\Big) \\
&= \xi\, \theta k(\rho)\, w(\rho,\xi) = 1,
\end{aligned}
$$

thanks to (2.10). Hence, (2.17) becomes

$$
r(\rho,\xi) = \theta \int_0^\rho K(\rho,s;\xi)\, \left(B(s)\, \frac{J_{\nu+1}\big(\xi k(s)\big)}{\xi k(s)} + g(s)\, r(s,\xi)\right) ds. \tag{2.18}
$$

Introduce now the notation

$$
G(\rho,\xi) := \int_0^\rho R\big(\xi\, k(s)\big)\, s^{-1+\theta(1+\epsilon)}\, ds.
$$

Clearly, for ρ within any given bounded set, we have

$$
|G(\rho,\xi)| \le C, \qquad \rho > 0,\ \xi \in \boldsymbol{R}. \tag{2.19}
$$

On the other hand, setting

$$
S(y) := \begin{cases} 1 & \text{for } |y| \le 1, \\ 1/y^2 & \text{for } |y| \ge 1 \end{cases}
$$

and using the assumption (2.1), it is not hard to check that, for some $C_3 > 0$,

$$
\int_0^\rho S\big(\xi\, k(s)\big)\, s^{-1+\theta(1+\epsilon)}\, ds \le \frac{C_3}{\xi^{\min(2,1+\epsilon)}}. \tag{2.20}
$$

We will also use below that

$$
Q_\nu^{-1}(y)\, R(y)\, |J_\nu(y)| \le S(y) \qquad \text{for } y > 0. \tag{2.21}
$$

We now apply a fixed point procedure to (2.18), defining a sequence of functions r_n by $r_0 = 0$ and, for $n \geq 0$,

$$r_{n+1}(\rho,\xi) = \theta \int_0^\rho K\big(\rho, s; \xi\big) \left(B(s) \frac{J_{\nu+1}\big(\xi k(s)\big)}{\xi k(s)} + r_n(s,\xi)\, g(s)\right) ds.$$

In view of (2.8) and, by (2.15), $|g(\rho)| \leq C_3\, \rho^{-1+\epsilon}$ for some $C_3 > 0$, we have for all $n \geq 1$

$$\begin{aligned}
&\big|r_{n+1}(\rho,\xi) - r_n(\rho,\xi)\big| \\
&\leq \theta\, C_2 \int_0^\rho Q_\nu\big(\xi\, k(\rho)\big)\, Q_\nu\big(\xi\, k(s)\big)^{-1} R\big(\xi\, k(s)\big) \big|r_n(r,\xi) - r_{n-1}(s,\xi)\big|\, |g(s)|\, ds \\
&\leq \theta\, C_2 C_3 \int_0^\rho Q_\nu\big(\xi\, k(\rho)\big)\, Q_\nu\big(\xi\, k(s)\big)^{-1} R\big(\xi\, k(s)\big) \big|r_n(s,\xi) - r_{n-1}(s,\xi)\big|\, s^{-1+\theta(1+\epsilon)}\, ds \\
&= \theta\, C_2 C_3\, Q_\nu\big(\xi\, k(\rho)\big) \int_0^\rho Q_\nu\big(\xi\, k(s)\big)^{-1} \big|r_n(s,\xi) - r_{n-1}(s,\xi)\big|\, G_\rho(s,\xi)\, ds.
\end{aligned}$$

In other words,

$$\begin{aligned}
&Q_\nu\big(\xi\, k(\rho)\big)^{-1} \big|r_{n+1}(\rho,\xi) - r_n(\rho,\xi)\big| \\
&\leq \theta\, C_2 C_3 \int_0^\rho Q_\nu\big(\xi\, k(s)\big)^{-1} \big|r^n(s,\xi) - r^{n-1}(s,\xi)\big|\, G_\rho(s,\xi)\, ds.
\end{aligned} \tag{2.22}$$

On the other hand, we estimate r_1 as follows. Using (2.8), then (2.7), (2.21), and $|B(\rho)| \leq C_4\, \rho^{-1+\theta(1+\epsilon)}$ for some $C_4 > 0$ thanks to (2.16) yields

$$\begin{aligned}
\big|r^1(\rho,\xi)\big| &\leq \theta \int_0^\rho \left| K\big(\rho, s; \xi\big)\, B(s) \frac{J_{\nu+1}\big(\xi\, k(s)\big)}{\xi k(s)} \right| ds \\
&\leq \theta\, C_2 \int_0^\rho Q_\nu\big(\xi\, k(\rho)\big)\, Q_\nu\big(\xi\, k(s)\big)^{-1} R\big(\xi\, k(s)\big) \left| \frac{J_{\nu+1}\big(\xi\, k(s)\big)}{\xi k(s)} \right| |B(s)|\, ds \\
&\leq \theta\, C_1 C_2 C_4\, Q_\nu\big(\xi\, k(\rho)\big) \int_0^\rho S\big(\xi\, k(s)\big)\, s^{-1+\theta(1+\epsilon)}\, ds \\
&\leq \theta\, C_1 C_2 C_3 C_4 \frac{Q_\nu\big(\xi\, k(\rho)\big)}{\xi^{\min(2,1+\epsilon)}} =: C_5 \frac{Q_\nu\big(\xi\, k(\rho)\big)}{\xi^{\min(2,1+\epsilon)}},
\end{aligned} \tag{2.23}$$

where we used (2.20) for the last inequality.

By induction on n, it follows from (2.21) and (2.23) that

$$\big|r_{n+1}(\rho,\xi) - r_n(\rho,\xi)\big| \leq C_5 \frac{Q_\nu\big(\xi\, k(\rho)\big)}{\xi^{\min(2,1+\epsilon)}} \frac{\big(C_6\, G(\rho,\xi)\big)^n}{n!}, \tag{2.24}$$

where $C_6 := \theta\, C_1 C_3$. We deduce from (2.24) that $\{r_n\}$ is a Cauchy sequence on the space of uniformly bounded and continuous functions on each compact subset in ρ and that

$$\begin{aligned}
|r(\rho,\xi)| &\leq \sum_{n=0}^{\infty} \big|r_{n+1}(\rho,\xi) - r_n(\rho,\xi)\big| \\
&\leq C_5 \frac{Q_\nu\big(\xi\, k(\rho)\big)}{\xi^{\min(2,1+\epsilon)}} \sum_{n=0}^{\infty} \frac{\big(C_6\, G(\rho,\xi)\big)^n}{n!} = C_5 \frac{Q_\nu\big(\xi\, k(\rho)\big)}{\xi^{\min(2,1+\epsilon)}}\, e^{C_6\, G(\rho,\xi)}.
\end{aligned}$$

Since, by (2.19), the function G is uniformly bounded and the first part of (2.12) is now established.
Finally, differentiating (2.18), we observe that the derivative r_ρ satisfies

$$r_\rho(\rho,\xi) = \theta \int_0^\rho K_\rho\big(\rho,s;\xi\big)\,\Big(B(s)\,\frac{J_{\nu+1}\big(\xi k(s)\big)}{\xi k(s)} + r(s,\xi)\,g(s)\Big)\,ds,$$

which has the same form as the formula satisfied by r. It is straightforward to check from (2.5) and (2.6) that, instead of (2.8), we now have the bound

$$|K_\rho\big(\rho,s;\xi\big)| \le C\,\frac{k'(\rho)}{k(\rho)}\,Q_\nu\big(\xi\,k(\rho)\big)\,Q_\nu\big(\xi\,k(s)\big)^{-1}\,R(\xi\,k(s)) \qquad \text{for } 0 \le s \le \rho,\ \xi \in \boldsymbol{R}.$$

By a direct estimate, we arrive at (2.12) for r_ρ. □

We now translate the above result into the original velocity variable. Because of the hyperbolic nature of the problem (2.1), the support of the entropy kernel should be the domain of dependence

$$\mathcal{K} := \big\{(\rho,v)\ :\ \rho \ge 0,\ |v| \le k(\rho)\big\}.$$

Furthermore, since the initial data is singular, the entropy kernel is not smooth but its derivatives contains some singularities localized on the boundary

$$\partial\mathcal{K} = \big\{(\rho,v)\ :\ v \pm k(\rho) = 0\big\}.$$

The expansion derived now for the entropy kernel is based on the two functions

$$(\rho,v) \mapsto \big[k(\rho)^2 - v^2\big]_+^\mu \qquad \text{for } \mu = \lambda, \lambda+1.$$

Theorem 2.2. *Assume that the pressure function $p = p(\rho)$ satisfies (1.2) (hyperbolicity) and (2.1) (behavior near the vacuum) for some $\gamma \in (1,\infty)$ and $\epsilon > 0$. Then the problem (2.1) admits a solution $\chi = \chi(\rho,v)$, supported in the set $\mathcal{K}$ and smooth in its interior. For each $\mu < \lambda$, the function $\big[k(\rho)^2 - v^2\big]_+^{-\mu}\,\chi(\rho,v)$ is Hölder continuous in (ρ,v) with*

$$\Big|\partial_v^{\lambda-\mu}\Big(\big[k(\rho)^2 - v^2\big]_+^{-\mu}\,\chi\Big)\Big| \le C\,k(\rho)^{2\,(\lambda-\mu)}. \tag{2.25}$$

More precisely, the entropy kernel admits the expansion

$$\chi(\rho,v) = a^\sharp(\rho)\,\big[k(\rho)^2 - v^2\big]_+^\lambda + a^\flat(\rho)\,\big[k(\rho)^2 - v^2\big]_+^{\lambda+1} + e(\rho,v), \tag{2.26}$$

where

$$a^\sharp := c_\lambda\,k^{-\lambda}\,k'^{-1/2} \qquad \text{with } \frac{1}{c_\lambda} = \frac{2\,\lambda}{\sqrt{2\,\lambda+1}}\int_{-1}^1 (1-y^2)^\lambda\,dy \tag{2.27}$$

and

$$a^\flat := -\frac{c_\lambda}{4(\lambda+1)}\,k^{-\lambda-1}k'^{-1/2}\int_0^\rho k^{-\lambda-1}\,k'^{-1/2}\,\Big(k^{\lambda+1}\,k'^{-1/2}\Big)''\,ds. \tag{2.28}$$

The remainder $e = e(\rho,v)$ is such that $\partial_v^\mu e$ is Hölder continuous in (ρ,v) for $\rho > 0$ for all μ with $-1 + \min(2, 1+\epsilon) < \mu < \lambda + \min(2,1+\epsilon)$ and, for all $0 < \beta < \mu$,

$$\big|\partial_v^\beta e(\rho,v)\big| \le C\,\rho^{-\mu\theta+\beta\theta+(1-\theta)/2}\,\big[k(\rho)^2 - v^2\big]_+^{\mu-\beta}. \tag{2.29}$$

Furthermore, if the genuine nonlinearity condition (1.6) *holds, the function* χ *is positive in the interior of its support.*

We point out that, when $\theta \in (0,1)$, that is, $\gamma \in (1,3)$ (which is the interval of interest in Theorem 1.1 and in [2]), the derivative $\partial_v^{\lambda+1} e(\rho,v)$ of the remainder is Hölder continuous up to the boundary $\rho = 0$. For instance, in the estimate (2.29) with the exponent $(1-\theta)/2 > 0$, we see that the factor $\rho^{-\mu\theta+\beta\theta+(1-\theta)/2}$ vanishes as $\rho \to 0$ when $\beta = \lambda+1$, provided one chooses μ sufficiently close to $\lambda+1$.

As a corollary, the family of weak entropies for the Euler equations is described by the formula

$$\eta(\rho,v) = \int_{\boldsymbol{R}} \chi(\rho, v-s)\,\psi(s)\,ds,$$

where $\psi : \boldsymbol{R} \to \boldsymbol{R}$ is arbitrary.

Proof of Theorem 2.2. In Theorem 2.1, we established the existence of the solution $\hat\chi = \hat\chi(\rho,\xi)$ of the problem (2.2), given by the expansion (2.11). According to (2.12), the following estimate holds for the remainder:

$$\big|r(\rho,\xi)\big| \le \begin{cases} C\,|\xi|^{-\min(2,1+\epsilon)}\,\big|\xi\,k(\rho)\big|^{\nu} & \text{for } k(\rho)\,|\xi| \le 1, \\ C\,|\xi|^{-\min(2,1+\epsilon)}\,\big|\xi\,k(\rho)\big|^{-1/2} & \text{for } k(\rho)\,|\xi| \ge 1. \end{cases} \tag{2.30}$$

Considering the expansion (2.11), one sees that the remainder $r(\rho,\xi)$ decays at infinity like

$$|\xi|^{-(1/2)-\min(2,1+\epsilon)}$$

and therefore faster than the second term $J_{\nu+1}\big(\xi k(\rho)\big)/\xi$ which is solely $|\xi|^{-3/2}$ according to (2.5). This allows us to treat $r(\rho,\xi)$ as a higher-order term.

Using the relation between J_ν and $\hat f_\lambda$ and observing that the corresponding function f_λ is closely related to the function $\big[k(\rho)^2 - v^2\big]_+^\lambda$, we now apply the inverse Fourier transform to (2.11). Referring to [2] for the details, we find exactly (2.26) in which $e(\rho,v)$ is determined from the remainder $r(\rho,\xi)$ by

$$e(\rho,v) = C\left(\frac{\theta\,k(\rho)}{k'(\rho)}\right)^{1/2} \int_{\boldsymbol{R}} \cos\big(\xi\,v\big)\,r(\rho,\xi)\,|\xi|^{-\nu}\,d\xi, \tag{2.31}$$

up to some multiplicative constant $C > 0$. Given some $\mu > 0$, by the definition of fractional derivatives we have

$$\big|\partial_v^\mu e(\rho,v)\big| \le C\left(\frac{\theta\,k(\rho)}{k'(\rho)}\right)^{1/2} \int_{\boldsymbol{R}} |\xi|^{\mu-\nu}\,|r(\rho,\xi)|\,d\xi.$$

Then, a straightforward calculation based on (2.30) yields the Hölder bound

$$\big|\partial_v^\mu e(\rho,v)\big| \le C\,\rho^{(1-\theta)/2}, \tag{2.32}$$

provided $-1 + \min(2,\epsilon/\theta) < \mu < \lambda + \min(2,\epsilon/\theta)$. Similarly, from (2.12), we can also obtain

$$\rho\,\big|\partial_\rho\partial_v^\mu e(\rho,v)\big| \le C\,\rho^{(1-\theta)/2}.$$

By a standard embedding theorem $W_\rho^{1,p} \subset C_\rho^{0,\alpha}$, we conclude that $\partial_v^\mu e$ are Hölder continuous in (ρ,v).

Finally, introducing the variable $z := v/k(\rho)$, we see that the function $[1-z^2]_+$ is positive on the support of $\mathcal{K}$. We have

$$\left|\partial_z^\mu e(\rho, v)\right| = k(\rho)^\mu \left|\partial_v^\mu e(\rho, v)\right| \leq C\, \rho^{\mu\theta+(1-\theta)/2}.$$

The function $e = e(\rho, v)$ is Hölder continuous with exponent μ and vanishes outside $\mathcal{K}$. Therefore, we have

$$\begin{aligned} |e(\rho, v)| &\leq C \sup_z \left|\partial_z^\mu e(\rho, v)\right| [1-z^2]_+^\mu \\ &\leq C_1\, \rho^{\mu\theta+(1-\theta)/2}\, [1-z^2]_+^\mu \\ &\leq C_2\, \rho^{-\mu\theta+(1-\theta)/2} \left[k(\rho)^2 - v^2\right]_+^\mu, \end{aligned}$$

which gives (2.29) for $\beta = 0$. More generally, for $\beta < \mu$, we obtain

$$\begin{aligned} |\partial_v^\beta e(\rho, v)| &\leq C\, \rho^{\beta\theta} \sup_z \left|\partial_z^\mu e(\rho, v)\right| [1-z^2]_+^{\mu-\beta} \\ &\leq C_1\, \rho^{-\beta\theta+\mu\theta+(1-\theta)/2} \left[k(\rho)^2 - v^2\right]_+^{\mu-\beta}, \end{aligned}$$

which gives (2.29). This completes the proof of Theorem 2.2. □

3. A Multiple-Term Expansion.

We now turn to the general situation of a pressure function $p = p(\rho)$ satisfying the following: There exists a sequence of exponents

$$1 < \gamma := \gamma_1 < \gamma_2 < \ldots < \gamma_N \leq (3\gamma - 1)/2 < \gamma_{N+1} \tag{3.1}$$

and a sufficiently smooth function $P = P(\rho)$ such that

$$\begin{gathered} p(\rho) = \sum_{n=1}^N \kappa_n\, \rho^{\gamma_n} + \rho^{\gamma_{N+1}}\, P(\rho), \\ P(\rho) \text{ and } \rho^3\, P'''(\rho) \text{ are bounded as } \rho \to 0, \end{gathered} \tag{3.2}$$

for some coefficients $\kappa_n \in \boldsymbol{R}$, where $\kappa_1 := (\gamma-1)^2/(4\gamma)$ after normalization. From the exponent γ we also define θ, λ, and ν as in Section 2, so that the reminder in (3.2) takes the same form as we had in (2.1).

The two-term expansion derived in Section 2 is no longer valid when, besides the singularity with exponent γ, the pressure contains singularities with exponents strictly less than $(3\gamma-1)/2$. Instead, we must introduce the Bessel functions of exponents ν_i, $i = 1, ..., I$, ordered in increasing order and determined in the following way. Set

$$\left\{\mu_1, \mu_2, ..., \mu_I\right\} = [0, \theta) \cap \left\{e_n\,(\gamma_n - \gamma_1) + e_m(\gamma_m - \gamma_1) \;:\; e_n, e_m = 0, 1, ...\right\}$$

and then

$$\nu_i - \frac{1}{2} := \lambda_i := \lambda + \frac{\mu_i}{\theta} \qquad \text{for } i = 1, \ldots, I. \tag{3.3}$$

Clearly, $\mu_1 = 0$ and

$$\lambda = \lambda_1, \quad \nu = \nu_1.$$

But, it is important to notice that the coefficients ν_i *are not* associated with the coefficients γ_i directly by the same formulas as in Section 2. Finally, we denote by μ_{I+1} the smallest of $e_n(\gamma_n - \gamma_1) + e_m(\gamma_m - \gamma_1)$ which is greater or equal to θ, and we set

$$\mu_{I+1} = \theta\,(1+\epsilon).$$

To motivate the introduction of the exponents μ_i, we point out that (3.2) implies that, for instance,

$$c(\rho) = \rho^\theta \left(\sum_{i=1}^{I} q_i\, \rho^{\mu_i} + \rho^{\mu_{I+1}}\, Q(\rho) \right),$$

where q_i, $1 \le i \le I$, are constants and Q is a more regular function. The function k defined as before by

$$k(\rho) := \int_0^\rho \frac{c(y)}{y}\, dy$$

admits a completely similar expansion.

We now prove the following theorem.

Theorem 3.1. *Suppose that the function $p = p(\rho)$ satisfies the assumption* (3.2) *for some sequence of exponents satisfying* (3.1)*. Then there exists a solution of the problem* (2.2)*, $\hat\chi = \hat\chi(\rho,\xi)$, defined for $\rho \ge 0$ and $\xi \in \boldsymbol{R}$. It is smooth for $\rho > 0$, continuous as $\rho \to 0$, and is given by the expansion*

$$\hat\chi(\rho,\xi) = \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} \left(\sum_{i=1}^{I} K_i\, \frac{J_{\nu_i}\big(|\xi|\, k(\rho)\big)}{|\xi|^{\nu_i}} + N(\rho)\, \frac{J_{\nu+1}\big(|\xi|\, k(\rho)\big)}{|\xi|^{\nu+1}} + r(\rho,\xi) \right), \tag{3.4}$$

where the function $N = N(\rho)$ was already defined in Theorem 2.1*, and the constants K_i are determined in the proof below with $K_1 = 1$. The remainder also satisfies the estimate* (2.12) *in Theorem* 2.1.

Proof. We follow the same strategy as in the proof of Theorem 2.1. As before, we take $\xi > 0$. Now, the remainder

$$\tilde r(\rho,\xi) := \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} r(\rho,\xi)$$

satisfies

$$\begin{aligned}
\tilde r_{\rho\rho} + \Big(k'(\rho)^2\,\xi^2 - \beta(\rho)\Big)\,\tilde r &= -\beta(\rho)\,\tilde r + H - \Big(\partial_{\rho\rho} + k'(\rho)^2\,\xi^2\Big) \sum_{i=2}^{I} K_i\, \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} \frac{J_{\nu_i}\big(|\xi|\, k(\rho)\big)}{|\xi|^{\nu_i}} \\
&= -\beta(\rho)\,\tilde r + H - \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} \sum_{i=2}^{I} K_i\, \beta_i(\rho)\, \frac{J_{\nu_i}\big(|\xi|\, k(\rho)\big)}{|\xi|^{\nu_i}} \\
&=: -\beta(\rho)\,\tilde r + \tilde H,
\end{aligned} \tag{3.5}$$

where the functions $\beta_i = \beta_i(\rho)$ are defined by

$$\beta_i := k^{-\lambda_i - 1}\, k'^{1/2} \left(k^{\lambda_i+1}\, k'^{-1/2}\right)''. \tag{3.6}$$

We now rely on the expression of the function H in the proof of Section 2. Using the assumption (3.2), we have

$$A(\rho) = \sum_{i=1}^{I} q_i\, \rho^{\mu_i - 1} + \rho^{\mu_{I+1}-1}\, Q(\rho) \tag{3.7}$$

for some q_i and Q. Since, by (2.5), all of $J_\delta\big(|\xi|\, k(\rho)\big)/|k(\rho)\xi|^\delta$ are equivalent to 1 as $\rho \to 0$, the behavior of the function $\tilde{H}(\rho, \xi)$ near the vacuum is given by

$$\begin{aligned} E(\rho) :&= \frac{A(\rho)}{c_{0,\lambda+1}} - \left(\frac{\theta\, k(\rho)}{k'(\rho)}\right)^{1/2} \sum_{i=2}^{I} K_i\, \beta_i(\rho)\, k(\rho)^{-\nu_i} \\ &= \frac{A(\rho)}{c_{0,\lambda+1}} - \theta^{1/2} \sum_{i=2}^{I} K_i \left(k^{\lambda_i+1}\, k'^{-1/2}\right)''. \end{aligned}$$

Now, again using the assumption (3.2), we find

$$\left(k^{\lambda_i+1}\, k'^{-1/2}\right)'' = \rho^{\theta\lambda_i + (\theta-3)/2} \left(\sum_{j=1}^{I} \kappa_j\, \rho^{\mu_j} + \rho^{\mu_{I+1}}\, \tilde{Q}(\rho)\right) \tag{3.8}$$

for some constants $\tilde{q}_j$ and some regular function $\tilde{Q}$, with in particular $\tilde{q}_1 = \theta^{-1/2}\big(\theta\lambda_i + (\theta - 1)/2\big)\, \big(\theta\lambda_i + (\theta-3)/2\big) \neq 0$. Furthermore, by our definition (3.3), we have exactly

$$\theta\lambda_i + (\theta - 3)/2 = \mu_i - 1.$$

That is, the exponents in the expansion (3.7) coincide with the principal exponent arising in (3.8) for $i = 1, \dots, I$. This implies that, by a suitable choice of the constants K_i, we can remove all of the singularities in the function E so that

$$E(\rho) = O(\rho^{\mu_{I+1}-1}). \tag{3.9}$$

Returning to the function $\tilde{H}$, we conclude that

$$|\tilde{H}(\rho, \xi)| \leq \begin{cases} C\, \rho^{\mu_{I+1}-1} & \text{for } |k(\rho)\xi| \leq 1, \\ C\, |k(\rho)\xi|^{-3/2} & \text{for } |k(\rho)\xi| \geq 1. \end{cases} \tag{3.10}$$

The rest of the proof follows as in Section 2, by using the following integral formulation for the solution of (2.14):

$$\tilde{r}(\rho, \xi) = \left(\frac{\theta k(\rho)}{k'(\rho)}\right)^{1/2} \int_0^\rho \frac{K(\rho, s; \xi)}{W(s, \xi)} \left(\frac{\theta k(s)}{k'(s)}\right)^{1/2} \left(\tilde{H}(s, \xi) - \beta(s)\, \tilde{r}(s, \xi)\right) ds,$$

where the kernel K was defined earlier. In other words, we have

$$r(\rho, \xi) = \theta \int_0^\rho K(\rho, s; \xi) \left(\frac{k(s)}{k'(s)}\, \tilde{H}(s, \xi) + r(s, \xi)\, g(s)\right) ds. \tag{3.11}$$

The only new feature is the treatment of the term r_1. We now use (3.10) and, in fact, arrive at exactly the same estimate. This completes the proof of Theorem 3.1. □

From Theorem 3.1, we deduce the existence of the entropy kernel.

Theorem 3.2. *Assume that the pressure-function $p = p(\rho)$ satisfies (1.2) (hyperbolicity) and (3.1)–(3.2) (behavior near the vacuum). Then the problem (2.2) admits a solution $\chi = \chi(\rho, v)$, supported in the set $\mathcal{K}$ and smooth in its interior. For each $\mu < \lambda$, the function $\big[k(\rho)^2 - v^2\big]_+^{-\mu} \chi(\rho, v)$ is Hölder continuous in (ρ, v) with*

$$\left|\partial_v^{\lambda-\mu}\Big(\big[k(\rho)^2 - v^2\big]_+^{-\mu} \chi\Big)\right| \le C\, k(\rho)^{2\,(\lambda-\mu)}. \tag{3.12}$$

More precisely, the entropy kernel admits the expansion

$$\chi(\rho, v) = \sum_{i=1}^{I} a_i^\sharp(\rho)\,\big[k(\rho)^2 - v^2\big]_+^{\lambda_i} + a^\flat(\rho)\,\big[k(\rho)^2 - v^2\big]_+^{\lambda+1} + e(\rho, v), \tag{3.13}$$

where

$$a_i^\sharp(\rho) := K_i\, k(\rho)^{-\lambda_i}\, k'(\rho)^{-1/2} \tag{3.14}$$

and $a^\flat(\rho)$ was defined in Theorem 2.2. Furthermore, all of the regularity properties stated in Theorem 2.2 hold.

Finally, we can also study the entropy kernel.

Theorem 3.3. *Under the assumption of Theorem 3.2, there exists an entropy flux kernel $\sigma = \sigma(\rho, v)$, supported in the set $\mathcal{K}$ and smooth in its interior. For each $\mu < \lambda$, the function $\big[k(\rho)^2 - v^2\big]_+^{-\mu} \sigma(\rho, v)$ is Hölder continuous in (ρ, v) with*

$$\left|\partial_v^{\lambda-\mu}\Big(\big[k(\rho)^2 - v^2\big]_+^{-\mu} \chi\Big)\right| \le C\, k(\rho)^{2\,(\lambda-\mu)}. \tag{3.15}$$

More precisely, the entropy kernel admits the expansion

$$\sigma(\rho, v) - v\,\chi(\rho, v) = \sum_{i=1}^{I} b_i^\sharp(\rho)\,\big[k(\rho)^2 - v^2\big]_+^{\lambda_i} + b^\flat(\rho)\,\big[k(\rho)^2 - v^2\big]_+^{\lambda+1} + f(\rho, v), \tag{3.16}$$

where

$$b_i^\sharp(\rho) := a_i^\sharp(\rho)\ \frac{\rho\, k'(\rho)}{k(\rho)} \tag{3.17}$$

and $b^\flat = b^\flat(\rho)$ is some function of ρ. Furthermore, the regularity results stated in Theorem 2.2 for the entropy kernel carry over to the entropy flux kernel.

4. Existence theory.

Our main existence result is the following.

Theorem 4.1. *Consider the isentropic Euler equations (1.1) under the assumptions (1.2) (hyperbolicity), (3.1)–(3.2) (behavior near the vacuum), and (1.7) (genuinely nonlinearity) with $\gamma \in (1, 3)$. Then, given any measurable and bounded initial data (ρ_0, v_0) at time $t = 0$, there exists a corresponding entropy solution $(\rho, m) = (\rho, \rho v)$ of the Cauchy problem associated with (1.1), globally defined in time, such that $0 \le \rho(x, t) \le C$ and $|m(x, t)| \le C\, \rho(x, t)$ for some constant $C > 0$ and for almost every (x, t).*

Recall that (2.1) is just a special case of (3.1)–(3.2). Note also that the condition $\gamma < 3$ is assumed in this section only.

Proof. Most of the arguments in [2] carry over with the more general expansion discovered in Section 3. Thus, we only need here to discuss some new observations required in the central part of the proof, i.e., the reduction of the Young measures. Denote by $\nu = \nu_{(x,t)}(\rho, v)$ a Young measure associated with the sequence of approximate solutions to (1.1). It is known that ν satisfies Tartar's commutation relations (at almost every point (x,t)):

$$\big\langle \nu, \eta_1\, q_2 - \eta_2\, q_1 \big\rangle \;=\; \big\langle \nu, \eta_1 \big\rangle \big\langle \nu, q_2 \big\rangle - \big\langle \nu, \eta_2 \big\rangle \big\langle \nu, q_1 \big\rangle \tag{4.1}$$

for any two weak entropy pairs (η_1, q_1) and (η_2, q_2) of (1.1). The objective is to deduce from (4.1) that the support of ν in the (ρ, v)-plane is either a single point or a subset of the vacuum line $\{\rho = 0\}$. To this end, the cancellation properties stated in Lemmas 4.2 and 4.3 in [2] play the central role and will be reconsidered here.

Using the notation in [2], in particular,

$$G_{\lambda_i}(\rho, v) := \big[k(\rho)^2 - v^2\big]_+^{\lambda_i}, \tag{4.1}$$

We now reconsider the distribution in the variables (ρ, v, s_1, s_2, s_3):

$$\begin{aligned}
E :=& \partial_2^{\lambda+1} \chi_2\, \partial_3^{\lambda+1} \sigma_3 - \partial_3^{\lambda+1} \chi_3\, \partial_2^{\lambda+1} \sigma_2 \\
=& \partial_2^{\lambda+1} \chi_2\, \partial_3^{\lambda+1} \big(\sigma_3 - v\, \chi_3\big) - \partial_3^{\lambda+1} \chi_3\, \partial_2^{\lambda+1} \big(\sigma_2 - v\, \chi_2\big) \\
=& \partial_2^{\lambda+1} \big(\sum_{i=1}^{I} a_i^\sharp G_{\lambda_i,2} + a^\flat G_{\lambda+1,2} + g_2\big)\, \partial_3^{\lambda+1} \Big((s_3 - v)\, \big(\sum_{i=1}^{I} b_i^\sharp G_{\lambda_i,3} + b^\flat G_{\lambda+1,3}\big) + h_3\Big) \\
& - \partial_3^{\lambda+1} \big(\sum_{i=1}^{I} a_i^\sharp G_{\lambda_i,3} + a^\flat G_{\lambda+1,3} + g_3\big)\, \partial_2^{\lambda+1} \Big((s_2 - v)\, \big(\sum_{i=1}^{I} b_i^\sharp G_{\lambda_i,2} + b^\flat G_{\lambda+1,2}\big) + h_2\Big),
\end{aligned}$$

where for instance $\chi_1 := \chi(\rho, v - s_1), \ldots$. Thus, we have

$$\begin{aligned}
E =& \big(\sum_{i=1}^{I} a_i^\sharp \partial_2^{\lambda+1} G_{\lambda_i,2} + a^\flat \partial_2^{\lambda+1} G_{\lambda+1,2} + \partial_2^{\lambda+1} g_2\big)\, \Big((s_3 - v)\, \big(\sum_{i=1}^{I} b_i^\sharp \partial_3^{\lambda+1} G_{\lambda_i,3} + b^\flat \partial_3^{\lambda+1} G_{\lambda+1,3}\big) \\
& \qquad + \partial_3^{\lambda+1} h_3 + (\lambda+1) \sum_{i=1}^{I} b_i^\sharp \partial_3^{\lambda} G_{\lambda_i,3} + (\lambda+1)\, b^\flat\, \partial_3^{\lambda} G_{\lambda+1,3}\Big) \\
& + \big(\sum_{i=1}^{I} a_i^\sharp \partial_3^{\lambda+1} G_{\lambda_i,3} + a^\flat \partial_3^{\lambda+1} G_{\lambda+1,3} + \partial_3^{\lambda+1} g_3\big)\, \Big((s_2 - v)\, \big(\sum_{i=1}^{I} b_i^\sharp \partial_2^{\lambda+1} G_{\lambda_i,2} + b^\flat \partial_2^{\lambda+1} G_{\lambda+1,2}\big) \\
& \qquad + \partial_2^{\lambda+1} h_2 + (\lambda+1) \sum_{i=1}^{I} b_i^\sharp \partial_2^{\lambda} G_{\lambda_i,2} + (\lambda+1)\, b^\flat\, \partial_2^{\lambda} G_{\lambda+1,2}\Big) \\
=:& \; E^I + E^{II} + E^{III}.
\end{aligned}$$

To decompose E, we rely on the crucial observation that, by (3.17), the ratios $b_i^\sharp / a_i^\sharp$ are *independent* of the index i. We define

$$E^I := (s_3 - s_2)\, \left(\sum_{i=1}^{I} a_i^\sharp \partial_2^{\lambda+1} G_{\lambda_i,2}\right) \left(\sum_{i=1}^{I} b_i^\sharp\, \partial_3^{\lambda+1} G_{\lambda_i,3}\right),$$

$$
\begin{aligned}
E^{II} := & \sum_{i=1}^{I} a_i^\sharp\, \partial_2^{\lambda+1} G_{\lambda_i,2} \Big((s_3 - v)\, b^\flat \partial_3^{\lambda+1} G_{\lambda+1,3} + (\lambda+1) \sum_{i=1}^{I} b_i^\sharp \partial_3^\lambda G_{\lambda_i,3} \Big) \\
& - \sum_{i=1}^{I} a_i^\sharp\, \partial_3^{\lambda+1} G_{\lambda_i,3} \Big((s_2 - v)\, b^\flat \partial_2^{\lambda+1} G_{\lambda+1,2} + (\lambda+1) \sum_{i=1}^{I} b_i^\sharp \partial_2^\lambda G_{\lambda_i,2} \Big) \\
& + a^\flat \sum_{i=1}^{I} b_i^\sharp \Big(\partial_2^{\lambda+1} G_{\lambda_i+1,2} (s_3 - v)\, \partial_3^{\lambda+1} G_{\lambda,3} - \partial_3^{\lambda+1} G_{\lambda+1,3} (s_2 - v)\, \partial_2^{\lambda_i+1} G_{\lambda,2} \Big),
\end{aligned}
$$

and E^{III} is the remainder.

We must determine the limit of the first two terms as $s_2, s_3 \to s_1$ in the sense of distributions. Dealing with E^{III} is easy since it involves only products of Hölder continuous functions by measures or more regular products. By symmetry, one gets

$$
E^{III} \rightharpoonup 0 \qquad \text{as } s_2, s_3 \to s_1 \tag{4.2}
$$

weakly in the sense of distributions, uniformly in (ρ, v).

On the other hand, E^I contains the favorable factor $s_2 - s_3$. Therefore, by the arguments developed in [2], we have again

$$
E^{I} \rightharpoonup 0 \qquad \text{as } s_2, s_3 \to s_1 \tag{4.3}
$$

weakly in the sense of distributions and uniformly in (ρ, v).

Finally, the term E^{II} contains products of functions of bounded variation by bounded measures. However, it is not hard to see that only the specific product treated in [2] generates a non-trivial contribution. The reason is that the terms containing G_{λ_i} for $i \neq 1$ are more regular than those associated with the first term $G_{\lambda_1} = G_\lambda$. This completes the proof of Lemma 4.3 in [2] for the multiple-term expansion and, in turn, this establishes that the reduction theorem for Young measures. Relying on the framework developed in [2], this also completes the proof of Theorem 4.1. □

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References.

[1] Chen G.-Q., Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III), Acta Math. Sci. **6** (1986), 75–120 (in English); **8** (1988), 243–276 (in Chinese).

[2] Chen G.-Q. and LeFloch P.G., Compressible Euler equations with general pressure law, Arch. Rational Mech. Anal. **153** (2000), 221–259.

[3] Chen G.-Q. and LeFloch P.G., Entropy and entropy-flux splittings for the isentropic Euler equations, Chinese Annals Math. **22B** (2001), 145–158.

[4] Chen G.-Q. and LeFloch P.G., in preparation.

[5] Ding X., Chen G.-Q., and Luo P., Convergence of the Lax-Friedrichs scheme for the isentropic gas dynamics (I)-(II), Acta Math. Sci. **5** (1985), 483-500, 501–540 (in English); **7** (1987), 467-480; **8** (1989), 61–94 (in Chinese).

[6] DiPerna R.J., Convergence of the viscosity method for isentropic gas dynamics, Commun. Math. Phys. **91** (1983), 1–30.

[7] Gelfand I.M. and Shilov G.E., *Generalized functions*, Vol. I, Academic Press, 1964.

[8] Lions P.-L., Perthame B., and Tadmor E., Kinetic formulation for the isentropic gas dynamics and p-system, Commun. Math. Phys. **163** (1994), 415–431.

[9] Lions P.-L., Perthame B., and Souganidis P.E., Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, Comm. Pure Appl. Math. **49** (1996), 599–638.

[10] Tartar L., The compensated compactness method applied to systems of conservation laws, in "Systems of Nonlinear Partial Differential Equations", J.M. Ball ed., NATO ASI Series, C. Reidel publishing Col., 1983, pp. 263–285.