GLOBAL SOLUTIONS TO THE NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE HEAT-CONDUCTING FLOW WITH SYMMETRY AND FREE BOUNDARY

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Abstract

Global solutions of the multidimensional Navier-Stokes equations for compressible heat-conducting flow are constructed, with spherically symmetric initial data of large oscillation between a static solid core and a free boundary connected to a surrounding vacuum state. The free boundary connects the compressible heat-conducting fluids to the vacuum state with free normal stress and zero normal heat flux. The fluids are initially assumed to fill with a finite volume and zero density at the free boundary, and with bounded positive density and temperature between the solid core and the initial position of the free boundary. One of the main features of this problem is the singularity of solutions near the free boundary. Our approach is to combine an effective difference scheme to construct approximate solutions with the energy methods and the pointwise estimate techniques to deal with the singularity of solutions near the free boundary and to obtain the bounded estimates of the solutions and the free boundary as time evolves. The convergence of the difference scheme is established. It is also proved that no vacuum develops between the solid core and the free boundary, and the free boundary expands with finite speed.

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1. Introduction

The Navier-Stokes equations for compressible heat-conducting flow in $\mathbb{R}^n\,(n\geq 2)$ read:

$$
\begin{cases}
\partial_t\rho+div(\rho\mathbf{v})=0,\\
\partial_t(\rho\mathbf{v})+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})+\nabla p=\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla(div\,\mathbf{v}),\\
\partial_t(\rho(e+\frac{|\mathbf{v}|^2}{2}))+div(\mathbf{v}(\rho(e+\frac{|\mathbf{v}|^2}{2})+p))\\
\qquad=\Delta(\kappa e+\frac{\mu}{2}|\mathbf{v}|^2)+\lambda\,div((div\,\mathbf{v})\mathbf{v})+\mu\,div((\nabla\mathbf{v})\mathbf{v}).
\end{cases}
\tag{1.1}
$$

Here ρ, p, e, and $\mathbf{v}=(v_1,\cdots,v_n)$ are the density, the pressure, the internal energy, and the velocity, respectively; λ and μ are the constant viscosity coefficients, $\mu>0, \lambda+2\mu/n\geq 0$; for polytropic ideal fluids,

$$p=R\rho e,\quad e=c_v\theta,$$

where $c_v>0$ is the heat capacity, and $R>0$ is the ratio of the ideal fluid constant over the heat capacity; and $\kappa>0$ is the ratio of the heat conductivity coefficient over the heat capacity.

In this paper we focus on the following initial-boundary value problem for compressible heat-conducting fluids between a static solid core and a free boundary connected to a surrounding vacuum state.

At $t=0$:

$$
\begin{aligned}
&(\rho,\mathbf{v},e)|_{t=0}=(\rho_0,\mathbf{v}_0,e_0)(\mathbf{x})=(\rho_0(r),v_0(r)\frac{\mathbf{x}}{r},e_0(r)),\qquad 1\leq r\leq r_0,\\
&(\rho_0,v_0,e_0)(r)\in W^{1,2}(0,r_0),\quad v_0(0)=0,\\
&0<\rho_0(r)\leq C_0,\quad e_0(r)\geq C_0^{-1},\qquad 1<r<r_0,
\end{aligned}
\tag{1.2}
$$

where $r=|\mathbf{x}|=\sqrt{x_1^2+\cdots+x_n^2}$.

On the fixed boundary $r=1$:

$$\mathbf{v}|_{r=1}=0,\quad \partial_r e|_{r=1}=0. \tag{1.3}$$

On the free boundary $r=r(t)$:

$$\partial_r e|_{r=r(t)}=0,\quad \sigma|_{r=r(t)}=0,\quad \rho_0|_{r=r_0}=0, \tag{1.4}$$

which models the surrounding vacuum state, where σ is the normal stress on the free boundary $r=r(t)$.

The initial-boundary conditions (1.2)–(1.4) allow that the free boundary connects the compressible heat-conducting fluids to the vacuum state with free normal stress and zero normal heat flux, which represent a natural physical situation that the fluids are free to expand into the vacuum region without normal resistance from the vacuum. The fluids are initially assumed to fill with a finite volume and zero density at the free boundary, and with bounded positive density and temperature (equivalently, internal energy) between the solid core and the initial position of the free boundary. One of the main analytical difficulties to handle this problem is the singularity of solutions near the free boundary. For physical significance and mathematical interests for such free boundaries, we refer to Nishida [26, 27], Antontsev-Kazhikhov-Monakhov [2], and the references cited therein.

Unlike the stability of solutions with vacuum states for the compressible Euler equations (see [4]), solutions with the vacuum states for the compressible Navier-Stokes equations are not stable in general (see Hoff-Serre [11]). Hence it is essential to estimate first a relative positive lower bound of the density between the solid core and the free boundary. Another physical interesting question is whether the free boundary expands with finite speed. The positive answer to these two questions is also essential for establishing the existence theorem of global solutions without the vacuum states between the solid core and the free boundary for this problem.

In this paper we establish the existence of global solutions of the initial-boundary value problem (1.1)–(1.4) for the Navier-Stokes equations for compressible heat-conducting flow with large spherically symmetric initial data between the solid core and the free boundary connected to the surrounding vacuum state. We also show that no vacuum develops between the solid core and the free boundary, and the free boundary expands with finite speed. Our approach is to combine an effective difference scheme to construct approximate solutions with the energy methods and the pointwise estimate techniques to deal with the singularity of solutions near the free boundary and to obtain the bounded estimates of the solutions and the free boundary as time evolves. One advantage of this approach is that the local existence theorem is not needed to obtain the global solutions, since the known local existence theorems of (1.1) do not apply directly for this problem (cf. [26, 27]). Another advantage of this approach is to allow the continuity of the density across the free boundary, that is, the density vanishes on the free boundary, even for spherically symmetric non-isentropic fluids.

For the one-dimensional or spherically symmetric solutions of the Navier-Stokes equations for *isentropic* compressible flow with a free boundary connected to a vacuum state, see Nishida [26, 27], Okada [28], Okada-Makino [29], MatusuNecasova-Okada-Makino [23], and the references cited therein. For related free boundary problems for isentropic fluids with density-dependent viscosity, see Liu-Xin-Yang [19], Luo-Xin-Yang [20], and the earlier references cited therein. Also see Lions [18] and Vaigant-Kazhikhov [32] for related references for multidimensional isentropic fluids.

For the Cauchy problem or the initial-boundary value problem for one-dimensional Navier-Stokes equations for compressible heat-conducting flow with strictly positive initial density and temperature, see Kazhikov [16], Kazhikov-Shelukhin [17], Antontsev-Kazhikhov-Monakhov [2], Nagasawa [24], Kawashima-Nishida [14], and references cited therein for smooth initial data of large oscillation; and Amosov-Zlotnick [1], Chen-Hoff-Trivisa [6], and the earlier references cited therein for discontinuous initial data of large oscillation. Also see Nikolaev [25] and Jiang [12] for spherically symmetric solutions with smooth initial data of large oscillation outside the solid core. Yashima and Benabidallah [37] dealt with nonnegative initial density and temperature whose boundary data are prescribed so that the uniform boundedness of the total energy of solutions is direct to achieve. For multidimensional Navier-Stokes equations with small initial data, see Matsumura-Nishida [21, 22] and Valli-Zajaczkowski [34] for smooth initial data and Hoff [10] for discontinuous initial data. As far as we are aware, the present paper is the first to handle the free-boundary problem with free normal stress on the free boundary for compressible heat-conducting flow with initial data of large oscillation, which is a natural physical situation for the fluids with the surrounding vacuum state. The convergence of an effective numerical scheme for this free boundary problem is also established.

In Section 2, we reformulate the initial-boundary value problem (1.1)–(1.4) in Lagrangian coordinates and state our Main Theorem.

Since the initial density vanishes at the free boundary to allow the continuity of the density across the boundary so that the known local existence theorems do not apply directly. In Section 3, we introduce a space-discrete difference scheme to construct approximate solutions to handle the singularity of solutions near the free boundary. An appropriate choice of a difference scheme is crucial for the fulfillment of the strategy (cf. Nishida [26]). Two basic lemmas are introduced in order to make the energy estimates and the pointwise estimates for the approximate solutions in Sections 4–8.

In Section 4, we make some key energy estimates and complete the global construction of the difference scheme and approximate solutions, in which the main difficulty is reduced to estimate the boundedness of the free boundary and its propagation speed. Some new techniques to achieve this are developed. Then, in Section 5, the relative lower and upper bounds of the density are obtained, which state that, for any fixed time, the lower and upper bounds of the density at any point in the region under consideration are controlled by the corresponding values of the initial density at that point in Lagrangian coordinates. These estimates can be achieved by combining some new techniques with an idea of Kazhikov-Shelukhin [17] for this discrete scheme, with the aid of the estimates in Sections 3–4, which are essential for the proof of our Main Theorem.

In Sections 6–8, some pointwise estimates for the velocity and the temperature, and some higher-order energy estimates for the difference scheme are made. We also make some boundary estimates for the approximate solutions. The main difficulty here is the singularity of solutions so that these estimates should be made very carefully with some new techniques.

In Sections 9–10, with the aid of the energy estimates and the pointwise estimates in Sections 4–8, we prove the strong convergence of the difference scheme and show that the limit of the approximate solutions is a global solution of the initial-boundary value problem of the Navier-Stokes equations for compressible heat-conducting flow. It is also proved that no vacuum develops between the solid core and the free boundary, and the free boundary expands with finite speed.

It would be interesting to study further the properties of global solutions of problem (1.1)–(1.4) and to analyze more general fluids such as the physical fluids with various constitutive equations of state and forms of heat conductivity and viscosity coefficients that may vary with temperature and density.

2. Reformulation and Main Theorem

For the initial-boundary value problem (1.1)–(1.4), we look for global spherically symmetric solutions $(\rho, \mathbf{v}, e)$:

$$\rho(\mathbf{x},t) = \rho(r,t), \quad \mathbf{v}(\mathbf{x},t) = v(r,t)\frac{\mathbf{x}}{r}, \quad e(\mathbf{x},t) = e(r,t). \tag{2.1}$$

Then $(\rho, v, e)(r,t)$ is determined by

$$\begin{cases} \partial_t\rho + \partial_r(\rho v) + \frac{n-1}{r}\rho v = 0, \\ \rho\left(\partial_t v + v\partial_r v\right) + \partial_r p = (\lambda+2\mu)\left(\partial_{rr}^2 v + \frac{n-1}{r}\partial_r v - \frac{n-1}{r^2}v\right), \\ \rho\left(\partial_t e + v\partial_r e\right) + p\left(\partial_r v + \frac{n-1}{r}v\right) \\ \quad = \kappa\left(\partial_{rr}^2 e + \frac{n-1}{r}\partial_r e\right) + \lambda\left(\partial_r v + \frac{n-1}{r}v\right)^2 + 2\mu\left((\partial_r v)^2 + \frac{n-1}{r^2}v^2\right). \end{cases} \tag{2.2}$$

To handle this problem, it is convenient to reduce the problem in Eulerian coordinates (r,t) to the problem in Lagrangian coordinates (x,t) moving with the fluid, via the transformation:

$$x = \int_1^{r(x,t)} s^{n-1}\rho(s,t)ds \quad \text{or,} \quad r(x,t) = r(x,0) + \int_0^t v(r(x,\tau),\tau)d\tau. \tag{2.3}$$

It is easy to check that $x = \int_1^{r(x,0)} s^{n-1}\rho_0(s)ds$.

Without loss of generality, we assume that $\int_1^a s^{n-1}\rho_0(s)dx = 1$, so that the region $\{(r,t)\,|\,1 \le r \le r(t), 0 \le t \le T\}$ under consideration is transformed into the region $\{(x,t)\,|\,0 \le x \le 1, 0 \le t \le T\}$.

With this change of coordinates, we can rewrite system (2.2) as the following system:

$$\begin{cases} \partial_t\rho = -\rho^2\partial_x(r^{n-1}v), \\ \partial_t v = r^{n-1}\partial_x\sigma, \\ \partial_t e = \kappa\partial_x\left(\rho r^{2n-2}\partial_x e\right) + \sigma\partial_x(r^{n-1}v) - 2\mu(n-1)\partial_x(r^{n-2}v^2), \end{cases} \tag{2.4}$$

where

$$\sigma = (\lambda+2\mu)\rho\partial_x(r^{n-1}v) - p. \tag{2.5}$$

The initial conditions can be written in the form

$$(\rho, v, e)|_{t=0} = (\rho_0, v_0, e_0)(x), \qquad 0 \le x \le 1. \tag{2.6}$$

The free boundary conditions and the solid boundary conditions then are

$$v(0,t) = 0, \quad (\rho\partial_x e)(0,t) = (\rho\partial_x e)(1,t) = 0, \quad \sigma(1,t) = 0. \tag{2.7}$$

To articulate the assumption that there is no initial cavity with the fluid residing in the bounded region, we assume that there exist a decreasing non-negative function $\Lambda(x)$, $\Lambda(1)=0$, $\int_0^1 \Lambda^{-1}(x)dx<\infty$, and a constant $C_0>0$ such that

$$C_0^{-1}\Lambda(x)\le \rho_0(x)\le C_0\Lambda(x) \tag{2.8}$$

holds uniformly for all $x\in[0,1]$. Then we have

Theorem 2.1 (Main Theorem)**.** *Suppose that initial data $(\rho_0,v_0,e_0)(x)\in W^{1,2}(0,1)$ in (2.6) satisfy (2.7)–(2.8) and*

$$v_0(0)=0, \qquad e_0(x)\ge C_0^{-1}. \tag{2.9}$$

Then the initial-boundary value problem (2.4)–(2.7) has a global solution $(\rho,v,e)(x,t)$ such that, for any $T\in(0,\infty)$, there exists $C>0$ so that, for any $x,y\in[0,1], t,s\in[0,T]$,

$$\begin{cases} C^{-1}\rho_0(x)\le\rho(x,t)\le C\rho_0(x),\ e(x,t)\ge C^{-1}>0,\\ |\rho(x,\cdot)-\rho(y,\cdot)|\le C\sqrt{|x-y|}, \quad |\rho(\cdot,t)-\rho(\cdot,s)|\le C\sqrt{|t-s|},\\ \rho\in C([0,T];L^2(0,1))\cap L^\infty([0,T];W^{1,2}(0,1)),\\ v\in L^\infty\cap W^{1,2}_{loc}([0,1)\times[0,T]),\ e\in L^2([0,1]\times[0,T]),\\ (\sqrt{\rho}e_x,\partial_t v,\sqrt{\rho}\partial_x v,\sqrt{\rho}\partial_x\sigma,\rho^2\partial_{xx}v)\in L^2((0,1)\times[0,T]),\\ \sqrt{\rho}\partial_x(r^{n-1}v)\in L^1([0,T];L^\infty(0,1))\cap L^\infty([0,T];L^2(0,1))\cap L^4([0,T];L^4(0,1)). \end{cases}$$

Remark 2.1. *It is easy to check by following our approach that Theorem 2.1 also holds, provided that condition (2.8) is replaced by*

$$C_0^{-1}\le\rho_0(x)\le C_0, \qquad x\in[0,1].$$

In this case, the density $\rho(x,t)$ in the global solution jumps across the free boundary.

3. Difference Scheme and Approximate Solutions

We now introduce a space-discrete difference scheme to handle the singularity of solutions near the free boundary.

Let h be an increment in x such that $Nh=1$ for $N\in\mathbb{Z}_+$, $x_j=jh$ for $j\in\{0,1,\cdots,N\}$. For each integer N, we construct the following time-dependent functions:

$$(\rho_j(t),v_j(t),e_j(t)), \quad j=0,\cdots,N, \tag{3.1}$$

that form a discrete approximation to $(\rho(x_j,t),v(x_j,t),e(x_j,t))$ for $j=0,\cdots,N$.

First, $(\rho_i(t),v_j(t),e_j(t)), i=0,\cdots,N-1, j=1,\cdots,N-1$, are determined by the following system of $3N-2$ differential equations:

$$\dot\rho_i=-\rho_i^2\delta(r_i^{n-1}v_i), \tag{3.2}$$

$$\dot v_j=r_j^{n-1}\delta\sigma_j, \tag{3.3}$$

$$\dot e_j=\kappa\delta(\rho_{j-1}r_j^{2n-2}\delta e_{j-1})+\sigma_j\delta(r_{j-1}^{n-2}v_{j-1})-2\mu(n-1)\delta(r_{j-1}^{n-2}v_{j-1}^2), \tag{3.4}$$

where δ is the operator defined by $\delta w_j = \frac{w_{j+1}-w_j}{h}$, and

$$\sigma_j(t) = (\lambda + 2\mu)\rho_{j-1}\delta(r_{j-1}^{n-1}v_{j-1}) - p(\rho_{j-1}, e_j), \tag{3.5}$$

$$r_0 = 1, \quad r_j^n(t) = 1 + n\sum_{i=0}^{j-1}\frac{h}{\rho_i}, \quad j = 1, \cdots, N. \tag{3.6}$$

The corresponding initial data are defined by

(3.7)

$$(\rho_i, v_j, e_j)(0) = (\frac{1}{h}\int_{ih}^{(i+1)h}\rho_0(x)dx,\ \frac{1}{h}\int_{(j-1)h}^{jh}v_0(x)dx,\ \frac{1}{h}\int_{(j-1)h}^{jh}e_0(x)dx),$$
$$i = 0, \cdots, N-2; j = 1, \cdots, N-1,$$

(3.8)

$$\rho_{N-1}(0) = \rho_{N-2}(0).$$

Then the initial data $(\rho_j, v_j, e_j)(0)$ satisfy

$$C_0^{-1}\Lambda(x_{j+1}) \le \rho_j(0) \le C_0\Lambda(x_j), \qquad e_j(0) \ge C_0^{-1}, \tag{3.9}$$

$$\sum_{j=0}^{N-1}(|(\rho_j, v_j, e_j)|^2 + |(\delta\rho_j, \delta v_j, \delta e_j)|^2)(0)h \le C_1, \tag{3.10}$$

which also implies $\sum_{j=0}^{N-1} v_j^4(0)h \le C_1$, where $C_1 > 0$ is independent of N.

These are coupled with additional five boundary conditions:

$$\begin{aligned} &\rho_N(t) = \rho_{N-1}(t), \qquad v_0(t) = 0, \\ &e_0(t) = e_1(t), \quad e_N(t) = e_{N-1}(t), \quad \sigma_N(t) = 0. \end{aligned} \tag{3.11}$$

These boundary conditions are consistent with the initial data. The condition $\sigma_N(t) = 0$ determines $v_N(t)$.

The basic theory of differential equations guarantees the local existence of smooth solutions $(\rho_i, v_j, e_j), i = 0, \cdots, N-1; j = 1, \cdots, N-1$, to the Cauchy problem (3.2)–(3.10) on some time interval $(0, T)$ such that

$$0 < \rho_j(t), e_j(t) < \infty, \quad |v_j(t)| < \infty, \qquad j = 0, 1, \cdots, N-1;\ 0 \le t < T,$$

with the aid of (3.9)–(3.10).

Let $[0, T_{max})$ be the maximal time interval on which the smooth solutions exist. Our first goal is to show that the solutions are actually globally defined in $[0, \infty)$, that is, $T_{max} = \infty$. We will achieve this by showing, for fixed $h > 0$, the boundedness of the density, the velocity, and the temperature, as well as the lower boundedness of the density and the temperature away from zero in Section 4.

In this way, we can globally define $(\rho_j, v_j, e_j), j = 0, 1, \cdots, N$.

With $(\rho_j, v_j, e_j), j = 0, \cdots, N$, we can define our approximate solutions $(\rho^N, v^N, e^N)(x, t)$ for the Cauchy problem (2.4)–(2.7). For each fixed N and

$t \geq 0$, we define piecewise linear continuous functions $(\rho^N, v^N, e^N)(x,t)$ and $r^N(x,t)$ with respect to x as follows: When $x \in ([xN], [xN]+1]$,

(3.12) $\rho^N(x,t) \equiv \rho_{[xN]}(t) + (xN - [xN])(\rho_{[xN]+1}(t) - \rho_{[xN]}(t)),$

(3.13) $v^N(x,t) \equiv v_{[xN]}(t) + (xN - [xN])(v_{[xN]+1}(t) - v_{[xN]}(t)),$

(3.14) $e^N(x,t) \equiv e_{[xN]}(t) + (xN - [xN])(e_{[xN]+1}(t) - e_{[xN]}(t)),$

(3.15) $r^N(x,t) = \left\{(r_{[xN]}(t))^n + (xN - [xN])((r_{[xN]+1}(t))^n - (r_{[xN]}(t))^n)\right)^{1/n}.$

We also introduce the corresponding step functions:

(3.16)
$$(\rho_h, v_h, e_h, r_h)(x,t) = (\rho_{[xN]}, v_{[xN]}, e_{[xN]}, r_{[xN]})(t), \quad x \in ([xN], [xN]+1].$$

Throughout this paper, we denote $C > 0$ a generic constant independent of N (i.e. h). Before we make energy estimates, we first introduce two lemmas.

Lemma 3.1. *There exist $C > 0$ and $a(\alpha) \to 0$, when $\alpha \to 0$, such that*

$$\sum_{j=0}^{N-1} \frac{h}{\rho_j(0)} \leq C, \qquad \sum_{i=j}^{N-1} \frac{h}{\rho_i(0)} \leq a(1 - j/N). \tag{3.17}$$

Since $\rho_{N-1} = \rho_{N-2}$, this can be seen easily as follows:

$$\sum_{j=0}^{N-1} \frac{h}{\rho_j(0)} = \sum_{j=0}^{N-2} \left(\frac{1}{h}\int_{jh}^{(j+1)h} \rho_0(x)dx\right)^{-1} h + \left(\frac{1}{h}\int_{(N-2)h}^{(N-1)h} \rho_0(x)dx\right)^{-1} h$$
$$\leq C\left(\int_0^{(N-1)h} \frac{dx}{\Lambda(x)} + 2\int_{(N-1)h}^{Nh} \frac{dx}{\Lambda(x)}\right) \leq 3C\int_0^1 \frac{dx}{\Lambda(x)} \leq C.$$

Similarly, we have $\sum_{i=j}^{N-1} \frac{h}{\rho_i(0)} \leq C\int_{j/N}^1 \frac{dx}{\Lambda(x)} \leq a(1 - j/N)$.

Lemma 3.2. *For $t \in [0, T_{max})$ and $j = 1, \cdots, N$, we have*

$$\dot{r}_j(t) = v_j(t), \quad 1 = r_0(t) < r_1(t) < \cdots < r_N(t). \tag{3.18}$$

To prove the lemma, we just take the time derivative in identity (3.6) and apply (3.2) to conclude

$$\dot{r}_j(t) = r_j^{1-n}\sum_{i=0}^{j-1} \frac{d}{dt}\left(\frac{1}{\rho_i}\right) h = r_j^{1-n}\sum_{i=0}^{j-1} \delta(r_i^{n-1} v_i)h = v_j.$$

The second claim follows from the positivity of ρ_j since

$$\delta r_j^n = \frac{n}{\rho_j} > 0, \qquad j = 1, \cdots, N-1. \tag{3.19}$$

4. Basic Estimates and Global Construction of the Difference Scheme

We first make the following key estimates and complete the global construction of the difference scheme.

Proposition 4.1. *For any $T \in (0,\infty)$, there exists $C > 0$ such that, for all $t \in [0,T]$,*

$$\sum_{j=0}^{N}\left(v_j^2+\Phi(e_j)\right)h+\sum_{j=0}^{N-1}\Phi(\frac{1}{\rho_j})h+\int_0^t\sum_{j=0}^{N-1}\frac{\rho_j r_{j+1}^{2n-2}}{e_je_{j+1}}(\delta e_j)^2 h d\tau \le C,$$

where $\Phi(w) = w - 1 - log(w)$ is a nonnegative convex function.

Proof: 1. Multiplying (3.2) by $R(\rho_{j-1}-1)h/\rho_{j-1}^2$, (3.3) by v_jh, and (3.4) by $(1-1/e_j)h$, and summing up, we have

$$\begin{aligned}
&\frac{d}{dt}\sum_{j=1}^{N-1}\left(\frac{1}{2}v_j^2+R\Phi(\frac{1}{\rho_{j-1}})+\Phi(e_j)\right)h+\kappa\sum_{j=1}^{N-1}\frac{r_{j+1}^{n-1}\rho_j}{e_je_{j+1}}(\delta e_j)^2h\\
=&Rr_{N-1}^{n-1}v_{N-1}-2\mu(n-1)r_{N-1}^{n-2}v_{N-1}^2+2\mu(n-1)\sum_{j=1}^{N-1}\frac{\delta(r_{j-1}^{n-2}v_{j-1}^2)}{e_j}h\\
&-(\lambda+2\mu)\sum_{j=1}^{N-1}\frac{\rho_{j-1}}{e_j}(\delta(r_{j-1}^{n-1}v_{j-1}))^2h\\
=&Rr_{N-1}^{n-1}v_{N-1}-2\mu(n-1)r_{N-1}^{n-2}v_{N-1}^2-(\lambda+\frac{2\mu}{n})\sum_{j=1}^{N-1}\frac{\rho_{j-1}}{e_j}(\delta(r_{j-1}^{n-1}v_{j-1}))^2\\
&-2\mu(1-\frac{1}{n})\sum_{j=1}^{N-1}\frac{\rho_{j-1}}{e_j}r_{j-1}^nr_j^n\left(\delta\left(\frac{v_{j-1}}{r_{j-1}}\right)\right)^2h\\
\le&Rr_{N-1}^{n-1}v_{N-1}-2\mu(n-1)r_{N-1}^{n-2}v_{N-1}^2.
\end{aligned}$$

Integrating both sides of the inequality above in $\tau \in [0,t] \subset [0,T)$ yields

$$\begin{aligned}
0 \ \le\ & E(t)+\kappa\int_0^t\sum_{j=1}^{N-1}\left(\frac{r_{j+1}^{2n-2}\rho_j}{e_je_{j+1}}(\delta e_j)^2h\right)d\tau \\
\le\ & E(0)+\int_0^t(Rr_{N-1}^{n-1}v_{N-1}-2\mu(n-1)r_{N-1}^{n-2}v_{N-1}^2)d\tau,
\end{aligned}\tag{4.1}$$

where $E(t)=\sum_{j=1}^{N-1}(\frac{1}{2}v_j^2+R\Phi(1/\rho_{j-1})+\Phi(e_j))h$. Notice $E(0)<\infty$ from assumption (3.9)–(3.10) on the initial data and Lemma 3.1.

2. Set $\beta(t) = \sqrt{r_{N-1}^n(t)/n}$. Then (4.1) implies

$$\begin{aligned} -C &\le -\frac{1}{R}E(0) = \int_0^t \left(2\beta\beta_t - 8\frac{\mu}{R}(1-\frac{1}{n})\beta_t^2\right) d\tau \\ &\le \int_0^t \left(\frac{1}{\alpha}\beta^2 - (\frac{8\mu(n-1)}{nR} - \alpha)\beta_t^2\right) d\tau. \end{aligned}$$

Choosing a positive constant $\alpha = \frac{4\mu(n-1)}{nR}$ yields

$$-C \le \int_0^t (\beta^2 - \alpha^2\beta_t^2) d\tau. \tag{4.2}$$

Noting that

$$\beta^2(t) = \left(\beta(0) + \int_0^t \beta_t(\tau)d\tau\right)^2 \le 2\beta^2(0) + 2t\int_0^t \beta_t^2(\tau)d\tau$$

and plugging it into (4.2) yield

$$\int_0^t \beta_t^2(\tau)d\tau \le \frac{C + 2t\beta^2(0)}{\alpha^2} + \int_0^t \frac{2\tau}{\alpha^2}\int_0^\tau \beta_t^2(s)dsd\tau.$$

Gronwall's inequality implies

$$\int_0^t \beta_t^2(\tau)d\tau \le C, \qquad \beta^2(t) \le 2\beta^2(0) + 2t\int_0^t \beta_t^2(\tau)d\tau \le C,$$

which yields

$$r_{N-1}(t) + \int_0^t v_{N-1}^2(\tau)d\tau \le C.$$

Therefore, we conclude

$$0 \le E(t) + \int_0^t \left(\sum_{j=1}^{N-1} \kappa\frac{r_{j+1}^{2n-2}\rho_j}{e_j e_{j+1}}(\delta e_j)^2 h\right) d\tau \le C. \tag{4.3}$$

3. Now we need to estimate $R\Phi(1/\rho_{N-1}(t))h + \frac{1}{2}v_N^2(t)h$. First, we have

$$\begin{aligned} \frac{d}{dt}\Phi(\frac{1}{\rho_{N-1}})(t)h &= (1-\rho_{N-1})(r_N^{n-1}v_N - r_{N-1}^{n-1}v_{N-1}) \\ &= \frac{R}{(\lambda+2\mu)}e_N(1-\rho_{N-1})h \le C, \end{aligned} \tag{4.4}$$

where we have used (4.3) and the boundary conditions $\sigma_N = 0$ and $e_N = e_{N-1}$.

Using the boundary conditions (3.11), we obtain the boundary velocity estimate

$$\begin{aligned} \frac{1}{2}v_N^2 h &\le \left(\frac{r_{N-1}^{n-1}}{r_N^{n-1}}\right)^2 v_{N-1}^2 h + \frac{1}{(\lambda+2\mu)^2}\left(\frac{p(\rho_{N-1}, e_{N-1})}{\rho_{N-1}}\right)^2 h \\ &\le C\left(1 + (e_{N-1}h)^2 h\right) \le C. \end{aligned} \tag{4.5}$$

Combining (4.3) with (4.4)–(4.5), we complete the proof of Proposition 4.1.

This proposition assures the existence of solutions to the Cauchy problem (3.2)–(3.10) for all time, that is, $T_{max} = \infty$. Indeed, for fixed $h > 0$, the bound $\Phi(1/\rho_j(t)) + v_j^2(t) + \Phi(e_j(t)) \le C/h$ implies the global bounds of functions (ρ_j, v_j, e_j):

$$0 < \Phi_-^{-1}(C/h) \le \rho_j(t), e_j(t) \le \Phi_+^{-1}(C/h) < \infty, \qquad |v_j(t)| \le C/\sqrt{h},$$

where $\Phi_\pm^{-1}$ denote the two branches of the inverse function of Φ defined on $(0,1]$ and $[1,\infty)$, respectively. In this case, the solutions can be locally extended beyond the maximal time interval $[0, T_{max})$ that is a contradiction unless $T_{max} = \infty$. Hence, we have completed our construction of the difference scheme $(\rho_j, v_j, e_j)(t)$, as well as the corresponding approximate solutions $(\rho^N, v^N, e^N)(x,t)$. In Sections 5–8, we make some energy estimates and pointwise estimates for the difference scheme in any interval $[0,T] \subset (0,\infty)$.

5. Relative Lower and Upper Bounds for the Density

Now we establish the uniform relative bounds for the density, which are essential for our proof of the main theorem.

Proposition 5.1. *There exists a constant $C > 0$ such that, for all $t \in [0,T]$,*

$$C^{-1}\rho_j(0) \le \rho_j(t) \le C\rho_j(0), \quad j = 0, 1, \cdots, N-1.$$

Proof: 1. From Proposition 4.1, we inherit the uniform estimate:

$$\sum_{j=0}^{N-1} \Phi(1/\rho_j(t))h + \sum_{j=1}^{N} \Phi(e_j(t))h \le C, \tag{5.1}$$

which implies that there exist $a_t \in \{0, \cdots, N-1\}$ and $b_t \in \{1, \cdots, [N/2]\}$ such that, for all $t \in [0,T]$,

$$C^{-1} \le \rho_{a_t - 1}(t) \le C, \quad C^{-1} \le e_{b_t}(t) \le C. \tag{5.2}$$

Equation (3.3) can be rewritten as

$$\frac{1}{\lambda + 2\mu} r_j^{1-n} \dot{v}_j = -\frac{d}{dt}\delta(log\, \rho_{j-1}) - \frac{1}{\lambda + 2\mu}\delta p(\rho_{j-1}, e_j).$$

After integration and summation, it takes the form

$$\begin{aligned} &\sum_{j=a_t}^{k} \int_0^t \frac{1}{(\lambda + 2\mu)} r_j^{1-n} \dot{v}_j d\tau h \\ = \quad & -log\, \rho_k(t) + log\, \rho_{a_t-1}(t) + log\, \rho_k(0) - log\, \rho_{a_t-1}(0) \\ & -\frac{1}{\lambda+2\mu}\left(\int_0^t p(\rho_k, e_{k+1})(\tau)d\tau - \int_0^t p(\rho_{a_t-1}, e_{a_t})(\tau)d\tau\right), \end{aligned} \tag{5.3}$$

where we have used a convention notation $\sum_{j=x}^{y} = -\sum_{j=y}^{x}$ for the case $y < x$.

2. Define the discrete Kazhikov-Shelukhin type functions as follows:

$$Y(t) = exp\left\{\frac{1}{\lambda+2\mu}\int_0^t p(\rho_{a_t-1}, e_{a_t})(\tau)d\tau\right\},$$

$$B_k(t) = exp\left\{-\frac{1}{\lambda+2\mu}\sum_{j=a_t}^{k}\int_0^t r_j^{1-n}(\tau)\dot{v}_j(\tau)d\tau h\right\},\ k=1,\cdots,N-1.$$

Take the exponential of both sides of (5.3) to get that, for $k=1,\cdots,N-1$,

$$\frac{\rho_k(0)}{\rho_k(t)}\frac{\rho_{a_t-1}(t)}{\rho_{a_t-1}(0)}Y(t)B_k(t) = exp\left\{\frac{1}{\lambda+2\mu}\int_0^t p(\rho_k, e_{k+1})d\tau\right\}. \tag{5.4}$$

Multiply this equality by $\frac{p(\rho_k,e_{k+1})}{\lambda+2\mu}$, integrate over $[0,t]$, and use (5.4) again to obtain

$$\int_0^t \frac{p(\rho_k,e_{k+1})(\tau)}{\lambda+2\mu}\frac{\rho_k(0)}{\rho_k(\tau)}\frac{\rho_{a_\tau-1}(\tau)}{\rho_{a_\tau-1}(0)}Y(\tau)B_k(\tau)d\tau = \frac{\rho_k(0)}{\rho_k(t)}\frac{\rho_{a_t-1}(t)}{\rho_{a_t-1}(0)}Y(t)B_k(t)-1. \tag{5.5}$$

Noting that

$$\begin{aligned}&|\sum_{j=a_t}^{k}\int_0^t r_j^{1-n}\dot{v}_j d\tau h|\\ &\le |\sum_{j=1}^{N}(r_j^{1-n}v_j)(t)h| + |\sum_{j=1}^{N}(r_j^{1-n}v_j)(0)h| + (n-1)\int_0^t\left(\sum_{j=1}^{N}(r_j^{-n}v_j^2)(\tau)h\right)d\tau \le C,\end{aligned}$$

we conclude

$$C^{-1} \le B_k(t) \le C, \quad k=0,\cdots,N-1. \tag{5.6}$$

Obviously, $Y(t)>1$. Set $\tilde{Y}(t) = Y(t)/\rho_{a_t-1}(0)$. Then equation (5.5) yields

$$\tilde{Y}(t)/\rho_k(t) \le C/\rho_k(0) + C\int_0^t e_{k+1}(\tau)\tilde{Y}(\tau)d\tau.$$

Multiplying both sides of the above inequality by h, summing up for $k=0,\cdots,N-1$, and using estimate (5.1) and Lemma 3.1, we have $\tilde{Y}(t) \le C + C\int_0^t \tilde{Y}(\tau)d\tau$, which yields $Y(t)/\rho_{a_t-1}(0) = \tilde{Y}(t) \le C$. Therefore, we have

$$1 \le Y(t) \le C\rho_{a_t-1}(0) \le C, \tag{5.7}$$

since $\rho_j(0) \le C$. Then, combining (5.4) with (5.2) and (5.6)–(5.7) implies

$$\begin{aligned}\rho_k(t) &\le \rho_k(t)\,exp\left\{\frac{1}{\lambda+2\mu}\int_0^t p(\rho_k,e_{k+1})d\tau\right\}\\ &= \rho_k(0)Y(t)B_k(t)\frac{\rho_{a_t-1}(t)}{\rho_{a_t-1}(0)} \le C\rho_k(0) \le C, \quad k=1,\cdots,N-1.\end{aligned} \tag{5.8}$$

3. From (5.5)–(5.7) and (5.2), we have, for some $C > 0$,

$$\frac{\rho_k(0)}{\rho_k(t)} \le C\frac{\rho_k(0)}{\rho_k(t)}\tilde{Y}(t)B_k(t)\rho_{a_t-1}(t) \le C\left(1+\rho_k(0)\int_0^t e_{k+1}(\tau)d\tau\right). \tag{5.9}$$

Notice that

$$e_{k+1}(t) = \left(\sqrt{e_{b_t}(t)} + \sum_{j=b_t}^{k}\delta(\sqrt{e_j})h\right)^2 \le C + C\sum_{j=0}^{N-1}\frac{\rho_j}{e_{j+1}e_j}(\delta e_j)^2 h\sum_{j=b_t}^{k}\frac{e_{j+1}}{\rho_j}h. \tag{5.10}$$

Combining (5.9) with (5.10), we have

$$\frac{\rho_k(0)}{\rho_k(t)} \le C + C\rho_k(0)\int_0^t \sum_{j=0}^{N-1}\frac{\rho_j}{e_{j+1}e_j}(\delta e_j)^2 h\sum_{j=b_t}^{k}\frac{e_{j+1}}{\rho_j}h\, d\tau. \tag{5.11}$$

4. If $k \le [N/2]$. Then, from (2.8), we have

$$\rho_k(0) \ge C^{-1}\rho_{[N/2]}(0) \ge C^{-1},$$

and thus inequality (5.11) yields

$$\frac{1}{\rho_k(t)} \le C + C\int_0^t (\max_{0\le j\le [N/2]}\frac{1}{\rho_j})\left(\sum_{j=0}^{N-1}\frac{\rho_j}{e_{j+1}e_j}(\delta e_j)^2 h\right)d\tau. \tag{5.12}$$

Since the right-hand side is independent of k, a direct application of Gronwall's inequality together with Proposition 4.1 yields

$$\max_{0\le j\le [N/2]}\frac{\rho_k(0)}{\rho_k(t)} \le C\max_{0\le j\le [N/2]}\frac{1}{\rho_k(t)} \le C. \tag{5.13}$$

If $k > [N/2]$, inequality (5.11) can be extended to

$$\frac{\rho_k(0)}{\rho_k(t)} \le C + C\int_0^t \sum_{j=0}^{N-1}\frac{\rho_j}{e_{j+1}e_j}(\delta e_j)^2 h\left(1 + \max_{[N/2]\le j\le N-1}\frac{\rho_j(0)}{\rho_j(\tau)}\right)d\tau$$

where we have used (5.13). Again, the right-hand side is independent of k, and the Gronwall inequality yields

$$\max_{[N/2]\le j\le N-1}\frac{\rho_j(0)}{\rho_j(\tau)} \le C.$$

This completes the proof of Proposition 5.1.

6. L^p Bounds for the Velocity and Temperature

We first introduce a discrete form of the summation by parts:

$$
\begin{aligned}
\sum_{j=1}^{N-1} a_j \delta b_{j-1} &= -\sum_{j=1}^{N-1} b_{j-1}\delta a_{j-1} + \sum_{j=1}^{N-1} \delta(a_{j-1}b_{j-1}) \\
&= -\sum_{j=1}^{N-1} b_{j-1}\delta a_{j-1} + \frac{a_{N-1}b_{N-1} - a_0 b_0}{h}.
\end{aligned} \tag{6.1}
$$

Denote the energy density $w_j = e_j + \frac{1}{2}v_j^2$. Multiply equation (3.3) by v_j, add it to (3.4), multiply both sides by w_j, sum up the resulting equality for $j = 1, \cdots, N-1$, and integrate over $[0, t]$. Then, using (6.1) and the boundary conditions (3.11), we obtain

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{N-1} w_j^2(t)h - \frac{1}{2}\sum_{j=1}^{N-1} w_j^2(0)h + 2\mu(n-1)\int_0^t w_{N-1} r_{N-1}^{n-2} v_{N-1}^2 d\tau \\
&= -\int_0^t \sum_{j=1}^{N-1} \sigma_j v_{j-1} r_{j-1}^{n-1} \delta w_{j-1} h d\tau - \kappa \int_0^t \sum_{j=1}^{N-1} \rho_{j-1} r_j^{2n-2} \delta w_{j-1} \delta e_{j-1} h d\tau \\
&\quad +2\mu(n-1)\int_0^t \sum_{j=1}^{N-1} r_{j-1}^{n-2} v_{j-1}^2 \delta w_{j-1} h d\tau.
\end{aligned} \tag{6.2}
$$

Using (3.19), we have

$$
\rho_{j-1}\delta(r_{j-1}^{n-1}v_{j-1}) = \rho_{j-1} r_j^{n-1}\delta v_{j-1} + n v_{j-1}\frac{\delta r_{j-1}^{n-1}}{\delta r_{j-1}^{n}}, \tag{6.3}
$$

$$
(1 - \frac{1}{n})\frac{1}{r_j} \le \frac{\delta r_{j-1}^{n-1}}{\delta r_{j-1}^{n}} \le \frac{2}{r_j + r_{j-1}}. \tag{6.4}
$$

Then, using ithe identities in (6.1) and (6.3)–(6.4) and applying inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$, we conclude

$$\tag{6.5} \begin{aligned} &\frac{1}{2}\sum_{j=1}^{N-1} w_j^2(t)h + 2\mu(n-1)\int_0^t w_{N-1} r_{N-1}^{n-2} v_{N-1}^2 d\tau \\ &\qquad + \frac{\kappa}{5}\int_0^t \sum_{j=1}^{N-1} \rho_{j-1} r_j^{2n-2} (\delta e_{j-1})^2 h d\tau \\ &\le \frac{1}{2}\sum_{j=1}^{N-1} w_j^2(0)h + C\int_0^t \sum_{j=1}^{N-1} \rho_{j-1}(v_j^2 + v_{j-1}^2)(\delta v_{j-1})^2 h d\tau \\ &\qquad + C\int_0^t \sum_{j=1}^{N-1} \left(\frac{v_{j-1}^4}{\rho_{j-1}} + \rho_{j-1} e_j^2 v_{j-1}^2\right) h d\tau. \end{aligned}$$

We now multiply equation (3.3) by v_j^3, sum up for $j = 1, \cdots, N-1$, integrate over $[0,t]$, and use (3.11) and (6.1) to get

$$\tag{6.6} \begin{aligned} \frac{1}{4}\sum_{j=1}^{N-1} v_j^4(t)h &= \frac{1}{4}\sum_{j=1}^{N-1} v_j^4(0)h + \int_0^t \sum_{j=1}^{N-1} \delta(v_{j-1}^3 r_{j-1}^{n-1}) p(\rho_{j-1}, e_j) h\, d\tau \\ &\quad -(\lambda+2\mu)\int_0^t \sum_{j=1}^{N-1} \rho_{j-1}\delta(v_{j-1}^3 r_{j-1}^{n-1})\delta(r_{j-1}^{n-1} v_{j-1}) h\, d\tau. \end{aligned}$$

Notice that

$$\begin{aligned} &\delta(v_{j-1}^3 r_{j-1}^{n-1}) p(\rho_{j-1}, e_j) \\ &\qquad \le \epsilon \rho_{j-1}(v_j^2 + v_{j-1}^2)(\delta v_{j-1})^2 + C_\epsilon\{\rho_{j-1} e_j^2 (v_j^2 + v_{j-1}^2) + v_{j-1}^4 \rho_{j-1}^{-1}\}, \\ &-\rho_{j-1}\delta(v_{j-1}^3 r_{j-1}^{n-1})\delta(r_{j-1}^{n-1} v_{j-1}) \\ &\qquad \le -\frac{1}{4} r_j^{2n-2} \rho_{j-1}(v_j^2 + v_{j-1}^2)(\delta v_{j-1})^2 + C\rho_{j-1}^{-1}(v_j^4 + v_{j-1}^4), \end{aligned}$$

for sufficiently small $\epsilon > 0$. Then, from (6.6), we have

$$\tag{6.7} \begin{aligned} &\sum_{j=1}^{N-1} v_j^4(t)h + \int_0^t \sum_{j=1}^{N-1} \rho_{j-1}(v_j^2 + v_{j-1}^2)(\delta v_{j-1})^2 h d\tau \\ &\le C\left(1 + \int_0^t \sum_{j=1}^{N-1} \{\rho_{j-1}^{-1}(v_j^4 + v_{j-1}^4) + e_j^2(v_j^2 + v_{j-1}^2)\rho_{j-1}\} h d\tau\right). \end{aligned}$$

Multiplying (6.7) by a large enough constant and adding it to (6.5), we obtain

$$
\begin{aligned}
(6.8) \qquad & \sum_{j=1}^{N-1} w_j^2 h + \sum_{j=1}^{N-1} v_j^4 h + \int_0^t w_{N-1} v_{N-1}^2 d\tau \\
& + \int_0^t \sum_{j=1}^{N-1} \{\rho_{j-1}(v_j^2 + v_{j-1}^2)(\delta v_{j-1})^2 + \rho_{j-1}(\delta e_{j-1})^2\} h d\tau \\
& \le C \left(1 + \int_0^t \sum_{j=1}^{N-1} \{\rho_{j-1}^{-1}(v_j^4 + v_{j-1}^4) + \rho_{j-1} e_j^2 (v_j^2 + v_{j-1}^2)\} h d\tau \right).
\end{aligned}
$$

The following lemma is quite crucial for establishing some higher-order estimates.

Lemma 6.1. *For any $\epsilon \in (0,1)$, there is a constant $C_\epsilon > 0$ such that*

$$
\sum_{j=1}^{N-1} (v_j^4 + v_{j-1}^4)\rho_{j-1}^{-1} h \le \epsilon \sum_{j=1}^{N-1} \rho_{j-1}(v_j^2 + v_{j-1}^2)(\delta v_{j-1})^2 h + C_\epsilon \sum_{j=1}^{N-1} (v_j^4 + v_{j-1}^4) h.
$$

Proof: Since $\Lambda(x)$ is nondecreasing and integrable, $\int_0^1 \Lambda(x)dx < \infty$, there exists $0 < \delta_\epsilon < \epsilon$ such that, for all $x \in (1-\delta_\epsilon, 1)$, $\int_x^1 \Lambda(y)dy \le \epsilon$. Then there is an index k such that $kh \le 1 - \delta_\epsilon \le (k+1)h$. We distinguish two cases.

Case $j \le k \le N-1$: We can easily see that

$$
(6.9) \qquad (v_j^4 + v_{j-1}^4)\rho_{j-1}^{-1}(t) \le C(v_j^4 + v_{j-1}^4)\Lambda(1-\delta_\epsilon)^{-1} \le C_\epsilon (v_j^4 + v_{j-1}^4).
$$

Case $k \le j \le N-1$: Using the standard techniques and the estimate (6.9), we obtain

$$
\begin{aligned}
\rho_{j-1}^{-1}(v_j^4 + v_{j-1}^4) =& \, 2\rho_{j-1}^{-1} v_0^4 + \rho_{j-1}^{-1} \sum_{i=1}^{j} \delta v_{i-1}^4 h + \rho_{j-1}^{-1} \sum_{i=1}^{j-1} \delta v_{i-1}^4 h \\
\le& \epsilon \rho_{j-1}^{-1} \sum_{i=1}^{N-1} \rho_{i-1}(v_i^2 + v_{i-1}^2)(\delta v_{i-1})^2 h + C_\epsilon \Lambda(1-\delta_\epsilon)\rho_{j-1}^{-1} \sum_{i=1}^{k-1} (v_i^4 + v_{i-1}^4) h \\
& + \frac{16}{\epsilon} \rho_{j-1}^{-1} \sum_{i=k}^{j} \rho_{i-1}^{-1}(v_i^4 + v_{i-1}^4) h.
\end{aligned}
$$

The discrete Gronwall inequality yields

$$
\rho_{j-1}^{-1}(v_j^4 + v_{j-1}^4) \le C\rho_{j-1}^{-1} \left\{ \epsilon \sum_{i=1}^{N-1} \rho_{i-1}(v_i^2 + v_{i-1}^2)(\delta v_{i-1})^2 h + C_\epsilon \sum_{i=1}^{N-1} (v_i^4 + v_{i-1}^4) h \right\}.
$$

Hence, we combine these two cases together to have

$$\begin{aligned}\sum_{j=1}^{N-1}\rho_{j-1}^{-1}(v_j^4+v_{j-1}^4)h=&\sum_{j=1}^{k-1}\rho_{j-1}^{-1}(v_j^4+v_{j-1}^4)h+\sum_{j=k}^{N-1}\rho_{j-1}^{-1}(v_j^4+v_{j-1}^4)h\\ \le& C_\epsilon\sum_{j=1}^{N-2}(v_j^4+v_{j-1}^4)h+\epsilon C\sum_{j=1}^{N-1}\rho_{j-1}(v_j^2+v_{j-1}^2)(\delta v_{j-1})^2h.\end{aligned}$$

By substituting ϵ/C for ϵ, we obtain the result of Lemma 6.1.

Now, using (6.8) and Lemma 6.1, together with $e_j^2\le w_j^2$, we have

$$\begin{aligned}&\sum_{j=1}^{N-1}(e_j^2+v_j^4+v_{j-1}^4)h+\int_0^t w_{N-1}v_{N-1}^2d\tau\\ &\quad+\int_0^t\sum_{j=1}^{N-1}\rho_{j-1}\{(v_j^2+v_{j-1}^2)(\delta v_{j-1})^2+(\delta e_{j-1})^2\}hd\tau\\ &\le C+C\int_0^t\sum_{j=1}^{N-1}\{(v_j^4+v_{j-1}^4)+\rho_{j-1}e_j^2(v_j^2+v_{j-1}^2)\}hd\tau.\end{aligned}\tag{6.10}$$

Noting from (5.10) and the Schwartz inequality that

$$\begin{aligned}&\int_0^t\sum_{j=1}^{N-1}\rho_{j-1}e_j^2(v_j^2+v_{j-1}^2)hd\tau\\ &\quad\le C\int_0^t\left(1+\sum_{j=1}^{N-1}\frac{\rho_{j-1}}{e_je_{j-1}}(\delta e_{j-1})^2h\right)\sum_{j=1}^{N-1}(e_j^2+v_j^4+v_{j-1}^4)hd\tau.\end{aligned}$$

and, using Proposition 4.1,

$$\int_0^t\sum_{j=1}^{N-1}\frac{\rho_{j-1}}{e_je_{j-1}}(\delta e_{j-1})^2hd\tau\le C,$$

we apply the Gronwall inequality to (6.10) to obtain

Proposition 6.1. *There exists $C > 0$ such that, for all $t \in [0,T]$,*

(6.11)
$$\sum_{j=0}^{N}(e_j^2+v_j^4)h+\int_0^t\left(\sum_{j=0}^{N}\rho_{j-1}(v_j^2+v_{j-1}^2)(\delta v_{j-1})^2h+\sum_{j=0}^{N-1}\rho_j(\delta e_j)^2h\right)d\tau\le C,$$

(6.12)
$$\sum_{j=1}^{N}(v_j^4+v_{j-1}^4)\rho_{j-1}^{-1}h+\int_0^t\sum_{j=1}^{N}e_j^2(v_j^2+v_{j-1}^2)\rho_{j-1}hd\tau\le C,$$

(6.13)
$$\int_0^t e_jd\tau\le C\rho_{j-1}^{-1}(0)\int_0^t\rho_{j-1}e_jd\tau\le C\rho_{j-1}^{-1}(0),$$

(6.14)
$$\int_0^t\sum_{j=1}^{N}\rho_{j-1}(\delta(r_{j-1}^{n-1}v_{j-1}))^2hd\tau\le C.$$

Proof: To complete the proof, it suffices to show the estimates for the terms near the free boundary, which can be achieved by the stress-free boundary condition.

Note that

$$\rho_{N-1}^{-1}v_N^4h\le C\rho_{N-1}^{-1}v_{N-1}^4h+C\rho_{N-1}^{-1}h(e_Nh)^4\le C+C\sum_{j=1}^{N}\rho_{j-1}^{-1}h\left(\sum_{j=1}^{N-1}e_jh\right)^4\le C.$$

Furthermore, $\delta v_{N-1}=\frac{v_{N-1}}{r_{N-1}^{n-1}}\delta r_{N-1}^{n-1}+\frac{1}{\lambda+2\mu}\frac{p(\rho_{N-1},e_N)}{\rho_{N-1}}$ implies

$$\begin{aligned}\rho_{N-1}(v_N^2+v_{N-1}^2)(\delta v_{N-1})^2h &\le C\rho_{N-1}\left(\frac{1}{\rho_{N-1}^2}+e_{N-1}^2\right)(v_{N-1}^2+e_{N-1}^2h^2)h\\ &\le C+Ce_{N-1}^2v_{N-1}^2\rho_{N-1}h\le C,\end{aligned}$$

which yields (6.11).

We now make some auxiliary a-priori estimates involving the velocity. Multiply (3.3) by v_i, sum up for $i=0,\cdots,N-1$, integrate over $[0,t]$, and use (3.11) to get

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{N-1}v_j^2(t)h\le&\frac{1}{2}\sum_{j=1}^{N-1}v_j^2(0)h-\frac{\lambda+2\mu}{2}\int_0^t\sum_{j=1}^{N-1}\rho_{j-1}(\delta(r_{j-1}^{n-1}v_{j-1}))^2hd\tau\\ &+C\int_0^t\sum_{j=1}^{N-1}\rho_{j-1}e_j^2hd\tau.\end{aligned}$$

Estimates (6.11) and Proposition 5.1 imply that the last integral is bounded from above, and hence we can reshuffle this inequality to get (6.14), except the term near the boundary. Notice that

$$e_j^2 = e_{b_t}^2 + \sum_{i=b_t}^{j} \delta(e_{i-1}^2)h \le C + \epsilon \max_{1\le i\le N} e_i^2 + C_\epsilon \sum_{i=1}^{N} \rho_{i-1}(\delta e_{i-1})^2 h.$$

The right-hand side is independent of index j, and ϵ can be chosen small enough, so as we can conclude the inequality

$$\int_0^t \max_{1\le j\le N} e_j^2 d\tau \le C\left(1 + \int_0^t \sum_{j=1}^{N} \rho_{j-1}(\delta e_{j-1})^2 h d\tau\right) \le C. \tag{6.15}$$

Then we have

$$\int_0^t \rho_{N-1}(\delta(r_{N-1}^{n-1}v_{N-1}))^2 h d\tau = \frac{1}{(\lambda+2\mu)^2}\int_0^t \rho_{N-1}e_N^2 h d\tau = C\int_0^t e_{N-1}^2 d\tau \le C.$$

Therefore, we can obtain inequality (6.14) by adding the last element.

Moreover, since $\rho_{j-1}r_j^{2n-2}(\delta v_{j-1})^2 \le \rho_{j-1}(\delta(r_{j-1}^{n-1}v_{j-1}))^2 + \frac{n^2}{r_j^2\rho_{j-1}}(v_j^4+1)$, we obtain

$$\int_0^t \sum_{j=1}^{N} \rho_{j-1}(\delta v_{j-1})^2 h d\tau \le C.$$

This completes the proof.

As a corollary, we have

(6.16)

$$\begin{aligned}
&\int_0^t \max_{0\le j\le N}|v_j| d\tau \le C + \int_0^t \max_{0\le j\le N}|v_j|^2 d\tau \le C + \int_0^t \max_{0\le j\le N}|v_j|^3 d\tau \\
&\le C + C\int_0^t \sum_{j=1}^{N}(v_j^2+v_{j-1}^2)|\delta v_{j-1}| h d\tau \\
&\le C\left(1 + \int_0^t \sum_{j=1}^{N}(v_j^4+v_{j-1}^4)\rho_{j-1}^{-1} h d\tau + \int_0^t \sum_{j=1}^{N}\rho_{j-1}(\delta v_{j-1})^2 h d\tau\right) \le C.
\end{aligned}$$

7. Further Bound for the Density

Proposition 7.1. *There exists $C>0$, independent of N, such that*

$$\sum_{j=1}^{N-1}(\delta\rho_{j-1})^2 h \le C\left(1+\sum_{j=1}^{N-1}(\delta\rho_{j-1})^2(0)h\right) \le C.$$

Proof: From (3.2)-(3.3), we have

$$\delta\rho_{j-1}(t) = \delta\rho_{j-1}(0) - \frac{1}{\lambda+2\mu}\left(\int_0^t \rho_j r_j^{1-n}\dot{v}_j d\tau + \int_0^t \delta(\rho_{j-1}^2 e_j) d\tau + \int_0^t \sigma_j \delta\rho_{j-1} d\tau\right).$$

Since $\int_0^t(|\sigma_j|+e_j)d\tau\le C$, the Gronwall inequality yields

$$|\delta\rho_{j-1}(t)|\le C\left(|\delta\rho_{j-1}(0)|+|\int_0^t\rho_j r_j^{1-n}\dot v_j d\tau|+|\int_0^t\delta(\rho_{j-1}^2e_j)d\tau|\right).$$

Take square of both sides and sum up to obtain

(7.1)

$$\begin{aligned}\sum_{j=1}^{N-1}(\delta\rho_{j-1})^2(t)\le&C\sum_{j=1}^{N-1}(\delta\rho_{j-1})^2(0)h+C\sum_{j=1}^{N-1}\int_0^t\left((\rho_{j-1}^2\delta e_{j-1})^2+(e_{j+1}\delta\rho_{j-1}^2)^2\right)d\tau h\\&+C\sum_{j=1}^{N-1}\left((\rho_jr_j^{1-n}v_j)(t)-(\rho_jr_j^{1-n}v_j)(0)-\int_0^tv_j\frac{d}{dt}(\rho_jr_j^{1-n})d\tau\right)^2h\\\le&C\left(1+\sum_{j=1}^{N-1}(\delta\rho_{j-1})^2(0)h+\sum_{j=1}^{N-1}\left(\int_0^tv_j\dot\rho_jd\tau\right)^2h+\sum_{j=1}^{N-1}\int_0^tv_j^4d\tau h\right.\\&\left.+\sum_{j=1}^{N-1}\int_0^t\rho_{j-1}(\delta e_{j-1})^2d\tau h+\int_0^t(\max_{1\le j\le N}e_j^2)\sum_{j=1}^{N-1}(\delta\rho_{j-1})^2hd\tau\right).\end{aligned}$$

Noting that

$$\sum_{j=1}^{N-1}\left(\int_0^tv_j\dot\rho_jd\tau\right)^2h\le\sum_{j=1}^{N-1}\left(\int_0^tv_j^2d\tau\int_0^t\rho_j^4(\delta(r_j^{n-1}v_j))^2d\tau\right)h\le C,$$

applying inequalities (6.14) and (6.16), and estimates (6.12) and (6.15), and using the Gronwall inequality, we conclude Proposition 7.1.

Remark 7.1. *If $\rho_0(x)$ satisfies only $\int_0^1|\rho_0'(x)|dx<\infty$, then we follow the same steps to conclude*

$$\sum_{j=1}^{N-1}|\delta\rho_{j-1}(t)|h\le C\left(1+\sum_{j=1}^{N-1}|\delta\rho_{j-1}(0)|h\right)\le C. \tag{7.2}$$

8. Further Bounds for the Velocity and Temperature

Proposition 8.1. *There exists $C>0$, independent of N, such that*

$$\int_0^t\sum_{j=1}^{N-1}(\dot v_j)^2hd\tau+\sum_{j=0}^{N-1}\rho_j(\delta(r_j^{n-1}v_j))^2h+\sum_{j=0}^{N-1}\rho_j^{-3}(\dot\rho_j)^2h\le C.$$

Proof: 1. Since the initial density vanishes at the free boundary,

$$\begin{aligned}\frac{\rho_{N-1}^2(t)}{h}&\le C\frac{\rho_{N-1}^2(0)}{h}=\frac{C}{h^3}\left(\int_{1-h}^1\rho_0(x)dx\right)^2\\&\le\frac{C}{h^3}\left(\int_{1-h}^1\int_x^1|\rho_0'(y)|dydx\right)^2\le C\int_{1-h}^1(\rho_0'(y))^2dy\le C.\end{aligned}\tag{8.1}$$

2. Multiplying (3.3) by $\dot{v}_j$, integrating in $\tau \in [0,t] \subset [0,T]$, summing up for $j = 1, \cdots, N-1$, and using (3.11), we have

(8.2)

$$
\begin{aligned}
&\int_0^t \sum_{j=1}^{N-1} (\dot{v}_j)^2 h d\tau + \frac{\lambda + 2\mu}{2} \sum_{j=1}^{N-1} \rho_{j-1} (\delta(r_{j-1}^{n-1} v_{j-1}))^2 h \\
=& \frac{\lambda + 2\mu}{2} \sum_{j=1}^{N-1} \rho_{j-1}(0) \left(\delta(r_{j-1}^{n-1} v_{j-1})\right)^2 (0) h + \int_0^t r_{N-1}^{n-1} \dot{v}_{N-1} p(\rho_{N-1}, e_N) d\tau \\
&+ (\lambda + 2\mu)(n-1) \int_0^t \sum_{j=1}^{N-1} \rho_{j-1} \delta(r_{j-1}^{n-2} v_{j-1}^2) \delta(r_{j-1}^{n-1} v_{j-1}) h d\tau \\
&+ \frac{\lambda + 2\mu}{2} \int_0^t \sum_{j=1}^{N-1} (\delta(r_{j-1}^{n-1} v_{j-1}))^2 \frac{d\rho_{j-1}}{dt} h d\tau - \int_0^t \sum_{j=1}^{N-1} r_j^{n-1} \dot{v}_j \delta p(\rho_{j-1}, e_j) h d\tau.
\end{aligned}
$$

We now estimate the last two integrals. First we apply (8.1) and Proposition 6.1 to obtain

$$
|\int_0^t r_N^{n-1} \dot{v}_{N-1} p(\rho_{N-1}, e_N) d\tau| \leq \frac{1}{4} \int_0^t (\dot{v}_{N-1})^2 h d\tau + C. \tag{8.3}
$$

Then we use Propositions 6.1 and 7.1, and estimate (6.15) to obtain

(8.4)

$$
\begin{aligned}
|\int_0^t \sum_{j=1}^{N-1} r_j^{n-1} \dot{v}_j \delta p(\rho_{j-1}, e_j) h d\tau| \leq& \frac{1}{4} \int_0^t \sum_{j=1}^{N-1} (\dot{v}_j)^2 h d\tau + C \int_0^t \sum_{j=1}^{N-1} \rho_j^2 (\delta e_j)^2 h d\tau \\
&+ C \int_0^t (\max_{1 \leq j \leq N-1} e_j^2) \sum_{j=1}^{N-1} (\delta \rho_{j-1})^2 h d\tau \\
\leq& \frac{1}{4} \int_0^t \sum_{j=1}^{N-1} (\dot{v}_j)^2 h d\tau + C.
\end{aligned}
$$

Noting from (3.19) that

$$
\delta(r_j^{n-2} v_j^2) = \delta(r_j^{n-1} v_j) \frac{r_{j+1}^{n-1} v_{j+1} + r_j^{n-1} v_j}{r_{j+1}^n} - \frac{n v_j^2}{\rho_j} \frac{r_j^{n-2}}{r_{j+1}^n}, \tag{8.5}
$$

and using (6.14) and the boundedness of r_j yield

$$
\begin{aligned}
&\int_0^t \sum_{j=0}^{N-2} \rho_j |\delta(r_j^{n-1} v_j)| |\delta(r_j^{n-2} v_j^2)| h d\tau \\
\leq& C + C \int_0^t (\max_{1 \leq j \leq N-1} v_j^2) \sum_{j=0}^{N-2} \rho_j (\delta(r_j^{n-1} v_j))^2 h d\tau + C \int_0^t \sum_{j=0}^{N-2} \frac{v_j^4}{\rho_j} h d\tau.
\end{aligned} \tag{8.6}
$$

Notice that

(8.7)

$$\begin{aligned}\max_{0\le j\le N-1}|\sqrt{\rho_j}\delta(r_j^{n-1}v_j)| \le& \frac{1}{\lambda+2\mu}\max_{0\le j\le N-1}\frac{|\sigma_{j+1}|}{\sqrt{\rho_j}}+\frac{R}{\lambda+2\mu}\max_{0\le j\le N-1}(e_{j+1}\sqrt{\rho_j})\\ \le& C\max_{0\le j\le N-1}\left(\frac{1}{\sqrt{\rho_j}}\sum_{i=j+1}^{N-1}|\delta\sigma_i|h\right)+C\max_{1\le j\le N}(e_j\sqrt{\rho_{j-1}})\\ \le& C\sum_{j=1}^{N-1}\frac{|\dot{v}_j|}{\sqrt{\rho_{j-1}}}h+C\max_{1\le j\le N}(e_j\sqrt{\rho_{j-1}})\\ \le& \epsilon(t)\sum_{j=1}^{N-1}(\dot{v}_j)^2h+C\max_{1\le j\le N}(e_j\sqrt{\rho_{j-1}})+\frac{C}{\epsilon(t)}.\end{aligned}$$

We have intentionally used $\epsilon(t)>0$, since we want to utilize the extra degree of freedom it offers with the time variable. Combining (8.2) with (8.3)–(8.7) yields

$$\begin{aligned}&\int_0^t\sum_{j=1}^{N-1}(\dot{v}_j)^2hd\tau+\sum_{j=0}^{N-2}\rho_j(\delta(r_j^{n-1}v_j))^2d\tau h\\ =&C+C\int_0^t(\max_{0\le j\le N-2}|\rho_j\delta(r_j^{n-1}v_j)|)\sum_{j=0}^{N-2}\rho_j^2(\delta(r_j^{n-1}v_j))^2hd\tau\\ &+C\int_0^t(\max_{1\le j\le N-1}v_j^2)\sum_{j=0}^{N-2}\rho_j^2(\delta(r_j^{n-1}v_j))^2hd\tau\\ =&C+C\int_0^t\left\{\epsilon(\tau)\sum_{j=1}^{N-1}(\dot{v}_j)^2h+\max_{1\le j<N}(e_j+v_j^2)+\frac{C}{\epsilon(\tau)}\right\}\left(\sum_{j=0}^{N-2}\rho_j(\delta(r_j^{n-1}v_j))^2h\right)d\tau.\end{aligned}$$

Then we have

$$\begin{aligned}&\int_0^t\sum_{j=1}^{N-1}(\dot{v}_j)^2hd\tau+\sum_{j=0}^{N-2}\rho_j(\delta(r_j^{n-1}v_j))^2d\tau h\\ =&C+C\int_0^t\left\{1+\max_{1\le j<N}(e_j+v_j^2)+\sum_{j=0}^{N-2}\rho_j(\delta(r_j^{n-1}v_j))^2h\right\}\left(\sum_{j=0}^{N-2}\rho_j(\delta(r_j^{n-1}v_j))^2h\right)d\tau,\end{aligned}$$

by choosing $\epsilon(\tau)^{-1}=2C\left(1+\sum_{j=0}^{N-2}\rho_j(\delta(r_j^{n-1}v_j))^2h\right)$.

Gronwall's inequality implies

$$\int_0^t\sum_{j=1}^{N-1}(\dot{v}_j)^2hd\tau+\sum_{j=1}^{N-1}\rho_{j-1}(\delta(r_{j-1}^{n-1}v_{j-1}))^2h\le C.$$

Since $\int_0^t \rho_{N-1}(\delta(r_{N-1}^{n-1}v_{N-1}))^2 hd\tau = \frac{R}{\lambda+2\mu}\int_0^t e_N^2\rho_{N-1}hd\tau \le C$, and $(\dot\rho_j)^2 = \rho_j^4(\delta(r_j^{n-1}v_j))^2$, we complete the proof of Proposition 8.1.

Corollary 8.1. *There exists $C>0$ such that*

$$\tag{8.8} \begin{aligned} &\int_0^t \max_{0\le j\le N-1}(\sqrt{\rho_j}|\delta(r_j^{n-1}v_j)|)d\tau + \int_0^t \max_{0\le j\le N-1}(e_{j+1}\sqrt{\rho_j}|\delta(r_j^{n-1}v_j)|)d\tau \\ &+\int_0^t \max_{0\le j\le N-1}(\rho_j(\delta(r_j^{n-1}v_j))^2)d\tau + \int_0^t \sum_{j=0}^{N-1}\rho_j^2(\delta(r_j^{n-1}v_j))^4 hd\tau \le C, \end{aligned}$$

$$\tag{8.9} \int_0^t \sum_{j=1}^{N-1}(\delta\sigma_j)^2(\tau)hd\tau \le C.$$

Indeed, using estimate (8.7), we get

$$\begin{aligned} &\int_0^t \max_{0\le j\le N-1}(\rho_j(\delta(r_j^{n-1}v_j))^2)d\tau \\ &\le C\int_0^t \sum_{j=1}^{N-1}\left((\dot v_j)^2+\frac{1}{\rho_j}\right)hd\tau + C\int_0^t(\max_{1\le j\le N} e_j^2)d\tau \le C. \end{aligned}$$

Noting

$$|\sqrt{\rho_j}\delta(r_j^{n-1}v_j)| + |e_{j+1}\sqrt{\rho_j}\delta(r_j^{n-1}v_j)| \le \rho_j(\delta(r_j^{n-1}v_j))^2 + e_{j+1}^2 + 1,$$

and

$$\int_0^t \sum_{j=0}^{N-1}\rho_j^2(\delta(r_j^{n-1}v_j))^4 hd\tau \le \int_0^t(\max_{0\le j\le N-1}\rho_j(\delta(r_j^{n-1}v_j))^2)\sum_{j=0}^{N-1}\rho_j(\delta(r_j^{n-1}v_j))^2 hd\tau,$$

and using Proposition 8.1, we arrive at the result. The result (8.9) is a direct corollary of Proposition 8.1.

Corollary 8.2. *There exists $C>0$ such that the following boundary estimates hold:*

$$\tag{8.10} \frac{1}{1-j/N}\int_0^t\sum_{i=j}^N \sigma_i^2(\tau)hd\tau \le C(1-j/N), \quad \frac{1}{j/N}\int_0^t\sum_{i=0}^j v_i^2(\tau)hd\tau \le Cj/N.$$

Since

$$\sigma_i^2(t) = \left(\sum_{k=i}^{N-1}\delta\sigma_k h\right)^2 = \left(\sum_{k=i}^{N-1} r_k^{2-2n}\dot v_k h\right)^2 \le C\frac{N-i}{N}\sum_{k=i}^{N-1}(\dot v_k)^2 h,$$

we have

$$\int_0^t \sigma_i^2(\tau)d\tau \le C\frac{N-i}{N}\int_0^t\sum_{k=i}^{N-1}(\dot v_k)^2 hd\tau \le C(1-i/N),$$

which implies the first estimate in (8.10). Similarly, we have

$$\int_0^t v_i^2 d\tau = \int_0^t (\sum_{k=1}^{i} \delta v_{k-1} h)^2 d\tau \le \int_0^t \sum_{k=1}^{i} \rho_{k-1}(\delta v_{k-1})^2 h \sum_{k=1}^{i} \frac{h}{\rho_{k-1}} d\tau \le Ci/N,$$

which implies the second estimate in (8.10).

Corollary 8.3. $|\rho_j(t) - \rho_j(s)| \le C\sqrt{|t-s|}$.

This can be achieved by using Corollary 8.1 and (3.2) as follows:

$$|\rho_j(t) - \rho_j(s)| = |\int_s^t \dot{\rho}_j(\tau) d\tau| \le |\int_s^t \rho_j^3 d\tau \int_s^t \rho_j(\delta(r_j^{n-1} v_j))^2 d\tau|^{1/2} \le C\sqrt{|t-s|}.$$

Also Proposition 8.1 implies

$$|v_j(t)| = r_j^{1-n} |\sum_{j=1}^{i} \delta(r_{j-1}^{n-1} v_{j-1})| \le r_j^{1-n} \sum_{j=1}^{i} \rho_{j-1}(\delta(r_{j-1}^{n-1} v_{j-1}))^2 h + r_j^{1-n} \sum_{j=1}^{i} \frac{h}{\rho_{j-1}} \le C.$$

Proposition 8.2. *For any $j = 1, 2, \cdots, N$, and $t \in [0,T]$, $|v_j(t)| \le C$.*

Since (6.3)–(6.4) yields $\rho_j(\delta v_j)^2 h \le C\rho_j(\delta(r_j^{n-1} v_j))^2 h + Ch/\rho_j$, Propositions 5.1, 8.1-8.2, and Lemma 3.1 imply

Corollary 8.4. $\sum_{j=0}^{N-1} \rho_j(\delta v_j)^2 h \le C$.

Before we make the estimates for the temperature (equivalently, the internal energy), we prove the following estimate involving the velocity.

Proposition 8.3. $\int_0^t \sum_{j=0}^{N-2} \rho_j^4 (\delta^2(r_j^{n-1} v_j))^2 h d\tau \le C$.

Proof. ¿From (3.3), we have

$$\begin{aligned}
&(\lambda + 2\mu) r_j^{n-1} \rho_{j-1} \delta^2(r_{j-1}^{n-1} v_{j-1}) \\
&\quad = -(\lambda + 2\mu) r_j^n \delta(r_j^{n-1} v_j) \delta\rho_{j-1} + \dot{v}_j + r_j^{n-1} e_{j+1} \delta\rho_{j-1} + R r_j^{n-1} \rho_{j-1} \delta e_j.
\end{aligned}$$

Multiply both sides by ρ_{j-1}, take squares, integrate, sum up, and use simple algebraic inequalities together with the uniform estimates $\rho_{j-1} + r_j^{n-1} \le C$ to obtain

$$\begin{aligned}
&\int_0^t \sum_{j=1}^{N-1} \rho_{j-1}^4 (\delta^2(r_{j-1}^{n-1} v_{j-1}))^2 h d\tau \\
&\le C \int_0^t \sum_{j=1}^{N-1} \rho_{j-1}^2 (\delta(r_j^{n-1} v_j))^2 (\delta\rho_{j-1})^2 h d\tau + C \int_0^t \sum_{j=1}^{N-1} \rho_{j-1}^2 (\dot{v}_j)^2 h d\tau \\
&\quad + \int_0^t \sum_{j=1}^{N-1} \rho_{j-1}^2 e_{j+1}^2 (\delta\rho_{j-1})^2 h d\tau + C \int_0^t \sum_{j=1}^{N-1} \rho_{j-1}^4 (\delta e_j)^2 h d\tau.
\end{aligned}$$

Notice that $\int_0^t \sum_{j=1}^{N-1} \rho_{j-1}^2 (\delta(r_j^{n-1} v_j))^2 (\delta\rho_{j-1})^2 h d\tau \le C \int_0^t \sum_{j=1}^{N-1} (\delta\rho_{j-1})^2 h d\tau \le C$. The other integrals have also uniformly upper bounds as above. This completes the proof.

Proposition 8.4. *For any $j = 0, \cdots, N$, and $t \in [0,T]$, $e_j(t) \geq C^{-1}$.*

Proof: Observe that

$$
\begin{aligned}
&-\frac{1}{e_j^2}\delta(\rho_{j-1}r_j^{2n-2}\delta e_{j-1}) \\
&\quad = \delta(\rho_{j-1}r_j^{2n-2}\delta e_{j-1}^{-1}) - \left(\frac{\rho_j r_{j+1}^{2n-2}}{e_{j+1}e_j^2}(\delta e_j)^2 + \frac{\rho_{j-1}r_j^{2n-2}}{e_j e_{j-1}^2}(\delta e_{j-1})^2\right), \\
&(\lambda+2\mu)\rho_{j-1}(\delta(r_{j-1}^{n-1}v_{j-1}))^2 - 2\mu(n-1)\delta(r_{j-1}^{n-2}v_{j-1}^2) \\
&\quad = (\lambda+\frac{2\mu}{n})\rho_{j-1}(\delta(r_{j-1}^{n-1}v_{j-1}))^2 + 2\mu(1-\frac{1}{n})\rho_{j-1}(r_j r_{j-1})^{n-2}(\delta(r_j v_{j-1}))^2 \geq 0.
\end{aligned}
$$

Multiplying (3.4) by e_j^{-1} and using the identities above yield

$$
\begin{aligned}
&\frac{d}{dt}(e_j^{-1}) + \kappa\left(\frac{\rho_j r_{j+1}^{2n-2}}{e_{j+1}e_j^2}(\delta e_j)^2 + \frac{\rho_{j-1}r_j^{2n-2}}{e_j e_{j-1}^2}(\delta e_{j-1})^2\right) \\
&\leq \kappa\delta(\rho_{j-1}r_j^{n-2}\delta e_{j-1}^{-1}) + \frac{\rho_{j-1}}{e_j}\delta(r_{j-1}^{n-1}v_{j-1}).
\end{aligned}
$$

Multiplying both sides by $ke_j^{1-k}, k > 1$, summing up, and integrating with respect to t, we get

$$
\begin{aligned}
&\int_0^t \sum_{j=1}^{N-1}\frac{d}{dt}(e_j^{-k})hd\tau + \kappa k\int_0^t\sum_{j=1}^{N-1}\left(\frac{\rho_j r_{j+1}^{2n-2}(\delta e_j)^2}{e_{j+1}e_j^k} + \frac{\rho_{j-1}r_j^{2n-2}(\delta e_{j-1})^2}{e_{j-1}e_j^k}\right)hd\tau \\
&\leq k\int_0^t\sum_{j=1}^{N-1}e_j^{1-k}\delta(\rho_{j-1}r_j^{2n-2}\delta e_{j-1}^{-1})hd\tau + k\int_0^t\sum_{j=1}^{N-1}\frac{\rho_{j-1}}{e_j^k}\delta(r_{j-1}^{n-1}v_{j-1})hd\tau \\
&\leq k\int_0^t\left(\sum_{j=1}^{N-1}\frac{e_j^{1-k}-e_{j-1}^{1-k}}{e_j e_{j-1}}\rho_{j-1}r_j^{2n-2}\delta e_{j-1} + (\max_{1\leq j\leq N}|\rho_{j-1}\delta(r_{j-1}^{n-1}v_{j-1})|)\sum_{j=1}^{N-1}e_j^{-k}\right)hd\tau.
\end{aligned}
$$

Since $(e_j^{1-k} - e_{j-1}^{1-k})(e_j - e_{j-1}) \leq 0$, we obtain

$$
\sum_{j=1}^{N-1}e_j^{-k}(t)h \leq \sum_{j=1}^{N-1}e_j^{-k}(0)h + k\int_0^t(\max_{1\leq j\leq N}|\rho_{j-1}\delta(r_{j-1}^{n-1}v_{j-1})|)\sum_{j=1}^{N-1}e_j^{-k}hd\tau.
$$

Using Corollary 8.1 and Gronwall's inequality, we obtain

$$
\sum_{j=1}^{N-1}e_j^{-k}(t)h \leq \sum_{j=1}^{N-1}e_j^{-k}(0)he^{kCt}, \quad \text{for any } k > 1,
$$

which yields that, in the Sobolev norm, $\|e^N(t)^{-1}\|_{L^k} \leq C\|e^N(0)^{-1}\|_{L^k}$. Letting $k \to \infty$, we conclude $\|e^N(t)^{-1}\|_{L^\infty} \leq C\|e^N(0)^{-1}\|_{L^\infty}$, which implies

$$
e_j(t) \geq C^{-1}\inf(e^N(0)) \geq C^{-1}.
$$

Proposition 8.5. *There exists $C > 0$ such that, for any $t, s \in [0, T], t < s$,*

$$|\int_t^s \rho_{j-1} r_j^{2n-2} \delta e_{j-1} d\tau| \le a(1 - \frac{j}{N}) + |\frac{1}{1 - j/N} \int_t^s \sum_{i=j}^{N} \rho_{i-1} r_i^{2n-2} \delta e_{i-1} h d\tau|$$

$$\le a(1 - \frac{j}{N}),$$

$$|\int_0^t \rho_{j-1} r_j^{2n-2} \delta e_{j-1} d\tau| \le C\sqrt{\frac{j}{N}} + |\frac{1}{j/N} \int_t^s \sum_{i=j}^{N} \rho_{i-1} r_i^{2n-2} \delta e_{i-1} h d\tau| \le C\sqrt{\frac{j}{N}},$$

where $a(\delta) \ge 0$ is a nondecreasing function depending on the function $\Lambda(x)$, related to the initial data, and satisfies $\lim_{\delta \to 0+} a(\delta) = 0$.

Proof: Integrate (3.4) over $[t, s]$ and sum up for $j = i, \cdots, N-1$ to get

$$\int_t^s \sum_{j=i}^{N-1} \dot{e}_j h dt = -\kappa \int_t^s \rho_{i-1} r_i^{2n-2} \delta e_{i-1} d\tau + (\lambda + 2\mu) \int_t^s \sum_{j=i}^{N-1} \rho_{j-1} (\delta(r_{j-1}^{n-1} v_{j-1}))^2 h d\tau$$

$$- \int_t^s \sum_{j=i}^{N-1} \rho_{j-1} e_j \delta(r_{j-1}^{n-1} v_{j-1}) h d\tau + 2\mu(n-1) \int_t^s \sum_{j=i}^{N-1} \frac{v_{j-1}^2}{\rho_{j-1}} \frac{r_{j-1}^{n-2}}{r_j^n} h d\tau$$

$$- 2\mu(n-1) \int_0^t \sum_{j=i}^{N-1} \delta(r_{j-1}^{n-1} v_{j-1}) \frac{r_j^{n-1} v_j + r_{j-1}^{n-1} v_{j-1}}{r_j^n} h d\tau,$$

where we applied identity (8.5). Therefore, the previous estimates yield

(8.11)

$$\begin{aligned}
&\kappa |\int_t^s \rho_{i-1} r_i^{2n-2} \delta e_{i-1} d\tau| \\
&\le \left(\sum_{j=i}^{N-1} e_j^2(s) h \sum_{j=i}^{N-1} h\right)^{1/2} + \left(\sum_{j=i}^{N-1} e_j^2(t) h \sum_{j=i}^{N-1} h\right)^{1/2} + C \int_0^t \sum_{j=i}^{N-1} \frac{v_{j-1}^2}{\rho_{j-1}} h d\tau \\
&\quad + C \int_0^t \max_{i \le j \le N-1} \left(\rho_{j-1} (\delta(r_{j-1}^{n-1} v_{j-1}))^2 + \rho_{j-1} e_j |\delta(r_{j-1}^{n-1} v_{j-1})|\right) d\tau \sum_{j=i}^{N-1} h \\
&\quad + C \int_0^t \sum_{j=i}^{N-1} \sqrt{\rho_{j-1}} |\delta(r_{j-1}^{n-1} v_{j-1})| |\frac{r_j^{n-1} v_j + r_{j-1}^{n-1} v_{j-1}}{r_j^n \sqrt{\rho_{j-1}}}| h d\tau \\
&\le C\sqrt{\frac{N-i}{N}} + C \int_0^t \sum_{j=i}^{N-1} \frac{h}{\rho_{j-1}(0)} d\tau \le C\sqrt{1 - \frac{i}{N}} + C \int_{1-\frac{i}{N}}^1 \frac{dx}{\Lambda(x)} = a(1 - \frac{i}{N}).
\end{aligned}$$

Similarly, we have

$$|\int_0^t \rho_{i-1} r_i^{2n-2} \delta e_{i-1} d\tau| \le C\sqrt{\frac{i}{N}} + C\frac{j}{N} \le C\sqrt{\frac{i}{N}}. \tag{8.12}$$

We sum up (8.11) for $i = j, \cdots, N$, and (8.12) for $i = 1, \cdots, j$, we obtain immediately the other part of Proposition 8.5.

9. Compactness of the Approximate Solutions

In this section we show the compactness of approximate solutions $(\rho^N, v^N, e^N)(x,t)$ (res. $(\rho_h, v_h, e_h)(x,t)$), and radii $r^N(x,t)$ (res. $r_h(x,t)$) defined by (3.12)-(3.15).

1. *Compactness of the Density:* ¿From Propositions 5.1, 7.1, and 8.1, there exists $C > 0$, independent of N, such that

$$|\rho^N(x,t)| + \int_0^1 (\partial_x \rho^N)^2(x,t)dx + \int_0^1 (\partial_t \rho^N)^2(x,t)dx \le C,$$

which implies that there exists a subsequence (still denoted) $\rho^N(x,t)$ such that, as $N \to \infty$,

$$\begin{aligned} &\rho^N(x,t) \to \rho(x,t), \quad a.e. \\ &(\partial_x \rho^N, \partial_t \rho^N)(x,t) \rightharpoonup (\partial_x \rho, \partial_t \rho)(x,t), \qquad \text{weak in } L^2((0,1)\times(0,T)), \\ &\rho \in C([0,T]; L^2(0,1)) \cap L^\infty((0,T); W^{1,2}(0,1)). \end{aligned}$$

Corollary 8.3 implies

$$|\rho(\cdot,t) - \rho(\cdot,s)| \le C\sqrt{t-s}.$$

Since

$$|\rho^N(x,\cdot) - \rho^N(y,\cdot)| \le C\sqrt{x-y},$$

we conclude

$$|\rho(x,\cdot) - \rho(y,\cdot)| \le C\sqrt{x-y}.$$

Noting that

$$\int_0^1 (\rho^N - \rho_h)^2 dx \le h^2 \int_0^1 (\partial_x \rho^N)dx, \quad \text{and} \quad \rho^N(x,t) \to \rho(x,t), \ a.e.$$

we conclude

$$\rho_h(x,t) \to \rho(x,t), \quad a.e.$$

Since $C^{-1}\rho^N(x,0) \le \rho^N(x,t) \le C\rho^N(x,0)$, we conclude

$$C^{-1}\rho(x,0) \le \rho(x,t) \le C\rho(x,0), \quad \text{for } x \in [0,1).$$

2. *Compactness of the Velocity*: ¿From Propositions 8.1-8.4, we have

$$\begin{aligned} &|v^N(x,t)| + \int_0^1 (\rho^N(\partial_x v^N)^2)(x,t)dx \\ &\quad + \int_0^T \int_0^1 ((\partial_t v^N)^2 + (\rho^N)^4(\partial_{xx} v^N)^2)(x,\tau)dxd\tau \le C, \end{aligned}$$

which implies that there exists a subsequence (still denoted) $v^N(x,t)$ such that, as $N\to\infty$,

$$v^N(x,t)\to v(x,t), \quad a.e.$$
$$(\sqrt{\rho^N}\partial_x v^N, \partial_t v^N, (\rho^N)^2\partial_{xx}v^N)(x,t) \rightharpoonup (\sqrt{\rho}\partial_x v, \partial_t v, \rho^2\partial_{xx}v)(x,t),$$
$$\text{weak in } L^2((0,1)\times(0,T)),$$
$$v\in L^\infty\cap W^{1,2}_{loc}([0,1)\times[0,T]), \qquad (\sqrt{\rho}\partial_x v, \partial_t v, \rho^2\partial_{xx}v)\in L^2((0,1)\times(0,T)).$$

Also, as $h\to 0$, $v_h(x,t)\to v(x,t)$, $\quad a.e.$

3. *Compactness of the Temperature (equivalently, the Internal Energy)*: Similarly, from Propositions 6.1 and 8.4, we have

$$C^{-1}\le e^N(x,t), \int_0^1 (e^N)^2(x,t)dx\le C, \qquad \int_0^T\int_0^1 (\rho^N(\partial_x e^N)^2)(x,\tau)dxd\tau\le C,$$

which implies that there exists a subsequence (still denoted) $e^N(x,t)$ such that, as $N\to\infty$,

$$e^N(x,t)\rightharpoonup e(x,t) \qquad \text{weak in } L^2((0,1)\times(0,T)),$$
$$(\sqrt{\rho^N}\partial_x e^N)(x,t)\rightharpoonup(\sqrt{\rho}\partial_x e)(x,t) \qquad \text{weak in } L^2((0,1)\times(0,T)),$$
$$e(x,t)\ge C^{-1}, \qquad (e,\sqrt{\rho}\partial_x e)\in L^2([0,1]\times[0,T]).$$

4. *Compactness of the Radii*: First, relation $\dot r_i(t)=v_i(t)$ implies

$$\partial_t(r^N)^n(x,t)=n\left(r^{n-1}_{[xN]}(t)v_{[xN]}(t)+(xN-[xN])r^{n-1}_{[xN]+1}(t)v_{[xN]+1}(t)-r^{n-1}_{[xN]}(t)v_{[xN]}(t))\right),$$
$$\partial_t r_h(t)=v_h(t), \quad \partial_{tx}(r^N)^n(x,t)=n(r^{n-1}_{[xN]+1}(t)v_{[xN]+1}(t)-r^{n-1}_{[xN]}(t)v_{[xN]}(t))/h, \quad a.e.$$

Also we notice (3.19) and $(r^N)^n(x,t)=r_0^n+n\int_0^x\frac{dx}{\rho_h(x,t)}$. We have

$$|r^N(x,t)|+|\partial_t r^N(x,t)|+|r_h(x,t)|+|\partial_t r_h(x,t)|+\int_0^1|\partial_x r^N(x,t)|dx\le C,$$
$$\int_0^t\int_0^1(\rho^N(\partial_x r^N)^2)(x,\tau)dxd\tau+\int_0^t\int_0^1\rho_h(\tau)(\partial_{tx}(r^N)^n)^2(x,\tau)dxd\tau\le C.$$

Since ρ_h converges pointwise almost everywhere to ρ, we use the Lebesgue dominated convergence theorem to have

$$(r^N,\partial_t r^N,\partial_x r^N)(x,t)\to(r,\partial_t r,\partial_x r)(x,t), \quad a.e.$$
$$(\partial_x r^N,\partial_{tx}(r^N)^n)(x,t)\rightharpoonup(\partial_x r,\partial_{tx}r^n)(x,t), \text{ weak in } L^2_{loc}([0,1)\times[0,T]),$$

when $N\to\infty$, where $r(x,t)=\left(r_0^n+n\int_0^x\frac{dy}{\rho(y,t)}\right)^{1/n}$. Also

$$r_h(x,t)\to r(x,t), \ a.e. \ h\to 0, \qquad r\in C([0,1]\times[0,T])\cap W^{1,2}_{loc}([0,1)\times[0,T]).$$

10. Completion of the Proof of Main Theorem

To complete the Main Theorem, in this section we show that the limit vector function $(\rho, v, e)(x,t)$ in Section 9 is a global solution of the Navier-Stokes equations with radius $r(x,t)$ as stated in the Main Theorem.

For any given test function $\phi(x,t) \in C_0^\infty((0,1)\times[0,T))$, we choose $N = 1/h$ is large enough so that the support of the test function ϕ is away enough from the boundaries, that is, $supp\,\phi \subset (h, 1-h)\times[0,T) = (1/N, 1-1/N)\times[0,T)$. Define

$$\phi_i(t) = \phi_h(x,t) \equiv \phi([xN]h, t), \quad ih \le x < (i+1)h. \tag{10.1}$$

We can see that $\phi_i(t) = 0$ for $i = 0, 1, N-1, N$.

Equation (3.2) can be viewed as written in the form

$$\partial_t \rho_h(x,t) = -\rho_h^2(x,t)\frac{1}{n}\partial_{tx}(r^N)^n(x,t).$$

Multiplying it by $\phi(x,t)$ and integrating, we obtain

$$\begin{aligned}\int_0^T\int_0^1 \rho_h(x,t)\phi_t(x,t)dxdt + \int_0^1 \rho_h(x,0)\phi(x,0)dx \\ - \int_0^t\int_0^1 (\phi\rho_h^2)(x,t)\frac{1}{n}\partial_{tx}(r^N)^n(x,t)dxdt = 0.\end{aligned} \tag{10.2}$$

Since the strong convergence of $\rho_h(x,t) \to \rho(x,t)$, $\rho_h(x,0) \to \rho_0(x)$, and the weak convergence of $\partial_{tx}(r^N)^n(x,t) \rightharpoonup \partial_{tx}r^n(x,t)$ in L^2, we can pass to the limit in (10.2) to obtain the first equation of (2.4).

Multiply equation (3.3) by ϕ_i, sum it up for $i = 1, \cdots, N-1$, and integrate over $[0,T]$ to get

$$\begin{aligned}0 =& \int_0^T \sum_{i=1}^{N-1} \phi_i\partial_t v_i h dt + \int_0^T \sum_{i=1}^{N-1} \phi_i r_i^{n-1}\delta(e_i\rho_{i-1})hdt \\ &- (\lambda+2\mu)\int_0^T \sum_{i=1}^{N-1} \phi_i r_i^{n-1}\delta(\rho_{i-1}\delta(r_{i-1}^{n-1}v_{i-1}))hdt \\ =& O(h) - \int_0^1 v_0(x)\phi(x,0)dx + \frac{n-1}{n}\int_0^T\int_0^1 \phi(n\frac{e_h}{r_h}\partial_{tx}(r^N)^n)dxdt \\ &- \int_0^T\int_0^1 v_h\partial_t\phi dxdt + \int_0^T\int_0^1 \partial_x\phi(e_h r_h^{n-1}\rho_h + r_h^{n-1}\rho_h\frac{1}{n}\partial_{tx}(r^N)^n)dxdt.\end{aligned}$$

We pass to the limit $h \to 0$ above to validate the second equation of (2.4).

Similarly, multiplying equation (3.2) by v_i and adding it to equation (3.3), and then multiplying the resulting equation by ϕ_i, summing up, and integrate over $[0,T]$ to get

(10.3)

$$O(h)=\int_0^1(\frac{1}{2}v_h^2(x,0)+e_h(x,0))\phi(x,0)dx+\int_0^T\int_0^1(\frac{1}{2}v_h^2+e_h)\phi_t dxdt$$
$$+\int_0^T\int_0^1(\rho_h r_h^{2n-2}\partial_x e^N+\frac{1}{n}r_h^{n-1}v_h\rho_h\partial_{tx}(r^N)^n+v_h r_h^{n-1}\rho_h e_h+r_h^{n-2}v_h^2)\phi_x dxdt.$$

Applying the strong convergence of $(r_h, v_h, \rho_h)\to(r,v,\rho)$ and the weak convergence of $(e_h,\partial_x e^N,\partial_{tx}(r^N)^n)\rightharpoonup(e,\partial_x e,\partial_{tx}r^n)$, and passing the limit $h\to 0$ in (10.3), we conclude the third equation of (2.4). This completes the proof of the Main Theorem.

Corollary 10.1. The global solution (ρ, v, e) in the Main Theorem satisfy the boundary conditions in the sense that, for any $t<s$,

$$\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\int_0^T\int_0^\epsilon v^2(x,\tau)dxd\tau+\left|\frac{1}{\epsilon}\int_t^s\int_0^\epsilon \rho r^{2n-2}e_x dxd\tau\right|\right)=0,$$
$$\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\int_0^T\int_{1-\epsilon}^1 \sigma^2(x,\tau)dxd\tau+\left|\frac{1}{\epsilon}\int_t^s\int_{1-\epsilon}^1 \rho r^{2n-2}e_x dxd\tau\right|\right)=0. \tag{10.4}$$

These can be seen as follows. First, Corollary 8.2 implies

$$\frac{1}{\epsilon}\int_t^s\int_0^\epsilon(v^N)^2(x,\tau)dxd\tau+\frac{1}{\epsilon}\int_t^s\int_{1-\epsilon}^1\sigma_h^2(x,\tau)dxd\tau\le a(\epsilon)\to 0,\ \ \epsilon\to 0,$$

where $\sigma_h\equiv(\lambda+2\mu)\rho_h\frac{1}{n}\partial_{tx}r^n-\rho_h e_h$. Then we can pass the limit to obtain the first two inequalities of the weak boundary condition (10.4).

Also, from Proposition 8.5, we have

$$|\frac{1}{\epsilon}\int_t^s\int_0^\epsilon \rho_h r_h^{2n-2}\partial_x e_N dxdt|+|\frac{1}{\epsilon}\int_t^s\int_{1-\epsilon}^1 \rho_h r_h^{2n-2}\partial_x e_N dxdt|\le a(\epsilon).$$

Then the estimate $\int_0^T\int_0^1\rho_h(\partial_x e_N)^2dxd\tau\le C$, the strong convergence

$$\sqrt{\rho_h}r_h^{2n-2}\to\sqrt{\rho}r^{2n-2},$$

and the Lebesgue theorem yield the last two weak boundary conditions in (10.4).

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