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Conservation Laws: Transonic Flow and Differential Geometry

Gui-Qiang Chen, Marshall Slemrod, and Dehua Wang

Abstract. The connection between gas dynamics and differential geometry is discussed. Some history of boundary value problems for systems of conservation laws is first given. Then the mathematical formulation of compressible gas dynamics, especially the subsonic and transonic flows past an obstacle (such as an airfoil), is provided. Some recent results on transonic flow from viscous approximation and compensated compactness are presented. Finally, a fluid dynamic formulation of the Gauss-Codazzi system for the isometric embedding problem in $\mathbb{R}^3$ is discussed.

1. History of Boundary Value Problems for Systems of Conservation Laws

A quick glance at post-World War II literature may surprise the reader. The subject of systems of conservation laws was dominated not by initial value problems but by boundary value problems. A little thought of course makes this state even clear: It was the underlying military role of gas dynamics, and the flow of compressible fluids over boundaries that was crucial in wartime and postwar research. However, taking a more civilian view, it is pleasant to quote A. Jameson [**17**] who wrote: *"The most important requirement for aeronautical applications of computational methods in fluid mechanics is the capability to predict the steady flow past a proposed configuration, so that key performance parameters such as the lift to drag ratio can be estimated. Even in maneuvering flight the time scales of the motion are large compared with those of the flow, so that unsteady effects are secondary".* In slightly less technical terms, we civilian fliers should note that, in most of the time we spend on our flight, we are flying at steady (not accelerating or decelerating) flow, and the engineer and mathematician need be concerned only with boundary

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value problems as opposed to initial value ones. This of course omits the minor issues of take off and landing.

The above remarks were reflected in the major postwar mathematical literature. For example, the classic monographs of Courant and Friedrichs [**9**] written around 1948 and Bers [**3**] published in 1958 pay scant attention to initial value problems. It is the multi-dimensionality of the boundary value problems that comes to the forefront and all the baggage (unfortunately in this case, not lost) that such problems bring with them. These books reflected the atmosphere of their times, from both the applied and analytical views. Courant and Friedrichs in [**9**] emphasized a myriad of applications and suggested where analysis could play a vital role in understanding boundary value problems in gas dynamics. On the other hand, Bers in [**3**] approached the gas dynamic problems as a motivating problem in nonlinear partial differential equations and summarized the results by himself and others on the existence and uniqueness of solutions for the relevant conservation laws. In fact, to quote Bers's introduction: *"These problems, while admittedly difficult, are exceedingly challenging and give us a glimpse of the long lost golden age of the unity of science. Indeed, physicists interested in them demand rigorous mathematical proofs, and mathematicians working on them need guidance from the research of experiments"*. With Bers's inspirational words, the subject of boundary value problems for systems of conservation laws seemed ready to begin a "golden age" in the late 1950's. Unfortunately, with the exception of a few brave souls (e.g., C. Morawetz, B. Keyfitz, S.-X. Chen), research in the area was quite limited in the following fifty years. Interest among experts was transferred to initial value problems where the fundamental ideas of P. D. Lax, J. Glimm, and O. Oleinik were documented and the necessity of facing issues of "multi-dimensionality" were less evident. However, the basic underlying mathematical problems described by Courant and Friedrichs, Bers, and others remained and were of no less importance than they had been in the 1950's.

2. Mathematical Formulation of Compressible Gas Dynamics

We first recall the basic balance laws of gas dynamics. To keep things simple, we restrict ourselves to two space dimensions (x, y). Then the conservation laws of mass, momentum, and energy can be written as

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, & \text{(mass)} \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, & \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, & \text{(momentum)} \\ (\rho e + \frac{1}{2}\rho q^2)_t + ((\rho e + \frac{1}{2}\rho q^2 + p)u)_x + ((\rho e + \frac{1}{2}\rho q^2)v + p)v)_y = 0, & \text{(energy)} \end{cases}$$

where ρ is the density, (u, v) is the velocity, $q^2 = u^2 + v^2$, e is the specific internal energy, and p is the pressure. This yields the four equations in the five unknown (ρ, u, v, p, e). The closure is accomplished by choosing constitutive relations: For polytropic gases,

$$p = R\rho\theta, \qquad e = c_v\theta, \qquad \gamma = 1 + \frac{R}{c_v},$$

where θ is the temperature, and c_v and R are constants.

If we presumed θ to be constant, the equations of mass and momentum would be satisfied; and the energy equation could be satisfied by the addition of an energy source term to force the balance law. On the other hand, setting p as

$$p = k\rho^\gamma e^{S/c_v} \qquad \text{for some constant } k,$$

combined with the above relation for p, yields a relation between (θ, ρ) and the specific entropy S. In this case, the energy equation becomes

$$(\rho S)_t + (\rho u S)_x + (\rho v S)_y = 0.$$

Notice that, if S is identically constant, the above energy equation becomes our conservation law of mass, and hence it yields the convenient (albeit over simplification) special case. In this special isentropic case,

$$p = \text{const.}\, \rho^\gamma, \quad \gamma > 1;$$

while in the special isothermal case,

$$p = \text{const.}\, \rho,$$

i.e., $\gamma = 1$. In the discussion given here, we normalize our constant so that

$$p = \rho^\gamma/\gamma, \quad \gamma \geq 1,$$

thus covering both the isothermal and isentropic cases. Of course, in both special cases, the energy equation is now assumed to be balanced automatically by either an energy source (or a fictitious entropic source in the case of discontinuous solutions in the isentropic case). Furthermore, in line with our goal to discuss only steady solutions, we now foucs on the three equations:

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, & \text{(mass)} \\ (\rho u^2 + p)_x + (\rho u v)_y = 0, & \\ (\rho u v)_x + (\rho v^2 + p)_y = 0, & \text{(momentum)} \end{cases}$$

where $p = \frac{\rho^\gamma}{\gamma}$, $\gamma \geq 1$.

So far, we have only used the classical balance laws of mass and momentum with an assumed energy balance combined with the assumption that we are studying a polytropic gas, e.g., air. In the next simplification, we make an assumption about the fluid flow itself that it is irrotational, which in our case means

$$v_x - u_y = 0. \qquad \text{(irrotationality)}$$

The combination of irrotationality with the balance laws of mass and momentum yields the four equations in the three dependent variables (ρ, u, v), which leads to a classic relation between ρ and q^2 $(= u^2 + v^2)$, the Bernoulli relation:

$$\rho = \Big(1 - \frac{\gamma - 1}{2} q^2\Big)^{\frac{1}{\gamma - 1}}.$$

This is a wonderful simplification, since the pair of equations for conservation of mass and irrotationality:

$$(\rho u)_x + (\rho v)_y = 0, \qquad \text{(mass)}$$

$$v_x - u_y = 0, \qquad \text{(irrotationality)}$$

combined with the Bernoulli equation provides two equations for the two unknowns (u, v). Furthermore, we note that, if ρ was constant (it is not for a compressible gas), we would have the Cauchy-Riemann equations and would reduce any boundary value problem to one of the classical elliptic partial differential equations. Fortunately, the elliptic theory still plays a role in the nonlinear case. Simply introduce the velocity potential φ so

$$(u, v) = \nabla\varphi.$$

Then the irrotationality equation is immediately satisfied, and our conservation law of mass becomes

$$(\rho\varphi_x)_x + (\rho\varphi_y)_y = 0 \tag{mass}$$

which is combined with the Bernoulli relation:

$$\rho = \Big(1 - \frac{\gamma - 1}{2}|\nabla\varphi|^2\Big)^{\frac{1}{\gamma-1}}.$$

The type of this equation is determined by the following relations. Introduce the sound speed c as

$$c^2 = p'(\rho) = 1 - \frac{\gamma - 1}{2}q^2,$$

so that, at the sonic value when $q = c$, we have $q = q_{cr}$ with

$$q_{cr} := \sqrt{\frac{2}{\gamma + 1}}.$$

Our equation of conservation of mass is then elliptic if $q < q_{cr}$ and hyperbolic when $q > q_{cr}$. Of course, there is an upper bound placed on q from the Bernoulli relation:

$$q \le q_{cav} := \sqrt{\frac{2}{\gamma - 1}},$$

where q_{cav} is the cavitation speed for which $\rho = 0$.

An appealing direction is to note that, if we define

$$G(q) = \int^{q^2} \Big(1 - \frac{\gamma - 1}{2}s\Big)^{\frac{1}{\gamma-1}} ds,$$

then the conservation law of mass corresponds to the Euler-Lagrange equation for the functional

$$\int_\Omega G(|\nabla\varphi|)dxdy.$$

An elementary computation shows

$$G''(q) \begin{cases} > 0 & q < q_{cr}, \\ < 0 & q > q_{cr}, \end{cases}$$

and hence the direct method of calculus of variations (e.g., Evans [**12**]) will provide the existence of weak solutions if it is known a priori that the flow is subsonic ($q < q_{cr}$) so that G is convex and the problem is elliptic. For example, this includes the fundamental problem of subsonic flow around a profile as formulated in Bers's book [**3**].

A profile $\mathcal{P}$ is prescribed by a smooth curve, except for a trailing edge with an opening $\varepsilon\pi$ at z_T, $0 \le \varepsilon \le 1$. If $\varepsilon = 0$, the profile has a tangent at the trailing edge. The tangent to $\mathcal{P}$ satisfies a uniform Hölder condition with respect to arc length. The velocity $w = (u, v)$ must attain a given subsonic limit at infinity. We enforce the Kutta-Joukowski condition: $q \to 0$ as $(x, y) \to z_T$ if $\varepsilon = 1$, $q = O(1)$ as

$(x, y) \to z_T$ if $0 \le \varepsilon < 1$, and define problem $P_1(w_\infty)$ with a prescribed constant velocity w_∞ at infinity. For a smooth profile, $\varepsilon = 1$, we define the circulation

$$\Gamma = \oint_{\mathcal{P}} (u, v) \cdot \mathbf{t}\, ds,$$

where $\mathbf{t}$ is the unit tangent to $\mathcal{P}$. In this case, we can also consider problem $P_2(w_\infty, \Gamma)$ where the circulation is prescribed, instead of the Kutta-Joukowski condition. In both problems, we require that the flow travel around the profile tangentially, that is,

$$(u, v) \cdot \mathbf{n} = 0 \qquad \text{on } \mathcal{P} \qquad \text{(boundary condition)}$$

where $\mathbf{n}$ is the exterior unit normal on P.

The first existence theorem for P_1 was given by Frankl and Keldysh [**14**] for sufficiently small speed at infinity. For a general gas, the first complete existence theorem for P_2 was given by Shiffman [**26**]. This was followed by a complete existence and uniqueness theorem by Bers [**2**] for P_1, a stronger uniqueness result of Finn and Gilbarg [**13**], and a higher dimensional result of Dong [**11**]. The basic result is as follows:

For a given constant velocity at infinity, there exists a number $\hat{q}$ depending upon the profile $\mathcal{P}$ and the equation of state for p such that the problem $P_1(w_\infty)$ has a unique solution for $0 < q_\infty := |w_\infty| < \hat{q}$. The maximum q_m of q takes on all values between 0 and q_{cr}, $q_m \to 0$ as $q_\infty \to 0$, and $q_m \to q_{cr}$ as $q_\infty \to \hat{q}$. A similar result holds for $P_2(w_\infty, \Gamma)$.

The main tool for the results is to know a priori that, if $q_\infty < \hat{q}$ (i.e., the speed at infinity is not only subsonic) but sufficiently subsonic, then $q < q_{cr}$ in whole flow domain. Subsonic flow at infinity itself does not guarantee that the flow remains subsonic, since the profile will produce flow orthogonal to the original flow direction. We also note that Shiffman's proof did use the direct method of the calculus of variations, while Bers's relied on both elliptic methods and the theory of pseudo-analytic functions. The existence of a critical point for the variational problem for the case when q_∞ is not restricted to be less than $\hat{q}$ would be a natural goal, since it would provide a direct proof of our boundary value problem. As of this writing, no such a proof has been given.

More recent investigations based on weak convergence methods start in the 1980's. DiPerna [**10**] suggested that the Murat-Tartar method of compensated compactness would be amenable to flows which exhibit both elliptic and hyperbolic regimes, and investigated an asymptotic approximation to our system (irrotation; mass conservation, Bernoulli) called the steady transonic small disturbance equation. He proved that, if a list of assumptions were satisfied (which then guaranteed the applicability of the compensated compactness method), then a weak solution exists to the steady transonic small disturbance equation. Unfortunately, no one has ever been able to show that DiPerna's list is indeed satisfied. In two significant papers [**20, 21**] written a decade apart (and surveyed by her another decade later [**22**]), Morawetz layed out a program for proving the existence of the steady flow problem about a bump profile in the upper half plane (which is equivalent to a symmetric profile in the whole plane). Similar to DiPerna's framework, Morawetz showed that, if the key hypotheses of the method of compensated compactness could be satisfied, now known as a "compactness framework" (see Chen [**4**]), then indeed there would exist a weak solution to the problem of flow over a bump which is exhibited by subsonic and supersonic regimes, i.e., transonic flow.

The "compactness framework" for our system is rather easy to state. Let a sequence of functions $w^\varepsilon(x,y) = (u^\varepsilon, v^\varepsilon)(x,y)$ defined on an open set $\Omega \subset \mathbb{R}^2$ satisfy the following set of conditions:

(A.1) $q^\varepsilon(x,y) = |w^\varepsilon(x,y)| \le q_*$ a.e. in Ω for some positive constant $q_* < q_{cav}$;

(A.2) $\partial_x Q_{1\pm}(w^\varepsilon) + \partial_y Q_{2\pm}(w^\varepsilon)$ are confined in a compact set in $H^{-1}_{\rm loc}(\Omega)$ for entropy-entropy flux pairs $(Q_{1\pm}, Q_{2\pm})$ ((Q_1, Q_2) is an entropy-entropy flux pair if $\partial_x Q_1(w^\varepsilon) + \partial_y Q_2(w^\varepsilon) = 0$ along smooth solutions of our system), where $(Q_{1\pm}(w^\varepsilon), Q_{2\pm}(w^\varepsilon))$ are confined to a bounded set uniformly in $L^\infty_{\rm loc}(\Omega)$.

In case (A.1) and (A.2) hold, then the Div-Curl lemma of Tartar [**27**] and Murat [**23**] and the Young measure representation theorem for a uniformly bounded sequence of functions (c.f. Tartar [**27**], Ball [**1**]) yield the following commutation identity:

$$
\begin{aligned}
&\langle \nu(w),\ Q_{1+}(w)Q_{2-}(w) - Q_{1-}(w)Q_{2+}(w)\rangle \\
&= \langle \nu(w),\ Q_{1+}(w)\rangle \langle \nu(w),\ Q_{2-}(w)\rangle - \langle \nu(w),\ Q_{1-}(w)\rangle \langle \nu(w),\ Q_{2+}(w)\rangle,
\end{aligned}
$$

where $\nu = \nu_{x,y}(w), w = (u,v)$, is the associated family of Young measures (probability measures) for the sequence $w^\varepsilon(x,y) = (u^\varepsilon, v^\varepsilon)(x,y)$.

The main point for the compensated compactness framework is to prove that ν is a Dirac measure by using entropy pairs, which implies the compactness of the sequence $w^\varepsilon(x,y) = (u^\varepsilon, v^\varepsilon)(x,y)$ in $L^1_{\rm loc}(\Omega)$. In this context, both DiPerna [**10**] and Morawetz [**20, 21**] needed to presume the existence of an approximating sequence parameterized by ε to their problems satisfying (A.1) and (A.2) so that they could exploit the commutation identity and obtain the strong convergence in $L^1_{\rm loc}(\Omega)$ to a weak solution of their problems.

As it turns out, there is one problem where (A.1) and (A.2) hold trivially, i.e., the sonic limit of subsonic flows. In that case, we return to the result by Bers [**2**] and Shiffman [**26**], which says that, if the speed at infinity q_∞ is less than some $\hat{q}$, there is a smooth unique solution to problems $P_1(w_\infty)$ and $P_2(w_\infty, \Gamma)$ and ask what happens as $q_\infty \nearrow \hat{q}$. In this case, the flow will develop sonic points and the governing equations become degenerate elliptic. Thus, if we set $\varepsilon = \hat{q} - q_\infty$ and examine a sequence of exact smooth solutions to our system, we see trivially that (A.1) is satisfied since $|q_\varepsilon| \le q_{cr}$, and (A.2) is satisfied since $\partial_x Q_\pm(w^\varepsilon) + \partial_y Q_\pm(w^\varepsilon) = 0$ along our solution sequence. The effort is in finding entropy-entropy flux pairs which will guarantee the Young measure ν reduces to a Dirac mass. Ironically, the search is very short. The original equations of momentum conservation provide two sets of entropy-entropy flux pairs, while the irrotationality and mass conservation equations provide another two sets. This observation has been explored in detail in Chen-Dafermos-Slemrod-Wang [**5**].

What then about the fully transonic problem of flow past an obstacle or bump where $q_\infty > \hat{q}$? A recent result of Chen-Slemrod-Wang [**6**] provides some of the ingredients to satisfying (A.1) and (A.2). In that paper, we introduced the usual flow angle $\theta = \tan^{-1}(\frac{v}{u})$ and wrote the irrotationality and mass conservation equation as an artificially viscous problem:

$$
\begin{aligned}
&v_x - u_y = \varepsilon \Delta \theta, \\
&(\rho u)_x + (\rho v)_y = \varepsilon \nabla \cdot (\sigma(\rho) \nabla \rho),
\end{aligned}
$$

where $\sigma(\rho)$ is suitably chosen, and appropriate boundary conditions are imposed for this regularized "viscous" problem. The crucial points are that a uniformly

L^∞ bound in q^ε is obtained when $1 \leq \gamma < 3$ which uniformly prevents cavitation. Unfortunately, in this formulation (and Morawetz's as well), a uniform bound in flow angle θ^ε must be assumed a priori to guarantee the (q, θ)–version of (A.1). On the other hand, (A.2) is easily obtained from the viscous formulation by using a special entropy-entropy flux pair of Hafez-Osher-Whitlow [**24**]. In fact, this entropy-entropy flux pair is very important: It guarantees that the inviscid limit of the above viscous system satisfies a physically meaningful "entropy" condition (Theorem 2 of Hafex-Osher-Whitlow [**24**]). With (A.1) and (A.2) satisfied (under the presumed θ^ε bound), Morawetz's theory [**20, 21**] then applies, and the strong convergence in $L^1_{\rm loc}(\Omega)$ of our approximating sequence is achieved. There is one more unfortunate technical detail, both in Morawetz [**20, 21**] and Chen-Slemord-Wang [**6**], which q^ε is assumed to be uniformly bounded away from zero (stagnation) in any fixed region disjoint from the profile.

3. Conservation Laws and Differential Geometry

The connection of the classical theory of surfaces with systems of conservation laws (more precisely, balance laws) can be traced back to Codazzi [**8**] in 1860, Mainardi [**19**] in 1856, and Peterson [**25**] in 1853. However, even for those who may have forgotten their undergraduate course in differential geometry, the issues are remarkably straightforward.

We have all derived the differential distance formula on a surface $z = f(x, y)$ via the Pythagorean theorem to obtain

$$ds^2 = (1 + f_x^2)dx^2 + 2f_x f_y dxdy + (1 + f_y^2)dy^2,$$

and hence defining a metric on the base (x, y)–plane yielding distances on our surface. Similarly, we have the computed curvature along any line on our surface and obtain the familiar expressions in terms of second derivatives of f. Geometers traditionally have confronted the inverse problem: Given a quadratic form for metric

$$ds^2 = g_{ij}dx_i dx_j,$$

does there exist a corresponding surface on $\mathbb{R}^3$? This is the problem of isometric embedding. In general, our quadratic form does not determine the curvature along lines or on surfaces but, if we look for the largest and smallest curvature by examining all lines through a point on the surface, we have determined the principal curvatures: Their average is the mean curvature and their product is the Gauss curvature. The "remarkable" result of Gauss's Theorem Egregium ("egregium" being Latin for "remarkable") is that our metric (g_{ij}) indeed yields a formula of the Gauss curvature. Thus, if the metric is prescribed, the Gauss curvature is known and, since the Gauss curvature is given in terms of second derivatives of our surface map f, we have established a second-order partial differential equation for f (the Darboux equation a.k.a. a Monge-Ampere equation). That is the good news. The bad news is that the Darboux equation is fully nonlinear. Here is where Codazzi, Mainardi, and Peterson came in. The Darboux equation is just a restatement of Gauss's Theorem Egregium and conveniently written as

$$\ell n - m^2 = \kappa, \tag{3.1}$$

where

$$\ell = \frac{h_{11}}{\sqrt{|g|}}, \quad m = \frac{h_{12}}{\sqrt{|g|}}, \quad n = \frac{h_{22}}{\sqrt{|g|}}, \qquad |g| = g_{11}g_{22} - g_{12}^2,$$

κ is the Gauss curvature, and the quadratic form (the second fundamental form)

$$II = h_{ij} dx_i dx_j$$

is a convenient way to keep track of curvature on our surface. Since h_{ij} are given in terms of second derivatives of f, they cannot be independent and must satisfy integrability conditions, i.e., the Codazzi-Mainardi-Peterson equations:

$$\begin{aligned} -\partial_y m + \partial_x n &= -\ell\Gamma^{(1)}_{22} + 2m\Gamma^{(1)}_{12} - n\Gamma^{(1)}_{11}, \\ \partial_y \ell - \partial_x m &= -\ell\Gamma^{(2)}_{22} + 2m\Gamma^{(2)}_{12} - n\Gamma^{(2)}_{11}, \end{aligned} \tag{3.2}$$

where Γ_{ij} are the Christoffel symbols which depend on g_{ij} and are known functions. System (3.2) may appear at first linear but, coupled with the Darboux equation (3.1), it becomes the Gauss-Codazzi-Peterson system of quasilinear balance laws. The advantage in this approach is that the fully nonlinear Darboux equation (3.1) has been replaced by the quasilinear system (3.2) with (3.1) playing a role of an "equation of state" or "constitutive equation".

The analogy with continuum mechanics becomes transparent if we make the identification with gas dynamics ([**3**]):

$$\ell = \rho v^2 + p, \qquad n = \rho u^2 + p, \qquad m = -\rho uv,$$

with the "pressure" p taken to be of Chaplygin type $p = -\frac{1}{\rho}$, (u, v) being the gas velocity, and $q^2 = u^2 + v^2$ (gas speed). Then the Darboux equation becomes

$$\rho = (q^2 + k)^{-1/2},$$

which implies $p = -(q^2 + k)^{1/2}$. Hence, the Gauss-Codazzi-Mainardi-Peterson system (3.2) becomes the equations of steady, two-dimensional gas dynamics with nonhomogenous geometric terms on the right-hand sides:

$$\begin{aligned} \partial_x(\rho u^2 + p) + \partial_y(\rho uv) &= -(\rho v^2 + p)\Gamma^{(1)}_{22} - 2\rho uv\Gamma^{(1)}_{12} - (\rho u^2 + p)\Gamma^{(1)}_{11}, \\ \partial_x(\rho uv) + \partial_y(\rho v^2 + p) &= -(\rho v^2 + p)\Gamma^{(2)}_{22} - 2\rho uv\Gamma^{(2)}_{12} - (\rho u^2 + p)\Gamma^{(2)}_{11}, \end{aligned} \tag{3.3}$$

and amended by a "Bernoulli relation":

$$\rho = (q^2 + k)^{-1/2}. \tag{3.4}$$

Now (3.3)–(3.4) not only has the form of steady gas dynamics, but also its type can be formulated in analogous terms to gas dynamics, i.e., if we set the "sound speed" $c^2 = p'(\rho)$ in our case, we have $c^2 = \frac{1}{\rho^2}$ and hence

$$\begin{aligned} &c^2 > q^2 \quad \text{and the "flow" is subsonic, when } \kappa > 0, \\ &c^2 < q^2 \quad \text{and the "flow" is supersonic, when } \kappa < 0, \\ &c^2 = q^2 \quad \text{and the "flow" is sonic, when } \kappa = 0, \end{aligned}$$

Just as in gas dynamics, one may formulate initial and/or boundary value problems. For example, in Chen-Slemrod-Wang [**7**], the initial value problem for the case

$$E(x) = G(x) = (\cosh(cx))^{\frac{2}{\beta^2 - 1}}, \quad F(x) = 0,$$

has been studied in the infinite region with initial data given on the line $x = 0$. The method of compensated compactness [**23, 27**] which has proven so useful in gas dynamics again proves to be a valuable tool for obtaining weak "viscosity" solutions.

Needless to say that this leaves open many other interesting problems, especially those where k changes sign and initial and/or boundary value problems will become "transonic". Here again special local solutions are known to exist for special data (cf. [**18, 15**]), however, the existence of global solutions is at the moment only a hope.

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Department of Mathematics, Northwestern University, Evanston, IL 60208, USA.
E-mail address: gqchen@math.northwestern.edu

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA.
E-mail address: slemrod@math.wisc.edu

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.
E-mail address: dwang@math.pitt.edu