Global Solutions of Nonlinear Magnetohydrodynamics with Large Initial Data

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A free boundary problem for nonlinear magnetohydrodynamics (MHD) with general large initial data is investigated. The existence, uniqueness, and regularity of global solutions are established with large initial data in H^1. It is showed that neither shock waves nor vacuum and concentration in the solutions are developed in a finite time, although there is a complex interaction between the hydrodynamic and magnetodynamic effects. An existence theorem of global solutions with large discontinuous initial data is also established.

Key Words: magnetohydrodynamics (MHD), free boundary, global solutions, existence, uniqueness, regularity, large discontinuities.

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1. INTRODUCTION

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids, such as gases, in an electromagnetic field. The applications of magnetohydrodynamics cover a very wide range of physical areas from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. If a conducting fluid moves in a magnetic field, electric fields are induced and an electric current flow is developed. The magnetic field exerts forces on these currents which considerably modify the hydrodynamic motion of the fluid. On the other hand, the development of electric currents effects a change in the magnetic field. There is a complex interaction between the magnetic and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. Plane magnetohydrodynamic flows are governed by the following equations (see Appendix):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \quad x \in \mathbf{R}, \quad t > 0, \\ (\rho u)_t + \left(\rho u^2 + p + \frac{1}{2}|\mathbf{b}|^2\right)_x = (\lambda u_x)_x, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_x = (\mu \mathbf{w}_x)_x, \\ \mathbf{b}_t + (u\mathbf{b} - \mathbf{w})_x = (\nu \mathbf{b}_x)_x, \\ \mathcal{E}_t + \left(u\left(\mathcal{E} + p + \frac{1}{2}|\mathbf{b}|^2\right) - \mathbf{w}\cdot\mathbf{b}\right)_x = (\lambda u u_x + \mu \mathbf{w}\cdot\mathbf{w}_x + \nu \mathbf{b}\cdot\mathbf{b}_x + \kappa\theta_x)_x, \end{cases} \tag{1.1}$$

where ρ denotes the density of the flow, $u \in \mathbf{R}$ the longitudinal velocity, $\mathbf{w} \in \mathbf{R}^2$ the transverse velocity, $\mathbf{b} \in \mathbf{R}^2$ the transverse magnetic field, and θ the temperature; the longitudinal magnetic field is a constant which is taken to be one in (1.1); the total energy of the plane magnetohydrodynamic flows is

$$\mathcal{E} = \rho\left(e + \frac{1}{2}(u^2 + |\mathbf{w}|^2)\right) + \frac{1}{2}|\mathbf{b}|^2,$$

with the internal energy e; both the pressure p and the internal energy e are related with the density and temperature of the flow via the equations of state:

$$p = p(\rho, \theta), \qquad e = e(\rho, \theta);$$

$\lambda = \lambda(\rho, \theta)$ and $\mu = \mu(\rho, \theta)$ are the viscosity coefficients of the flow, $\nu = \nu(\rho, \theta)$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, $\kappa = \kappa(\rho, \theta)$ is the heat conductivity, and all these kinetic coefficients and the magnetic diffusivity are independent of the magnitude and direction of the magnetic field (cf. [23]). The equations in (1.1) describe the macroscopic behavior of the magnetohydrodynamic flow with dissipative mechanisms. This is a three-dimensional magnetohydrodynamic flow which is uniform in the transverse directions.

There have been a lot of studies on magnetohydrodynamics by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenging; see [1, 3, 4, 6, 9, 15, 16, 21, 23, 24, 25, 30, 31] and the references cited therein. The initial-boundary value problems with fixed or free boundaries for this system are important and fundamental. In this paper we consider such a fundamental problem for the magnetohydrodynamic fluid flow with the pressure, internal energy, and heat conductivity satisfying certain physical growth conditions on the temperature. In particular, the case of perfect gases with

$$p = R\rho\theta, \qquad e = c_v\theta, \tag{1.2}$$

is included in the class of fluids investigated in this paper, where R is the gas constant, $c_v = R/(\gamma - 1)$ is the heat capacity of the gas at constant volume, and γ is the adiabatic exponent, We are interested in the well-posedness and regularity of global solutions of the initial-boundary value problem. For perfect gases with small smooth initial data, the existence of global solutions was proved in [15], and the large-time behavior was studied in [25]. For large initial data, these problems have additional difficulties because of the presence of the magnetic field and its interaction with the hydrodynamic motion of the flow of large oscillation.

In this paper we focus on a free boundary problem of (1.1) for $x \in \Omega_t = (0, x(t))$, $t > 0$, with the following initial-boundary conditions:

$$\begin{aligned} &(\rho, u, \mathbf{w}, \mathbf{b}, \theta)|_{t=0} = (\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)(x), \quad x \in \Omega_0 = (0,1), \\ &(\mathbf{w}, \mathbf{b}, \theta_x)|_{\partial\Omega_t} = 0, \quad u|_{x=0} = 0, \quad (p - \lambda u_x)|_{x=x(t)} = 1, \end{aligned} \tag{1.3}$$

where $x = x(t)$ is the free boundary defined by $x'(t) = u(x(t), t)$ with $x(0) = 1$, some outer pressure (or stress) (taken to be one without loss of generality) is applied along the free boundary, and the initial values are compatible with the boundary conditions. The main point in our analysis is whether shock waves, vacuum, and concentration are developed in the solutions in a finite time under the outer pressure, provided the initial data is bounded, smooth, and does not contain vacuum. As well-known, for the inviscid case, such phenomena may occur.

To treat the free boundary, we introduce a Lagrangian variable and transform the initial-boundary value problem (1.1) and (1.3) with the free boundary in Euler coordinates into the corresponding initial-boundary value problem with some fixed boundary in Lagrangian coordinates. These two problems are equivalent for the solutions under consideration.

We first establish the existence and uniqueness of global solutions to the initial-boundary value problem with general large initial data in H^1 (as well as in a Hölder space) and show that neither shock waves nor vacuum and concentration are developed in a finite time for such initial data under outer pressure. There were some similar fundamental results on the nonlinear thermoviscoelasticity [7, 8] and on viscous heat-conductive real gases [17]. Also see [13] and the references cited therein for some recent discussions. We consider the real magnetohydrodynamic flows with general pressure and internal energy, and permit the generation of heat by the magnetic field, as well as its interaction with the fluid motion. The existence and uniqueness of local solutions can be obtained by using the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval (cf. Nash [26]). The existence of global solutions is proved by extending the local solutions globally in time based on the global a-priori estimates of solutions.

We first obtain an entropy-type energy estimate involving the dissipative effects of viscosity, magnetic diffusion, and heat diffusion, which is essential to deduce the lower and upper bounds of the density. Some new techniques are developed to achieve these bounds. With these bounds, all the required a-priori estimates are obtained subsequently by our careful analysis and techniques with the aid of some arguments and ideas in [7, 8, 17], and the existence of global solutions in H^1 (as well as in a Hölder space) is established. In particular, the estimates of the first derivatives of u and the temperature θ are complicated because of the complexity of the system for the real flow, and will be achieved by developing some detailed analysis of the energy equation. The boundedness of the temperature will be proved by a maximum principle. For the case of discontinuous initial data, the initial discontinuities propagate as time evolves (see [5] and [12]). We establish the existence of global weak solutions with large discontinuous initial data for the perfect gases (1.2) by mollifying the initial data and using the estimates obtained above for continuous solutions and a convergence argument in [28, 29].

We state our main results in Section 2, prove the existence of global solutions with initial data in H^1 (and in a Hölder space) in Section 3, and show the existence of global weak solutions with large discontinuous initial data in Section 4. Section 5 is an appendix for the physical background of the magnetohydrodynamic equations and the derivation of the equations in (1.1).

2. MAIN RESULTS

We consider the free boundary problem (1.1) and (1.3) with positive lower and upper bounds of the initial density and temperature: $C_0^{-1} \le \rho_0 \le C_0, M_0^{-1} \le \theta_0 \le M_0$, for some constants $C_0, M_0 > 0$. Without loss of generality, we assume $\int_0^1 \rho_0(x)dx = 1$. We first suppose that $\rho, \theta > 0$, then we will prove the positive lower bounds of ρ and θ later. Introduce the Lagrangian variables (y,t) with $y = y(x,t)$ by

$$y_x = \rho, \quad y_t = -\rho u; \qquad y(0,0) = 0.$$

Then

$$y = y(x,t) = \int_0^x \rho(\xi,t)d\xi,$$

since $u(0,t) = 0$. Using the first equation in (1.1) and the free boundary condition, one has $y(x(t),t) = y(x(0),0) = \int_0^1 \rho_0 dx = 1$. Thus $0 \le y \le 1$. The free boundary problem (1.1) and (1.3) in Euler coordinates (x,t) is transformed into the following problem in Lagrangian coordinates (y,t) for $y \in \Omega = [0,1]$, $t > 0$:

$$v_t - u_y = 0, \tag{2.1a}$$

$$u_t + \left(p + \frac{1}{2}|\mathbf{b}|^2\right)_y = \left(\frac{\lambda u_y}{v}\right)_y, \tag{2.1b}$$

$$\mathbf{w}_t - \mathbf{b}_y = \left(\frac{\mu \mathbf{w}_y}{v}\right)_y, \tag{2.1c}$$

$$(v\mathbf{b})_t - \mathbf{w}_y = \left(\frac{\nu \mathbf{b}_y}{v}\right)_y, \tag{2.1d}$$

$$E_t + \left(u\left(p + \frac{1}{2}|\mathbf{b}|^2\right) - \mathbf{w}\cdot\mathbf{b}\right)_y = \left(\frac{\lambda u u_y}{v} + \frac{\mu \mathbf{w}\cdot\mathbf{w}_y}{v} + \frac{\nu \mathbf{b}\cdot\mathbf{b}_y}{v} + \frac{\kappa\theta_y}{v}\right)_y, \tag{2.1e}$$

with the following initial-boundary conditions:

$$\begin{aligned} &(v, u, \mathbf{w}, \mathbf{b}, \theta)|_{t=0} = (v_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)(y), \quad y \in \Omega, \\ &(\mathbf{w}, \mathbf{b}, \theta_y)|_{\partial\Omega} = 0, \quad u|_{y=0} = 0, \quad \left(p - \lambda v^{-1} u_y\right)|_{y=1} = 1, \quad t > 0, \end{aligned} \tag{2.2}$$

where $v = 1/\rho$ is the specific volume, $p = p(v, \theta)$, $e = e(v, \theta)$, and

$$E = e + \frac{1}{2}(u^2 + |\mathbf{w}|^2 + v|\mathbf{b}|^2).$$

The second law of thermodynamics states the relation between p and e:

$$e_v(v, \theta) + p(v, \theta) = \theta p_\theta(v, \theta). \tag{2.3}$$

We assume that p and e are continuously differentiable and κ is twice continuously differentiable in $v > 0$ and $\theta \geq 0$; λ, μ, and ν depend only on v with continuous first derivatives in $v > 0$ such that $\lambda_1 \leq \lambda \leq \lambda_2$, $\mu_1 \leq \mu \leq \mu_2$, $\nu_1 \leq \nu \leq \nu_2$ for $v > 0$ and for some positive constants λ_i, μ_i, ν_i $(i = 1, 2)$. We also assume the growth conditions with exponents $r \in [0, 1]$ and $q \geq 2 + 2r$ such that

(1). there exists a constant $e_0 > 0$ so that, for $v > 0$ and $\theta \geq 0$,

$$p_v(v, \theta) \leq 0, \quad e(v, \theta) \geq e_0(1 + \theta^{1+r}); \tag{2.4}$$

(2). for any given $v_1 > 0$, there exist positive constants $\kappa_0 = \kappa_0(v_1)$, $p_0 = p_0(v_1)$, $e_1 = e_1(v_1)$ such that, for $v \geq v_1$, $\theta \geq 0$,

$$0 \leq p(v, \theta) \leq p_0 \frac{1 + \theta^{1+r}}{v}, \ \kappa(v, \theta) \geq \kappa_0(1 + \theta^q), \ e_\theta(v, \theta) \geq e_1(1 + \theta^r); \tag{2.5}$$

(3). for any given $v_2 > v_1 > 0$, there exist positive constants $p_i = p_i(v_1, v_2)$ $(i = 1, 2, 3)$, $e_j = e_j(v_1, v_2)$ $(j = 2, 3)$, and $\kappa_1 = \kappa_1(v_1, v_2)$ so that, for any $v \in [v_1, v_2]$, $\theta \geq 0$,

$$|p_\theta(v,\theta)| \leq p_1 \frac{1+\theta^r}{v}, \quad -p_3 \frac{1+\theta^{1+r}}{v^2} \leq p_v(v,\theta) \leq -p_2 \frac{1+\theta^{1+r}}{v^2}, \tag{2.6}$$

$$|e_v(v,\theta)| \leq e_2(1+\theta^{1+r}), \quad e_\theta(v,\theta) \leq e_3(1+\theta^r), \tag{2.7}$$

$$\kappa(v,\theta) + |\kappa_v(v,\theta)| + |\kappa_{vv}(v,\theta)| \leq \kappa_1(1+\theta^q). \tag{2.8}$$

These growth conditions are motivated by the physical facts: $e \propto \theta^{1+r}$ with $r \approx 0.5$ and $\kappa \propto \theta^{5/2}$ for important physical regimes (see [2, 6, 30, 32]). The perfect gases are included which correspond to the special case $r = 0$. For the problem (2.1) and (2.2), we will see that, if the initial data is in H^1 (or a Hölder space), then the solution will be at least in H^1 (or the Hölder space), and neither shock waves nor vacuum and concentration are developed in a finite time. Precisely, we have the following results.

THEOREM 2.1. *Suppose that there are two positive constants C_0, M_0 such that*

$$C_0^{-1} \leq v_0(y) \leq C_0, \quad M_0^{-1} \leq \theta_0(y) \leq M_0, \quad y \in \Omega,$$
$$\|(u_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^4} + \|\theta_0\|_{L^2} \leq C_0, \quad \|(v_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)\|_{H^1} \leq M_0,$$

and $v_0 \in W^{1,\infty}(\Omega)$. Then the problem (2.1) *and* (2.2) *has a unique global solution $(v, u, \mathbf{w}, \mathbf{b}, \theta)(y,t)$ such that, for any fixed $T > 0$,*

$$v \in L^\infty(0,T; H^1 \cap W^{1,\infty}(\Omega)), \qquad (u, \mathbf{w}, \mathbf{b}, \theta) \in L^\infty(0,T; H^1(\Omega)),$$

and, for each $(y,t) \in \Omega \times [0,T]$,

$$\begin{aligned} &C^{-1} \leq v(y,t) \leq C, \quad M^{-1} \leq \theta(y,t) \leq M, \quad |(u, \mathbf{w}, \mathbf{b})(y,t)| \leq M, \\ &\|(u, \mathbf{w}, \mathbf{b})\|_{L^2(0,T;L^4 \cap H^1)} + \|\theta\|_{L^2(0,T;L^2 \cap H^1)} \leq C, \\ &\|(v, u, \mathbf{w}, \mathbf{b}, \theta)\|_{H^1}^2(t) + \int_0^t \|(v_{yt}, u_{yy}, \mathbf{w}_{yy}, \mathbf{b}_{yy}, \theta_{yy})\|_{L^2}^2(s)ds \leq M, \end{aligned} \tag{2.9}$$

where $C > 0$ is some constant depending only on C_0 and T, and $M = M(C_0, M_0, T) > 0$. If $v_0 \in C^{1+\alpha}(\Omega)$ and $(u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in C^{2+\alpha}(\Omega)$ for some $\alpha \in (0,1)$, there exists a unique classical solution $v \in C^{1+\alpha, 1+\alpha/2}(\Omega \times [0,T])$ and $(u, \mathbf{w}, \mathbf{b}, \theta) \in C^{2+\alpha, 1+\alpha/2}(\Omega \times [0,T])$ for any fixed $T > 0$ to the problem (2.1) *and* (2.2) *satisfying* (2.9).

The existence of local solutions is known from the standard method based on the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval (cf. [26]). The global existence of solutions will be proved by the method of extending the local solutions with respect to time based on a-priori global estimates.

Moreover, we consider the perfect gases in the sense of the state equations (1.2) with constant λ, μ, ν and with κ independent of v satisfying (2.5) and (2.8) for $q \geq 2$. Then we have the following theorem for large discontinuous initial data.

THEOREM 2.2. *Suppose that there exists* $C_0 > 0$ *such that*

$$C_0^{-1} \le v_0(y) \le C_0, \quad \theta_0(y) \ge 0, \quad \|(u_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^4} + \|\theta_0\|_{L^2} \le C_0.$$

Then there exists a global weak solution $(v, u, \mathbf{w}, \mathbf{b}, \theta)(y, t)$ *to the problem* (2.1) *and* (2.2) *for perfect gases, satisfying* (2.1) *in the sense of distributions such that, for any fixed* $T > 0$,

$$(u, \mathbf{w}, \mathbf{b}) \in L^2(0, T; L^4), \quad \theta \in L^2(0, T; L^2),$$

with $\theta(\cdot, t) \rightharpoonup \theta_0$ *weakly in* L^2 *as* $t \to 0$. *Furthermore, there is a constant* C *depending on* C_0 *and* T *such that, for each* $(y, t) \in \Omega \times [0, T]$,

$$\begin{aligned} &C^{-1} \le v(y, t) \le C, \quad \theta(y, t) \ge 0, \\ &\|(u, \mathbf{w}, \mathbf{b})\|_{L^2(0,T;L^4 \cap H^1)} + \|\theta\|_{L^2(0,T;L^2 \cap H^1)} \le C. \end{aligned} \tag{2.10}$$

Remark 2.1. The solutions in Theorem 2.2 can be improved to the case where the solution $(u, \mathbf{w}, \mathbf{b}) \in C([0, T]; L^4(\Omega)) \cap C((0, T]; H^1(\Omega))$ and $\theta \in C^2([0, T]; L^2(\Omega)) \cap C((0, T]; H^1(\Omega))$.

The results for the problem (2.1) and (2.2) in Theorems 2.1 and 2.2 in Lagrangian coordinates can easily be converted to equivalent statements for the corresponding results for the problem (1.1) and (1.3) in Euler coordinates. In particular, we have the following results.

THEOREM 2.3. *For the problem* (1.1) *and* (1.3), *the following holds:* (1). *Suppose that there are some constants* $C_0, M_0 > 0$ *such that*

$$\begin{aligned} &C_0^{-1} \le \rho_0(x) \le C_0, \quad M_0^{-1} \le \theta_0(x) \le M_0, \quad x \in (0, 1), \\ &\|(u_0, \mathbf{w}_0, \mathbf{b}_0)\|_{L^4(0,1)} + \|\theta_0\|_{L^2(0,1)} \le C_0, \\ &\|(\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)\|_{H^1(0,1)} \le M_0, \end{aligned}$$

and $\rho_0 \in W^{1,\infty}(0, 1)$. *Then the problem* (1.1) *and* (1.3) *has a global solution* $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)(x, t)$ *such that, for any fixed* $T > 0$,

$$\rho \in L^\infty(0, T; H^1 \cap W^{1,\infty}), \quad (u, \mathbf{w}, \mathbf{b}, \theta) \in L^\infty(0, T; H^1),$$

and, for each $(x, t), x \in \Omega_t, t \in [0, T]$,

$$\begin{aligned} &C^{-1} \le \rho(x, t) \le C, \quad M^{-1} \le \theta(x, t) \le M, \quad |(u, \mathbf{w}, \mathbf{b})(x, t)| \le M, \\ &\|(u, \mathbf{w}, \mathbf{b})\|_{L^2(0,T;L^4 \cap H^1)} + \|\theta\|_{L^2(0,T;L^2 \cap H^1)} \le C, \\ &\|(v, u, \mathbf{w}, \mathbf{b}, \theta)\|^2_{H^1(\Omega_t)}(t) + \int_0^t \|(v_{xt}, u_{xx}, \mathbf{w}_{xx}, \mathbf{b}_{xx}, \theta_{xx})\|^2_{L^2(\Omega_s)}(s) ds \le M, \end{aligned} \tag{2.11}$$

where $C > 0$ *is a constant independent of* M_0, *and* $M = M(C_0, M_0, T) > 0$. *If* $\rho_0 \in C^{1+\alpha}(0, 1)$ *and* $(u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0) \in C^{2+\alpha}(0, 1)$ *for some* $\alpha \in$

$(0,1)$, *there exists a classical solution* $\rho \in C^{1+\alpha,1+\alpha/2}$ *and* $(u,\mathbf{w},\mathbf{b},\theta) \in C^{2+\alpha,1+\alpha/2}$ *for any fixed* $T>0$ *to the problem* (1.1) *and* (1.3) *satisfying* (2.11).

(2). *Suppose that there exists* $C_0>0$ *such that, for* $x\in(0,1)$,

$$C_0^{-1} \le \rho_0(x) \le C_0,\ \theta_0(x) \ge 0,\ \|(u_0,\mathbf{w}_0,\mathbf{b}_0)\|_{L^4(0,1)} + \|\theta_0\|_{L^2(0,1)} \le C_0.$$

Then the problem (1.1) *and* (1.3) *for perfect gases has a global weak solution* $(\rho,u,\mathbf{w},\mathbf{b},\theta)(x,t)$ *satisfying* (1.1) *and* (1.3) *in the sense of distributions such that, for any fixed* $T>0$,

$$(u,\mathbf{w},\mathbf{b}) \in L^2(0,T;L^4), \quad \theta \in L^2(0,T;L^2),$$

with $\theta(\cdot,t) \rightharpoonup \theta_0$ *weakly in* L^2 *as* $t\to 0$. *Furthermore, there is a constant* C *depending on* C_0 *and* T *such that, for each* $(x,t), x\in\Omega_t, t\in[0,T]$,

$$\begin{aligned} &C^{-1} \le \rho(x,t) \le C, \quad \theta(x,t) \ge 0, \\ &\|(u,\mathbf{w},\mathbf{b})\|_{L^2(0,T;L^4\cap H^1)} + \|\theta\|_{L^2(0,T;L^2\cap H^1)} \le C. \end{aligned} \tag{2.12}$$

Remark 2.2. If the constant outer pressure in (1.3) is replaced by a general outer pressure $R(t)>0$, the results in Theorem 2.3 are still true, and the proof follows directly from the techniques developed here. The case of free outer pressure $R(t)=0$ needs new techniques which is handled in a forthcoming paper. For the initial-boundary value problem of (1.1) for $x\in\Omega=(0,1)$ and $t>0$ with the simple fixed boundary condition: $(u,\mathbf{w},\mathbf{b},\theta_x)|_{\partial\Omega}=0$, similar results can be obtained by using the techniques of this paper with certain modifications.

3. A-PRIORI ESTIMATES

In order to establish the global existence, we need a-priori estimates of the solutions for $(y,t)\in\Omega\times[0,T]$ for any fixed $T>0$. We denote $C>0$ the generic constant, which may depend on C_0 and T but independent of M_0, and denote $M>0$ the generic constant, which may depend on C_0, M_0, and T.

To obtain the complete global a-priori estimates, we first prove the positive lower and upper bounds of the density and some entropy-type energy estimates, then we estimate the first derivatives of v, $\mathbf{w}$, and $\mathbf{b}$. The estimates on the first derivatives of u and θ are complicated because of the complexity of the system of the real flow, which will be achieved by detailed analysis of the energy equation. The boundedness of the temperature will be proved by a maximum principle.

First, from the equations (2.1) and the initial-boundary conditions (2.2), we have the following entropy-type energy estimates and the lower and upper bounds of the density.

LEMMA 3.1.

$$C^{-1} \le v(y,t) \le C, \tag{3.1}$$

$$\int_0^1 \left(|(u,\mathbf{w},\sqrt{v}\mathbf{b})|^2 + v + \theta + \theta^{1+r} + (\theta - 1 - \ln\theta) + e\right) dy \le C, \tag{3.2}$$

$$\int_0^t\int_0^1 \left(\frac{\kappa\theta_y^2}{\theta^2} + \lambda u_y^2 + \mu|\mathbf{w}_y|^2 + \nu|\mathbf{b}_y|^2 + \theta_y^2\right) dy\, ds \le C. \tag{3.3}$$

Proof. We first prove the lower bound of v. Integrating the energy equation (2.1e), and using the initial-boundary condition (2.2), integration by parts, and $u(1,t) = \int_0^1 u_y dy = \int_0^1 v_t dy$, we have

$$\int_0^1 E(y,t)dy - \int_0^1 E(y,0)dy = -\int_0^t \left(\int_0^1 v dy\right)_t ds$$
$$= -\int_0^1 v(y,t)dy + \int_0^1 v_0(y)dy,$$

and then

$$\int_0^1 (E+v)(y,t)dy = \int_0^1 (E(y,0) + v_0(y))\, dy \le C,$$

that is,

$$\int_0^1 \left(e + |(u,\mathbf{w},\sqrt{v}\mathbf{b})|^2 + v\right) dy \le C.$$

From (2.4), $\int_0^1 \left(\theta + \theta^{1+r}\right) dy \le C + C\int_0^1 e\, dy \le C$. Set $V(v) \equiv \int_1^v \frac{\lambda(\xi)}{\xi} d\xi$, and then $\lambda_1 \ln v \le V(v) \le \lambda_2 \ln v$, $V_t = \frac{\lambda}{v} u_y$, and (2.1b) becomes $u_t = \left(V_t - p - \frac{1}{2}|\mathbf{b}|^2\right)_y$. Integrating this equation over $[y,1]\times[0,t]$ yields

$$V(v(y,t)) = V(v_0(y)) + \int_0^t \left(p + \frac{1}{2}|\mathbf{b}|^2\right) ds - \int_y^1 (u(\xi,t) - u_0(\xi))d\xi - t. \tag{3.4}$$

Then $V(v(y,t)) \ge -C$ for any $(y,t) \in [0,1]\times[0,T]$ since $p \ge 0$ and

$$\left|\int_y^1 (u(\xi,t) - u_0(\xi))d\xi\right| \le \left(2\int_0^1 (u^2 + u_0^2)dy\right)^{1/2} \le C,$$

which implies the positive lower bound of v, i.e. $v \ge C^{-1}$.

In order to prove the upper bound of v, we need some entropy-type energy estimates. Define $\eta(v,\theta)$ by the relations: $e_\theta = \theta\eta_\theta$, $\eta_v = p_\theta$. Set $\psi(v,\theta) = e(v,\theta) - \theta\eta(v,\theta)$. Then $e_\theta = -\theta\psi_{\theta\theta}$ and $\psi_v = -p$. Set

$$W(v,\theta) = \psi(v,\theta) - \psi(v,1) - \psi_\theta(v,\theta)(\theta - 1).$$

Then

$$\left(W+\frac{1}{2}(u^2+|\mathbf{w}|^2+v|\mathbf{b}|^2)\right)_t+\frac{\kappa\theta_y^2}{v\theta^2}+\frac{\lambda u_y^2}{v\theta}+\frac{\mu|\mathbf{w}_y|^2}{v\theta}+\frac{\nu|\mathbf{b}_y|^2}{v\theta}$$
$$=p(v,1)v_t+\left(\frac{\lambda uu_y}{v}+\frac{\mu\mathbf{w}\cdot\mathbf{w}_y}{v}+\frac{\nu\mathbf{b}\cdot\mathbf{b}_y}{v}+\frac{\kappa\theta_y}{v}\right)_y$$
$$-\left(u\left(p+\frac{1}{2}|\mathbf{b}|^2\right)-\mathbf{w}\cdot\mathbf{b}+\frac{\kappa\theta_y}{v\theta}\right)_y,$$

where $p(v,1)$ is the value of $p(v,\theta)$ at $\theta=1$. Integrating the above equation over $[0,1]\times[0,t]$ and using (2.5) yield

$$\int_0^1\left(W+\frac{1}{2}(u^2+|\mathbf{w}|^2+v|\mathbf{b}|^2)\right)dy$$
$$+\int_0^t\int_0^1\left(\frac{\kappa\theta_y^2}{v\theta^2}+\frac{\lambda u_y^2}{v\theta}+\frac{\mu|\mathbf{w}_y|^2}{v\theta}+\frac{\nu|\mathbf{b}_y|^2}{v\theta}\right)dy\,ds$$
$$\le C+C\int_0^t\int_0^1\frac{|u_y|}{v}\,dy\,ds\le C+\frac{1}{2}\int_0^t\int_0^1\frac{\lambda u_y^2}{v\theta}\,dy\,ds+C\int_0^t\int_0^1\theta\,dy\,ds.$$

Then

$$\int_0^1\left(W+(u^2+|\mathbf{w}|^2+v|\mathbf{b}|^2)\right)dy$$
$$+\int_0^t\int_0^1\left(\frac{\kappa\theta_y^2}{v\theta^2}+\frac{\lambda u_y^2}{v\theta}+\frac{\mu|\mathbf{w}_y|^2}{v\theta}+\frac{\nu|\mathbf{b}_y|^2}{v\theta}\right)dy\,ds\le C.$$

By (2.5), one has

$$W(v,\theta)=\psi(v,\theta)-\psi(v,1)-\psi_\theta(v,\theta)(\theta-1)$$
$$\ge(\theta-1)^2\int_0^1\frac{e_\theta(v,1+s(\theta-1))}{1+s(\theta-1)}\,s\,ds$$
$$\ge\frac{1}{2}e_1(\theta-1-\ln\theta)-C_1(\theta+\theta^{1+r})-C_2,$$

where C_1 and C_2 are some positive constants. Then

$$\int_0^1(\theta-1-\ln\theta)\,dy+\int_0^t\int_0^1\left(\frac{\kappa\theta_y^2}{v\theta^2}+\frac{\lambda u_y^2}{v\theta}+\frac{\mu|\mathbf{w}_y|^2}{v\theta}+\frac{\nu|\mathbf{b}_y|^2}{v\theta}\right)dy\,ds\le C.$$

Now we estimate the upper bound of v. For any $t\in[0,T]$, there exists $y(t)\in[0,1]$ such that $\bar\theta=\theta(y(t),t)=\int_0^1\theta dy\le C$. From (2.5) and the estimates above, we have

$$\begin{aligned}\int_0^t p ds \leq & C \int_0^t (1+\theta^{1+r}) ds \leq C \int_0^t (1+\theta^2) ds \\ \leq & C + C \int_0^t \left(\bar{\theta} + \int_{y(s)}^y \theta_y(\xi, s) d\xi\right)^2 ds \\ \leq & C + C \int_0^t \int_0^1 \frac{\theta_y^2}{v} dy \int_0^1 v\, dy\, ds \leq C + C \int_0^t \int_0^1 \frac{\kappa \theta_y^2}{v\theta^2}\, dy\, ds \leq C.\end{aligned}$$

From (2.1), we have

$$e_t + p u_y = \left(\frac{\kappa \theta_y}{v}\right)_y + \frac{\lambda u_y^2}{v} + \frac{\mu |\mathbf{w}_y|^2}{v} + \frac{\nu |\mathbf{b}_y|^2}{v}. \tag{3.5}$$

Integrating the equation (3.5) and using (2.5), we obtain

$$\begin{aligned}&\int_0^t \int_0^1 \left(\frac{\lambda u_y^2}{v} + \frac{\mu |\mathbf{w}_y|^2}{v} + \frac{\nu |\mathbf{b}_y|^2}{v}\right) dy\, ds \\ &\quad \leq C + \frac{1}{2} \int_0^t \int_0^1 \frac{\lambda u_y^2}{v}\, dy\, ds + C \int_0^t \int_0^1 \frac{1+\theta^{2+2r}}{v}\, dy\, ds,\end{aligned}$$

and then

$$\begin{aligned}&\int_0^t \int_0^1 \left(\frac{\lambda u_y^2}{v} + \frac{\mu |\mathbf{w}_y|^2}{v} + \frac{\nu |\mathbf{b}_y|^2}{v}\right) dy\, ds \\ &\quad \leq C + C \int_0^t \int_0^1 \left(\bar{\theta}^{1+r} + \int_0^1 \theta^r |\theta_y| dy\right)^2 dy\, ds \\ &\quad \leq C + C \int_0^t \int_0^1 \frac{\kappa \theta_y^2}{v\theta^2} dy \int_0^1 v\, dy\, ds \leq C.\end{aligned}$$

Thus, from the boundary condition (2.2) on $\mathbf{b}$ and the estimates above,

$$\begin{aligned}&\int_0^t \frac{1}{2} |\mathbf{b}|^2 ds = \int_0^t \int_0^y \mathbf{b} \cdot \mathbf{b}_y ds \\ &\quad \leq \int_0^t \int_0^1 v |\mathbf{b}|^2\, dy\, ds + C \int_0^t \int_0^1 \frac{\nu |\mathbf{b}_y|^2}{v}\, dy\, ds \leq C.\end{aligned}$$

Therefore, $V(v(y,t)) \leq C$, which implies the upper bound of v, that is, $v \leq C$. The proof of Lemma 3.1 is completed. ∎

We now make estimates on the first derivatives of v, $\mathbf{w}$, and $\mathbf{b}$.

LEMMA 3.2.

$$\begin{aligned} &\int_0^1 \left(v_y^2 + |\mathbf{w}_y|^2 + |\mathbf{b}_y|^2\right) dy \\ &\quad + \int_0^t\int_0^1 \left(|\mathbf{w}_{yy}|^2 + |\mathbf{b}_{yy}|^2 + u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{b}_y|^4\right) dy\, ds \le M. \end{aligned} \tag{3.6}$$

Proof. Set $h(y,t) = \int_y^1 u(\xi,t)d\xi$. Equation (2.1b) yields

$$h_t - \frac{\lambda}{v}h_{yy} = p + \frac{1}{2}|\mathbf{b}|^2 - 1$$

with the following initial-boundary conditions:

$$h(y,0) = \int_y^1 u_0(\xi)d\xi, \quad h(0,t) = \int_0^1 u(y,t)dy, \quad h(1,t) = 0.$$

Then, from (2.5), we have the following estimate (see [8, 22]):

$$\begin{aligned} \int_0^t\int_0^1 u_y^4\, dy\, ds = \int_0^t\int_0^1 h_{yy}^4\, dy\, ds &\le M + M\int_0^t\int_0^1 \left(p^4 + |\mathbf{b}|^8\right) dy\, ds \\ &\le M + M\int_0^t\int_0^1 \left(|\mathbf{b}|^8 + \theta^{4+4r}\right) dy\, ds. \end{aligned}$$

Rewrite (2.1b) as:

$$(V_y - u)_t = \left(p + \frac{1}{2}|\mathbf{b}|^2\right)_y.$$

Multiplying the above equation by $V_y - u$ and integrating it, we have

$$\frac{1}{2}\int_0^1 |V_y - u|^2\, dy \le M + \int_0^t\int_0^1 \left(p_v v_y + p_\theta\theta_y + \mathbf{b}\cdot\mathbf{b}_y\right)(V_y - u)\, dy\, ds.$$

Using (2.6), $V_y = \lambda v_y/v$, and the following observation

$$\begin{aligned} &\int_0^t \max_{y\in\Omega}\left(1+\theta^{1+r}\right) ds + \int_0^t \max_{y\in\Omega}\left(1+\theta^{2+2r}\right) ds + \int_0^t \max_{y\in\Omega}\left(1+\theta^{1+r}\right)^2 ds \\ &\le M\int_0^t \max_{y\in\Omega}\left(1+\theta^{1+r}\right)^2 ds \le M\int_0^t \left(1 + \bar{\theta}^{1+r} + (1+r)\int_0^1 \theta^r|\theta_y|dy\right) ds \\ &\le M + M\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2}\, dy\, ds \le M, \end{aligned} \tag{3.7}$$

we obtain that there exists some constant $M_1 > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 |V_y - u|^2\, dy \\
&\le M - M_1 \int_0^t\int_0^1 \left(1+\theta^{1+r}\right) v_y^2\, dy\, ds + M\int_0^t\int_0^1 \left(1+\theta^{1+r}\right)|v_y u|\, dy\, ds \\
&\quad + M\int_0^t\int_0^1 (1+\theta^r)\,|\theta_y|\,(|V_y| + |u|)\, dy\, ds + \int_0^t\int_0^1 \mathbf{b}\cdot\mathbf{b}_y\,(V_y - u)\, dy\, ds \\
&\le M - \frac{M_1}{2}\int_0^t\int_0^1 \left(1+\theta^{1+r}\right) v_y^2\, dy\, ds + M\int_0^t \max_{y\in\Omega}\left(1+\theta^{1+r}\right)\int_0^1 u^2\, dy\, ds \\
&\quad + M\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2}\, dy\, ds + M\int_0^t\int_0^1 |\mathbf{b}\cdot\mathbf{b}_y|^2\, dy\, ds + M\int_0^t\int_0^1 u^2\, dy\, ds \\
&\le M - \frac{M_1}{2}\int_0^t\int_0^1 \left(1+\theta^{1+r}\right) v_y^2\, dy\, ds + M\int_0^t\int_0^1 |\mathbf{b}\cdot\mathbf{b}_y|^2\, dy\, ds.
\end{aligned}
$$

Then

$$
\int_0^1 |V_y - u|^2\, dy + \int_0^t\int_0^1 \left(1+\theta^{1+r}\right) v_y^2\, dy\, ds \le M + M\int_0^t\int_0^1 |\mathbf{b}\cdot\mathbf{b}_y|^2\, dy\, ds,
$$

and

$$
\begin{aligned}
\int_0^1 v_y^2 dy \le M\int_0^1 V_y^2 dy \le M\int |V_y - u|^2\, dy + M\int_0^1 u^2 dy \\
\le M + M\int_0^t\int_0^1 |\mathbf{b}\cdot\mathbf{b}_y|^2\, dy\, ds.
\end{aligned}
\tag{3.8}
$$

Multiplying (2.1e) by $K_1 E$ and (2.1b) by $K_2 u^3$, taking the inner product of (2.1c) with $K_3|\mathbf{w}|^2\mathbf{w}$ and (2.1d) with $K_4|v\mathbf{b}|^2 v\mathbf{b}$, multiplying (3.8) by K_5, respectively, for proper positive constants $K_j, 1 \le j \le 5$, integrating them over $[0,1]\times[0,t]$, adding them all together, and using Gronwall's inequality, we have $\int_0^1 v_y^2 dy + \int_0^t\int_0^1 |\mathbf{b}\cdot\mathbf{b}_y|^2\, dy\, ds \le M$ by tedious calculations. Using

Young's inequality, we have

$$
\begin{aligned}
&\int_0^t\int_0^1 \left(|\mathbf{b}|^8+\theta^{4+4r}\right)\,dy\,ds\\
&\le \int_0^t \max_{y\in\Omega}|\mathbf{b}|^6\int_0^1|\mathbf{b}|^2\,dy\,ds+\int_0^t\max_{y\in\Omega}\theta^{3+3r}\int_0^1\theta^{1+r}\,dy\,ds\\
&\le M\int_0^t\left(\bar\theta^{3+3r}+\int_0^1|\mathbf{b}|^6dy+\int_0^1|\mathbf{b}|^4|\mathbf{b}\cdot\mathbf{b}_y|+\int_0^1\theta^{2+2r}\cdot\theta^r|\theta_y|dy\right)ds\\
&\le \frac{1}{2}\int_0^t\int_0^1|\mathbf{b}|^8\,dy\,ds+\frac{1}{2}\int_0^t\int_0^1\theta^{4+4r}\,dy\,ds\\
&\quad+M\left(1+\int_0^t\int_0^1|\mathbf{b}\cdot\mathbf{b}_y|^2\,dy\,ds+\int_0^t\int_0^1\frac{\kappa\theta_y^2}{\theta^2}\,dy\,ds\right)\\
&\le M+\frac{1}{2}\int_0^t\int_0^1|\mathbf{b}|^8\,dy\,ds+\frac{1}{2}\int_0^t\int_0^1\theta^{4+4r}\,dy\,ds,
\end{aligned}
$$

and then

$$\int_0^t\int_0^1\left(|\mathbf{b}|^8+\theta^{4+4r}\right)\,dy\,ds\le M,$$

and thus $\int_0^t\int_0^1 u_y^4\,dy\,ds\le M$. Multiplying (2.1c) by $\mathbf{w}_{yy}$ and then integrating, and using the interpolation inequality:

$$|\mathbf{w}_y|^2\le(1+\delta^{-1})\int_0^1|\mathbf{w}_y|^2dy+\delta\int_0^1|\mathbf{w}_{yy}|^2dy,\quad\text{for any}\quad\delta>0,$$

we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^1|\mathbf{w}_y|^2dy=\frac{1}{2}\int_0^1|\mathbf{w}_y(y,0)|^2dy-\int_0^t\int_0^1\left(\mathbf{b}+\frac{\mu\mathbf{w}_y}{v}\right)_y\cdot\mathbf{w}_{yy}\,dy\,ds\\
&\le M-M_2\int_0^t\int_0^1|\mathbf{w}_{yy}|^2\,dy\,ds+M\int_0^t\int_0^1\left(|\mathbf{b}_y|+|v_y||\mathbf{w}_y|\right)|\mathbf{w}_{yy}|\,dy\,ds\\
&\le M-\frac{3M_2}{4}\int_0^t\int_0^1|\mathbf{w}_{yy}|^2\,dy\,ds+M\int_0^t\int_0^1|\mathbf{b}_y|^2\,dy\,ds\\
&\quad+M\int_0^t\max_{y\in\Omega}|\mathbf{w}_y|^2\int_0^1v_y^2\,dy\,ds\\
&\le M-\frac{M_2}{2}\int_0^t\int_0^1|\mathbf{w}_{yy}|^2\,dy\,ds+M\int_0^t\int_0^1|\mathbf{w}_y|^2\,dy\,ds\\
&\le M-\frac{M_2}{2}\int_0^t\int_0^1|\mathbf{w}_{yy}|^2\,dy\,ds,
\end{aligned}
$$

and then

$$\int_0^1|\mathbf{w}_y|^2dy+\int_0^t\int_0^1|\mathbf{w}_{yy}|^2\,dy\,ds\le M.$$

As a consequence,

$$\int_0^t\int_0^1 |\mathbf{w}_y|^4\,dy\,ds \le \int_0^t \max_{y\in\Omega}|\mathbf{w}_y|^2\int_0^1 |\mathbf{w}_y|^2\,dy\,ds$$
$$\le M\int_0^t\int_0^1 \left(|\mathbf{w}_y|^2+|\mathbf{w}_{yy}|^2\right)\,dy\,ds \le M.$$

Using $v_t = u_y$, we rewrite (2.1d) into:

$$\mathbf{b}_t = \frac{u_y}{v}\mathbf{b} + \frac{1}{v}\left(\mathbf{w}+\frac{\nu\mathbf{b}_y}{v}\right)_y. \tag{3.9}$$

Multiplying the above equation by $\mathbf{b}_{yy}$, integrating it, and using Lemma 3.1 and the interpolation inequalities: for any $\delta > 0$,

$$|\mathbf{b}|^2 \le (1+\delta^{-1})\int_0^1 |\mathbf{b}|^2dy + \delta\int_0^1 |\mathbf{b}_y|^2dy,$$
$$|\mathbf{b}_y|^2 \le (1+\delta^{-1})\int_0^1 |\mathbf{b}_y|^2dy + \delta\int_0^1 |\mathbf{b}_{yy}|^2dy$$

yield

$$\begin{aligned}
&\frac{1}{2}\int_0^1 |\mathbf{b}_y|^2dy\\
&\le M - M_3\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2\,dy\,ds\\
&\quad + M\int_0^t\int_0^1 \left(|u_y||\mathbf{b}|+|\mathbf{w}_y|+|v_y||\mathbf{b}_y|\right)|\mathbf{b}_{yy}|\,dy\,ds\\
&\le M - \frac{3M_3}{4}\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2\,dy\,ds + M\max_{y,t}|\mathbf{b}|^2\int_0^t\int_0^1 u_y^2\,dy\,ds\\
&\quad + M\int_0^t \max_{y\in\Omega}|\mathbf{b}_y|^2\int_0^1 v_y^2\,dy\,ds\\
&\le M - \frac{M_3}{2}\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2\,dy\,ds + M\int_0^1 |\mathbf{b}|^2dy + \frac{1}{4}\int_0^1 |\mathbf{b}_y|^2dy\\
&\quad + M\int_0^t\int_0^1 |\mathbf{b}_y|^2\,dy\,ds\\
&\le M - \frac{M_3}{2}\int_0^t\int_0^1 |\mathbf{b}_{yy}|^2\,dy\,ds + \frac{1}{4}\int_0^1 |\mathbf{b}_y|^2dy.
\end{aligned}$$

Then

$$\int_0^1 |\mathbf{b}_y|^2dy + \int_0^t\int_0^1 |\mathbf{b}_{yy}|^2\,dy\,ds \le M.$$

Thus

$$\int_0^t\int_0^1 |\mathbf{b}_y|^4\,dy\,ds \le \int_0^t \max_{y\in\Omega}|\mathbf{b}_y|^2\int_0^1|\mathbf{b}_y|^2\,dy\,ds$$
$$\le M\int_0^t\int_0^1\left(|\mathbf{b}_y|^2+|\mathbf{b}_{yy}|^2\right)\,dy\,ds\le M.$$

The proof of Lemma 3.2 is completed. ∎

We now make estimates on the first derivatives of u and θ.

LEMMA 3.3.

$$\int_0^1\left(|u_y|^2+|u_{yy}|^2+\theta_y^2\right)dy+\int_0^t\int_0^1\left(\theta_t^2+|\mathbf{b}_t|^2\right)\,dy\,ds\le M. \tag{3.10}$$

Proof. Due to the complicated structures of the pressure, internal energy, and heat conductivity, as well as the strong coupling of the equations, the estimates of u_y and θ_y are quite complex. Set

$$\Theta=\max_{y\in\Omega,t\in[0,T]}\theta(y,t),\quad X=\int_0^t\int_0^1\left(1+\theta^{q+r}\right)\theta_t^2\,dy\,ds,$$
$$Y=\max_{t\in[0,T]}\int_0^1\left(1+\theta^{2q}\right)\theta_y^2dy,\quad Z=\max_{t\in[0,T]}\int_0^1 u_{yy}^2dy.$$

We will show that Θ, X, and Y can be controlled by Z and then we derive an inequality on Z which yields the upper bound of Z and thus the upper bounds of Θ, X, and Y. The complete proof can be divided into four steps.
Step 1. Using the following interpolation inequality (cf. [10])

$$\int_0^1 u_y^2dy\le M\int_0^1u^2dy+M\left(\int_0^1u^2dy\right)^{1/2}\left(\int_0^1u_{yy}^2dy\right)^{1/2},$$

one has

$$\int_0^1u_y^2dy\le M+MZ^{1/2}, \tag{3.11}$$

and

$$u_y^2\le\int_0^1u_y^2dy+2\left(\int_0^1u_y^2dy\right)^{1/2}\left(\int_0^1u_{yy}^2dy\right)^{1/2}\le M+MZ^{1/2}+MZ^{3/4},$$

which implies

$$|u_y|\le M+MZ^{3/8}. \tag{3.12}$$

From Lemma 3.1, we have

$$\theta^{(2q+3+r)/2}(y,t)=\bar\theta^{(2q+3+r)/2}+\frac{2q+3+r}{2}\int_{y(t)}^y\theta^{(2q+1+r)/2}\theta_y d\xi$$
$$\le M+M\left(\int_0^1\theta^{2q}\theta_y^2dy\right)^{1/2}\left(\int_0^1\theta^{1+r}dy\right)^{1/2}\le M+MY^{1/2},$$

and thus

$$\Theta \le M + MY^{\beta_1}, \tag{3.13}$$

where $\beta_1 = 1/(2q+3+r)$.
Step 2. We now show that X and Y can be controlled by Z. Use (2.3) and rewrite the equation (3.5) as:

$$e_\theta \theta_t + \theta p_\theta u_y = \left(\frac{\kappa\theta_y}{v}\right)_y + \frac{\lambda u_y^2}{v} + \frac{\mu|\mathbf{w}_y|^2}{v} + \frac{\nu|\mathbf{b}_y|^2}{v}. \tag{3.14}$$

Set $H(v,\theta) = v^{-1}\int_0^\theta \kappa(v,\xi)\,d\xi$. Multiplying (3.14) by H_t, using integration by parts and the boundary condition (2.2), we have

$$\begin{aligned} &\int_0^t\int_0^1 \left(e_\theta\theta_t + \theta p_\theta u_y - \frac{\lambda}{v}u_y^2 - \frac{\mu}{v}|\mathbf{w}_y|^2 - \frac{\nu}{v}|\mathbf{b}_y|^2\right) H_t\,dy\,ds \\ &\quad + \int_0^t\int_0^1 \frac{\kappa}{v}\theta_y H_{ty}\,dy\,ds = 0, \end{aligned} \tag{3.15}$$

where

$$H_t = H_v u_y + \frac{\kappa}{v}\theta_t, \quad H_{ty} = H_v u_{yy} + H_{vv}u_y v_y + \left(\frac{\kappa}{v}\right)_v \theta_t v_y + \left(\frac{\kappa}{v}\theta_y\right)_t.$$

We now estimate all the terms in the above equality. First we have, from (2.4) and (2.5),

$$\int_0^t\int_0^1 e_\theta\theta_t\frac{\kappa}{v}\theta_t\,dy\,ds \ge M_4 X, \quad \int_0^T\int_0^1 \frac{\kappa}{v}\theta_y\left(\frac{\kappa}{v}\theta_y\right)_t\,dy\,ds \ge M_5 Y - M,$$

for some positive constants M_4 and M_5. From (2.8),

$$|H_v| + |H_{vv}| \le M\int_0^\theta (|\kappa| + |\kappa_v| + |\kappa_{vv}|)\,d\xi \le M\left(1+\theta^{q+1}\right).$$

Using (2.6), (2.7), (3.12), (3.13), Lemmas 3.1 and 3.2, and Young's inequality, we have

$$\begin{aligned} \int_0^t\int_0^1 e_\theta\theta_t H_v u_y\,dy\,ds &\le M\int_0^t\int_0^1 \left(1+\theta^{q+r+1}\right)|\theta_t||u_y|\,dy\,ds \\ &\le \frac{M_4}{8}X + M\left(1+\Theta^{q+2+r}\right)\int_0^t\int_0^1 u_y^2\,dy\,ds \\ &\le \frac{M_4}{8}X + MY^{(q+2+r)\beta_1} + M \le \frac{M_4}{8}X + \frac{M_5}{8}Y + M, \end{aligned}$$

and

$$
\begin{aligned}
&\int_0^t\int_0^1 \left(\theta p_\theta u_y - \frac{\lambda}{v}u_y^2 - \frac{\mu}{v}|\mathbf{w}_y|^2 - \frac{\nu}{v}|\mathbf{b}_y|^2\right) H_v u_y \, dy\, ds \\
&\le M\left(1+\Theta^{q+1}\right)\max_{y,t}|u_y|\times \\
&\qquad \int_0^t\int_0^1 \left(\left(1+\theta^{1+r}\right)|u_y| + u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{b}_y|^2\right)\, dy\, ds \\
&\le M\left(1+\Theta^{q+2+r}\right)\left(1+Z^{3/8}\right)\int_0^t\int_0^1 \left(|u_y| + u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{b}_y|^2\right)\, dy\, ds \\
&\le M + \frac{M_5}{8}Y + MZ^{\beta_2}
\end{aligned}
$$

with $\beta_2 = 3(2q+3+r)/(8q+8)$, and

$$
\begin{aligned}
&\int_0^t\int_0^1 \left(\theta p_\theta u_y - \frac{\lambda}{v}u_y^2 - \frac{\mu}{v}|\mathbf{w}_y|^2 - \frac{\nu}{v}|\mathbf{b}_y|^2\right) \frac{\kappa}{v}\theta_t \, dy\, ds \\
&\le M\int_0^t\int_0^1 \left(\left(1+\theta^{1+r}\right)|u_y| + u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{b}_y|^2\right)\left(1+\theta^q\right)|\theta_t|\, dy\, ds \\
&\le \frac{M_4}{8}X + M\left(1+\Theta^{q+2+r}\right)\int_0^t\int_0^1 u_y^2\, dy\, ds \\
&\quad + M\left(1+\Theta^{q-r}\right)\int_0^t\int_0^1 \left(u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{b}_y|^4\right)\, dy\, ds \\
&\le \frac{M_4}{8}X + M\Theta^{q+2+r} + M \le \frac{M_4}{8}X + \frac{M_5}{8}Y + M.
\end{aligned}
$$

Since $\theta^q(y,t) = \bar{\theta}^q + \int_{y(t)}^y q\theta^{q-1}\theta_y d\xi$, we use Lemma 3.1 and Young's inequality to obtain

$$
\begin{aligned}
&\int_0^t \max_{y\in\Omega}\theta^q ds \le M + M\int_0^t\int_0^1 \theta^{q-1}|\theta_y|\, dy\, ds \\
&\le M + \int_0^t \max_{y\in\Omega}\theta^{q-1}\int_0^1 \theta\, dy\, ds + M\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2}\, dy\, ds \\
&\le M + \int_0^t \max_{y\in\Omega}\theta^{q-1} ds \le M + \frac{1}{2}\int_0^t \max_{y\in\Omega}\theta^q ds,
\end{aligned}
$$

and then

$$
\int_0^t \max_{y\in\Omega}\theta^q ds \le M. \tag{3.16}
$$

From (2.4)–(2.8), (3.12), (3.13), (3.16), Lemmas 3.1 and 3.2, and Young's inequality, we have

$$
\begin{aligned}
&\int_0^t\int_0^1 \frac{\kappa}{v}\theta_y H_{vv} u_y v_y \, dy \, ds \le M\left(1+\Theta^{q+1}\right)\max_{y,t}|u_y|\int_0^t\int_0^1 \kappa|\theta_y||v_y| \, dy \, ds \\
&\le M\left(1+\Theta^{q+2}\right)\max_{y,t}|u_y|\left(\int_0^t \max_y\left(1+\theta^q\right)\int_0^1 v_y^2 \, dy \, ds\right)^{1/2}\times \\
&\qquad\qquad \left(\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2} \, dy \, ds\right)^{1/2} \\
&\le M\left(1+Z^{3/8}\right)\left(1+Y^{(q+2)\beta_1}\right) \le \frac{M_6}{8}Y + MZ^{\beta_2} + M.
\end{aligned}
$$

Using (2.8), (3.13), (3.14), Lemma 3.2, and Young's inequality yields

$$
\begin{aligned}
&\int_0^t\int_0^1 \frac{\kappa}{v}\theta_y \left(\frac{\kappa}{v}\right)_v \theta_t v_y \, dy \, ds \le M \int_0^t\int_0^1 (1+\theta^q)\left|\theta_t\right| \left|\frac{\kappa}{v}\theta_y\right| |v_y| \, dy \, ds \\
&\le \frac{M_5}{16}X + M\left(1+\Theta^{q-r}\right)\int_0^t \max_y \left|\frac{\kappa}{v}\theta_y\right|^2 \int_0^1 v_y^2 \, dy \, ds \\
&\le \frac{M_5}{16}X + M\left(1+\Theta^{q-r}\right) \times \\
&\qquad \times \int_0^t \left(\int_0^1 \left|\frac{\kappa}{v}\theta_y\right|^2 dy + 2\int_0^1 \left|\frac{\kappa}{v}\theta_y\right| \left|\left(\frac{\kappa}{v}\theta_y\right)_y\right| dy\right) ds \\
&\le \frac{M_5}{16}X + M\left(1+\Theta^{2q+2-r}\right)\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2} \, dy \, ds \\
&\quad + M\left(1+\Theta^{q-r}\right)\int_0^t\int_0^1 \left|\frac{\kappa}{v}\theta_y\right| \left|\left(\frac{\kappa}{v}\theta_y\right)_y\right| dy \, ds \\
&\le \frac{M_5}{16}X + M + MY^{(2q+2-r)\beta_1} \\
&\quad + M\left(1+\Theta^{q+1-r}\right)\left(\int_0^t\int_0^1 \kappa\left(\frac{\kappa}{v}\theta_y\right)_y^2 dy \, ds\right)^{1/2}\left(\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2} \, dy \, ds\right)^{1/2} \\
&\le \frac{M_5}{16}X + \frac{M_6}{16}Y + M + M\left(1+\Theta^{q+1-r}\right) \times \\
&\qquad \times \left(\int_0^t\int_0^1 \kappa\left(e_\theta^2\theta_t^2 + \theta^2 p_\theta^2 u_y^2 + u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{b}_y|^4\right) dy \, ds\right)^{1/2} \\
&\le \frac{M_5}{16}X + \frac{M_6}{16}Y + M + M\left(1+\Theta^{(2q+2-r)/2}\right)X^{1/2} \\
&\quad + M\left(1+\Theta^{(3q+4)/2}\right)\left(\int_0^t\int_0^1 u_y^2 \, dy \, ds\right)^{1/2} \\
&\quad + M\left(1+\Theta^{(3q+2-2r)/2}\right)\left(\int_0^t\int_0^1 \left(u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{b}_y|^4\right) dy \, ds\right)^{1/2} \\
&\le \frac{M_5}{16}X + \frac{M_6}{16}Y + M + M\left(1+\Theta^{(2q+2-r)/2}\right)X^{1/2} + M\Theta^{(3q+4)/2} + M \\
&\le \frac{M_5}{8}X + \frac{M_6}{16}Y + M\Theta^{2q+2-r} + M \le \frac{M_5}{8}X + \frac{M_6}{8}Y + M.
\end{aligned}
$$

From (2.4), (2.8), (3.12), (3.13), (3.16), Lemma 3.1, and Young's inequality, we have

$$\int_0^t\int_0^1 \frac{\kappa}{v}\theta_y H_v u_{yy}\,dy\,ds \le M\int_0^t\int_0^1 \left(1+\theta^{q+1}\right)\kappa|\theta_y||u_{yy}|\,dy\,ds$$

$$\le M\left(1+\Theta^{(4q+5)/4}\right)\left(\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2}\,dy\,ds\right)^{1/2}\times$$

$$\times\left(Z\int_0^t\left(1+\max_y\theta^{(2q+3)/2}\right)ds\right)^{1/2}$$

$$\le M\left(1+Y^{\beta_1(4q+5)/4}\right)Z^{1/2}\le\frac{M_6}{8}Y+MZ^{\beta_3}+M,$$

where $\beta_3 = 2(2q+3+r)/(4q+7+4r)\in(0,1)$ and the following estimate is employed:

$$\int_0^t \max_y\theta^{(2q+3)/2}ds\le M, \tag{3.17}$$

which will be shown in the next step. Assuming the estimate (3.17), we conclude, from (3.15), that

$$X+Y\le M+MZ^{\beta_4}, \tag{3.18}$$

where $0<\beta_4=\max\{\beta_2,\beta_3\}<1$.

Step 3. The proof of (3.17) is as follows. For some small δ, from Lemma 3.1,

$$\int_0^t \max_y\theta^{(2q+3)/2}ds\le\int_0^t\left(\bar\theta^{(2q+3)/2}+\int_0^1\theta^{(2q+1)/2}|\theta_y|dy\right)ds$$

$$\le M+\delta\int_0^t\max_y\theta^{(2q+3)/2}\int_0^1\theta^{1+r}\,dy\,ds+M\int_0^t\int_0^1\theta^{(2q-3-2r)/2}\theta_y^2\,dy\,ds$$

$$\le M+\frac{1}{2}\int_0^t\max_y\theta^{(2q+3)/2}ds+M\int_0^t\int_0^1\theta^{(2q-3-2r)/2}\theta_y^2\,dy\,ds,$$

and then

$$\int_0^t\max_y\theta^{(2q+3)/2}ds\le M+M\int_0^t\int_0^1\theta^{(2q-3-2r)/2}\theta_y^2\,dy\,ds.$$

For $r=1$, using Lemma 3.1, we obtain (3.17) since

$$\int_0^t\max_y\theta^{(2q+3)/2}ds\le M+M\int_0^t\int_0^1\theta^{(2q-5)/2}\theta_y^2\,dy\,ds$$

$$\le M+M\int_0^t\int_0^1\frac{\kappa\theta_y^2}{\theta^2}\,dy\,ds\le M.$$

For $0 < r < 1$, define $G(v,\theta) = \int_0^\theta \xi^{-r} e_\theta(v,\xi)d\xi$. Then, from (2.4)–(2.8), Lemma 3.1, and (3.5),

$$\int_0^1 G(v,\theta)dy \le M\int_0^1 \left(\theta + \frac{1}{1-r}\theta^{1-r}\right) dy \le M\int_0^1 (\theta+1)dy \le M,$$

and

$$G_t + (1-r)u_y \int_0^\theta \xi^{-r} p_\theta(v,\xi)d\xi = \frac{\lambda u_y^2}{v\theta^r} + \frac{r\kappa\theta_y^2}{v\theta^{r+1}} + \left(\frac{\kappa\theta_y}{v\theta^r}\right)_y.$$

Integrating the above equation and using (2.4)–(2.8) and Lemma 3.1, we have

$$\begin{aligned}
&\int_0^t\int_0^1 \left(\frac{\lambda u_y^2}{v\theta^r} + \frac{r\kappa\theta_y^2}{v\theta^{r+1}}\right) dy\,ds \\
&\le M + M\int_0^t\int_0^1 u_y^2\,dy\,ds + M\int_0^t\int_0^1 (1+\theta)^2\,dy\,ds \\
&\le M + M\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^2}\,dy\,ds \le M.
\end{aligned}$$

Then, from the above estimate, we have

$$\begin{aligned}
\int_0^t \max_y \theta^{(2q+3)/2}ds &\le M + M\int_0^t\int_0^1 \theta^{(2q-3-2r)/2}\theta_y^2\,dy\,ds \\
&\le M + M\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^{1+r}}\,dy\,ds \le M.
\end{aligned}$$

If $r = 0$, replace the above definition of G by $G(v,\theta) = \int_0^\theta \xi^{-1/2}e_\theta(v,\xi)d\xi$ and follow the same procedure to obtain

$$\int_0^t\int_0^1 \left(\frac{\lambda u_y^2}{v\theta^{1/2}} + \frac{\kappa\theta_y^2}{2v\theta^{3/2}}\right) dy\,ds \le M.$$

Then

$$\begin{aligned}
\int_0^t \max_y \theta^{(2q+3)/2}ds &\le M + M\int_0^t\int_0^1 \theta^{(2q-3)/2}\theta_y^2\,dy\,ds \\
&\le M + M\int_0^t\int_0^1 \frac{\kappa\theta_y^2}{\theta^{3/2}}\,dy\,ds \le M.
\end{aligned}$$

Step 4. Now we estimate Z. Differentiate the equation (2.1b) with respect to t, multiply it by u_t, and then integrate to obtain

$$\frac{1}{2}\int_0^1 u_t^2 dy + \int_0^t\int_0^1 \left(\frac{\lambda}{v}u_{yt} + \left(\frac{\lambda}{v}\right)_v u_y^2 - p_t - \mathbf{b}\cdot\mathbf{b}_t\right) u_{yt}\,dy\,ds \le M,$$

from integration by parts and the initial-boundary condition (2.2). Using Lemmas 3.1 and 3.2 and (3.13), one has the following estimates:

$$\int_0^t\int_0^1 \frac{\lambda}{v}u_{yt}^2\,dy\,ds \ge M_7\int_0^t\int_0^1 u_{yt}^2\,dy\,ds,$$

$$\begin{aligned}\int_0^t\int_0^1 \left(\frac{\lambda}{v}\right)_v u_y^2u_{yt}\,dy\,ds &\le \frac{M_7}{8}\int_0^t\int_0^1 u_{yt}^2\,dy\,ds + M\int_0^t\int_0^1 u_y^4\,dy\,ds\\ &\le \frac{M_7}{8}\int_0^t\int_0^1 u_{yt}^2\,dy\,ds + M,\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\int_0^1 p_tu_{yt}\,dy\,ds \le \int_0^t\int_0^1 \left(|p_v||u_y| + |p_\theta||\theta_t|\right)|u_{yt}|\,dy\,ds\\ &\le \frac{M_7}{8}\int_0^t\int_0^1 u_{yt}^2\,dy\,ds + M\left(1+\Theta^{2+2r}\right)\int_0^t\int_0^1 u_y^2\,dy\,ds\\ &\quad + M\int_0^t\int_0^1 \left(1+\theta^{2r}\right)\theta_t^2\,dy\,ds\\ &\le \frac{M_7}{8}\int_0^t\int_0^1 u_{yt}^2\,dy\,ds + M(X+Y+1)\\ &\le \frac{M_7}{8}\int_0^t\int_0^1 u_{yt}^2\,dy\,ds + M + MZ^{\beta_4}.\end{aligned}$$

From the equation (3.9),

$$|\mathbf{b}_t|^2 \le M\left(|\mathbf{b}|^2u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{b}_{yy}|^2 + |\mathbf{b}_y|^2v_y^2\right),$$

and then, by Lemmas 3.1–3.2,

$$\begin{aligned}&\int_0^t\int_0^1 |\mathbf{b}_t|^2\,dy\,ds \le M\int_0^t\int_0^1 \left(|\mathbf{b}|^2u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{b}_{yy}|^2 + |\mathbf{b}_y|^2v_y^2\right)\,dy\,ds\\ &\le M\max_t\int_0^1 |\mathbf{b}_y|^2dy\int_0^t\int_0^1 u_y^2\,dy\,ds + M\int_0^t\int_0^1 \left(|\mathbf{w}_y|^2 + |\mathbf{b}_{yy}|^2\right)\,dy\,ds\\ &\quad + M\int_0^t \max_y|\mathbf{b}_y|^2\int_0^1 v_y^2\,dy\,ds\\ &\le M + M\int_0^t\int_0^1 \left(|\mathbf{b}_y|^2 + |\mathbf{b}_{yy}|^2\right)\,dy\,ds \le M.\end{aligned} \tag{3.19}$$

Thus, we have

$$\begin{aligned}\int_0^t\int_0^1 u_{yt}\mathbf{b}\cdot\mathbf{b}_t\,dy\,ds &\le \frac{M_7}{8}\int_0^t\int_0^1 u_{yt}^2\,dy\,ds + M\int_0^t\int_0^1 |\mathbf{b}_t|^2\,dy\,ds\\ &\le \frac{M_7}{8}\int_0^t\int_0^1 u_{yt}^2\,dy\,ds + M,\end{aligned}$$

and then

$$\int_0^1 u_t^2 dy + \int_0^t\int_0^1 u_{yt}^2\, dy\, ds \le M + MZ^{\beta_4}.$$

From the equation (2.1b),

$$u_{yy} = \frac{v}{\lambda}\left(u_t - p_y - \mathbf{b}\cdot\mathbf{b}_y - \left(\frac{\lambda}{v}\right)_v v_y u_y\right),$$

and then, by Lemma 3.2 and (3.12),

$$\begin{aligned}
&\int_0^1 u_{yy}^2 dy \le \int_0^1 \left(u_t^2 + p_v^2 v_y^2 + p_\theta^2\theta_y^2 + |\mathbf{b}_y|^2 + v_y^2 u_y^2\right) dy\\
&\le M + MZ^{\beta_4} + M\Theta^{2+2r}\int_0^1 v_y^2 dy + MY + M\max_{y,t} u_y^2 \int_0^1 v_y^2 dy\\
&\le M + MZ^{3/4} + MZ^{\beta_4},
\end{aligned}$$

that is, $Z \le M + MZ^{3/4} + MZ^{\beta_4}$. Therefore $Z \le M$ since the exponents of Z on the right side of the above inequality are both less than one.

Then (3.18) implies $X + Y \le M$ and the lemma follows from (3.11). ∎

We now prove the positive lower bound of the temperature with the aid of a maximum principle.

Lemma 3.4.

$$\begin{aligned}
&|(u, \mathbf{w}, \mathbf{b}, u_y, \mathbf{w}_y, \mathbf{b}_y, v_y)| + \int_0^1 \left(|\mathbf{w}_{yy}|^2 + |\mathbf{b}_{yy}|^2 + \theta_{yy}^2\right) \le M,\\
&M^{-1} \le \theta(y,t) \le M.
\end{aligned} \tag{3.20}$$

Proof. From Lemmas 3.1–3.3,

$$\begin{aligned}
&u^2 + |\mathbf{w}|^2 + |\mathbf{b}|^2\\
&\le M\int_0^1 \left(u^2 + |\mathbf{w}|^2 + |\mathbf{b}|^2\right) dy + M\int_0^1 \left(u_y^2 + |\mathbf{w}_y|^2 + |\mathbf{b}_y|^2\right) dy \le M,
\end{aligned}$$

and $|u_y| \le M$ follows from (3.12). Differentiate (2.1c) with respect to t and then multiply by $\mathbf{w}_t$ and integrate it to obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 |\mathbf{w}_t|^2 dy + \int_0^t\int_0^1 \frac{\mu}{v}|\mathbf{w}_{yt}|^2\, dy\, ds\\
&\le M - \int_0^t\int_0^1 \left(\frac{\mu}{v}\right)_v v_t \mathbf{w}_y\cdot\mathbf{w}_{yt}\, dy\, ds - \int_0^t\int_0^1 \mathbf{b}_t\cdot\mathbf{w}_{yt}\, dy\, ds\\
&\le \frac{1}{2}\int_0^t\int_0^1 \frac{\mu}{v}|\mathbf{w}_{yt}|^2 dyds + M\int_0^t\int_0^1 \left(u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{b}_t|^2\right) dy\, ds\\
&\le \frac{1}{2}\int_0^t\int_0^1 \frac{\mu}{v}|\mathbf{w}_{yt}|^2 dyds + M,
\end{aligned}$$

from integration by parts, the initial-boundary condition (2.2), Lemma 3.2, and (3.19). Then

$$\int_0^1 |\mathbf{w}_t|^2 dy + \int_0^t\int_0^1 \frac{\mu}{v}|\mathbf{w}_{yt}|^2\,dy\,ds \le M.$$

From the equation (2.1c),

$$\mathbf{w}_{yy} = \frac{v}{\mu}\left(\mathbf{w}_t - \mathbf{b}_y - \left(\frac{\mu}{v}\right)_v v_y \mathbf{w}_y\right),$$

and then

$$\begin{aligned}\int_0^1 |\mathbf{w}_{yy}|^2 dy &\le M\int_0^1 \left(|\mathbf{w}_t|^2 + |\mathbf{b}_y|^2 + v_y^2|\mathbf{w}_y|^2\right) dy\\ &\le M + M\max_y|\mathbf{w}_y|^2\int_0^1 v_y^2 dy\\ &\le M + M\int_0^1 |\mathbf{w}_y|^2 dy + \frac{1}{2}\int_0^1 |\mathbf{w}_{yy}|^2 dy,\end{aligned}$$

which implies that

$$\int_0^1 |\mathbf{w}_{yy}|^2 dy \le M + M\int_0^1 |\mathbf{w}_y|^2 dy \le M,$$

and

$$|\mathbf{w}_y|^2 \le M\int_0^1 |\mathbf{w}_y|^2 dy + M\int_0^1 |\mathbf{w}_{yy}^2| dy \le M.$$

Rewrite the equation (3.14) into:

$$e_\theta\theta_t + \theta p_\theta u_y - \frac{\lambda}{v}u_y^2 - \frac{\mu}{v}|\mathbf{w}_y|^2 - \frac{\nu}{v}|\mathbf{b}_y|^2 = \left(\frac{\kappa}{v}\right)_v v_y\theta_y + \frac{\kappa_\theta}{v}\theta_y^2 + \frac{\kappa}{v}\theta_{yy}.$$

Then, by Lemmas 3.2 and 3.3, and an interpolation inequality on θ_y^2,

$$\begin{aligned}&\int_0^t\int_0^1 \frac{\kappa^2}{v^2}\theta_{yy}^2\,dy\,ds\\ &\le M\int_0^t\int_0^1 \left(\theta_t^2 + u_y^2 + u_y^4 + |\mathbf{w}_y|^4 + |\mathbf{b}_y|^4 + v_y^2\theta_y^2 + \theta_y^4\right) dy\,ds\\ &\le M + M\int_0^t \max_y \theta_y^2 \int_0^1 \left(v_y^2 + \theta_y^2\right) dy\,ds\\ &\le M + M\int_0^t\int_0^1 \theta_y^2\,dy\,ds + \frac{1}{2}\int_0^t\int_0^1 \frac{\kappa^2}{v^2}\theta_{yy}^2\,dy\,ds,\end{aligned}$$

which implies

$$\int_0^t\int_0^1 \frac{\kappa^2}{v^2}\theta_{yy}^2\,dy\,ds \le M + M\int_0^t\int_0^1 \theta_y^2\,dy\,ds \le M. \tag{3.21}$$

From (3.4),

$$V(v(y,t))_y = V(v(y,0))_y + u(y,t) - u_0(y) + \int_0^t (p_y + \mathbf{b}\cdot\mathbf{b}_y)dy.$$

Then, using the interpolation inequalities on $|\mathbf{b}_y|^2$ and θ_y^2, Lemma 3.1–3.3, and (3.21), we have

$$\begin{aligned}
v_y^2 &\le M(V(v)_y)^2 \le M + M\int_0^t \left(|\mathbf{b}_y|^2 + p_v^2 v_y^2 + p_\theta^2\theta_y^2\right) ds \\
&\le M + M\int_0^t\int_0^1 \left(|\mathbf{b}_y|^2 + |\mathbf{b}_{yy}|^2 + \theta_y^2 + \theta_{yy}^2\right) dy\, ds + M\int_0^t v_y^2 ds \\
&\le M + M\int_0^t v_y^2 ds,
\end{aligned}$$

which yields, from Gronwall's inequality, $v_y^2 \le M$. Differentiate the equation (2.1d) with respect to t and then multiply by $(v\mathbf{b})_t$ and integrate to obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (v\mathbf{b})_t^2 dy \\
&\le M - \frac{1}{2}\int_0^t\int_0^1 \nu\mathbf{b}_{yt}^2\, dy\, ds \\
&\quad + M\int_0^t\int_0^1 \left(|\mathbf{w}_t|^2 + u_y^2|\mathbf{b}_y|^2 + u_{yy}^2 + v_y^2|\mathbf{b}_t|^2\right) dy\, ds \\
&\le M - \frac{1}{2}\int_0^t\int_0^1 \nu\mathbf{b}_{yt}^2\, dy\, ds + M\int_0^t \max_y |\mathbf{b}_y|^2 \int_0^1 u_y^2\, dy\, ds \\
&\quad + M\int_0^t\int_0^1 |\mathbf{b}_t|^2\, dy\, ds \\
&\le M - \frac{1}{2}\int_0^t\int_0^1 \nu\mathbf{b}_{yt}^2\, dy\, ds + M\int_0^t\int_0^1 \left(|\mathbf{b}_y|^2 + |\mathbf{b}_{yy}|^2\right) dy\, ds \\
&\le M - \frac{1}{2}\int_0^t\int_0^1 \nu\mathbf{b}_{yt}^2\, dy\, ds,
\end{aligned}$$

which yields

$$\int_0^1 (v\mathbf{b})_t^2 dy + \int_0^t\int_0^1 \nu\mathbf{b}_{yt}^2\, dy\, ds \le M.$$

From the equation (2.1d),

$$\begin{aligned}
\int_0^1 |\mathbf{b}_{yy}|^2 dy &\le M\int_0^1 \left((v\mathbf{b})_t^2 + |\mathbf{w}_y|^2 + v_y^2|\mathbf{b}_y|^2\right) dy \\
&\le M + M\int_0^1 |\mathbf{b}_y|^2 dy \le M,
\end{aligned}$$

and then

$$|\mathbf{b}_y|^2 \leq M \int_0^1 |\mathbf{b}_y|^2 dy + M \int_0^1 |\mathbf{b}_{yy}|^2 dy \leq M.$$

As a consequence, we have $M^{-1} \leq \theta \leq M$ from the maximum principle (see [27]) applied to (3.14) and the boundedness of $(v_y, u_y, \mathbf{w}_y, \mathbf{b}_y)$, $0 < \theta \leq M$ (from (3.13)), and the positive lower bound of θ_0.

If the initial data is Hölder continuous, following the standard method and argument in [14, 19, 22] and the Schauder estimates (cf. [20, 11]), we obtain that, for any $T \in (0, \infty)$,

$$\|v\|_{C^{1+\alpha,1+\alpha/2}([0,1]\times[0,T])} \leq M, \quad \|(u, \mathbf{w}, \mathbf{b}, \theta)\|_{C^{2+\alpha,1+\alpha/2}([0,1]\times[0,T])} \leq M,$$

from the above apriori estimates. To achieve these estimates, we differentiate the equations in (2.1) and make the energy estimates. Using the embedding theorem and the estimates of parabolic equations in the class of Hölder functions, we obtain the desired Hölder continuity of the solutions indicated in the theorem (see [14, 19, 22] for the details). This completes the proof of Theorem 2.1. ∎

4. WEAK SOLUTIONS WITH LARGE DISCONTINUOUS INITIAL DATA

Before we prove Theorem 2.2, we first introduce a well-known lemma (see [28, 29]).

LEMMA 4.1. *Let X_0, X, and X_1 be Banach spaces such that*

$$X_0 \subset X \subset X_1,$$

where the injections are continuous and

(1) the injection $X_0 \to X$ is compact;

(2) $X_j, j = 0, 1$, are reflexive.

Let $T > 0$ and $\alpha_0, \alpha_1 > 1$ are constants. Consider the space

$$Y \equiv \{v \in L^{\alpha_0}(0, T; X_0), \frac{dv}{dt} \in L^{\alpha_1}(0, T; X_1)\}$$

under the norm

$$\|v\|_Y = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|\frac{dv}{dt}\|_{L^{\alpha_1}(0,T;X_1)}.$$

Then the injection of Y into $L^{\alpha_0}(0, T; X)$ is compact.

Proof of Theorem 2.2. For the discontinuous initial data, we first approximate them by

$$(v_0^\epsilon, u_0^\epsilon, \mathbf{w}_0^\epsilon, \mathbf{b}_0^\epsilon, \theta_0^\epsilon) \in C^\infty(0, 1),$$

via mollification such that

$$C_0^{-1} \leq v_0^\epsilon(y) \leq C_0, \quad \theta_0^\epsilon(y) \geq M_{0\epsilon}^{-1}, \quad \|(u_0^\epsilon, \mathbf{w}_0^\epsilon, \mathbf{b}_0^\epsilon)\|_{L^4} + \|\theta_0^\epsilon\|_{L^2} \leq C_0,$$

where $C_0 > 0$ is independent of ϵ. Theorem 2.1 implies that there exist global smooth solutions $(v^\epsilon, u^\epsilon, \mathbf{w}^\epsilon, \mathbf{b}^\epsilon, \theta^\epsilon)(y,t)$, and Lemmas 3.1–3.4 imply that there exists $C > 0$ independent of ϵ and $M_\epsilon > 0$ such that

$$\begin{aligned} &C^{-1} \leq 1/v^\epsilon(y,t) \leq C, \quad \theta^\epsilon(y,t) \geq M_\epsilon^{-1}, \\ &\|(u^\epsilon, \mathbf{w}^\epsilon, \mathbf{b}^\epsilon)\|_{L^2(0,T;L^4\cap H^1)} + \|\theta^\epsilon\|_{L^2(0,T;L^2\cap H^1)} \leq C, \end{aligned} \tag{4.1}$$

where $C > 0$ is independent of ϵ.

Take $X_0 = H_0^1(\Omega), X = L^4(\Omega), X_1 = H^{-1}(\Omega)$ for $(u, \mathbf{w}, \mathbf{b})$, and $X_0 = H_0^1(\Omega), X = L^2(\Omega), X_1 = H^{-1}(\Omega)$ for θ, and $\alpha_0, \alpha_1 = 2$. Then Lemma 4.1 and (4.1) imply that there exists a subsequence (still denoted by) $(v^\epsilon, u^\epsilon, \mathbf{w}^\epsilon, \mathbf{b}^\epsilon, \theta^\epsilon)$ such that

$$\begin{aligned} &(u^\epsilon, \mathbf{w}^\epsilon, \mathbf{b}^\epsilon) \to (u, \mathbf{w}, \mathbf{b}) \quad \text{in } L^2(0,T;L^4); \quad \theta^\epsilon \to \theta \geq 0 \quad \text{in } L^2(0,T;L^2); \\ &1/v^\epsilon \overset{*}{\rightharpoonup} 1/v \quad \text{in } L^\infty; \quad (u^\epsilon, \mathbf{w}^\epsilon, \mathbf{b}^\epsilon, \theta^\epsilon) \rightharpoonup (u, \mathbf{w}, \mathbf{b}, \theta) \quad \text{in } L^2(0,T;H^1). \end{aligned}$$

Then it is easy to check that the limit function $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$ is a weak solution satisfying the equations in (2.1) in the sense of distributions such that $C^{-1} \leq v \leq C$, $\theta \geq 0$, $(u, \mathbf{w}, \mathbf{b}) \in L^2(0,T;L^4 \cap H^1)$, $\theta \in L^2(0,T;L^2 \cap H^1)$. This completes the proof of Theorem 2.2.

5. APPENDIX: EQUATIONS OF MAGNETOHYDRODYNAMICS

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids (cf. gases) in an electromagnetic field with a very broad range of applications. The dynamic motion of the fluids and the magnetic field interacts strongly on each other. We must consider both the hydrodynamic and electrodynamic effects. The equations of magnetohydrodynamic flows have the following form ([4, 21, 23]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \qquad x \in \mathbf{R}^3, \quad t > 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div}(\lambda'(\operatorname{div}\mathbf{u})\mathbf{I}) \\ \qquad\qquad\qquad\qquad +\operatorname{div}(\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)), \\ \mathcal{E}_t + \operatorname{div}(\mathbf{u}(\mathcal{E} + p)) = \operatorname{div}((\mathbf{u} \times \mathbf{H}) \times \mathbf{H} + \nu\mathbf{H} \times (\nabla \times \mathbf{H})) \\ \qquad\qquad +\operatorname{div}(\mathbf{u}(\lambda'(\operatorname{div}\mathbf{u})\mathbf{I} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top))) + \operatorname{div}(\kappa\nabla\theta), \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu\nabla \times \mathbf{H}), \qquad \operatorname{div}\mathbf{H} = 0, \end{cases} \tag{5.1}$$

where ρ is the density, $\mathbf{u} \in \mathbf{R}^3$ the velocity, $\mathbf{H} \in \mathbf{R}^3$ the magnetic field, and θ the temperature; the total energy is

$$\mathcal{E} = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) + \frac{1}{2}|\mathbf{H}|^2,$$

with e being the internal energy and $\frac{1}{2}|\mathbf{H}|^2$ the magnetic energy; the equations of state

$$p = p(\rho, \theta), \qquad e = e(\rho, \theta)$$

relate the pressure p and the internal energy e with the density and temperature of the flow; $\mathbf{I}$ is the 3×3 identity matrix, and $(\nabla \mathbf{u})^\top$ is the transpose of the matrix $\nabla \mathbf{u}$; $\lambda' = \lambda'(\rho, \theta)$ and $\mu = \mu(\rho, \theta)$ are the viscosity coefficients of the flow satisfying $\lambda' + 2\mu > 0$, $\nu = \nu(\rho, \theta)$ is the magnetic diffusivity (see [1]) acting as a magnetic diffusion coefficient of the magnetic field, $\kappa = \kappa(\rho, \theta)$ is the heat conductivity, and all these kinetic coefficients and the magnetic diffusivity are independent of the magnitude and direction of the magnetic field (see [23]). The magnetic permeability differs only slightly from unity (the difference is not important in magnetohydrodynamics) and therefore is taken to be 1, which does not appear in the equations. The fluid under consideration is a Newtonian fluid, i.e., the stress tensor

$$-p\mathbf{I} + \big(\mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) + \lambda'(\mathrm{div}\mathbf{u})\mathbf{I}\big)$$

is a linear function of the deformation tensor $\frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$. The viscosity and heat conduction terms describe the dissipative processes in magnetohydrodynamics.

Equations (5.1) of magnetohydrodynamics are based on neglecting the displacement current in Maxwell's equation. As a consequence, the last equation in (5.1)

$$\mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}) \tag{5.2}$$

is called the induction equation. The Maxwell equations governing the electromagnetic fields are:

$$\varepsilon_0 \mathbf{E}_t - \nabla \times \mathbf{H} + \mathbf{J} = 0, \tag{5.3}$$

$$\mathbf{H}_t + \nabla \times \mathbf{E} = 0, \qquad \mathrm{div}\, \mathbf{H} = 0, \tag{5.4}$$

where ε_0 is the dielectric constant, and the current density $\mathbf{J}$ is given by Oham's law:

$$\mathbf{J} = \frac{1}{\nu}(\mathbf{E} + \mathbf{u} \times \mathbf{H}).$$

In magnetohydrodynamics, the displacement current $\mathbf{E}_t$ can be neglected (see [21, 23]), that is, we can set $\varepsilon_0 \mathbf{E}_t = 0$ in (5.3), and then $\mathbf{J} = \nabla \times \mathbf{H}$. Therefore, the electric field can be written in terms of the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$,

$$\mathbf{E} = \nu \nabla \times \mathbf{H} - \mathbf{u} \times \mathbf{H}. \tag{5.5}$$

Then (5.2) is obtained by substituting (5.5) into (5.4). We see now, although the electric field $\mathbf{E}$ does not appear in (5.1), that it is indeed induced according to (5.5) by the moving conductive flow in the magnetic field.

Consider a three-dimensional MHD flow with spatial variables $\mathbf{x} = (x, x_2, x_3)$ which is moving in the x direction and uniform in the transverse direction (x_2, x_3):

$$\begin{aligned} &\rho = \rho(x,t),\, \theta = \theta(x,t); \quad \mathbf{u} = (u, \mathbf{w})(x,t),\, \mathbf{w} = (u_2, u_3); \\ &\mathbf{H} = (b_1, \mathbf{b})(x,t),\, \mathbf{b} = (b_2, b_3), \end{aligned} \tag{5.6}$$

where u and b_1 are the longitudinal velocity and longitudinal magnetic field, respectively; $\mathbf{w}$ and $\mathbf{b}$ are the transverse velocity and transverse magnetic field, respectively. The magnetohydrodynamic flow satisfies (5.1). With this special structure (5.6), equations (5.1) are reduced to (1.1) with the constant longitudinal magnetic field b_1 and $\lambda = \lambda' + 2\mu > 0$. We take $b_1 = 1$. The term $\frac{1}{2}|\mathbf{b}|^2$ appears in the second equation of (1.1) because of this equality: $(\nabla \times \mathbf{H}) \times \mathbf{H} = (-\frac{1}{2}|\mathbf{b}|^2, \mathbf{b})_x^\top$. System (1.1) can be verified by direct calculations.

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REFERENCES

[1] R. Balescu, "Transport Processes in Plasmas I: Classical Transport Theory", North-Holland, New York, 1988.

[2] E. Becker, "Gasdynamik", Teubner, Stuttgart, 1966.

[3] M. Brio and C. C. Wu, An upwind differencing scheme for the equations of magnetohydrodynamics, *J. Comput. Phys.* 75 (1988). 400-422.

[4] H. Cabannes, "Theoretical Magnetofluiddynamics", Academic Press, New York, 1970.

[5] G.-Q. Chen, D. Hoff, and K. Trivisa, Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data, *Commun. Partial Diff. Eqs.* 25 (2000), 2233–2257.

[6] P. C. Clemmow and J. P. Dougherty, "Electrodynamics of Particles and Plasmas", Addison-Wesley, New York, 1990.

[7] C. M. Dafermos, Global smooth solutions to the initial-boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity, *SIAM J. Math. Anal.* 13 (1982), 397-408.

[8] C. M. Dafermos and L. Hsiao, Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity, *Nonlinear Anal., Theory Methods Appl.* 6 (1982), 435-454.

[9] H. Freistühler and P. Szmolyan, Existence and bifurcation of viscous profiles for all intermediate magnetohydrodynamic shock waves, *SIAM J. Math. Anal.* 26 (1995), 112-128.

[10] A. Friedman, "Partial Differential Equations", Krieger, New York, 1976.

[11] D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order", Springer, Berlin, 1994.

[12] D. Hoff, Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data, *J. Diff. Eqs.* 95 (1992), 33-74.

[13] L. Hsiao and T. Luo, Large-time behavior of solutions for the outer pressure problem of a viscous heat-conductive one-dimensional real gas, *Proc. Royal Soc. Edin. Sec A* 126 (1996), 1277-1296.

[14] Y. I. Kanel, On a model system of equations of one-dimensional gas motion, *Differential Equations*, 4 (1968), 374-380.

[15] S. Kawashima and M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, *Proc. Japan Acad. Ser. A, Math. Sci.* 58 (1982), 384–387.

[16] S. Kawashima and Y. Shizuta, Magnetohydrodynamic approximation of the complete equations for an electromagnetic fluid, *Tsukuba J. Math.* 10 (1986), 131-149.

[17] B. Kawohl, Global existence of large solutions to initial boundary value problems for a viscous, heat-conducting, one-dimensional real gas, *J. Diff. Eqs.* 58 (1985), 76-103.

[18] V. Kazhikhov, Cauchy problem for viscous gas equations, *Siberian Math. J.* 23 (1982), 44-49.

[19] V. Kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of initial-boundary-value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* 41 (1977), 273-282.

[20] S. Kruzhkov, A priori estimates for the derivative of a solution to a parabolic equation and some of its applications, *Dokl. Akad. Nauk. SSSR*, 170 (1966), 501-504.

[21] A. G. Kulikovskiy and G. A. Lyubimov, "Magnetohydrodynamics", Addison-Wesley, Reading, Massachusetts, 1965.

[22] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs 23, American Mathematical Society, Providence, Rhode Island, 1988.

[23] L. D. Laudau and E. M. Lifshitz, "Electrodynamics of Continuous Media", 2nd ed., Pergamon, New York, 1984.

[24] R. J. LeVeque, "Nonlinear Conservation Laws and Finite Volume Methods for Astrophysical Fluid Flow", 27th Sass-Fee Advanced Course Lecture Notes, 1998.

[25] T.-P. Liu and Y. Zeng, "Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws", Memoirs Amer. Math. Soc. 599, 1997.

[26] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, *Bull. Soc. Math. France* 90 (1962), 487-497.

[27] M. H. Protter and H. F. Weinberger, "Maximum Principles in Differential Equations", Prentice-Hall, New Jersey, 1967.

[28] L. Tartar, "Topics in Nonlinear Analysis", Publications Mathematiques d'Orsay 78, 13, Université de Paris-Sud, Dipartement de Mathimatique, Orsay, 1978.

[29] R. Temam, "Navier-Stokes Equations: Theory and Numerical Analysis", 3rd Edition, Elsevier Science Publishers, B. V.: Amsterdam, 1984.

[30] L. C. Woods, "Principles of Magnetoplasma Dynamics", Oxford University Press, New York, 1987.

[31] C. C. Wu, Formation, structure, and stability of MHD intermediate shocks, *J. Geophys. Res.* 95 (1990), 8149-8175.

[32] Y. B. Zel'dovich and Y. P. Raizer, "Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena", Vol 2, Academic Press, New York, 1967.