

Shock Reflection-Diffraction Phenomena and Multidimensional Conservation Laws

Gui-Qiang Chen and Mikhail Feldman

ABSTRACT. When a plane shock hits a wedge head on, it experiences a reflection-diffraction process, and then a self-similar reflected shock moves outward as the original shock moves forward in time. The complexity of reflection-diffraction configurations was first reported by Ernst Mach in 1878, and experimental, computational, and asymptotic analysis has shown that various patterns of shock reflection-diffraction configurations may occur, including regular reflection and Mach reflection. In this paper we start with various shock reflection-diffraction phenomena, their fundamental scientific issues, and their theoretical roles as building blocks and asymptotic attractors of general solutions in the mathematical theory of multidimensional hyperbolic systems of conservation laws. Then we describe how the global problem of shock reflection-diffraction by a wedge can be formulated as a free boundary problem for nonlinear conservation laws of mixed-composite hyperbolic-elliptic type. Finally we discuss some recent developments in attacking the shock reflection-diffraction problem, including the existence, stability, and regularity of global regular reflection-diffraction solutions. The approach includes techniques to handle free boundary problems, degenerate elliptic equations, and corner singularities, which is highly motivated by experimental, computational, and asymptotic results. Further trends and open problems in this direction are also addressed.

1. Introduction

Shock waves occur in many physical situations in nature. For example, shock waves can be produced by solar winds (bow shocks), supersonic or near sonic aircrafts (transonic shocks around the body), explosions (blast waves), and various

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natural processes. When such a shock hits an obstacle (steady or flying), shock reflection-diffraction phenomena occur. One of the most important problems in mathematical fluid dynamics is the problem of shock reflection-diffraction by a wedge. When the plane shock hits a wedge head on, it experiences a reflection-diffraction process, and then a fundamental question is what types of wave patterns of reflection-diffraction configurations it may form around the wedge.

The complexity of reflection-diffraction configurations was first reported by Ernst Mach [109] in 1878, who first observed two patterns of reflection-diffraction configurations: regular reflection (two-shock configuration) and Mach reflection (three-shock configuration); also see [5, 96, 118]. The problem remained dormant until the 1940's when von Neumann, Friedrichs, Bethe, as well as many experimental scientists, among others, began extensive research into all aspects of shock reflection-diffraction phenomena, due to its importance in applications. See von Neumann [142, 143] and Ben-Dor [5]; also see [6, 69, 86, 92, 95, 108, 131, 132] and the references cited therein. It has been found that the situation is much more complicated than what Mach originally observed: The Mach reflection can be further divided into more specific sub-patterns, and various other patterns of shock reflection-diffraction may occur such as the von Neumann reflection and the Guderley reflection; see [5, 54, 77, 80, 91, 129, 135, 136, 137, 139, 142, 143] and the references cited therein.

Then the fundamental scientific issues include:

- (i) *Structure of the shock reflection-diffraction configurations;*
- (ii) *Transition criteria between the different patterns of shock reflection-diffraction configurations;*
- (iii) *Dependence of the patterns upon the physical parameters such as the wedge angle θ_w , the incident-shock-wave Mach number M_s , and the adiabatic exponent $\gamma \geq 1$.*

Careful asymptotic analysis has been made for various reflection-diffraction configurations in Lighthill [104], Keller-Blank [93], Hunter-Keller [90], Morawetz [116], [67, 134, 84, 87, 142, 143], and the references cited therein; also see Glimm-Majda [77]. Large or small scale numerical simulations have been also made; see, e.g. [5, 77], [57, 58, 85, 98, 120, 126], [8, 52, 71, 72, 73, 92, 144], and the references cited therein.

However, most of the fundamental issues for shock reflection-diffraction phenomena have not been understood, especially the global structure and transition of different patterns of shock reflection-diffraction configurations. This is partially because physical and numerical experiments are hampered by various difficulties and have not been able to select the correct transition criteria between different patterns. In particular, numerical dissipation or physical viscosity smear the shocks and cause boundary layers that interact with the reflection-diffraction patterns and can cause *spurious Mach stems*; cf. Woodward-Colella [144]. Furthermore, some difference between two different patterns are only fractions of a degree apart (e.g., see Fig. 5 below), a resolution even by sophisticated modern experiments (e.g. [107]) has been unable to reach, as pointed out by Ben-Dor in [5]: “*For this reason it is almost impossible to distinguish experimentally between the sonic and detachment criteria*” (cf. Section 5 below). In this regard, it seems that the ideal approach to understand fully the shock reflection-diffraction phenomena, especially the transition criteria, is still via rigorous mathematical analysis. To achieve this, it is

essential to establish first the global existence, regularity, and structural stability of solutions of the shock reflection-diffraction problem.

On the other hand, shock reflection-diffraction configurations are the core configurations in the structure of global solutions of the two-dimensional Riemann problem for hyperbolic conservation laws; while the Riemann solutions are building blocks and local structure of general solutions and determine global attractors and asymptotic states of entropy solutions, as time tends to infinity, for multidimensional hyperbolic systems of conservation laws. See [18, 19, 20, 21, 77, 76, 97, 99, 101, 121, 124, 123, 156] and the references cited therein. In this sense, we have to understand the shock reflection-diffraction phenomena, in order to understand fully entropy solutions to multidimensional hyperbolic systems of conservation laws.

In this paper, we first formulate the shock reflection-diffraction problem into an initial-boundary value problem in Section 2. Then we employ the essential feature of self-similarity of the initial-boundary value problem to reformulate the problem into a boundary value problem in the unbounded domain in Section 3. In Section 4, we present the unique solution of normal reflection for this problem when the wedge angle is $\pi/2$. In Section 5, we exhibit the local theory of regular reflection-diffraction, introduce a stability criterion to determine state (2) at the reflection point on the wedge, and present the von Neumann's detachment and sonic conjectures. Then we discuss the role of the potential flow equation in the shock reflection-diffraction problem even in the level of the full Euler equations in Section 6. Based on the local theory, we reduce the boundary value problem into a free boundary problem in the context of potential flow in Section 7. In Section 8, we describe a global theory for regular reflection-diffraction for potential flow, established in Chen-Feldman [33, 34, 35] and Bae-Chen-Feldman [3]. In Section 9, we discuss some open problems and new mathematics required for further developments, which are also essential for solving multidimensional problems in conservation laws and other areas in nonlinear partial differential equations.

2. Mathematical Formulation I: Initial-Boundary Value Problem

The full Euler equations for compressible fluids in $\mathbb{R}_+^3 := \mathbb{R}_+ \times \mathbb{R}^2$, $t \in \mathbb{R}_+ := (0, \infty)$, $\mathbf{x} \in \mathbb{R}^2$, are of the following form:

$$(2.1) \quad \begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \\ \partial_t (\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \partial_t (\frac{1}{2} \rho |\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2} \rho |\mathbf{v}|^2 + \rho e + p) \mathbf{v}) = 0, \end{cases}$$

where ρ is the density, $\mathbf{v} = (u, v)$ the fluid velocity, p the pressure, and e the internal energy. Two other important thermodynamic variables are the temperature θ and the energy S . The notation $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of the vectors \mathbf{a} and \mathbf{b} .

Choosing (ρ, S) as the independent thermodynamical variables, then the constitutive relations can be written as $(e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S))$ governed by

$$\theta dS = de + p d\tau = de - \frac{p}{\rho^2} d\rho.$$

For a polytropic gas,

$$(2.2) \quad p = (\gamma - 1)\rho e, \quad e = c_v \theta, \quad \gamma = 1 + \frac{R}{c_v},$$

or equivalently,

$$(2.3) \quad p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \quad e = e(\rho, S) = \frac{\kappa}{\gamma - 1} \rho^{\gamma-1} e^{S/c_v},$$

where $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas, $c_v > 0$ is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent, and $\kappa > 0$ is any constant under scaling.

When a flow is potential, that is, there is a velocity potential Φ such that

$$\mathbf{v} = \nabla_{\mathbf{x}} \Phi,$$

then the Euler equations for the flow take the form:

$$(2.4) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \nabla_{\mathbf{x}} \Phi) = 0, & (\text{conservation of mass}) \\ \partial_t \Phi + \frac{1}{2} |\nabla_{\mathbf{x}} \Phi|^2 + i(\rho) = B_0, & (\text{Bernoulli's law}) \end{cases}$$

where

$$i(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1} \quad \text{when } \gamma > 1,$$

especially, $i(\rho) = \ln \rho$ when $\gamma = 1$, by scaling and B_0 is the Bernoulli constant, which is usually determined by the boundary conditions if such conditions are prescribed. From the second equation in (2.4), we have

$$(2.5) \quad \rho(D\Phi) = i^{-1}(B_0 - (\partial_t \Phi + \frac{1}{2} |\nabla_{\mathbf{x}} \Phi|^2)).$$

Then system (2.4) can be rewritten as the following time-dependent potential flow equation of second order:

$$(2.6) \quad \partial_t \rho(D\Phi) + \nabla \cdot (\rho(D\Phi) \nabla \Phi) = 0$$

with (2.5).

For a steady solution $\Phi = \varphi(\mathbf{x})$, i.e., $\partial_t \Phi = 0$, we obtain the celebrated steady potential flow equation in aerodynamics:

$$(2.7) \quad \nabla_{\mathbf{x}} \cdot (\rho(\nabla_{\mathbf{x}} \Phi) \nabla_{\mathbf{x}} \Phi) = 0.$$

In applications in aerodynamics, (2.4) or (2.6) is used for discontinuous solutions, and the empirical evidence is that entropy solutions of (2.4) or (2.6) are fairly close to entropy solutions for (2.1), provided the shock strengths are small, the curvature of shock fronts is not too large, and the amount of vorticity is small in the region of interest. Furthermore, we will show in Section 6 that, for the shock reflection-diffraction problem, the Euler equations for potential flow is actually *exact* in an important region of the solution (see Theorem 6.1 below).

Then the problem of shock reflection-diffraction by a wedge can be formulated as follows:

Problem 2.1 (Initial-boundary value problem). *Seek a solution of system (2.1) satisfying the initial condition at $t = 0$:*

$$(2.8) \quad (\mathbf{v}, p, \rho) = \begin{cases} (0, 0, p_0, \rho_0), & |x_2| > x_1 \tan \theta_w, x_1 > 0, \\ (u_1, 0, p_1, \rho_1), & x_1 < 0; \end{cases}$$

and the slip boundary condition along the wedge boundary:

$$(2.9) \quad \mathbf{v} \cdot \nu = 0,$$

where ν is the exterior unit normal to the wedge boundary, and state (0) and (1) satisfy

$$(2.10) \quad u_1 = \sqrt{\frac{(p_1 - p_0)(\rho_1 - \rho_0)}{\rho_0 \rho_1}}, \quad \frac{p_1}{p_0} = \frac{(\gamma + 1)\rho_1 - (\gamma - 1)\rho_0}{(\gamma + 1)\rho_0 - (\gamma - 1)\rho_1}, \quad \rho_1 > \rho_0.$$

That is, given ρ_0, p_0, ρ_1 , and $\gamma > 1$, the other variables u_1 and p_1 are determined by (2.10). In particular, the Mach number $M_1 = u_1/c_1$ is determined by

$$(2.11) \quad M_1^2 = \frac{2(\rho_1 - \rho_0)^2}{\rho_0((\gamma + 1)\rho_1 - (\gamma - 1)\rho_0)},$$

where $c_1 = \sqrt{\gamma p_1/\rho_1}$ is the sonic speed of fluid state (1).

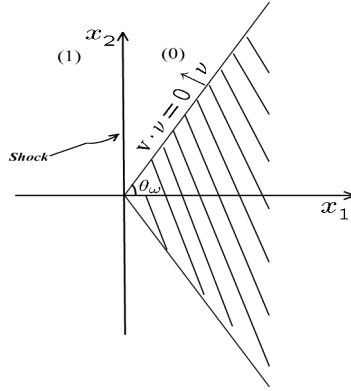


FIGURE 1. Initial-boundary value problem

3. Mathematical Formulation II: Boundary Value Problem

Notice that the initial-boundary value problem (**Problem 2.1**) is invariant under the self-similar scaling:

$$(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha \mathbf{x}) \quad \text{for any } \alpha \neq 0.$$

Therefore, we seek self-similar solutions:

$$(\mathbf{v}, p, \rho)(t, \mathbf{x}) = (\mathbf{v}, p, \rho)(\xi, \eta), \quad (\xi, \eta) = \frac{\mathbf{x}}{t}.$$

Then the self-similar solutions are governed by the following system:

$$(3.1) \quad \begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p)_\xi + (\rho UV)_\eta + 3\rho U = 0, \\ (\rho UV)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\ (U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0, \end{cases}$$

where $q = \sqrt{U^2 + V^2}$, and $(U, V) = (u - \xi, v - \eta)$ is the pseudo-velocity.

The eigenvalues of system (3.1) are

$$\lambda_0 = \frac{V}{U} \text{ (repeated),} \quad \lambda_{\pm} = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2},$$

where $c = \sqrt{\gamma p/\rho}$ is the sonic speed.

When the flow is pseudo-subsonic, i.e., $q < c$, the eigenvalues λ_{\pm} become complex and thus the system consists of two transport equations and two nonlinear equations of hyperbolic-elliptic mixed type. Therefore, system (3.1) is *hyperbolic-elliptic composite-mixed* in general.

Since the problem is symmetric with respect to the axis $\eta = 0$, it suffices to consider the problem in the half-plane $\eta > 0$ outside the half-wedge:

$$\Lambda := \{\xi < 0, \eta > 0\} \cup \{\eta > \xi \tan \theta_w, \xi > 0\}.$$

Then the initial-boundary value problem (**Problem 2.1**) in the (t, \mathbf{x}) -coordinates can be formulated as the following boundary value problem in the self-similar coordinates (ξ, η) :

Problem 3.1 (Boundary value problem in the unbounded domain).

Seek a solution to system (3.1) satisfying the slip boundary condition on the wedge boundary and the matching condition on the symmetry line $\eta = 0$:

$$(U, V) \cdot \nu = 0 \quad \text{on } \partial\Lambda = \{\xi \leq 0, \eta = 0\} \cup \{\xi > 0, \eta \geq \xi \tan \theta_w\},$$

the asymptotic boundary condition as $\xi^2 + \eta^2 \rightarrow \infty$:

$$(U + \xi, V + \eta, p, \rho) \longrightarrow \begin{cases} (0, 0, p_0, \rho_0), & \xi > \xi_0, \eta > \xi \tan \theta_w, \\ (u_1, 0, p_1, \rho_1), & \xi < \xi_0, \eta > 0. \end{cases}$$

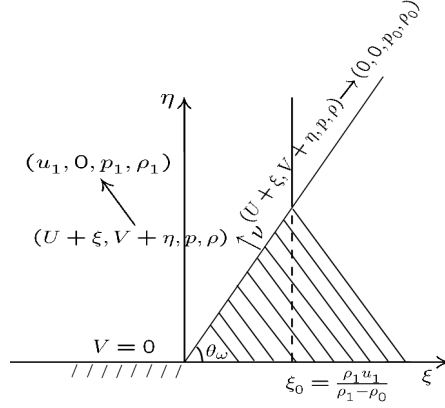


FIGURE 2. Boundary value problem in the unbounded domain Λ

It is expected that the solutions of **Problem 3.1** contain all possible patterns of shock reflection-diffraction configurations as observed in physical and numerical experiments; cf. [5, 54, 77, 80, 95, 108, 109, 129, 137] and the references cited therein.

4. Normal Reflection

The simplest case of the shock reflection-diffraction problem is when the wedge angle θ_w is $\pi/2$. In this case, the reflection-diffraction problem simply becomes the normal reflection problem, for which the incident shock normally reflects, and the reflected shock is also a plane. It can be shown that there exist a *unique* state (p_2, ρ_2) , $\rho_2 > \rho_1$, and a *unique* location of the reflected shock

$$(4.1) \quad \xi_1 = -\frac{\rho_1 u_1}{\rho_2 - \rho_1} \quad \text{with} \quad u_1 = \sqrt{\frac{(p_2 - p_1)(\rho_2 - \rho_1)}{\rho_1 \rho_2}}$$

such that state (2) $= (-\xi, -\eta, p_2, \rho_2)$ is subsonic inside the sonic circle with center at the origin and radius $c_2 = \sqrt{\gamma p_2 / \rho_2}$, and is supersonic outside the sonic circle (see Fig. 3). That is, in this case, the normal reflection solution is unique.

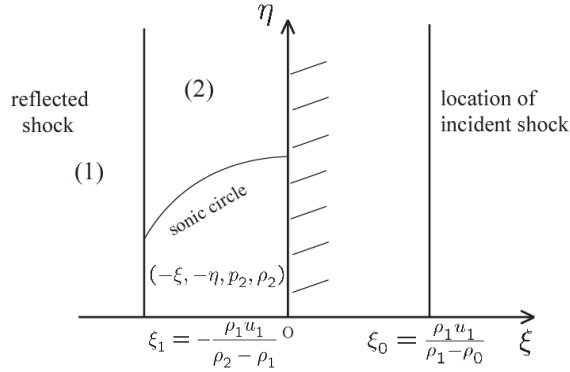


FIGURE 3. Normal reflection solution

In this case,

$$(4.2) \quad M_1^2 = \frac{2(\rho_2 - \rho_1)^2}{\rho_2((\gamma + 1)\rho_1 - (\gamma - 1)\rho_2)}, \quad \frac{p_2}{p_1} = \frac{(\gamma + 1)\rho_2 - (\gamma - 1)\rho_1}{(\gamma + 1)\rho_1 - (\gamma - 1)\rho_2},$$

and $\frac{p_2}{\rho_1} = t > 1$ is the unique root of

$$\left(1 + \frac{\gamma - 1}{2} M_1^2\right) t^2 - \left(2 + \frac{\gamma + 1}{2} M_1^2\right) t + 1 = 0,$$

that is,

$$(4.3) \quad \frac{\rho_2}{\rho_1} = \frac{4 + (\gamma + 1)M_1^2 + M_1 \sqrt{16 + (\gamma + 1)^2 M_1^2}}{2(2 + (\gamma - 1)M_1^2)}.$$

In other words, given ρ_0, p_0, ρ_1 , and $\gamma > 1$, state (2) $= (-\xi, -\eta, p_2, \rho_2)$ is uniquely determined through (2.10)–(2.11) and (4.1)–(4.3).

5. Local Theory and von Neumann's Conjectures for Regular Reflection-Diffraction Configuration

For a wedge angle $\theta_w \in (0, \pi/2)$, different reflection-diffraction patterns may occur. Various criteria and conjectures have been proposed for the existence of configurations for the patterns (cf. Ben-Dor [5]). One of the most important

conjectures made by von Neumann [142, 143] in 1943 is the *detachment conjecture*, which states that the regular reflection-diffraction configuration may exist globally whenever the two shock configuration (one is the incident shock and the other the reflected shock) exists locally around the point P_0 (see Fig. 4).

The following theorem was rigorously shown in Chang-Chen [18] (also see Sheng-Yin [127], Bleakney-Taub [12], Neumann [142, 143]).

THEOREM 5.1 (Local theory). *There exists $\theta_d = \theta_d(M_s, \gamma) \in (0, \pi/2)$ such that, when $\theta_w \in (\theta_d, \pi/2)$, there are two states (2) = $(U_2^a, V_2^a, p_2^a, \rho_2^a)$ and $(U_2^b, V_2^b, p_2^b, \rho_2^b)$ such that*

$$|(U_2^a, V_2^a)| > |(U_2^b, V_2^b)| \quad \text{and} \quad |(U_2^b, V_2^b)| < c_2^b,$$

where $c_2^b = \sqrt{\gamma p_2^b / \rho_2^b}$ is the sonic speed.

Then the conjecture can be stated as follows:

The von Neumann's Detachment Conjecture: *There exists a global regular reflection-diffraction configuration whenever the wedge angle θ_w is in $(\theta_d, \pi/2)$.*

It is clear that the regular reflection-diffraction configuration is not possible without a local two-shock configuration at the reflection point on the wedge, so this is the weakest possible criterion. In this case, the local theory indicates that there are two possible states for state (2). There had been a long debate to determine which one is more physical for the local theory; see Courant-Friedrichs [54], Bendor [5], and the references cited therein.

Since the reflection-diffraction problem is not a local problem, we take a different point of view that the selection of state (2) should be determined by the global features of the problem, more precisely, by the stability of the configuration with respect to the wedge angle θ_w , rather than the local features of the problem.

Stability Criterion to Select the Correct State (2): *Since the solution is unique when the wedge angle $\theta_w = \pi/2$, it is required that our global regular reflection-diffraction configuration should be stable and converge to the unique normal reflection solution when $\theta_w \rightarrow \pi/2$, provided that such a global configuration can be constructed.*

We employ this stability criterion to conclude that our choice for state (2) must be $(U_2^a, V_2^a, p_2^a, \rho_2^a)$. In general, $(U_2^a, V_2^a, p_2^a, \rho_2^a)$ may be supersonic or subsonic. If it is supersonic, the propagation speeds are finite and state (2) is completely determined by the local information: state (1), state (0), and the location of the point P_0 . This is, any information from the reflected region, especially the disturbance at the corner P_3 , cannot travel towards the reflection point P_0 . However, if it is subsonic, the information can reach P_0 and interact with it, potentially altering the reflection-diffraction type. This argument motivated the second conjecture as follows:

The von Neumann's Sonic Conjecture: *There exists a regular reflection-diffraction configuration when $\theta_w \in (\theta_s, \pi/2)$ for $\theta_s > \theta_d$ such that $|(U_2^a, V_2^a)| > c_2^a$ at P_0 .*

This sonic conjecture is based on the following fact: If state (2) is sonic when $\theta_w = \theta_s$, then $|(U_2^a, V_2^a)| > c_2^a$ for any $\theta_w \in (\theta_s, \pi/2)$. This conjecture is stronger than the detachment one. In fact, the regime between the angles θ_s and θ_d is very narrow and is only fractions of a degree apart; see Fig. 5 from Sheng-Yin [127].

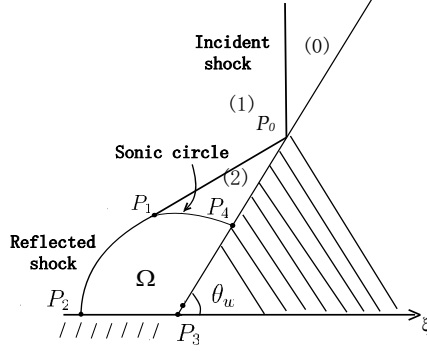
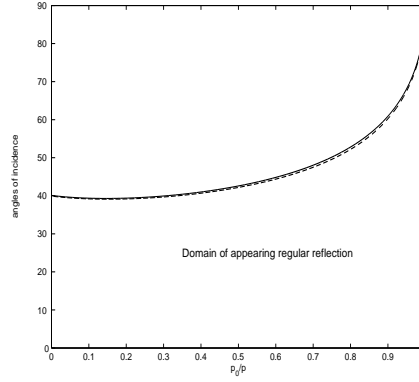


FIGURE 4. Regular reflection-diffraction configuration

FIGURE 5. The von Neumann's sonic criterion vs the detachment criterion $\theta_s > \theta_d$ when $\gamma = 1.4$.

6. The Potential Flow Equation

In this section, we discuss the role of the potential flow equation in the shock reflection-diffraction problem for the full Euler equations.

Under the Hodge-Helmoltz decomposition $(U, V) = \nabla\varphi + W$ with $\nabla \cdot W = 0$, the Euler equations (3.1) become

$$(6.1) \quad \nabla \cdot (\rho \nabla \varphi) + 2\rho + \nabla \cdot (\rho W) = 0,$$

$$(6.2) \quad \nabla \left(\frac{1}{2} |\nabla \varphi|^2 + \varphi \right) + \frac{1}{\rho} \nabla p = (\nabla \varphi + W) \cdot \nabla W + (\nabla^2 \varphi + I)W,$$

$$(6.3) \quad (\nabla \varphi + W) \cdot \nabla \omega + (1 + \Delta \varphi) \omega = 0,$$

$$(6.4) \quad (\nabla \varphi + W) \cdot \nabla S = 0,$$

where $\omega = \text{curl } W = \text{curl}(U, V)$ is the vorticity of the fluid, $S = c_v \ln(p\rho^{-\gamma})$ is the entropy, and the gradient ∇ is with respect to the self-similar variables (ξ, η) from now on.

When $\omega = 0$, $S = \text{const.}$, and $W = 0$ on a curve Γ transverse to the fluid direction, we first conclude from (6.3) that, in the domain Ω_1 determined by the

fluid trajectories:

$$\frac{d}{dt}(\xi, \eta) = (\nabla\varphi + W)(\xi, \eta)$$

past Γ ,

$$\omega = 0, \quad \text{i.e.} \quad \text{curl } W = 0.$$

This implies that $W = \text{const.}$ since $\nabla \cdot W = 0$. Then we conclude that

$$W = 0 \quad \text{in } \Omega_1,$$

since $W|_\Gamma = 0$, which yields that the right-hand side of equation (6.2) vanishes. Furthermore, from (6.4),

$$S = \text{const.} \quad \text{in } \Omega_1,$$

which implies that

$$p = \text{const.} \rho^\gamma.$$

By scaling, we finally conclude that the solutions of system (6.1)–(6.4) in the domain Ω_1 is determined by the following system for self-similar solutions:

$$(6.5) \quad \begin{cases} \nabla \cdot (\rho \nabla \varphi) + 2\rho = 0, \\ \frac{1}{2}|\nabla \varphi|^2 + \varphi + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}. \end{cases}$$

or the potential flow equation for self-similar solutions:

$$(6.6) \quad \nabla \cdot (\rho(\nabla \varphi, \varphi) \nabla \varphi) + 2\rho(\nabla \varphi, \varphi) = 0,$$

with

$$(6.7) \quad \rho(|\nabla \varphi|^2, \varphi) = (\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|\nabla \varphi|^2))^{\frac{1}{\gamma-1}}.$$

Then we have

$$(6.8) \quad c^2 = c^2(|\nabla \varphi|^2, \varphi, \rho_0^{\gamma-1}) = \rho_0^{\gamma-1} - (\gamma-1)(\frac{1}{2}|\nabla \varphi|^2 + \varphi).$$

For our problem (see Fig. 4), we note that, for state (2),

$$(6.9) \quad \omega = 0, \quad W = 0, \quad S = S_2.$$

Then, if our solution (U, V, p, ρ) is $C^{0,1}$ and the gradient of the tangential component of the velocity is continuous across the sonic arc Γ_{sonic} , we still have (6.9) along Γ_{sonic} on the side of Ω . Thus, we have

THEOREM 6.1. *Let (U, V, p, ρ) be a solution of our **Problem 3.1** such that (U, V, p, ρ) is $C^{0,1}$ in the open region $P_0P_1P_2P_3$ and the gradient of the tangential component of (U, V) is continuous across the sonic arc Γ_{sonic} . Let Ω_1 be the subregion of Ω formed by the fluid trajectories past the sonic arc Γ_{sonic} , then, in Ω_1 , the potential flow equation (6.6) with (6.7) coincides with the full Euler equations (6.1)–(6.4), that is, equation (6.6) with (6.7) is exact in the domain Ω_1 for **Problem 3.1**.*

REMARK 6.1. The regions such as Ω_1 also exist in various Mach reflection-diffraction configurations. Theorem 6.1 applies to such regions whenever the solution (U, V, p, ρ) is $C^{0,1}$ and the gradient of the tangential component of (U, V) is continuous. In fact, Theorem 8.3 indicates that, for the solutions φ of (6.6) with (6.7), the $C^{1,1}$ regularity of φ and the continuity of the tangential component of the velocity field $(U, V) = \nabla \varphi$ are optimal across the sonic arc Γ_{sonic} .

REMARK 6.2. The importance of the potential flow equation (2.6) with (2.5) in the time-dependent Euler flows was also observed by Hadamard [83] through a different argument.

Furthermore, when the wedge angle θ_w is close to $\pi/2$, it is expected that the curvature of the reflected shock is small so that, in the other part Ω_2 of Ω , the vorticity ω is small and the entropy is close to the constant. Then, in the reflection-diffraction domain $\Omega = \Omega_1 \cup \Omega_2$, the potential flow equation (6.6) with (6.7) dominates, provided that the exact state along the reflected shock is given.

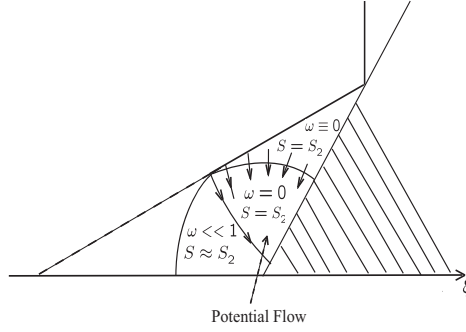


FIGURE 6. The potential flow equation dominates the domain Ω

Equation (6.6) with (6.7) is a nonlinear equation of mixed elliptic-hyperbolic type. It is elliptic if and only if

$$(6.10) \quad |\nabla\varphi| < c(|\nabla\varphi|^2, \varphi, \rho_0^{\gamma-1}),$$

which is equivalent to

$$(6.11) \quad |\nabla\varphi| < c_*(\varphi, \rho_0, \gamma) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}.$$

The study of partial differential equations of mixed hyperbolic-elliptic type can date back 1940s (cf. [11, 22, 24, 146, 149]). Linear models of partial differential equations of mixed hyperbolic-elliptic type include the Lavrentyev-Betsadze equation:

$$\partial_{xx}u + \text{sign}(x)\partial_{yy}u = 0,$$

the Tricomi equation:

$$u_{xx} + xu_{yy} = 0 \quad (\text{hyperbolic degeneracy at } x = 0),$$

and the Keldysh equation:

$$xu_{xx} + u_{yy} = 0 \quad (\text{parabolic degeneracy at } x = 0).$$

Nonlinear models of mixed-type equations for (6.6) with (6.7) include the transonic small disturbance equation:

$$\left((u-x)u_x + \frac{u}{2}\right)_x + u_{yy} = 0$$

or, for $v = u - x$,

$$(vv_x)_x + v_{yy} + \frac{3}{2}v_x + \frac{1}{2} = 0,$$

which has been studied in [15, 17, 87, 88, 89, 116] and the references cited therein. Also see [16, 130, 157, 158] for the models for self-similar solutions from the pressure gradient system and nonlinear wave equations.

7. Mathematical Formulation III: Free Boundary Problem for Potential Flow

For the potential equation (6.6) with (6.7), shocks are discontinuities in the pseudo-velocity $\nabla\varphi$. That is, if D^+ and $D^- := D \setminus \overline{D^+}$ are two nonempty open subsets of $D \subset \mathbb{R}^2$ and $S := \partial D^+ \cap D$ is a C^1 -curve where $D\varphi$ has a jump, then $\varphi \in W_{loc}^{1,1}(D) \cap C^1(D^\pm \cup S) \cap C^2(D^\pm)$ is a global weak solution of (6.6) with (6.7) in D if and only if φ is in $W_{loc}^{1,\infty}(D)$ and satisfies equation (6.6) in D^\pm and the Rankine-Hugoniot condition on S :

$$(7.1) \quad [\rho(|\nabla\varphi|^2, \varphi)\nabla\varphi \cdot \nu]_S = 0,$$

where the bracket $[\cdot]$ denotes the difference of the values of the quantity along the two sides of S .

Then the plane incident shock solution in the (t, \mathbf{x}) -coordinates with states $(\rho, \nabla_{\mathbf{x}}\Phi) = (\rho_0, 0, 0)$ and $(\rho_1, u_1, 0)$ corresponds to a continuous weak solution φ of (6.5) in the self-similar coordinates (ξ, η) with the following form:

$$(7.2) \quad \varphi_0(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) \quad \text{for } \xi > \xi_0,$$

$$(7.3) \quad \varphi_1(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) + u_1(\xi - \xi_0) \quad \text{for } \xi < \xi_0,$$

respectively, where

$$(7.4) \quad u_1 = \sqrt{\frac{2(\rho_1 - \rho_0)(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{(\gamma-1)(\rho_1 + \rho_0)}} > 0,$$

$$(7.5) \quad \xi_0 = \rho_1 \sqrt{\frac{2(\rho_1^{\gamma-1} - \rho_0^{\gamma-1})}{(\gamma-1)(\rho_1^2 - \rho_0^2)}} = \frac{\rho_1 u_1}{\rho_1 - \rho_0} > 0$$

are the velocity of state (1) and the location of the incident shock, uniquely determined by (ρ_0, ρ_1, γ) through (7.1). Then $P_0 = (\xi_0, \xi_0 \tan \theta_w)$ in Fig. 2, and **Problem 3.1** in the context of the potential flow equation can be formulated as:

Problem 7.1 (Boundary value problem) (see Fig. 2). *Seek a solution φ of equation (6.6) with (6.7) in the self-similar domain Λ with the boundary condition on $\partial\Lambda$:*

$$(7.6) \quad \nabla\varphi \cdot \nu|_{\partial\Lambda} = 0,$$

and the asymptotic boundary condition at infinity:

$$(7.7) \quad \varphi \rightarrow \bar{\varphi} := \begin{cases} \varphi_0 & \text{for } \xi > \xi_0, \eta > \xi \tan \theta_w, \\ \varphi_1 & \text{for } \xi < \xi_0, \eta > 0, \end{cases} \quad \text{when } \xi^2 + \eta^2 \rightarrow \infty,$$

where (7.7) holds in the sense that $\lim_{R \rightarrow \infty} \|\varphi - \bar{\varphi}\|_{C(\Lambda \setminus B_R(0))} = 0$.

For our problem, since φ_1 does not satisfy the slip boundary condition (7.6), the solution must differ from φ_1 in $\{\xi < \xi_0\} \cap \Lambda$, thus a shock diffraction-diffraction by the wedge vertex occurs. In Chen-Feldman [33, 34], we first followed the von Neumann criterion and the stability criterion introduced in Section 5 to establish

a local existence theory of regular shock reflection near the reflection point P_0 in the level of potential flow, when the wedge angle is large and close to $\pi/2$. In this case, the vertical line is the incident shock $S = \{\xi = \xi_0\}$ that hits the wedge at the point $P_0 = (\xi_0, \xi_0 \tan \theta_w)$, and state (0) and state (1) ahead of and behind S are given by φ_0 and φ_1 defined in (7.2) and (7.3), respectively. The solutions φ and φ_1 differ only in the domain $P_0P_1P_2P_3$ because of shock diffraction by the wedge vertex, where the curve $P_0P_1P_2$ is the reflected shock with the straight segment P_0P_1 . State (2) behind P_0P_1 can be computed explicitly with the form:

$$(7.8) \quad \varphi_2(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) + u_2(\xi - \xi_0) + (\eta - \xi_0 \tan \theta_w)u_2 \tan \theta_w,$$

which satisfies $\nabla \varphi \cdot \nu = 0$ on $\partial \Lambda \cap \{\xi > 0\}$; the constant velocity u_2 and the angle θ_s between P_0P_1 and the ξ -axis are determined by $(\theta_w, \rho_0, \rho_1, \gamma)$ from the two algebraic equations expressing (7.1) and continuous matching of state (1) and state (2) across P_0P_1 , whose existence is exactly guaranteed by the condition on $(\theta_w, \rho_0, \rho_1, \gamma)$ under which regular shock reflection-diffraction is expected to occur as in Theorem 5.1. Moreover, φ_2 is the unique solution in the domain $P_0P_1P_4$, as argued in [18, 123]. Denote

$$P_1P_4 := \Gamma_{sonic} = \partial \Omega \cap \partial B_{c_2}(u_2, u_2 \tan \theta_w)$$

the sonic arc of state (2) with center $(u_2, u_2 \tan \theta_w)$ and radius c_2 . Also we introduce the following notation for the other parts of $\partial \Omega$:

$$\Gamma_{shock} := P_1P_2; \quad \Gamma_{wedge} := \partial \Omega \cap \partial \Lambda \cap \{\eta > 0\} \equiv P_3P_4; \quad \Gamma_{symm} := \{\eta = 0\} \cap \partial \Omega.$$

Then **Problem 7.1** can be formulated as:

Problem 7.2. *Seek a solution φ in Ω to equation (6.6) with (6.7) subject to the boundary condition (7.6) on $\partial \Omega \cap \partial \Lambda$, the Rankine-Hugoniot conditions on the shock Γ_{shock} :*

$$(7.9) \quad [\varphi]_{\Gamma_{shock}} = 0,$$

$$(7.10) \quad [\rho(\nabla \varphi, \varphi, \rho_0) \nabla \varphi \cdot \nu]_{\Gamma_{shock}} = 0,$$

and the Dirichlet boundary condition on the sonic arc Γ_{sonic} :

$$(7.11) \quad (\varphi - \varphi_2)|_{\Gamma_{sonic}} = 0.$$

It should be noted that, in order that the solution φ in the domain Ω is a part of the global solution to **Problem 7.1**, that is, φ satisfies the equation in the sense of distributions in Λ , especially across the sonic arc Γ_{sonic} , it is required that

$$\nabla(\varphi - \varphi_2) \cdot \nu|_{\Gamma_{sonic}} = 0.$$

That is, we have to match our solution with state (2), which is the necessary condition for our solution in the domain Ω to be a part of the global solution. To achieve this, we have to show that our solution is at least C^1 with $\nabla(\varphi - \varphi_2) = 0$ across Γ_{sonic} .

Then the problem can be reformulated as the following free boundary problem:

Problem 7.3 (Free boundary problem). *Seek a solution φ and a free boundary $\Gamma_{shock} = \{\xi = f(\eta)\}$ such that*

(i) $f \in C^{1,\alpha}$ and

$$(7.12) \quad \Omega_+ = \{\xi > f(\eta)\} \cap D = \{\varphi < \varphi_1\} \cap D;$$

- (ii) φ satisfies the free boundary condition (7.10) along Γ_{shock} ;
- (iii) $\varphi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+)$ solves (6.5) in Ω_+ , is subsonic in Ω_+ , and satisfies

$$(7.13) \quad (\varphi - \varphi_2, \nabla(\varphi - \varphi_2) \cdot \nu)|_{\Gamma_{sonic}} = 0,$$

$$(7.14) \quad \nabla \varphi \cdot \nu|_{\Gamma_{wedge} \cup \Gamma_{symm}} = 0.$$

The boundary condition on Γ_{symm} implies that $f'(0) = 0$ and thus ensures the orthogonality of the free boundary with the ξ -axis. Formulation (7.12) implies that the free boundary is determined by the level set $\varphi = \varphi_1$, which is a convenient formulation to apply useful free boundary techniques. The free boundary condition (7.10) along Γ_{shock} is the conormal boundary condition on Γ_{shock} . Condition (7.13) ensures that the solution of the free boundary problem in Ω is a part of the global solution as pointed out earlier. Condition (7.14) is the slip boundary condition.

Problem 7.3 involves two types of transonic flow: one is a continuous transition through the sonic arc Γ_{sonic} as a fixed boundary from the pseudo-supersonic region (2) to the pseudo-subsonic region Ω ; the other is a jump transition through the transonic shock as a free boundary from the supersonic region (1) to the subsonic region Ω .

8. Global Theory for Regular Reflection-Diffraction for Potential Flow

In this section, we describe a global theory for regular reflection-diffraction established in Chen-Feldman [33, 34, 35] and Bae-Chen-Feldman [3].

8.1. Existence and stability of regular reflection-diffraction configurations. In Chen-Feldman [33, 34], we have developed a rigorous mathematical approach to solve **Problem 7.3** and established a global theory for solutions of regular reflection-diffraction, which converge to the unique solution of the normal shock reflection when θ_w tends to $\pi/2$.

Introduce the polar coordinates (r, θ) with respect to the center $(u_2, u_2 \tan \theta_w)$ of the sonic arc Γ_{sonic} of state (2), that is,

$$(8.1) \quad \xi - u_2 = r \cos \theta, \quad \eta - u_2 \tan \theta_w = r \sin \theta.$$

Then, for $\varepsilon \in (0, c_2)$, we denote by

$$\Omega_\varepsilon := \Omega \cap \{(r, \theta) : 0 < c_2 - r < \varepsilon\}$$

the ε -neighborhood of the sonic arc $P_1 P_4$ within Ω ; see Fig. 4. In Ω_ε , we introduce the coordinates:

$$(8.2) \quad x = c_2 - r, \quad y = \theta - \theta_w.$$

Then $\Omega_\varepsilon \subset \{0 < x < \varepsilon, y > 0\}$ and $P_1 P_4 \subset \{x = 0, y > 0\}$.

THEOREM 8.1 (Chen-Feldman [33, 34]). *There exist $\theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \pi/2)$ and $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, 1/2)$ such that, when $\theta_w \in (\theta_c, \pi/2)$, there exists a global self-similar solution*

$$\Phi(t, \mathbf{x}) = t \varphi\left(\frac{\mathbf{x}}{t}\right) + \frac{|\mathbf{x}|^2}{2t} \quad \text{for } \frac{\mathbf{x}}{t} \in \Lambda, t > 0$$

with $\rho(t, \mathbf{x}) = (\rho_0^{\gamma-1} - \Phi_t - \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2)^{\frac{1}{\gamma-1}}$ of **Problem 7.1** (equivalently, **Problem 7.2**) for shock reflection-diffraction by the wedge. The solution φ satisfies that, for

$$\begin{aligned}
(\xi, \eta) &= \mathbf{x}/t, \\
\varphi &\in C^{0,1}(\Lambda), \\
\varphi &\in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega}), \\
(8.3) \quad \varphi &= \begin{cases} \varphi_0 & \text{for } \xi > \xi_0 \text{ and } \eta > \xi \tan \theta_w, \\ \varphi_1 & \text{for } \xi < \xi_0 \text{ and above the reflected shock } P_0 P_1 P_2, \\ \varphi_2 & \text{in } P_0 P_1 P_4. \end{cases}
\end{aligned}$$

Moreover,

- (i) equation (6.5) is elliptic in Ω ;
- (ii) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω ;
- (iii) the reflected shock $P_0 P_1 P_2$ is C^2 at P_1 and C^∞ except P_1 ;
- (iv) there exists $\varepsilon_0 \in (0, c_2/2)$ such that $\varphi \in C^{1,1}(\bar{\Omega}_{\varepsilon_0}) \cap C^2(\bar{\Omega}_{\varepsilon_0} \setminus \overline{\Gamma_{sonic}})$; in particular, in the coordinates (8.2),

$$(8.4) \quad \|\varphi - \varphi_2\|_{2,0,\Omega_{\varepsilon_0}}^{(par)} := \sum_{0 \leq k+l \leq 2} \sup_{(x,y) \in \Omega_{\varepsilon_0}} (x^{k+\frac{1}{2}-2} |\partial_x^k \partial_y^l (\varphi - \varphi_2)(x, y)|) < \infty;$$

- (v) there exists $\delta_0 > 0$ so that, in the coordinates (8.2),

$$(8.5) \quad |\partial_x (\varphi - \varphi_2)(x, y)| \leq \frac{2 - \delta_0}{\gamma + 1} x \quad \text{in } \Omega_{\varepsilon_0};$$

- (vi) there exist $\omega > 0$ and a function $y = \hat{f}(x)$ such that, in the coordinates (8.2),

$$\begin{aligned}
(8.6) \quad \Omega_{\varepsilon_0} &= \{(x, y) : x \in (0, \varepsilon_0), \ 0 < y < \hat{f}(x)\}, \\
\Gamma_{shock} \cap \partial\Omega_{\varepsilon_0} &= \{(x, y) : x \in (0, \varepsilon_0), \ y = \hat{f}(x)\},
\end{aligned}$$

and

$$(8.7) \quad \|\hat{f}\|_{C^{1,1}([0, \varepsilon_0])} < \infty, \quad \frac{d\hat{f}}{dx} \geq \omega > 0 \quad \text{for } 0 < x < \varepsilon_0.$$

Furthermore, the solution φ is stable with respect to the wedge angle θ_w in $W_{loc}^{1,1}$ and converges in $W_{loc}^{1,1}$ to the unique solution of the normal reflection as $\theta_w \rightarrow \pi/2$.

The existence of a solution φ of **Problem 7.1** (equivalently, **Problem 7.2**), satisfying (8.3) and property (iv), follows from [34, Main Theorem]. Property (i) follows from Lemma 5.2 and Proposition 7.1 in [34]. Property (ii) follows from Proposition 7.1 and Section 9 in [34] which assert that $\varphi - \varphi_2 \in \mathcal{K}$, where the set \mathcal{K} is defined by (5.15) in [34]. Property (v) follows from Propositions 8.1–8.2 and Section 9 in [34]. Property (vi) follows from (5.7) and (5.25)–(5.27) in [34] and the fact that $\varphi - \varphi_2 \in \mathcal{K}$.

We remark that estimate (8.4) above confirms that our solutions satisfy the assumptions of Theorem 6.1 for the velocity field $(U, V) = \nabla\varphi$.

One of the main difficulties for the global existence is that the ellipticity condition (6.10) for (6.6) with (6.7) is hard to control, in comparison to our work on steady flow [29, 30, 31, 32]. The second difficulty is that the ellipticity degenerates along the sonic arc Γ_{sonic} . The third difficulty is that, on Γ_{sonic} , the solution in Ω has to be matched with φ_2 at least in C^1 , i.e., the two conditions on the fixed boundary Γ_{sonic} : the Dirichlet and conormal conditions, which are generically overdetermined for an elliptic equation since the conditions on the other parts

of boundary have been prescribed. Thus, one needs to prove that, if φ satisfies (6.6) in Ω , the Dirichlet continuity condition on the sonic arc, and the appropriate conditions on the other parts of $\partial\Omega$ derived from Problem 7.3, then the normal derivative $\nabla\varphi \cdot \nu$ automatically matches with $\nabla\varphi_2 \cdot \nu$ along Γ_{sonic} . Indeed, equation (6.6), written in terms of the function $\psi = \varphi - \varphi_2$ in the (x, y) -coordinates defined near Γ_{sonic} such that Γ_{sonic} becomes a segment on $\{x = 0\}$, has the form:

$$(8.8) \quad (2x - (\gamma + 1)\psi_x)\psi_{xx} + \frac{1}{c_2^2}\psi_{yy} - \psi_x = 0 \quad \text{in } x > 0 \text{ and near } x = 0,$$

plus the *small* terms that are controlled by $\pi/2 - \theta_w$ in appropriate norms. Equation (8.8) is *elliptic* if $\psi_x < 2x/(\gamma + 1)$. Hence, it is required to obtain the $C^{1,1}$ estimates near Γ_{sonic} to ensure $|\psi_x| < 2x/(\gamma + 1)$ which in turn implies both the ellipticity of the equation in Ω and the match of normal derivatives $\nabla\varphi \cdot \nu = \nabla\varphi_2 \cdot \nu$ along Γ_{sonic} . Taking into account the *small* terms to be added to equation (8.8), one needs to make the stronger estimate $|\psi_x| \leq 4x/(3(\gamma + 1))$ and assume that $\pi/2 - \theta_w$ is suitably small to control these additional terms. Another issue is the non-variational structure and nonlinearity of this problem which makes it hard to apply directly the approaches of Caffarelli [13] and Alt-Caffarelli-Friedman [1, 2]. Moreover, the elliptic degeneracy and geometry of the problem makes it difficult to apply the hodograph transform approach in Chen-Feldman [31] and Kinderlehrer-Nirenberg [94] to fix the free boundary.

For these reasons, one of the new ingredients in our approach is to develop further the iteration scheme in [29, 31] to a partially modified equation. We modified equation (6.6) in Ω by a proper Shiffmanization (i.e. a cutoff) that depends on the distance to the sonic arc, so that the original and modified equations coincide when φ satisfies $|\psi_x| \leq 4x/(3(\gamma + 1))$, and the modified equation $\mathcal{N}\varphi = 0$ is elliptic in Ω with elliptic degeneracy on P_1P_4 . Then we solved a free boundary problem for this modified equation: The free boundary is the curve Γ_{shock} , and the free boundary conditions on Γ_{shock} are $\varphi = \varphi_1$ and the Rankine-Hugoniot condition (7.1).

On each step, an *iteration free boundary* curve Γ_{sonic} is given, and a solution of the modified equation $\mathcal{N}\varphi = 0$ is constructed in Ω with the boundary condition (7.1) on Γ_{shock} , the Dirichlet condition $\varphi = \varphi_2$ on the degenerate arc Γ_{sonic} , and $\nabla\varphi \cdot \nu = 0$ on P_2P_3 and Γ_{wedge} . Then we proved that φ is in fact $C^{1,1}$ up to the boundary Γ_{sonic} , especially $|\nabla(\varphi - \varphi_2)| \leq Cx$, by using the nonlinear structure of elliptic degeneracy near Γ_{sonic} which is modeled by equation (8.8) and a scaling technique similar to Daskalopoulos-Hamilton [56] and Lin-Wang [105]. Furthermore, we modified the *iteration free boundary* curve Γ_{shock} by using the Dirichlet condition $\varphi = \varphi_1$ on Γ_{shock} . A fixed point φ of this *iteration procedure* is a solution of the free boundary problem for the modified equation. Moreover, we proved the precise gradient estimate: $|\psi_x| < 4x/(3(\gamma + 1))$ for ψ , which implies that φ satisfies the original equation (6.5).

This global theory for large-angle wedges has been extended in Chen-Feldman [35] to the sonic angle $\theta_s \leq \theta_c$, for which state (2) is sonic, such that, as long as $\theta_w \in (\theta_s, \pi/2]$, the global regular reflection-diffraction configuration exists.

THEOREM 8.2 (von Neumann's Sonic Conjecture (Chen-Feldman [35])). *The global existence result in Theorem 8.1 can be extended up to the sonic wedge-angle*

θ_s for any $\gamma \geq 1$ and $u_1 \leq c_1$. Moreover, the solutions satisfy the properties (i)–(vi) in Theorem 8.1.

The condition $u_1 \leq c_1$ depends explicitly only on the parameters $\gamma > 1$ and $\rho_1 > \rho_0 > 0$. For the case $u_1 > c_1$, we have been making substantial progress as well, and the final detailed results can be found in Chen-Feldman [35].

8.2. Optimal regularity. By Theorem 8.1(iv), the solution φ constructed there is at least $C^{1,1}$ near the sonic arc Γ_{sonic} . The next question is to analyze the behavior of solutions $\varphi(\xi, \eta)$ to regular reflection-diffraction, especially the optimal regularity of the solutions.

We first define the class of regular reflection-diffraction solutions.

DEFINITION 8.3. Let $\gamma > 1$, $\rho_1 > \rho_0 > 0$, and $\theta_w \in (0, \pi/2)$ be constants, let u_1 and ξ_0 be defined by (7.4) and (7.5). Let the incident shock $S = \{\xi = \xi_0\}$ hits the wedge at the point $P_0 = (\xi_0, \xi_0 \tan \theta_w)$, and let state (0) and state (1) ahead of and behind Γ_{shock} be given by (7.2) and (7.3), respectively. The function $\varphi \in C^{0,1}(\Lambda)$ is a regular reflection-diffraction solution if φ is a solution to **Problem 7.1** such that

- (a) there exists state (2) of form (7.8) with $u_2 > 0$, satisfying the entropy condition $\rho_2 > \rho_1$ and the Rankine-Hugoniot condition (7.1) along the line $S_1 := \{\varphi_1 = \varphi_2\}$ which contains the points P_0 and P_1 , such that $P_1 \in \Lambda$ is on the sonic circle of state (2), and state (2) is supersonic along P_0P_1 ;
- (b) there exists an open, connected domain $\Omega := P_1P_2P_3P_4 \subset \Lambda$ such that (8.3) holds and equation (6.5) is elliptic in Ω ;
- (c) $\varphi \geq \varphi_2$ on the part $P_1P_2 = \Gamma_{shock}$ of the reflected shock.

REMARK 8.1. The global solution constructed in [33, 34, 35] is a regular reflection-diffraction solution, which is a part of the assertions in Theorems 8.1–8.2.

REMARK 8.2. If state (2) exists and is supersonic, then the line $S_1 = \{\varphi_1 = \varphi_2\}$ necessarily intersects the sonic circle of state (2); see the argument in [33, 34] starting from (3.5) there. Thus, the only assumption regarding the point P_1 is that S_1 intersects the sonic circle within Λ .

REMARK 8.3. We note that, in the case $\theta_w = \frac{\pi}{2}$, the regular reflection becomes the normal reflection, in which $u_2 = 0$ and the solution is smooth across the sonic line of state (2); see [34, Section 3.1]. Condition $\theta_w \in (0, \frac{\pi}{2})$ in Definition 8.3 rules out this case. Moreover, for $\theta_w \in (0, \frac{\pi}{2})$, the property $u_2 > 0$ in part (a) of Definition 8.3 is always true for state (2) of form (7.8), satisfying the entropy condition $\rho_2 > \rho_1$ and the Rankine-Hugoniot condition (7.1) along the line $S_1 := \{\varphi_1 = \varphi_2\}$ which contains the point P_0 . These are readily derived from the calculations in [34, Section 3.2].

REMARK 8.4. There may exist a global regular reflection-diffraction configuration when state (2) is subsonic which is a very narrow regime [54, 142, 143]. Such a case does not involve the difficulty of elliptic degeneracy, which we are facing for the configurations in the class of solutions in the sense of Definition 8.3.

REMARK 8.5. Since $\varphi = \varphi_1$ on Γ_{shock} by (8.3), condition (c) in Definition 8.3 is equivalent to

$$\Gamma_{shock} \subset \{\varphi_2 \leq \varphi_1\},$$

that is, Γ_{shock} is below S_1 .

In Bae-Chen-Feldman [3], we have developed a mathematical approach to establish the regularity of solutions of the regular reflection-diffraction problem in the sense of Definition 8.3.

First, we have shown that any regular reflection-diffraction solutions cannot be C^2 across the sonic arc $\Gamma_{sonic} := P_1 P_4$.

THEOREM 8.4 (Bae-Chen-Feldman [3]). *There does not exist a global regular reflection-diffraction solution in the sense of Definition 8.3 such that φ is C^2 across the sonic arc Γ_{sonic} .*

Now we study the one-sided regularity up to Γ_{sonic} from the elliptic side, i.e., from Ω . For simplicity of presentation, we now use a localized version of Ω_ε : For a given neighborhood $\mathcal{N}(\Gamma_{sonic})$ of Γ_{sonic} and $\varepsilon > 0$, define

$$\Omega_\varepsilon := \Omega \cap \mathcal{N}(\Gamma_{sonic}) \cap \{x < \varepsilon\}.$$

Since $\mathcal{N}(\Gamma_{sonic})$ will be fixed in the following theorem, we do not specify the dependence of Ω_ε on $\mathcal{N}(\Gamma_{sonic})$.

THEOREM 8.5 (Bae-Chen-Feldman [3]). *Let φ be a regular reflection-diffraction solution in the sense of Definition 8.3 and satisfy the following properties: There exists a neighborhood $\mathcal{N}(\Gamma_{sonic})$ of Γ_{sonic} such that*

- (a) φ is $C^{1,1}$ across the sonic arc Γ_{sonic} : $\varphi \in C^{1,1}(\overline{P_0 P_1 P_2 P_3} \cap \mathcal{N}(\Gamma_{sonic}))$;
- (b) there exists $\delta_0 > 0$ so that, in the coordinates (8.2),

$$(8.9) \quad |\partial_x(\varphi - \varphi_2)(x, y)| \leq \frac{2 - \delta_0}{\gamma + 1} x \quad \text{in } \Omega \cap \mathcal{N}(\Gamma_{sonic});$$

- (c) there exists $\varepsilon_0 > 0$, $\omega > 0$, and a function $y = \hat{f}(x)$ such that, in the coordinates (8.2),

$$(8.10) \quad \begin{aligned} \Omega_{\varepsilon_0} &= \{(x, y) : x \in (0, \varepsilon_0), \ 0 < y < \hat{f}(x)\}, \\ \Gamma_{shock} \cap \partial\Omega_{\varepsilon_0} &= \{(x, y) : x \in (0, \varepsilon_0), \ y = \hat{f}(x)\}, \end{aligned}$$

and

$$(8.11) \quad \|\hat{f}\|_{C^{1,1}([0, \varepsilon_0])} < \infty, \quad \frac{d\hat{f}}{dx} \geq \omega > 0 \quad \text{for } 0 < x < \varepsilon_0.$$

Then we have

- (i) φ is $C^{2,\alpha}$ up to Γ_{sonic} away from P_1 for any $\alpha \in (0, 1)$. That is, for any $\alpha \in (0, 1)$ and any given $(\xi_0, \eta_0) \in \overline{\Gamma_{sonic}} \setminus \{P_1\}$, there exists $K < \infty$ depending only on $\rho_0, \rho_1, \gamma, \varepsilon_0, \alpha, \|\varphi\|_{C^{1,1}(\Omega_{\varepsilon_0})}$, and $d = \text{dist}((\xi_0, \eta_0), \Gamma_{shock})$ so that

$$\|\varphi\|_{2,\alpha;\overline{B_{d/2}(\xi_0, \eta_0)} \cap \Omega} \leq K;$$

- (ii) For any $(\xi_0, \eta_0) \in \Gamma_{sonic} \setminus \{P_1\}$,

$$\lim_{\substack{(\xi, \eta) \rightarrow (\xi_0, \eta_0) \\ (\xi, \eta) \in \Omega}} (D_{rr}\varphi - D_{rr}\varphi_2) = \frac{1}{\gamma + 1};$$

(iii) $D^2\varphi$ has a jump across Γ_{sonic} : For any $(\xi_0, \eta_0) \in \Gamma_{sonic} \setminus \{P_1\}$,

$$\begin{aligned} \lim_{\substack{(\xi, \eta) \rightarrow (\xi_0, \eta_0) \\ (\xi, \eta) \in \Omega}} D_{rr}\varphi - \lim_{\substack{(\xi, \eta) \rightarrow (\xi_0, \eta_0) \\ (\xi, \eta) \in \Lambda \setminus \Omega}} D_{rr}\varphi &= \frac{1}{\gamma + 1}, \\ \lim_{\substack{(\xi, \eta) \rightarrow (\xi_0, \eta_0) \\ (\xi, \eta) \in \Omega}} (D_{r\theta}, D_{\theta\theta})\varphi &= \lim_{\substack{(\xi, \eta) \rightarrow (\xi_0, \eta_0) \\ (\xi, \eta) \in \Lambda \setminus \Omega}} (D_{r\theta}, D_{\theta\theta})\varphi = 0; \end{aligned}$$

(iv) The limit $\lim_{\substack{(\xi, \eta) \rightarrow P_1 \\ (\xi, \eta) \in \Omega}} D^2\varphi$ does not exist.

We remark that the solutions established in [33, 34, 35] satisfy the assumptions of Theorem 8.5. In particular, we proved that the $C^{1,1}$ -regularity is *optimal* for the solution across the open part P_1P_4 of the sonic arc (the degenerate elliptic curve) and at the point P_1 where the sonic circle meets the reflected shock (as a free boundary).

To achieve the optimal regularity, one of the main difficulties is that the sonic arc Γ_{sonic} is the transonic boundary separating the elliptic region from the hyperbolic region, near where the solution is governed by the nonlinear degenerate elliptic equation (8.8) for $\psi = \varphi - \varphi_2$. We carefully analyzed the features of equation (8.8) and established the $C^{2,\alpha}$ regularity of solutions in the elliptic region up to the open sonic arc P_1P_4 . As a corollary, we showed that the $C^{1,1}$ -regularity is actually optimal across the transonic boundary P_1P_2 from the elliptic to hyperbolic region. Since the reflected shock P_1P_2 is regarded as a free boundary connecting the hyperbolic region (1) with the elliptic region Ω for the nonlinear second-order equation of mixed type, another difficulty for the optimal regularity of the solution is that the point P_1 is exactly the point where the degenerate elliptic arc P_1P_4 meets a transonic free boundary for the nonlinear partial differential equation of second order. As far as we know, this is the first optimal regularity result for solutions to a free boundary problem of nonlinear degenerate elliptic equations at the point where an elliptic degenerate curve meets the free boundary. To achieve this, we carefully constructed two sequences of points on where the corresponding sequences of values of ψ_{xx} have different limits at P_1 ; this has been done by employing the one-sided $C^{2,\alpha}$ regularity of the solution up to the open arc P_1P_4 and by studying detailed features of the free boundary conditions on the free boundary P_1P_2 , i.e., the Rankine-Hugoniot conditions.

We remark that some efforts were also made mathematically for the reflection-diffraction problem via simplified models. One of these models, the unsteady transonic small-disturbance (UTSD) equation, was derived and used in Keller-Blank [93], Hunter-Keller [90], Hunter [87], and Morawetz [116] for asymptotic analysis of shock reflection-diffraction. Also see Zheng [158] for the pressure gradient equation and Canic-Keyfitz-Kim [15] for the UTSD equation and the nonlinear wave system. Furthermore, in order to deal with the reflection-diffraction problem, some asymptotic methods have been also developed. Lighthill [104] studied shock reflection-diffraction under the assumption that the wedge angle is either very small or close to $\pi/2$. Keller-Blank [93], Hunter-Keller [90], Harabetian [84], and Gamba-Rosales-Tabak [67] considered the problem under the assumption that the shock is so weak that its motion can be approximated by an acoustic wave. For a weak incident shock and a wedge with small angle for potential flow, by taking the jump of the incident shock as a small parameter, the nature of the shock reflection-diffraction pattern was explored in Morawetz [116] by a number

of different scalings, a study of mixed equations, and matching the asymptotics for the different scalings. Also see Chen [39] for a linear approximation of shock reflection-diffraction when the wedge angle is close to $\pi/2$ and Serre [123] for an apriori analysis of solutions of shock reflection-diffraction and related discussions in the context of the Euler equations for isentropic and adiabatic fluids.

Another related recent effort has been on various important physical problems in steady potential flow, as well as steady fully Euler flow, for which great progress has been made. The problems for global subsonic flow past an obstacle and for local supersonic flow past an obstacle with sharp head are classical, due to the works of Shiffman [128], Bers [9], Finn-Gilbarg [64], Dong [59], and others for the pure elliptic case and to the works of Gu [79], Shaeffer [119], Li [102], and others for the pure hyperbolic case. Recent progress has been made on transonic flow past nozzles (e.g. [4, 25, 26, 29, 30, 32, 37, 43, 49, 148, 150, 151]), transonic flow past a wedge or conical body (e.g. [28, 47, 63]), and transonic flow past a smooth obstacle (e.g. [36, 66, 115, 117]). Also see [27, 147] for the existence of global subsonic-sonic flow, [38, 40, 41, 42, 44, 48, 103, 155] for global supersonic flow past an obstacle with sharp head, and the references cited therein.

For some of other recent related developments, we refer the reader to Chen [45, 46] for the local stability of Mach configuration, Elling-Liu [62] for physicality of weak Prandtl-Meyer reflection for supersonic potential flow around a ramp, Serre [125] for multidimensional shock interaction for a Chaplygin gas, Canic-Keyfitz-Kim [15, 16] for semi-global solutions for the shock reflection problem, Glimm-Ji-Li-Li-Zhang-Zhang-Zheng [75] for the formation of a transonic shock in a rarefaction Riemann problem for polytropic gases, Zheng [156, 157, 158] for various solutions to some two-dimensional Riemann problems, Gues-Métivier-Williams-Zumbrun [81, 82] and Benzoni-Gavage and Serre [7] for revisits of the local stability of multidimensional shocks and phase boundaries, among many others.

9. Shock Reflection-Diffraction vs New Mathematics

As we have seen from the previous discussion, the shock reflection-diffraction problem involves several core challenging difficulties that we have to face in the study of nonlinear partial differential equations. These nonlinear difficulties include *free boundary problems*, *oblique derivative problems for nonsmooth domains*, *degenerate elliptic equations*, *degenerate hyperbolic equations*, *transport equations with rough coefficients*, *mixed and/or composite equations of hyperbolic-elliptic type*, *behavior of solutions when a free boundary meets an elliptic degenerate curve*, and *compressible vortex sheets*. More efficient numerical methods are also required for further understanding of shock reflection-diffraction phenomena.

Furthermore, the wave patterns of shock reflection-diffraction configurations are the core patterns and configurations for the global solutions of the two-dimensional Riemann problem; these solutions are building blocks and local structure of general entropy solutions and determine global attractors and asymptotic states of entropy solutions, as time goes infinity, for two-dimensional systems of hyperbolic conservation laws.

Therefore, a successful solution to the shock reflection-diffraction problem not only provides our understanding of shock reflection-diffraction phenomena and behavior of entropy solutions to multidimensional conservation laws, but also provides

important new ideas, insights, techniques, and approaches for our developments of more efficient analytical techniques and methods to overcome the core challenging difficulties in multidimensional problems in conservation laws and other areas in nonlinear partial differential equations. The shock reflection-diffraction problem is also an excellent test problem to examine our capacity and ability to solve rigorously various challenging problems for nonlinear partial differential equations and related applications.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208-2730,
USA

E-mail address: `ggchen@math.northwestern.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706-1388, USA

E-mail address: `feldman@math.wisc.edu`