ON THE ZERO-ROSSBY LIMIT FOR THE PRIMITIVE EQUATIONS OF THE ATMOSPHERE

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In Memory of Jacques-Louis Lions

ABSTRACT. The zero-Rossby limit for the primitive equations (PE) governing the atmospheric motions is analyzed. The limit is important in geophysics for large scale models (cf. Lions [13]) and is in the level of the zero relaxation limit for nonlinear partial differential equations (cf. [5]). It is proved that, if the initial data appropriately approximate data of geostrophic type, the corresponding solutions of the simplified primitive equations approximate the solutions of the quasigeostrophic equations (QG) with order ϵ accuracy as the Rossby number ϵ goes to zero.

1. INTRODUCTION

The primitive equations (PE) governing the atmospheric motions are

$$\left\{\begin{array}{l} \partial_t v + v\cdot\nabla v - \operatorname{div} V(v)\,\partial_\eta v - \mu_1\triangle_3 v + \frac{1}{\epsilon}(v^\perp + \nabla(\Theta(\theta)+\Phi)) = 0, \\ \partial_t\theta + v\cdot\nabla\theta - \operatorname{div} V(v)\,\partial_\eta\theta - \mu_2\triangle_3\theta - \frac{1}{\epsilon}\operatorname{div} V(v) = 0, \\ \operatorname{div}\int_0^1 v(t,x,\eta)\,d\eta = 0, \qquad x=(x_1,x_2)\in R^2, \end{array}\right. \tag{1.1}$$

where $V(v)=\int_0^\eta v(t,x,s)\,ds$, $\Theta(\theta) = -\int_\eta^1 \theta(t,x,s)ds$; $v=(v_1,v_2)$ is the horizontal velocity and $v^\perp = (-v_2, v_1)$; θ is the temperature of the atmosphere; $\Phi(t,x)$ is a scalar pressure function (depending only on the time variable t and the horizontal variables x); $(\operatorname{div}, \nabla, \triangle)$ are the divergence, gradient, and Laplace operators in x, respectively; $\eta\in[0,1]$ is the ratio of pressure difference; $\Delta_3 = \Delta + \partial_\eta^2$; ϵ is the Rossby number; μ_1 is the fluid viscosity, and μ_2 is the thermal conduction coefficient. The equations for v in (1.1) describe the hydrodynamics of the atmosphere, and the equation for θ describes the thermodynamics of the atmosphere, under the so called "Boussinesq" and hydrostatic assumptions.

For concreteness, in this paper we focus on the primitive equations on $\Omega = \mathbb{T}^2\times(0,1)$, with the two-dimensional torus $\mathbb{T}^2$, to understand the qualitative behavior of solutions of (1.1). We are mainly interested in the limiting behavior of solutions of (1.1), especially in the possibility of uniform convergence of solutions of (1.1) up to the boundaries, as the Rossby number $\epsilon\to 0$. This limit is important in geophysics for large scale models (cf. Lions [13]) and is in the level of the zero relaxation limit which has been addressed for hyperbolic systems of conservation laws in Chen-Levermore-Liu [5]. The zero-Rossby limit addressed here is also interesting from the point of view of nonlinear partial differential equations, since the equations involve the nonlocal terms.

In Lions-Teman-Wang [9], the Leray method [7, 8] was applied to constructing the global L^2 weak solutions $(v^\epsilon, \theta^\epsilon, \Phi^\epsilon)$ of an initial-boundary value problem for (1.1); and, in [11],

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the solutions were asymptotically expanded by

$$\begin{aligned} v^\epsilon &= v + \epsilon v^1 + \cdots, \\ \theta^\epsilon &= \theta + \epsilon\theta^1 + \cdots, \\ \Phi^\epsilon &= \Phi + \epsilon\Phi^1 + \cdots, \end{aligned}$$

respectively. Then, from the ϵ^{-1} terms,

$$v^\perp + \nabla(\Theta(\theta) + \Phi) = 0, \qquad \operatorname{div} V(v) = 0, \tag{1.2}$$

and, from the ϵ^0 terms,

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu_1 \triangle_3 v + (v^1)^\perp + \nabla(\Theta(\theta) + \Phi^1) = 0, \\ \partial_t \theta + v \cdot \nabla\theta - \mu_2 \triangle_3 \theta - \operatorname{div} V(v^1) = 0. \end{cases} \tag{1.3}$$

A direct computation from (1.2) and (1.3) yields the quasigeostrophic equations (QG):

$$\begin{cases} \partial_t \omega + v \cdot \nabla\omega - \mu_1 \triangle_3 \omega + (\mu_1 - \mu_2)\partial_\eta^2 \omega = 0, \\ \triangle_3 \phi = \omega, \quad v = (-\partial_{x_2}\phi, \partial_{x_1}\phi), \quad \theta = \partial_\eta \phi, \end{cases} \tag{1.4}$$

where $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1 + \partial_\eta \theta$ is the potential vorticity.

For system (1.4), Bourgeois-Beale [2] established its well-posedness in the torus $\mathbb{T}^3$ for the inviscid case, and Chemin [4] showed the well-posedness of the Cauchy problem. In Section 3, we will show the existence and uniqueness of global smooth solutions for the initial-boundary value problem for (1.4) in $\Omega = \mathbb{T}^2 \times (0,1)$.

In [13], Lions conjectured that the solutions of the quasigeostrophic equations (1.4) are the valid approximation of the solutions of the primitive equations (1.1), provided certain restrictions on the initial data are satisfied.

In [4], Chemin verified that (1.4) is the valid approximation of the following system:

$$\begin{cases} \frac{Du^\epsilon}{Dt} - Lu^\epsilon + \frac{1}{\epsilon} A u^\epsilon = \frac{1}{\epsilon}(-\nabla\Phi^\epsilon, 0), \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} u^\epsilon = 0, \\ u^\epsilon|_{t=0} = u_0^\epsilon, \end{cases} \tag{1.5}$$

as ϵ tends to zero, under the assumption that u_0^ϵ is appropriately close to geostrophic data: there exists some pressure function ϕ_0 such that u_0^ϵ approximates $(-\partial_2\triangle^{-1}\phi_0, \partial_1\triangle^{-1}\phi_0, 0, \partial_3\triangle^{-1}\phi_0)$ and μ_2 is sufficiently close to μ_1, where A is an anti-symmetric matrix and

$$\frac{D}{Dt} = \partial_t + \sum_{i=1}^3 v_i^\epsilon \partial_{x_i}, \qquad u^\epsilon = (v_1^\epsilon, v_2^\epsilon, v_3^\epsilon, \theta^\epsilon), \qquad Lu^\epsilon = (\mu_1 \triangle v^\epsilon, \mu_2 \triangle \theta^\epsilon).$$

In [2], Bourgeois-Beale successfully obtained a similar result in the torus $\mathbb{T}^3$ for the inviscid case of (1.5).

In this paper we are concerned with the problem raised by Lions in [13] to analyze whether the quasigeostropic equations (1.4) are the valid approximation of the primitive equations (1.1). This problem for (1.1) has different features from (1.5): In particular, the relaxation terms are nonlocal and do not have the feature as the matrix A in (1.5), which causes additional difficulties. For concreteness, we focus on the following initial-boundary problem of (1.1) in the pseudo-geometrical domain $\Omega = \mathbb{T}^2 \times [0,1]$:

$$(v^\epsilon, \theta^\epsilon)|_{t=0} = (v_0^\epsilon, \theta_0^\epsilon)(x,\eta), \qquad (v^\epsilon, \partial_\eta \theta^\epsilon)|_{\eta=l} = 0, \quad \text{for } l = 0, 1, \tag{1.6}$$

to solve this limit problem. Since $\eta = 0, 1$ correspond to the bottom and the top of the atmosphere, respectively, the boundary conditions $\theta|_{\eta=l} = 0, l = 0, 1$, represent the case that there is no thermal transition at the bottom and the top of the atmosphere, and $v|_{\eta=1} = 0$ is the assumption that the horizontal velocity is zero at the top of the atmosphere under consideration. In general, at $\eta = 0$, the boundary conditions are much more complicated;

they depend upon whether the atmosphere is above solid earth or above solid sea or liquid sea.

The analysis that we present in this paper is another step forward in dealing with the difficulties involving in the zero-Rossby limit problem. One of the main concerns for the zero-Rossby limit in the atmospheric motions is whether the zero-Rossby limit converges strongly even when the Prandtl number (the ratio of the thermal conduction coefficient and the fluid viscosity) is not one. We give a positive answer for (1.1) and (1.6). Another concern is that, in some situations, Ekman-type boundary layers occur in the atmospheric motions. In this paper, we rigorously prove that, for (1.1) and (1.6), the Ekman-type boundary layers do not occur. It would be interesting to study further the zero-Rossby limit problem for various initial-boundary value problems involving the Ekman-type boundary layers.

For the sequel presentation, we need the following spaces of functions:

$$\begin{aligned}
&V_1 = \{a \in C^\infty(\Omega) \mid a|_{\eta=l} = 0, \partial_\eta^{2k} a|_{\eta=0} = \partial_\eta^{2k} a|_{\eta=1}, \operatorname{div}\int_0^1 a(x,\eta)d\eta = 0,\ l=0,1;\ k=1,2,\cdots\},\\
&V_2 = \{b \in C^\infty(\Omega) \mid \partial_\eta^{2k+1} b|_{\eta=l} = 0,\ l=0,1;\ k=0,1,\cdots\},\\
&V_3 = \{c \in C^\infty(\Omega) \mid \partial_\eta^{2k} c|_{\eta=l} = 0,\ l=0,1;\ k=0,1,\cdots\},\\
&J^\alpha = J_1^\alpha \times J_2^\alpha,
\end{aligned}$$

where J_j^α are the completion of $V_j, j=1,2$, in $H^\alpha(\Omega)$, respectively. We denote J_3^α the completion of V_3 in $H^\alpha(\Omega)$. It has been shown in [9] and [11] that

$$(J_1^0)^\perp = \{P = \nabla\Psi \mid \Psi \in H^1(\mathbb{T}^2)\}. \tag{1.8}$$

Theorem 1.3 below is our main theorem of this paper. Some related results are formulated in Theorems 1.1 and 1.2.

Theorem 1.1. *Let* $(v_0^\epsilon, \theta_0^\epsilon) \in J^2, \nabla\int_0^1 \theta_0^\epsilon(x,\eta)\,d\eta = 0$. *Then there exists* $T^* > 0$ *such that the initial-boundary value problem* (1.1) *and* (1.6) *has a strong solution:*

$$u^\epsilon \equiv (v^\epsilon,\theta^\epsilon) \in L^\infty([0,T^*);J^2) \cap C([0,T^*);H^\alpha(\Omega)) \cap L^2([0,T^*);H^3(\Omega)),\ \textit{for any}\ \alpha < 2,$$

such that

$$u_t^\epsilon \in L^2([0,T^*);L^2(\Omega)),\ \ \nabla\Phi^\epsilon \in L^2([0,T^*);L^2(\mathbb{T}^2)),\ \ \lim_{t\to 0}\|u^\epsilon(t,\cdot) - u_0^\epsilon(\cdot)\|_{H^2(\Omega)} = 0.$$

Theorem 1.2. *Let* $(v_0,\theta_0) = (-\partial_2\phi_0, \partial_1\phi_0, \partial_\eta\phi_0)$, $\phi_0 \in J_3^5$, *and* $0 < \mu_2 < 2\mu_1$. *Then there exists a unique global solution* $(\phi,\omega,v,\theta)(t,x,\eta)$ *of the initial-boundary value problem* (1.4) *with*

$$(v,\theta)|_{t=0} = (v_0,\theta_0), \qquad (\phi,\omega,v,\partial_\eta\theta)|_{\eta=l} = 0, \quad \textit{for}\ l=0,1, \tag{1.9}$$

such that

$$\begin{aligned}
&\phi \in L^\infty_{loc}(\mathbb{R}_+;J_3^5) \cap L^2_{loc}(\mathbb{R}_+;H^6(\Omega)), \qquad \omega \in L^\infty_{loc}(\mathbb{R}_+;J_3^3) \cap L^2_{loc}(\mathbb{R}_+;H^4(\Omega)),\\
&(v,\theta) \in L^\infty_{loc}(\mathbb{R}_+;J^4) \cap L^2_{loc}(\mathbb{R}_+;H^5(\Omega)), \qquad \partial_t(v,\theta) \in L^2_{loc}(\mathbb{R}_+;H^3(\Omega)).
\end{aligned}$$

Theorem 1.3. *Let*

$$v_0^\epsilon = v_0 + \epsilon v_0^a, \quad \theta_0^\epsilon = \theta_0 + \epsilon\theta_0^a, \qquad (v_0,\theta_0) = (-\partial_{x_2}\phi_0, \partial_{x_1}\phi_0, \partial_\eta\phi_0),$$

for some pressure function $\phi_0 \in J_3^5$, $(v_0^a,\theta_0^a) \in J^2, \nabla\int_0^1 \theta_0^a(x,\eta)\,d\eta = 0$, *and* $0 < \mu_2 < 2\mu_1$. *Then, for any* $T > 0$, *there exists some* $\epsilon_0 = \epsilon_0(T) > 0$ *such that, for any* $\epsilon \le \epsilon_0$, *the solutions constructed in Theorem 1.1 can be extended to the time interval* $[0,T]$. *Moreover,*

$$v^\epsilon = v + \epsilon v^{\epsilon,a}, \qquad \theta^\epsilon = \theta + \epsilon\theta^{\epsilon,a},$$

where (v,θ) is the solution corresponding to (1.4) *constructed in Theorem 1.2, and*

$$\|(v^{\epsilon,a},\theta^{\epsilon,a})(t,\cdot)\|^2_{H^2(\Omega)}+\int_0^T\|\nabla_3(v^{\epsilon,a},\theta^{\epsilon,a})(t,\cdot)\|^2_{H^2(\Omega)}\,dt\le K(T),$$

where $K(T)$ is independent of ϵ.

Observe that the results in Theorems 1.2 and 1.3 hold even when the Prandtl number is not one, that is, $\mu_1\neq\mu_2$.

Throughout this paper, ∇_3 denotes the gradient and derivative with respect to (x,η), $(\cdot,\cdot)_\alpha$ denotes the inner product in $H^\alpha(\Omega)$ and $\|\cdot\|_\alpha$ the norm in $H^\alpha(\Omega)$, and $C>0$ is a generic constant which may be different at each occurrence.

To prove our main theorem (Theorem 1.3), we first show Theorem 1.1 in §2 and Theorem 1.2 in §3.

2. Proof of Theorem 1.1: Solutions of the Primitive Equations

In this section, we use the Galerkin method to establish Theorem 1.1. We first construct the Galerkin approximate solutions for (1.1) and (1.6), and then we make the L^2 and high-order apriori estimates for the approximate solutions, which yield the convergence of the approximate solutions to the solutions of the initial-boundary value problem (1.1) and (1.6). Although the approach is standard in this section, we will use some ideas and techniques in the proof to show Theorem 1.2; and we hence outline the steps of the proof for self-containedness. Since we only use $(\cdot,\cdot)_0$ in this section, we directly denote $(\cdot,\cdot)$ as $(\cdot,\cdot)_0$. Before our proof, we recall a lemma due to Lions-Teman-Wang [9, 12].

Consider the Stokes problem with nonlocal constraint:

$$\begin{cases} -\Delta_3 v+\nabla\Phi=f, \qquad \Omega=\mathbb{T}^2\times(0,1),\\ \operatorname{div}\int_0^1 v(x,\eta)d\eta=0,\\ v|_{\eta=l}=0, \qquad \text{for } l=0,1.\end{cases} \tag{2.1}$$

Lemma 2.1. *Let $f\in H^k, k\ge 0$. Then there exists a unique (weak) solution $(v,\Phi)\in J_1^1\times L^2(\mathbb{T}^2)$ of* (2.1) *such that*

$$v\in H^{k+2}(\Omega)\cap J_1,\quad \Phi\in H^{k+1}(\mathbb{T}^2),\quad \|v\|_{H^{k+2}}+\|\Phi\|_{H^{k+1}}\le C\|f\|_{H^k}, \tag{2.2}$$

where C is a constant, independent of v,Φ, and f.

Lemma 2.1 indicates that Φ in the solution depends only on the first two independent variables (x_1,x_2), independent of $\eta\in[0,1]$. As a corollary, if $f|_{\eta=0}=f|_{\eta=1}$, then $\Delta_3 v|_{\eta=0}=\Delta_3 v|_{\eta=1}$. Now we first prove Theorem 1.1.

Step 1: Construction of the Galerkin approximate solutions. Let P be the projection operator from $L^2(\Omega)$ to $J_1^0(\Omega)$. As discussed in [6], we use the Galerkin method with normalized eigenfunctions of the Stokes and Laplace operators as the basis functions in L^2 to construct the Galerkin approximate solutions. The eigenfunctions $\{a^k\}$ and $\{b^k\}$ are taken orthonormally in J_1^0 and J_2^0, respectively, with the corresponding eigenvalues $\{\lambda_k\}$ and $\{\nu_k\}$:

$$P\triangle_3 a^k=\lambda_k a^k,\quad \triangle_3 b^k=\nu_k b^k,\quad \int_0^1 b^k(x,\eta)\,d\eta=0.$$

Then Lemma 2.1 and the regularity of the Laplace operator indicate that $\{a^k\}$ and $\{b^k\}$ are in $H^\infty(\Omega)$. Now we take the n^{th} approximate solutions:

$$v^{\epsilon,n}(t,x,\eta)=\sum_{k=1}^n c^\epsilon_{kn}(t)a^k(x,\eta),\qquad \theta^{\epsilon,n}(t,x,\eta)=\sum_{k=1}^n d^\epsilon_{kn}(t)b^k(x,\eta), \tag{2.3}$$

where $c^\epsilon_{kn}(t)$ and $d^\epsilon_{kn}(t), 1 \le k \le n$, are determined by the following initial value problem:

$$\begin{aligned}\frac{d}{dt}(v^{\epsilon,n},a^k) &+ ((v^{\epsilon,n}\cdot\nabla - \operatorname{div} V(v^{\epsilon,n})\partial_\eta)v^{\epsilon,n},a^k) - \mu_1(\triangle_3 v^{\epsilon,n},a^k)\\ &+ \frac{1}{\epsilon}\{((v^{\epsilon,n})^\perp,a^k)+(\nabla\Theta(\theta^{\epsilon,n}),a^k)\}=0,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\frac{d}{dt}(\theta^{\epsilon,n},b^k) &+ ((v^{\epsilon,n}\cdot\nabla - \operatorname{div} V(v^{\epsilon,n})\,\partial_\eta)\theta^{\epsilon,n},b^k)\\ &- \mu_2(\triangle_3\theta^{\epsilon,n},b^k) - \frac{1}{\epsilon}(\operatorname{div} V(v^{\epsilon,n}),b^k)=0,\end{aligned} \tag{2.5}$$

$$c^\epsilon_{kn}(0) = (v^\epsilon_0,a^k),\quad d^\epsilon_{kn}(0)=(\theta^\epsilon_0,b^k),\qquad 1\le k\le n. \tag{2.6}$$

Step 2: L^2 apriori estimates for the approximate solutions. Since $\|P\triangle_3 a\|_0$ (resp. $\|\triangle_3 b\|_0$) is equivalent to $\|a\|_2$ (resp. $\|b\|_2$) for $a\in J^2_1$ (resp. $b\in J^2_2$) (cf. Lemma 2.1), as a convention, $\|P\triangle_3 a\|_0$ (resp. $\|\triangle_3 b\|_0$) in this section can be regarded as $\|a\|_2$ (resp. $\|b\|_2$).

Now, we multiply (2.4) by $c^\epsilon_{ln}(t)$ and (2.5) by $d^\epsilon_{ln}(t)$, respectively. Then, summing up and noting that

$$(\nabla\Theta(\theta^{\epsilon,n}),v^{\epsilon,n}) + (\operatorname{div} V(v^{\epsilon,n}),\theta^{\epsilon,n}) = 0, \tag{2.7}$$

we have

$$\frac{1}{2}\frac{d}{dt}\|(v^{\epsilon,n},\theta^{\epsilon,n})\|_0^2 + \mu_1\|\nabla_3 v^{\epsilon,n}\|_0^2 + \mu_2\|\nabla_3\theta^{\epsilon,n}\|_0^2 = 0. \tag{2.8}$$

On the other hand, since $\{a^k\}$ and $\{b^k\}$ are orthonormal eigenfunctions of $P\triangle_3$ and $\triangle_3$, respectively, $\{P\triangle_3 a^k\}\times\{\triangle_3 b^k\}$ are still orthogonal in J^0. Hence, the Bessel inequality implies

$$\|(v_0^{\epsilon,n},\theta_0^{\epsilon,n})\|_0 \le \|(v_0^\epsilon,\theta_0^\epsilon)\|_0,\qquad \|(v_0^{\epsilon,n},\theta_0^{\epsilon,n})\|_2 \le \|(v_0^\epsilon,\theta_0^\epsilon)\|_2. \tag{2.9}$$

Therefore, integrating (2.8) with respect to t yields

$$\|(v^{\epsilon,n},\theta^{\epsilon,n})(t,\cdot)\|_0^2 + \int_0^t \left(\mu_1\|\nabla_3 v^{\epsilon,n}(\tau,\cdot)\|_0^2 + \mu_2\|\nabla_3\theta^{\epsilon,n}(\tau,\cdot)\|_0^2\right)d\tau \le \|(v_0^\epsilon,\theta_0^\epsilon)\|_0^2. \tag{2.10}$$

Step 3. High-order apriori estimates for the approximate solutions. Multiply (2.4) and (2.5) by $\lambda_k^2 c^\epsilon_{kn}(t)$ and $\nu_k^2 d^\epsilon_{kn}(t)$, respectively, and then sum up. Since

$$((v^{\epsilon,n})^\perp,(P\triangle_3)^2 v^{\epsilon,n}) = 0, \tag{2.11}$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(v^{\epsilon,n},\theta^{\epsilon,n})\|_2^2 + \mu_1\|\nabla_3 v^{\epsilon,n}\|_2^2 + \mu_2\|\nabla_3\theta^{\epsilon,n}\|_2^2\\ &= -\,(v^{\epsilon,n}\cdot\nabla v^{\epsilon,n} - \operatorname{div} V(v^{\epsilon,n})\,\partial_\eta v^{\epsilon,n},(P\triangle_3)^2 v^{\epsilon,n}) - \frac{1}{\epsilon}(\nabla\Theta(\theta^{\epsilon,n}),(P\triangle_3)^2 v^{\epsilon,n})\\ &\quad - (v^{\epsilon,n}\cdot\nabla\theta^{\epsilon,n} - \operatorname{div} V(v^{\epsilon,n})\partial_\eta\theta^{\epsilon,n},\triangle_3^2\theta^{\epsilon,n}) + \frac{1}{\epsilon}(\operatorname{div} V(v^{\epsilon,n}),\triangle_3^2\theta^{\epsilon,n}).\end{aligned} \tag{2.12}$$

To complete the estimates, we will make repeated use of the interpolation inequality:

$$\|a\|_\alpha \le C\|a\|_{\alpha_1}^\gamma\|a\|_{\alpha_2}^{1-\gamma},\qquad \alpha = \gamma\alpha_1 + (1-\gamma)\alpha_2, \tag{2.13}$$

and the Sobolev inequality:

$$\|a\|_{L^q} \le C\|a\|_\alpha,\qquad \text{if } \alpha<\frac{3}{2},\ \frac{1}{q}=\frac{1}{2}-\frac{\alpha}{3}. \tag{2.14}$$

By integration by parts,

$$
\begin{aligned}
&((v^{\epsilon,n}\cdot\nabla-\operatorname{div}V(v^{\epsilon,n})\partial_\eta)v^{\epsilon,n},(P\triangle_3)^2v^{\epsilon,n})\\
\le&\|\nabla_3v^{\epsilon,n}\|_{L^4}^2\|\nabla_3v^{\epsilon,n}\|_2+\|\operatorname{div}v^{\epsilon,n}\|_{L^6}\|\partial_\eta\nabla_3v^{\epsilon,n}\|_{L^3}\|\nabla_3v^{\epsilon,n}\|_2\\
&+\|v^{\epsilon,n}\|_{L^6}\|\nabla_3\nabla v^{\epsilon,n}\|_{L^3}\|\nabla_3v^{\epsilon,n}\|_2+\|\partial_\eta v^{\epsilon,n}\|_{L^6}\|\nabla_3\operatorname{div}v^{\epsilon,n}\|_{L^3}\|\nabla_3v^{\epsilon,n}\|_2.
\end{aligned}\tag{2.15}
$$

Inequalities (2.13) and (2.14) yield that, for any $\beta>0$, there exists $C=C_\beta>0$ such that

$$
\begin{aligned}
&\|\nabla_3v^{\epsilon,n}\|_{L^4}^2\|\nabla_3v^{\epsilon,n}\|_2\le\|\nabla_3v^{\epsilon,n}\|_{\frac34}^2\|\nabla_3v^{\epsilon,n}\|_2\\
&\quad\le C\|v^{\epsilon,n}\|_0^{\frac14}\|v^{\epsilon,n}\|_2^{\frac74}\|\nabla_3v^{\epsilon,n}\|_2\le C\|v^{\epsilon,n}\|_2^{\frac72}+\beta\|\nabla_3v^{\epsilon,n}\|_2^2,
\end{aligned}\tag{2.16}
$$

$$
\begin{aligned}
&\|v^{\epsilon,n}\|_{L^6}\|\nabla_3\nabla v^{\epsilon,n}\|_{L^3}\|\nabla_3v^{\epsilon,n}\|_2\le C\|v^{\epsilon,n}\|_0^{\frac12}\|v^{\epsilon,n}\|_2\|\nabla_3v^{\epsilon,n}\|_2^{\frac32}\\
&\quad\le C\|v^{\epsilon,n}\|_2^4+\beta\|\nabla_3v^{\epsilon,n}\|_2^2,
\end{aligned}\tag{2.17}
$$

$$
\begin{aligned}
&\|\partial_\eta v^{\epsilon,n}\|_{L^6}\|\nabla_3\operatorname{div}v^{\epsilon,n}\|_{L^3}\|\nabla_3v^{\epsilon,n}\|_2\le C\|v^{\epsilon,n}\|_2^{\frac32}\|\nabla_3v^{\epsilon,n}\|_2^{\frac32}\\
&\quad\le C\|v^{\epsilon,n}\|_2^6+\beta\|\nabla_3v^{\epsilon,n}\|_2^2,
\end{aligned}\tag{2.18}
$$

$$
\|\operatorname{div}v^{\epsilon,n}\|_{L^6}\|\partial_\eta\nabla_3v^{\epsilon,n}\|_{L^3}\|\nabla_3v^{\epsilon,n}\|_2\le C\|v^{\epsilon,n}\|_2^6+\beta\|\nabla_3v^{\epsilon,n}\|_2^2,\tag{2.19}
$$

where we used the inequality: $|ab|\le\frac{|a|^p}{p}+\frac{|b|^q}{q},\frac1p+\frac1q=1,p,q>1$.

Then, summing up (2.16)-(2.19) yields

$$
\begin{aligned}
&((v^{\epsilon,n}\cdot\nabla-\operatorname{div}V(v^{\epsilon,n})\partial_\eta)v^{\epsilon,n},(P\triangle_3)^2v^{\epsilon,n})\\
\le&C\left(\|v^{\epsilon,n}\|_2^{\frac72}+\|v^{\epsilon,n}\|_2^4+\|v^{\epsilon,n}\|_2^6\right)+4\beta\|\nabla_3v^{\epsilon,n}\|_2^2.
\end{aligned}\tag{2.20}
$$

By the same argument,

$$
\begin{aligned}
&((v^{\epsilon,n}\cdot\nabla-\operatorname{div}V(v^{\epsilon,n})\partial_\eta)\theta^{\epsilon,n},\triangle_3^2\theta^{\epsilon,n})\\
\le&\frac{3\beta}{2}\|\nabla_3\theta^{\epsilon,n}\|_2^2+\frac{\beta}{2}\|\nabla_3v^{\epsilon,n}\|_2^2\\
&+C\left(\|v^{\epsilon,n}\|_2^{\frac74}\|\theta^{\epsilon,n}\|_2^{\frac74}+\|v^{\epsilon,n}\|_2^2\|\theta^{\epsilon,n}\|_2^2+\|\theta^{\epsilon,n}\|_2^4\|v^{\epsilon,n}\|_2^2+\|v^{\epsilon,n}\|_2^4\|\theta^{\epsilon,n}\|_2^2\right).
\end{aligned}\tag{2.21}
$$

By the construction of the approximate solutions, $\nabla\int_0^1\theta^{\epsilon,n}(t,x,\eta)\,d\eta=0$. Thus, by integration by parts and (2.10), there is a constant C depending only on $\|(v_0^\epsilon,\theta_0^\epsilon)\|_0$ such that

$$
|(\nabla\Theta(\theta^{\epsilon,n}),(P\triangle_3)^2v^{\epsilon,n})|\le C\|\theta\|_2^2+\beta(\|\nabla_3v^{\epsilon,n}\|_2^2+C),\tag{2.22}
$$

and

$$
|(\operatorname{div}V(v^{\epsilon,n}),\triangle_3^2\theta^{\epsilon,n})|\le C\|v^{\epsilon,n}\|_2^2+\beta(\|\nabla_3\theta^{\epsilon,n}\|_2^2+C).\tag{2.23}
$$

Set $Y(t)=\|(v^{\epsilon,n},\theta^{\epsilon,n})(t,\cdot)\|_2^2$. Then, using (2.12), (2.20)-(2.23), and taking $\beta=\frac{\mu}{12}$, we have

$$
Y'(t)+\mu_1\|\nabla_3v^{\epsilon,n}(t,\cdot)\|_2^2+\mu_2\|\nabla_3\theta^{\epsilon,n}(t,\cdot)\|_2^2\le C(1+Y(t)+Y^{\frac72}(t)+Y^4(t)+Y^6(t)),\tag{2.24}
$$

where the constant C may depend on ϵ.

The comparison principle of ordinary differential equations yields

$$
Y(t)\le F(t,y(0)),\quad\int_0^t\|\nabla_3(v^{\epsilon,n},\theta^{\epsilon,n})(\tau,\cdot)\|_2^2\,d\tau<\infty,\ \text{ for }0\le t<T^*,\tag{2.25}
$$

where $F(t,y(0))$ and T^* are respectively the solution and lifespan of the following Cauchy problem:

$$
\frac{dy}{dt}=C(1+y+y^{\frac72}+y^4+y^6),\qquad y|_{t=0}=y(0)\equiv\|(v_0^\epsilon,\theta_0^\epsilon)\|_2^2.
$$

Step 4. Further apriori estimates for the approximate solutions. Similar to the arguments in establishing (2.8) and (2.12), we multiply (2.4) and (2.5) by $\frac{d}{dt}c^\epsilon_{kn}(t)$ and $\frac{d}{dt}d^\epsilon_{kn}(t)$, respectively, and sum up to obtain

$$\begin{aligned}\|\partial_t(v^{\epsilon,n},\theta^{\epsilon,n})\|_0^2 =&\mu_1(P\triangle_3 v^{\epsilon,n},\partial_t v^{\epsilon,n})+\mu_2(\triangle_3\theta^{\epsilon,n},\partial_t\theta^{\epsilon,n})\\ &-((v^{\epsilon,n}\cdot\nabla-\operatorname{div}V(v^{\epsilon,n})\partial_\eta)v^{\epsilon,n},\partial_t v^{\epsilon,n})\\ &-((v^{\epsilon,n}\cdot\nabla-\operatorname{div}V(v^{\epsilon,n})\partial_\eta)\theta^{\epsilon,n},\partial_t\theta^{\epsilon,n}),\end{aligned}$$

which implies

$$\begin{aligned}\|\partial_t(v^{\epsilon,n},\theta^{\epsilon,n})\|_0 \le&\mu_1\|v^{\epsilon,n}\|_2+\mu_2\|\theta^{\epsilon,n}\|_2+\|(v^{\epsilon,n}\cdot\nabla-\operatorname{div}V(v^{\epsilon,n})\partial_\eta)(v^{\epsilon,n},\theta^{\epsilon,n})\|_0\\ \le&\mu_1\|v^{\epsilon,n}\|_2+\mu_2\|\theta^{\epsilon,n}\|_2+C\|v^{\epsilon,n}\|_0^{\frac12}\|v^{\epsilon,n}\|_2^{\frac12}(\|v^{\epsilon,n}\|_0^{\frac14}\|v^{\epsilon,n}\|_2^{\frac34}+\|\theta^{\epsilon,n}\|_0^{\frac14}\|\theta^{\epsilon,n}\|_2^{\frac34})\\ &+C\|v^{\epsilon,n}\|_0^{\frac18}\|v^{\epsilon,n}\|_2^{\frac78}(\|v^{\epsilon,n}\|_0^{\frac18}\|v^{\epsilon,n}\|_2^{\frac78}+\|\theta^{\epsilon,n}\|_0^{\frac18}\|\theta^{\epsilon,n}\|_2^{\frac78}),\end{aligned}$$

where we have used (2.13), (2.14), and the Hölder inequality. Then, by (2.25), we have

$$\|\partial_t(v^{\epsilon,n},\theta^{\epsilon,n})(t,\cdot)\|_0\le G(t),\qquad 0\le t<T^*, \tag{2.26}$$

for some continuous function $G(t)$ defined in $[0,T^*)$.

Step 5. Construction of solutions. Estimates (2.25) and (2.26) imply that $(v^{\epsilon,n},\theta^{\epsilon,n})$ is compact in $L^p_{loc}([0,T^*);H^\alpha(\Omega))$, for any $1<p<\infty,\alpha<3$. Let $(v^\epsilon,\theta^\epsilon)$ be a limit of $(v^{\epsilon,n},\theta^{\epsilon,n})$. Then $(v^\epsilon,\theta^\epsilon)$ must be a weak solution of (1.1). By (2.25) and (2.26), $(v^\epsilon,\theta^\epsilon)\in L^\infty([0,T^*);J^2)\cap L^2([0,T^*);H^3(\Omega))$ and $(v^\epsilon_t,\theta^\epsilon_t)\in L^\infty([0,T^*);L^2(\Omega))$. Moreover, by Lemma 2.1, there exists some $\Phi^\epsilon\in L^2_{loc}([0,T^*);H^1(\mathbb{T}))$ such that $(v^\epsilon,\theta^\epsilon)$ and Φ^ϵ satisfy (1.1) in the weak sense.

For arbitrary $0\le t_1<t_2<T^*$,

$$\|v^\epsilon(t_1,\cdot)-v^\epsilon(t_2,\cdot)\|_0=\|\int_{t_1}^{t_2}\frac{\partial v^\epsilon}{\partial t}(t,\cdot)\,dt\|_0\le\|\frac{\partial v^\epsilon}{\partial t}\|_{L^2([t_2,t_1]\times\Omega)}|t_1-t_2|^{\frac12}. \tag{2.27}$$

By the Rellich Theorem, $H^2(\Omega)\hookrightarrow H^\alpha(\Omega)$ is compact, for arbitrary $\alpha<2$. Thus, by the Lions-Aubin Lemma, for any $\delta>0$, there exists $C_\delta>0$ such that

$$\|v^\epsilon(t_1,\cdot)-v^\epsilon(t_2,\cdot)\|_\alpha\le\delta\|v^\epsilon\|_{L^\infty([t_1,t_2];H^2(\Omega))}+C_\delta\|v^\epsilon(t_1,\cdot)-v^\epsilon(t_2,\cdot)\|_0. \tag{2.28}$$

Then we conclude from (2.27) and (2.28), similarly for θ^ϵ, that

$$(v^\epsilon,\theta^\epsilon)\in C([0,T^*);H^\alpha(\Omega)),\quad\text{for any }\alpha<2. \tag{2.29}$$

Step 6. Claim: $\lim_{t\to0}\|(v^\epsilon,\theta^\epsilon)(t,\cdot)-(v^\epsilon_0,\theta^\epsilon_0)(\cdot)\|_2=0$.
Taking $t=0$ in (2.25), we have

$$\lim_{t\to0}\|(v^\epsilon,\theta^\epsilon)(t,\cdot)\|_2\le\lim_{t\to0}\overline{\lim}_{n\to\infty}\|(v^{\epsilon,n},\theta^{\epsilon,n})(t,\cdot)\|_2\le\|(v^\epsilon_0,\theta^\epsilon_0)\|_2.$$

Then, taking $t_1=0$ in (2.27), we conclude

$$\lim_{t\to0}v^\epsilon(t,\cdot)=v^\epsilon_0(\cdot)\text{ weakly in }H^2(\Omega),$$

which implies

$$\lim_{t\to0}\|v^\epsilon(t,\cdot)-v^\epsilon_0(\cdot)\|_2=0.$$

Similarly, we have

$$\lim_{t\to0}\|\theta^\epsilon(t,\cdot)-\theta^\epsilon_0(\cdot)\|_2=0.$$

This completes the proof of Theorem 1.1.

Remark 2.1. *Following the proof of Step 6, we can show*

$$\lim_{t\to t'+0} \|(v^\epsilon,\theta^\epsilon)(t,\cdot)-(v^\epsilon,\theta^\epsilon)(t',\cdot)\|_2=0.$$

One can also establish the corresponding case of Theorem 1.1 for initial data $(v_0^\epsilon,\theta_0^\epsilon)(x,\eta)\in J^\alpha$, *with* $\alpha>2$ *an integer.*

Remark 2.2. *By the construction of the approximate solutions for (1.1), we have*

$$\triangle_3 v^{\epsilon,n}(t,x,0)=\triangle_3 v^{\epsilon,n}(t,x,1). \tag{2.30}$$

Thus, using the Trace Theorem and the arguments at the beginning of Step 5, we obtain for all $0<t<T^*$ *that*

$$\begin{aligned}
&\int_0^t \|\triangle_3 v^\epsilon(\tau,\cdot,1)-\triangle_3 v^\epsilon(\tau,\cdot,0)\|_0\,d\tau\\
&=\lim_{n\to\infty}\int_0^t \{\|\triangle_3(v^\epsilon-v^{\epsilon,n})(\tau,\cdot,1)\|_0+\|\triangle_3(v^\epsilon-v^{\epsilon,n})(\tau,\cdot,0)\|_0\}\,d\tau\\
&\le C\lim_{n\to\infty}\int_0^t \|(v^\epsilon-v^{\epsilon,n})(\tau,\cdot,\cdot)\|_{\frac52}\,d\tau=0,
\end{aligned}$$

which implies

$$\int_0^t \|P\triangle_3 v^\epsilon(\tau,\cdot,\eta)|_{\eta=l}\|_0^q\,d\tau=0,\quad \text{for } l=0,1,\ 0\le t<T^*,\ q<\infty. \tag{2.31}$$

3. Proof of Theorem 1.2: Solutions of the Quasigeostrophic Equations

The proof of Theorem 1.2 is similar to that of Theorem 1.1; and we only outline it here. A special care is taken since the Prandtl number is not one.

We first take the n^{th} approximate solutions:

$$\begin{aligned}
&\phi^n(t,x,\eta)=\sum_{k=1}^n g_{kn}(t)c^k(x,\eta),\\
&v_1^n(t,x,\eta)=-\sum_{k=1}^n g_{kn}(t)\partial_2 c^k(x,\eta),\quad v_2^n(t,x,\eta)=\sum_{k=1}^n g_{kn}(t)\partial_1 c^k(x,\eta),\\
&\theta^n(t,x,\eta)=\sum_{k=1}^n g_{kn}(t)\partial_\eta c^k(x,\eta),\quad \omega^n(t,x,\eta)=\sum_{k=1}^n g_{kn}(t)\triangle_3 c^k(x,\eta),
\end{aligned} \tag{3.1}$$

satisfying the boundary conditions: $(v^n,\partial_\eta\theta^n)|_{\eta=l}=0, l=0,1,$ where $c^k(x,\eta)$ are the normalized eigenfunctions of the Laplace operator, $\triangle_3 c^k=\gamma_k c^k$, which are orthonormal in J_3^0, and $g_{kn}(t), 1\le k\le n$, are determined by the following system:

$$\begin{cases}
\frac{d}{dt}(\omega^n,c^k)_0+(v^n\cdot\nabla\omega^n,c^k)_0-\mu_1(\triangle_3\omega^n,c^k)_0+(\mu_1-\mu_2)(\partial_\eta^2\omega^n,c^k)_0=0,\\
g_{kn}(0)=(\phi_0,c^k)_0.
\end{cases} \tag{3.2}$$

The regularity of the Laplace operator ensures that $\{c^k\}$ are in $H^\infty(\Omega)$. Then we will obtain some a-priori estimates for the approximations $(\phi^n,\omega^n,v^n,\theta^n)$.

Now we multiply (3.2) by $\gamma_k g_{kn}(t)$ and $\gamma_k^3 g_{kn}(t)$, respectively, and sum up to obtain

$$\frac12\frac{d}{dt}\|\omega^n\|_0^2+\mu_1\|\nabla_3\omega^n\|_0^2=(\mu_2-\mu_1)(\partial_\eta^2\omega^n,\omega^n)_0, \tag{3.3}$$

$$\frac12\frac{d}{dt}\|\triangle_3\omega^n\|_0^2+\mu_1\|\nabla_3\triangle_3\omega^n\|_0^2=(v^n\cdot\nabla\omega^n,\triangle_3^2\omega^n)_0+(\mu_2-\mu_1)(\partial_\eta^2\omega^n,\triangle_3^2\omega^n)_0. \tag{3.4}$$

On the other hand, similar to the proof of (2.10), the Bessel inequality implies

$$\|\omega_0^n\|_0\le\|\phi_0\|_2,\qquad \|\triangle_3\omega_0^n\|_0\le\|\phi_0\|_4. \tag{3.5}$$

Since

$$(v^n\cdot\nabla\omega^n,\triangle_3^2\omega^n)_0\le C\|(v^n,\nabla v^n)\|_{L^6}\|\omega^n\|_2\|\nabla_3\triangle_3\omega^n\|_0\le C\|(v^n,\nabla v^n)\|_1\|\omega^n\|_2\|\nabla_3\triangle_3\omega^n\|_0, \tag{3.6}$$

we follow (3.1) to have

$$\|v^n(t,\cdot)\|_\alpha\le\|\omega^n(t,\cdot)\|_{\alpha-1}\le\|\phi^n(t,\cdot)\|_{\alpha+1},\qquad \|\nabla_3v^n(t,\cdot)\|_\alpha\le\|\nabla_3\omega(t,\cdot)\|_{\alpha-1}. \tag{3.7}$$

Hence

$$(v^n\cdot\nabla\omega^n,\triangle_3^2\omega^n)_0\le C\|(\omega^n,\nabla_3\omega^n)\|_0^2\|\omega^n\|_2^2+\frac{\mu}{2}\|\nabla_3\omega^n\|_2^2, \tag{3.8}$$

with $\mu=\mu_1-|\mu_1-\mu_2|>0$. Furthermore,

$$\begin{aligned}|(\partial_\eta^2\omega^n,\omega^n)_0|&=\|\partial_\eta\omega^n\|_0^2\le\|\nabla_3\omega^n\|_0^2,\\ |(\partial_\eta^2\triangle_3\omega^n,\triangle_3^2\omega^n)_0|&\le\|\nabla_3\triangle_3\omega^n\|_0^2.\end{aligned} \tag{3.9}$$

Thus, by (3.3), (3.4), and (3.8),

$$\frac{d}{dt}\|\omega^n(t,\cdot)\|_0^2+\mu\|\nabla_3\omega^n(t,\cdot)\|_0^2\le 0, \tag{3.10}$$

$$\frac{d}{dt}\|\triangle_3\omega^n(t,\cdot)\|_0^2+\mu\|\nabla_3\triangle_3\omega^n(t,\cdot)\|_0^2\le C\|(\omega^n,\nabla_3\omega^n)(t,\cdot)\|_0^2\|\triangle_3\omega^n(t,\cdot)\|_0^2. \tag{3.11}$$

Using the Gronwall inequality and integrating (3.10) yield

$$\|\triangle_3\omega(t,\cdot)\|_0^2\le C\exp\left(C\int_0^t\|(\omega^n,\nabla_3\omega^n)(\tau,\cdot)\|_0^2\,d\tau\right)\le C\exp(Ct). \tag{3.12}$$

Now, multiplying (3.2) by $\gamma_k g'_{kn}(t)$ and summing up yield

$$\|\omega_t^n(t,\cdot)\|_0\le\mu_1\|\triangle_3\omega^n(t,\cdot)\|_0+\mu_2\|\triangle_3\partial_\eta\theta^n(t,\cdot)\|_0+\|v^n(t,\cdot)\|_1\|\omega^n(t,\cdot)\|_0^{\frac14}\|\omega^n(t,\cdot)\|_2^{\frac34}, \tag{3.13}$$

with a similar argument as in the proof of (2.26). Similarly, we can establish

$$\frac{d}{dt}\|\nabla_3\triangle_3\omega^n(t,\cdot)\|_0^2+\mu_1\|\triangle_3^2\omega^n(t,\cdot)\|_0^2=(v^n\cdot\nabla\omega^n,\triangle_3^3\omega^n)_0+(\mu_1-\mu_2)(\triangle_3\partial_\eta\theta^n,\triangle_3^3\omega^n)_0. \tag{3.14}$$

Trivially, by (3.1), we have

$$|(\triangle_3\partial_\eta\theta^n,\triangle_3^3\omega^n)_0|\le\|\triangle_3^2\omega^n\|_0^2, \tag{3.15}$$

and

$$\begin{aligned}&(v^n\cdot\nabla\omega^n,\triangle_3^3\omega^n)_0=-([\nabla_3\triangle_3,v^n]\nabla\omega^n,\nabla_3\triangle_3\omega^n)_0\\ \le&C(\|\nabla v^n(t,\cdot)\|_{L^\infty}\|D^3\omega^n(t,\cdot)\|_0+\|\nabla\omega^n(t,\cdot)\|_{L^\infty}\|D^3v^n(t,\cdot)\|_0)\|\nabla_3\triangle_3\omega^n(t,\cdot)\|_0,\end{aligned} \tag{3.16}$$

where we have used the inequality of Moser type (see [14]).

Thus, by the Sobolev and Poincaré inequalities, we have

$$(v^n\cdot\nabla\omega^n,\triangle_3^3\omega^n)_0\le(\|\nabla v^n(t,\cdot)\|_{L^\infty}+\|D^3v^n(t,\cdot)\|_0)\|\nabla_3\triangle_3\omega^n(t,\cdot)\|_0^2. \tag{3.17}$$

Using (3.7), (3.12), and $\|\nabla v^n(t,\cdot)\|_{L^\infty}+\|D^3v^n(t,\cdot)\|_0\le Ce^{Ct}$, we obtain

$$\|\nabla_3\triangle_3\omega^n(t,\cdot)\|_0^2+\mu\int_0^t\|\triangle_3\omega(\tau,\cdot)\|_0^2d\tau\le Ce^{e^{Ct}}. \tag{3.18}$$

Thus, by (3.12), (3.13), and (3.18), $\{\omega^n\}$ is compact in $L^p_{loc}([0,\infty);H^\alpha(\Omega))$, and $\{(v^n,\theta^n)\}$ is compact in $L^p_{loc}([0,\infty);H^{\alpha+1}(\Omega))$, for any $1<p<\infty,\alpha<4$. Without loss of generality, we assume that $(\phi,\omega,v,\theta)(t,x,\eta)$ is a limit of $(\phi^n,\omega^n,v^n,\theta^n)(t,x,\eta)$. Then $\omega=\triangle_3\phi$, and $(\phi,\omega,v,\theta)(t,x,\eta)$ is a smooth solution of (1.4) and

$$\|(v,\theta)(t,\cdot)\|_4+\mu\int_0^t\|\nabla_3(v,\theta)(\tau,\cdot)\|_4^2\,d\tau<\infty,\qquad \mu=\mu_1-|\mu_1-\mu_2|,\quad\text{for any}\quad t<\infty. \tag{3.19}$$

Hence, $(\phi, \omega, v, \theta)(t, x, \eta)$ satisfies all the properties in Theorem 1.2. The uniqueness is straightforward, and hence we omit it here.

Remark 3.1. *As Theorem 1.1, we can obtain further regularity results for $(\phi, \omega, v, \theta)(t, x, \eta)$ with respect to t.*

Remark 3.2. *By the same proof of Theorem 1.1, we can prove the local existence of strong solution $(v, \theta)(t, x, \eta) \in L^\infty((0, T^*); J^4) \cap L^2([0, T^*); H^5(\Omega))$, for some positive constant $T^* \leq \infty$, to the following equations:*

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu_1 \triangle_3 v + \nabla \Phi(t, x) = 0, \\ div\, v = 0, \\ \partial_t \theta + v \cdot \nabla \theta - \mu_2 \triangle_3 \theta = 0, \\ (v, \theta)|_{t=0} = (v_0, \theta_0) = (-\partial_{x_2} \phi_0, \partial_{x_1} \phi_0, \partial_\eta \phi_0), \\ (v, \theta_\eta)|_{\eta = l} = 0, \qquad for\ l = 0, 1, \end{cases} \tag{3.20}$$

with $\phi_0 \in J_3^5$. Moreover, it is trivial to calculate that, if we set $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1 + \partial_\eta \theta$, and $\triangle_3 \phi = \omega$, then $v = \nabla^\perp \phi, \theta = \partial_\eta \phi$, and, by the first equation of (3.20),

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega - \mu_1 \triangle_3 \omega + (\mu_1 - \mu_2) \triangle_3 \partial_\eta \theta = 0, \\ \omega|_{\eta = l} = 0, \quad for \quad l = 0, 1. \end{cases} \tag{3.21}$$

This shows that the solution $(v, \theta)(t, x, \eta)$ of (1.4) *and* (1.9) *must be a solution of* (3.20). *On the other hand, by* (1.4) *and* (1.9), *we have $\triangle_3 v|_{\eta = l} = \nabla^\perp \omega|_{\eta = l} = 0$ for $l = 0, 1$. Thus, by the first equation of* (3.20), *we have $\nabla \Phi(t, x) = 0$, which implies*

$$\partial_t v + v \cdot \nabla v - \mu_1 \triangle_3 v = 0. \tag{3.22}$$

Furthermore, by a direct calculation, we have

$$v^\perp + \nabla \Theta(\theta) = 0. \tag{3.23}$$

4. Proof of Theorem 1.3: Zero-Rossby Limit

We rewrite the solutions $(\phi^\epsilon, v^\epsilon, \theta^\epsilon)(t, x, \eta)$ of (1.1) as follows:

$$(\phi^\epsilon, v^\epsilon, \theta^\epsilon)(t, x, \eta) = (\phi, v, \theta)(t, x, \eta) + \epsilon(\overline{\phi}^{\epsilon, a}, v^{\epsilon, a}, \theta^{\epsilon, a})(t, x, \eta),$$

where $(\phi, v, \theta)(t, x, \eta)$ is the same as the correspondence in Theorem 1.2, and

$$\phi^\epsilon(t, x, \eta) = \int_1^\eta \theta^\epsilon(t, x, s)\, ds + \Phi^\epsilon(t, x).$$

Using Theorem 1.2, it suffices for Theorem 1.3 to prove that, for every $T > 0$, there exist $\epsilon_0 > 0$ and some continuous function $K(t)$ defined on $[0, T]$ depending only on T and ϵ_0 such that, when $\epsilon \leq \epsilon_0$,

$$\|v^{\epsilon, a}(t, \cdot)\|_2 + \|\theta^{\epsilon, a}(t, \cdot)\|_2 \leq K(t), \qquad \text{for } 0 \leq t < T. \tag{4.1}$$

To achieve this, we first use Theorems 1.1 and 1.2 and Remarks 2.2 and 3.2 to derive the equations and boundary conditions of $(v^{\epsilon, a}, \theta^{\epsilon, a})$. Then we perform the energy estimates by carefully employing the boundary conditions for $(v^{\epsilon, a}, \theta^{\epsilon, a})$ to derive an integral inequality for the H^2-norm of $(v^{\epsilon, a}, \theta^{\epsilon, a})$. Finally, using the comparison principle of ordinary differential equations yields (4.1).

Proof of (4.1). In the sequel, we will omit ϵ in $(v^{\epsilon, a}, \theta^{\epsilon, a})$, and we may regard $\|\triangle_3 a\|_0$ as $\|a\|_2$ and $(\cdot, \cdot)$ as $(\cdot, \cdot)_2$.

By Theorems 1.1 and 1.2 and Remark 3.2, we have

$$
\begin{cases}
\theta^a = \partial_\eta \phi^a, \\
\frac{Dv}{Dt} + \epsilon \frac{Dv^a}{Dt} - \mu_1 \triangle_3 (v + \epsilon v^a) + ((v^a)^\perp + \nabla \overline{\phi}^a) = 0, \\
\frac{D\theta}{Dt} + \epsilon \frac{D\theta^a}{Dt} - \mu_2 \triangle_3 (\theta + \epsilon \theta^a) - \operatorname{div} V(v^a) = 0, \\
(v^a, \partial_\eta \theta^a)|_{\eta=l} = 0, \qquad l = 1, 2,
\end{cases}
\tag{4.2}
$$

where the material derivative $\frac{D}{Dt} = \partial_t + v^\epsilon \cdot \nabla - \operatorname{div} V(v^\epsilon)\partial_\eta$. On the other hand, notice that, by (1.8), there is some $\Psi^a(t,x) \in L^2((0,T_\epsilon); H^2(\mathbb{T}^2))$ such that

$$
\triangle_3 v^a = P\triangle_3 v^a + \nabla \Psi^a. \tag{4.3}
$$

Set $\phi^a(t,x,\eta) = \overline{\phi^a}(t,x,\eta) - \epsilon\mu\Psi^a(t,x)$. Then we can write the second equation of (4.2) in the following form:

$$
\frac{Dv}{Dt} + \epsilon \frac{Dv^a}{Dt} - \mu_1 (\triangle_3 v + \epsilon P \triangle_3 v^a) + ((v^a)^\perp + \nabla \phi^a) = 0. \tag{4.4}
$$

Thus, by Remarks 2.2 and 3.2, we have

$$
\nabla \phi^a(t,x,\eta)|_{\eta=l} = 0, \qquad \text{for} \quad l = 0, 1, \quad (t,x) \in (0,T^*) \times \mathbb{T}^2. \tag{4.5}
$$

Taking $H^2(\Omega)$ inner product of (4.4) with v^a and the third equation in (4.2) with θ^a, respectively, yields

$$
\begin{aligned}
&\epsilon\{(\frac{Dv^a}{Dt} - \mu_1 P \triangle_3 v^a, v^a) + (\frac{D\theta^a}{Dt} - \mu_2 \triangle_3 \theta^a, \theta^a)\} + (\frac{Dv}{Dt} - \mu_1 \triangle_3 v, v^a) \\
&\quad + (\frac{D\theta}{Dt} - \mu_2 \triangle_3 \theta, \theta^a) + (\nabla \phi^a, v^a) - (\operatorname{div} V(v^a), \theta^a) = 0.
\end{aligned}
\tag{4.6}
$$

Now we perform the following integration by parts:

$$
\begin{aligned}
(\operatorname{div} V(v^a), \theta^a) &= \int_0^1 \int_{\mathbb{T}^2} \operatorname{div} \triangle_3 \int_0^\eta v^a(t,x,s)\, ds \triangle_3 \theta^a \, dx\, d\eta \\
&= \int_0^1 \int_{\mathbb{T}^2} \partial_\eta \left(\operatorname{div} \triangle_3 \int_0^\eta v^a(t,x,s)\, ds \triangle_3 \phi^a \right) dx\, d\eta - (\operatorname{div} v^a, \phi^a) = (v^a, \nabla \phi^a),
\end{aligned}
\tag{4.7}
$$

since $\phi^a(t,x,\eta) = \partial_\eta \theta^a(t,x,\eta)$ and $\triangle_3 \phi^a|_{\eta=l} = \triangle \phi^a|_{\eta=l} + \partial_\eta \theta^a|_{\eta=l} = 0$ by (4.5). Then

$$
\begin{aligned}
&(\frac{Dv}{Dt} - \mu_1 \triangle_3 v, v^a) + (\frac{D\theta}{Dt} - \mu_2 \triangle_3 \theta, \theta^a) \\
=&(\frac{Dv_1}{Dt} - \mu_1 \triangle_3 v_1, -\partial_{x_2}\phi^a + \mu_1 \triangle_3 v_2 - \frac{Dv_2}{Dt} - \epsilon(\frac{Dv_2^a}{Dt} - \mu_1 \triangle_3 v_2^a)) \\
&+ (\frac{Dv_2}{Dt} - \mu_1 \triangle_3 v_2, \partial_{x_1}\phi^a - \mu_1 \triangle_3 v_1 + \frac{Dv_1}{Dt} + \epsilon(\frac{Dv_1^a}{Dt} - \mu_1 \triangle_3 v_1^a)) + (\frac{D\theta}{Dt} - \mu_2 \triangle_3 \theta, \partial_\eta \phi^a) \\
=&(\frac{Dv_1}{Dt}, -\partial_{x_2}\phi^a) + (\frac{Dv_2}{Dt}, \partial_{x_1}\phi^a) + (\frac{D\theta}{Dt}, \partial_\eta \phi^a) + \mu_1(\triangle_3 v_1, \partial_{x_2}\phi^a) \\
&- \mu_1(\triangle_3 v_2, \partial_{x_1}\phi^a) - \mu_2(\triangle_3 \theta, \partial_\eta \phi^a) + \mu_1(\frac{Dv_1}{Dt}, \triangle_3 v_2) - \mu_1(\frac{Dv_2}{Dt}, \triangle_3 v_1) \\
&- \mu_1^2(\triangle_3 v_1, \triangle_3 v_2) + \mu_1^2(\triangle_3 v_2, \triangle_3 v_1) - (\frac{Dv_1}{Dt}, \frac{Dv_2}{Dt}) + (\frac{Dv_2}{Dt}, \frac{Dv_1}{Dt}) \\
&+ \mu_1(\triangle_3 v_1, \frac{Dv_2}{Dt}) - \mu_1(\triangle_3 v_2, \frac{Dv_1}{Dt}) + \epsilon(\frac{Dv_1}{Dt} - \mu_1 \triangle_3 v_1, \frac{Dv_2^a}{Dt} - \mu_1 \triangle_3 v_2^a) \\
&- \epsilon(\frac{Dv_2}{Dt} - \mu_1 \triangle_3 v_2, \frac{Dv_1^a}{Dt} - \mu_1 \triangle_3 v_1^a),
\end{aligned}
\tag{4.8}
$$

where we have substituted (v^a,θ^a) in (4.2) and (4.4) to $(\frac{Dv}{Dt}-\mu_1\triangle_3 v, v^a)$ and $(\frac{D\theta}{Dt}-\mu_2\triangle_3\theta,\theta^a)$. Note that

$$
\begin{aligned}
&(\frac{Dv_1}{Dt},-\partial_{x_2}\phi^a)+(\frac{Dv_2}{Dt},\partial_{x_1}\phi^a)+(\frac{D\theta}{Dt},\partial_\eta\phi^a)\\
&=-(d_g v_1+\epsilon d_a v_1,\partial_{x_2}\phi^a)+(d_g v_2+\epsilon d_a v_2,\partial_{x_1}\phi^a)+(d_g\theta+\epsilon d_a\theta,\partial_\eta\phi^a)\\
&=-(d_g v_1,\partial_{x_2}\phi^a)+(d_g v_2,\partial_{x_1}\phi^a)+(d_g\theta,\partial_\eta\phi^a)\\
&\quad-\epsilon\{-(d_a v_1,\partial_{x_2}\phi^a)+(d_a v_2,\partial_{x_1}\phi^a)+(d_a\theta,\partial_\eta\phi^a)\},
\end{aligned}
\tag{4.9}
$$

where we have adopted the notations $d_g=\partial_t+v\cdot\nabla$ and $d_a=v^a\cdot\nabla-\operatorname{div}V(v^a)\partial_\eta$. Moreover, it is obvious to see

$$
(d_g v_1,\partial_{x_2}\phi^a)=-(\partial_{x_2}d_g v_1,\phi^a),\qquad (d_g v_2,\partial_{x_1}\phi^a)=-(\partial_{x_1}d_g v_1,\phi^a),
\tag{4.10}
$$

and

$$
\begin{aligned}
(d_g\theta,\partial_\eta\phi^a)&=\int_0^1\int_{\mathbb{T}^2}\triangle_3(d_g\theta)\triangle_3(\partial_\eta\phi^a)\,dx\,d\eta\\
&=\int_0^1\int_{\mathbb{T}^2}\partial_\eta(\triangle_3(d_g\theta)\triangle_3\phi^a)\,dx\,d\eta-(\partial_\eta d_g\theta,\phi^a)=-(\partial_\eta d_g\theta,\phi^a),
\end{aligned}
\tag{4.11}
$$

where we have used again the observation following (4.7). Summing up (4.10) and (4.11) and using (3.18) lead to

$$
\begin{aligned}
&-(d_g v_1,\partial_{x_2}\phi^a)+(d_g v_2,\partial_{x_1}\phi^a)+(d_g\theta,\partial_\eta\phi^a)\\
&=(d_g(\partial_{x_2}v_1-\partial_{x_1}v_2-\partial_\eta\theta),\phi^a)=(\mu_1\triangle_3(\partial_{x_2}v_1-\partial_{x_1}v_2)-\mu_2\triangle_3\partial_\eta\theta,\phi^a)\\
&=-\mu_1(\triangle_3 v_1,\partial_{x_2}\phi^a)+\mu_1(\triangle_3 v_2,\partial_{x_1}\phi^a)+\mu_2(\triangle_3\theta,\partial_\eta\phi^a).
\end{aligned}
\tag{4.12}
$$

Using (4.7)-(4.12), we finally rewrite (4.6) as

$$
\begin{aligned}
&(\frac{Dv^a}{Dt}-\mu_1 P\triangle_3 v^a,v^a)+(\frac{D\theta^a}{Dt}-\mu_2\triangle_3\theta^a,\theta^a)-(d_a v_1,\partial_{x_2}\phi^a)+(d_a v_2,\partial_{x_1}\phi^a)+(d_a\theta,\partial_\eta\phi^a)\\
&+(\frac{Dv_1}{Dt}-\mu_1\triangle_3 v_1,\frac{Dv_2^a}{Dt}-\mu_1\triangle_3 v_2^a)+(\frac{Dv_2}{Dt}-\mu_1\triangle_3 v_2,\frac{Dv_1^a}{Dt}-\mu_1\triangle_3 v_1^a)=0,
\end{aligned}
\tag{4.13}
$$

while

$$
\begin{aligned}
&(\frac{Dv_1}{Dt}-\mu_1\triangle_3 v_1,\frac{Dv_2^a}{Dt}-\mu_1\triangle_3 v_2^a)+(\frac{Dv_2}{Dt}-\mu_1\triangle_3 v_2,\frac{Dv_1^a}{Dt}-\mu_1\triangle_3 v_1^a)\\
&=-\epsilon(d_a v_1,\frac{Dv_2^a}{Dt}-\mu_1\triangle_3 v_2^a)+\epsilon(d_a v_2,\frac{Dv_1^a}{Dt}-\mu_1\triangle_3 v_1^a)\\
&\quad-(d_g v_1-\mu_1\triangle_3 v_1,\frac{Dv_2^a}{Dt}-\mu_1\triangle_3 v_2^a)-(d_g v_2-\mu_1\triangle_3 v_2,\frac{Dv_1^a}{Dt}-\mu_1\triangle_3 v_1^a).
\end{aligned}
\tag{4.14}
$$

Using (4.4) again yields

$$
\begin{aligned}
&-(d_a v_1,\partial_{x_2}\phi^a)+(d_a v_2,\partial_{x_1}\phi^a)\\
&=\epsilon(d_a v_1,\frac{Dv_2^a}{Dt}-\mu_1\triangle_3 v_2^a)+(d_a v_1,\frac{Dv_2}{Dt}-\mu_1\triangle_3 v_2+v_1^a)\\
&\quad-\epsilon(d_a v_2,\frac{Dv_1^a}{Dt}-\mu_1\triangle_3 v_1^a)-(d_a v_2,\frac{Dv_1}{Dt}-\mu_1\triangle_3 v_1+v_2^a).
\end{aligned}
\tag{4.15}
$$

Thus, (4.13) turns to the following equality:

$$
\begin{aligned}
&(\frac{Dv^a}{Dt} - \mu_1 P\triangle_3 v^a, v^a) + (\frac{D\theta^a}{Dt} - \mu_2\triangle_3\theta^a, \theta^a)\\
&= -(d_a v_1, v_1^a) - (d_a v_2, v_2^a) - (d_a\theta, \theta^a) + (d_g v_1 - \mu_1\triangle_3 v_1, \frac{Dv_2^a}{Dt} - \mu_1\triangle_3 v_2^a)\\
&\quad - (d_g v_2 - \mu_1\triangle_3 v_2, \frac{Dv_1^a}{Dt} - \mu_1\triangle_3 v_1^a)\\
&\quad - (d_a v_1, \frac{Dv_2}{Dt} - \mu_1\triangle_3 v_2) + (d_a v_2, \frac{Dv_1}{Dt} - \mu_1\triangle_3 v_1).
\end{aligned}
\tag{4.16}
$$

On the other hand, noticing the fact from Remark 3.2 that $d_g v - \mu_1\triangle_3 v = 0$ and integrating (4.16) with respect to t, we obtain

$$
\begin{aligned}
&\|(v^a,\theta^a)(t,\cdot)\|_2^2 + 2\int_0^t (\mu_1\|\nabla_3 P\triangle_3 v^a(\tau,\cdot)\|_0^2 + \mu_2\|\nabla_3\theta^a(\tau,\cdot)\|_2^2)\,d\tau\\
\le&\|(v_0^a,\theta_0^a)\|_2^2 + 2|\int_0^t \{-(d_a v_1, v_1^a) + (d_a v_2, v_2^a) - (d_a\theta, \theta^a)\\
&\quad + ((\operatorname{div} V(v^\epsilon)\partial_\eta - v^\epsilon\cdot\nabla)v^a, v^a) + ((\operatorname{div} V(v^\epsilon)\partial_\eta - v^\epsilon\cdot\nabla)\theta^a, \theta^a)\\
&\quad - (d_a v_1, \frac{Dv_2}{Dt} - \mu_1\triangle_3 v_2) + (d_a v_2, \frac{Dv_1}{Dt} - \mu_1\triangle_3 v_1)\}\,d\tau|.
\end{aligned}
\tag{4.17}
$$

Next we estimate term by term in (4.17). First, from the discussion at the beginning of this section and the proof of (2.20), we find

$$
\begin{aligned}
&((\operatorname{div} V(v^\epsilon)\partial_\eta - v^\epsilon\cdot\nabla)v^a, v^a)\\
&\le\|\partial_\eta v^a\|_{L^6}\|\nabla_3 \operatorname{div} V(v^\epsilon)\|_{L^3}\|\nabla_3 v^a\|_2 + \|\operatorname{div} V(v^\epsilon)\|_{L^6}\|\nabla_3\partial_\eta v^a\|_{L^3}\|\nabla_3 v^a\|_2\\
&\quad + \|\nabla_3 v^\epsilon\|_{L^4}\|\nabla v^a\|_{L^4}\|\nabla_3 v^a\|_2 + \|v^\epsilon\|_{L^6}\|\nabla_3\nabla v^a\|_{L^3}\|\nabla_3 v^a\|_2\\
&\le C(3\epsilon\|v^a\|_2^{\frac32}\|\nabla_3 v^a\|_2^{\frac32} + \epsilon\|v^a\|_{\frac74}^2\|\nabla_3 v^a\|_2 + \|\nabla v^\epsilon\|_{L^4}\|v^a\|_{\frac74}\|\nabla v^a\|_2\\
&\quad + \|v\|_{L^6}\|v^a\|_2\|\nabla_3 v^a\|_2^{\frac32})\\
&\le C(\|v^a\|_2^2 + \epsilon^2\|v^a\|_2^4 + \epsilon^4\|v^a\|_2^6) + \alpha\|\nabla_3 v^a\|_2^2.
\end{aligned}
\tag{4.18}
$$

Similarly,

$$
\begin{aligned}
&((\operatorname{div} V(v^\epsilon)\partial_\eta - v^\epsilon\cdot\nabla)\theta^a, \theta^a)\\
&\quad\le C\left(\|\theta^a\|_2^2 + \epsilon^2\|v^a\|_2^2\|\theta^a\|_2^2 + \epsilon^4(\|\theta^a\|_2^4\|v^a\|_2^2 + \|v^a\|_2^4\|\theta^a\|_2^2)\right) + \alpha\|\nabla_3\theta^a\|_2^2,
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
(d_a v_1, v_1^a) &= (\triangle_3(v^a\cdot\nabla - \operatorname{div} V(v^a)\partial_\eta)v_1, \triangle_3 v_1^a)_0\\
&\le\|\nabla_3 v^a\|_{L^6}\|\nabla v_1\|_{L^3}\|\nabla v_1^a\|_2 + \|v^a\|_{L^\infty}\|\nabla_3\nabla v_1\|_0\|\nabla_3 v_1^a\|_2\\
&\quad + \|\partial_\eta v_1\|_{L^6}\|\operatorname{div} V(v^a)\|_{L^3}\|\nabla v_1^a\|_2 + \|\operatorname{div} V(v^a)\|_{L^\infty}\|\nabla_3\partial_\eta v_1\|_0\|\nabla_3 v_1^a\|_2\\
&\le C\|v_1\|_2^2\|v^a\|_2^2 + \alpha\|\nabla_3 v_1^a\|_2^2.
\end{aligned}
\tag{4.20}
$$

Similar estimates for $(d_a v_2, v_2^a)$ and $(d_a\theta, \theta^a)$ can be obtained.

By the definition of $\frac{D}{Dt}$, d_g, and d_a, we have

$$
\begin{aligned}
&-(d_a v_1, \frac{Dv_2}{Dt} - \mu_1\triangle_3 v_2) + (d_a v_2, \frac{Dv_1}{Dt} - \mu_1\triangle_3 v_1)\\
&= -(d_a v_1, d_g v_2 - \mu_1\triangle_3 v_2) - \epsilon(d_a v_1, d_a v_2) + (d_a v_2, d_g v_1 - \mu_1\triangle_3 v_1) + \epsilon(d_a v_2, d_a v_1) = 0.
\end{aligned}
\tag{4.21}
$$

Summing up (4.17)-(4.21), together with Theorem 1.2, Lemma 2.1, and taking $\alpha = \frac{\min(\mu_1,\mu_2)}{6}$, we finally have

$$\begin{aligned} &\|(v^a,\theta^a)(t,\cdot)\|_2^2 + \min(\mu_1,\mu_2)\int_0^t \|\nabla_3(v^a,\theta^a)(\tau,\cdot)\|_2^2\,d\tau \\ \le& C + C\int_0^t (\|v^a\|_2^2 + 2\epsilon^2\|v^a\|_2^4 + \epsilon^4\|v^a\|_2^6)\,d\tau, \qquad \text{for } 0 \le t < T^*. \end{aligned} \tag{4.22}$$

Set $y(t) = \|(v^a,\theta^a)(t,\cdot)\|_2^2$. Then

$$y(t) \le C\left(1 + \int_0^t \left(y(\tau) + 2\epsilon^2 y(\tau)^2 + \epsilon^4 y(\tau)^3\right)\,d\tau\right), \quad \text{for } 0 \le t < T^*. \tag{4.23}$$

Now we compare (4.23) with the following initial value problem:

$$\frac{df(t)}{dt} = C(f(t) + 2\epsilon^2 f(t)^2 + \epsilon^4 f(t)^3), \qquad f|_{t=0} = C. \tag{4.24}$$

For this equation, we can choose ϵ small enough to suppress the nonlinear growth terms so that its solution exists up to the pre-specified time T. In fact, if $f(t)$ is the solution of (4.24), then

$$t = \frac{1}{C}\ln\left(\frac{f(t)}{1+\epsilon^2 f(t)}\right)\left(\frac{1+\epsilon^2 C}{C}\right) + \frac{1}{1+\epsilon^2 f(t)} - \frac{1}{1+C\epsilon^2}.$$

Hence, if we can take $\epsilon_0 \ll 1$ such that $T \le \frac{1}{C}\ln\frac{1}{2\epsilon_0^2 C} - 1$, then, for any $\epsilon \in (0,\epsilon_0)$, $f(t)$ exists at least up to T. Thus, by the comparison principle of ordinary differential equation, (4.23), and (4.24),

$$\begin{aligned} &\|(v^a,\theta^a)(t,\cdot)\|_2^2 + \min(\mu_1,\mu_2)\int_0^t \|\nabla_3(v^a,\theta^a)(\tau,\cdot)\|_2^2\,d\tau \\ \le& C\left(1 + \int_0^t (f(\tau) + 2\epsilon^2 f(\tau)^2 + \epsilon^4 f(\tau)^3)\,d\tau\right) \equiv K(t), \end{aligned}$$

which implies (4.1). Thus, we complete the proof of Theorem 1.3.

Remark 4.1. *To justify* (4.17) *from* (4.2) *rigorously, we need a mollifying process; and, for simplicity of the presentation, we omit it here.*

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References

[1] J. T. Beale, T. Kato, and A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Commun. Math. Phys.* **94** (1984), 61-66.

[2] A. Bourgeois and J. T. Beale, Validity of the quasi-geostrophic model for large scale flow in the atmosphere and ocean, *SIAM. J. Math. Anal.* **25** (1994), 1023-1068.

[3] J. Y. Chemin, Remarque sur l'existence global pour le system de Navier-Stokes incompressible, *SIAM. J. Math. Anal.* **23** (1992), 20-28.

[4] J. Y. Chemin, A Propos d'un probléme de pènalisation de type antisymètrique, *C. R. Acad. Sci. Paris,Sér. I Math.* **321** (1995), 861-864.

[5] G.-Q. Chen, T.-P. Liu, and D. Levermore, Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure Appl. Math.* **47** (1994), 787–830.

[6] J. G. Heywood, The Navier-Stokes equations: On the existence, regularity and decay of solutions, *Indinia U. Math. J.* **29** (1980), 1-63.

[7] J. Leray, Etudes de diverses équations intégrables non linéaires et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pure Appl.* **12** (1933), 1-82.
[8] J. Leray, Essais sur le movement plan d'un liquid visqueux que limitent des parois, *J. Math. Pure Appl.* **13** (1934), 331-418.
[9] J. L. Lions, R. Teman, and S. Wang, New formulations of the primitive equations of the atmosphere and applications, *Nonlinearity*, **5** (1992), 237-288.
[10] J. L. Lions, R. Teman, and S. Wang, On the equations of the large scale ocean, *Nonlinearity*, **5** (1992), 1007-1053.
[11] J. L. Lions, R. Teman, and S. Wang, Geostrophic asympotics of the primitive equations of the atmosphere, *Topological Methods in Nonlinear Analysis*, **4** (1994), 1-35.
[12] J. L. Lions, R. Teman, and S. Wang, Mathematical theory for the coupled atmosphere-ocean model (CAO III), *J. Math. Pures. Appl.* **74** (1995), 105-163.
[13] J. L. Lions, Remark on some mathematical problems arising in Climatology, Proc. Int. Congress Indust. Appl. Math. (ICIAM 95) (Hamburg, 1995), 177-212, Math. Res. **87**, Akademie Verlag, Berlin, 1996.
[14] A. Majda, *Compressible Fluid Flow and System of Conservation Laws in Several Space Variables*, Springer-Verlag: Berlin-Heidelberg-New York-Tokyo, 1984.
[15] H. Trieble, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, 1978.

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