INITIAL LAYERS AND UNIQUENESS OF WEAK ENTROPY SOLUTIONS TO HYPERBOLIC CONSERVATION LAWS

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ABSTRACT. We consider initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws through the scalar case. The entropy solutions we address assume their initial data *only* in the sense of weak-star in L^∞ as $t \to 0_+$ and satisfy the entropy inequality in the sense of distributions for $t > 0$. We prove that, if the flux function has weakly genuine nonlinearity, then the entropy solutions are always unique and the initial layers do not appear. We also discuss its applications to the zero relaxation limit for hyperbolic systems of conservation laws with relaxation.

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Proposed Running Title: Initial Layers and Uniqueness

1991 *Mathematics Subject Classification.* Primary:35L65,35B30,35L45; Secondary:35B35,76E30.

Key words and phrases. Weak entropy solutions, Kruzkov solutions, uniqueness, initial layers, traces, conservation laws.

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1. Introduction

We are concerned with initial layers and uniqueness of weak entropy solutions for the Cauchy problem of scalar hyperbolic conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \tag{1.1}$$

$$u(x,0) = u_0(x). \tag{1.2}$$

The weak entropy solutions we address are defined in the following sense.

Definition 1.1. (1). We say that a function $u(x,t) \in L^\infty$ is a *weak entropy solution* of (1.1)-(1.2) if $u : (x,t) \to u(x,t)$ satisfies the following.

(a). u is a weak solution: For any function $\phi \in C_0^\infty(\mathbb{R}_+^2)$, $\mathbb{R}_+^2 \equiv \mathbb{R} \times [0,\infty)$,

$$\int_0^\infty \int_{-\infty}^\infty (u\, \partial_t \phi + f(u)\, \partial_x \phi) dx dt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0. \tag{1.3}$$

(b). For any nonnegative function $\phi \in C_0^\infty(\mathbb{R}_+^2 - \{t = 0\})$ and any convex entropy pair $(\eta(u), q(u)), \eta''(u) \geq 0, q'(u) = \eta'(u)f'(u)$,

$$\int_0^\infty \int_{-\infty}^\infty (\eta(u)\partial_t \phi + q(u)\, \partial_x \phi) dx\, dt \geq 0. \tag{1.4}$$

(2). In contrast, we say that a function $u(x,t) \in L^\infty$ is a Kruzkov solution if $u(x,t)$ satisfies, besides (1.3)-(1.4), the following property of (weak) L^1 continuity in time: For any $R > 0$,

$$\frac{1}{T}\int_0^T \int_{|x|\leq R} |u(x,t) - u_0(x)|\, dxdt \ \to\ 0, \quad \text{when } T \to 0. \tag{1.5}$$

(3). We say that a function $u(x,t)$ satisfies the strong entropy inequality if, for any convex entropy pair $(\eta(u), q(u))$ and any $\phi \in C_0^\infty(\mathbb{R}_+^2), \phi \geq 0$,

$$\int_0^\infty \int_{-\infty}^\infty (\eta(u)\partial_t \phi + q(u)\, \partial_x \phi) dx\, dt + \int_{-\infty}^\infty \eta(u_0)(x)\phi(x,0)dx \ \geq 0. \tag{1.6}$$

It is easy to check that any function $u(x,t)$ satisfying the strong entropy inequality (1.6) is a Kruzkov solution. This fact can be easily achieved with the aid of basic properties of divergence-measure fields, especially the normal traces, which will be discussed in Section 3. First, we pick a trivial entropy $\eta(u) = \pm u$ in (1.6) to conclude that u satisfies (1.3) and has a trace $u(\cdot, 0_+) = u_0$ on the set $t = 0$, defined at least in the weak-star sense in L^∞. Then, using the Gauss-Green formula of divergence-measure fields, we conclude from (1.6) that, for any strictly convex entropy η, the trace $\eta(u)(\cdot, 0_+)$ of $\eta(u)$ at $t = 0_+$ satisfies

$$\eta(u)(\cdot, 0_+) \leq \eta(u_0) = \eta(u(\cdot, 0_+)).$$

Then the strict convexity of η implies that the trace of u on the set $t = 0$ is in fact defined in the strong sense in L^1, which immediately implies (1.5).

The main objective of this paper is to establish the uniqueness and L^1 strong continuity in time of solutions satisfying only (1.3)-(1.4), provided that equation (1.1) has weakly genuine nonlinearity, that is,

(1.7) There exists no nontrivial interval on which f is affine.

Remark that the solutions defined in (1.3)-(1.4) are in general weaker than the Kruzkov solutions. It has been proved ([20, 10], see also [35] for the extension to the L^p case) that the Kruzkov solutions are *uniquely* defined.

For approximate solutions generated by either the vanishing viscosity method or a total variation diminishing (TVD) numerical scheme, e.g. a monotone conservative scheme, one can easily show that the limit function $u(x,t)$ satisfies (1.6), even if the initial data $u_0(x)$ are only in L^∞. Then, by the above arguments, there is no initial layer, which implies that the solution $u(x,t)$ is unique and stable in L^1.

However, when we consider the limit behavior of other physical regularizing effects, especially the zero relaxation limit, there is definitely an initial layer, unless the initial data are already at the equilibrium. Therefore, the uniqueness of limit functions becomes a crucial problem, as observed in [25] (also see [19]). To be more specific, we will recall in Section 2 a particular case (cf. [16]), in which the relaxing system is a semilinear 2×2 system. In this case one can show that, for any strictly convex entropy pair $(\eta, q)(u)$ of (1.1), there exists a strictly convex entropy pair $(\eta_*, q_*)(u,v)$ of (2.1) such that

$$(\eta, q)(u) \equiv (\eta_*, q_*)(u, f(u)),$$

by using the theory developed in [6, 7]. Then it is easy to show by compensated compactness (see [9, 6, 7]) that the limit function $u(x,t)$ satisfies the following inequality: For any nonnegative function $\phi \in C_0^\infty(I\!R^2_+)$ and any convex entropy pair $(\eta, q)(u)$ of (1.1),

$$\int_0^\infty \int_{-\infty}^\infty (\eta(u)\partial_t\phi + q(u)\,\partial_x\phi)dxdt + \int_{-\infty}^\infty \eta_*(u_0, v_0)(x)\phi(x,0)dx \geq 0. \tag{1.8}$$

Inequality (1.8) implies that u satisfies (1.3)-(1.4).

Now, if the initial data of the relaxing system (2.1) are at the equilibrium, i.e. $v_0(\cdot) = f(u_0(\cdot))$, then $u(x,t)$ satisfies (1.6), and hence (1.3)-(1.5), by the above convexity arguments (also see [10]). Therefore, $u(x,t)$ *is* the unique Kruzkov solution. In contrast, if the initial data of (2.1) are *away from the equilibrium*, i.e. $v_0(\cdot) \neq f(u_0(\cdot))$, then there is obviously an initial layer. Then the problem is whether the limit function $u(x,t)$ satisfies *not only* (1.8), but *also* (1.6). In other words, we have to show that the limit function $u(x,t)$ is strongly continuous in time in L^1 and is indeed the unique Kruzkov solution of (1.1)-(1.2), that is, the initial layer has entirely collapsed to the line $t = 0$. In general, this is not the case.

With this type of motivations, in this paper we establish a more general result: *any* weak entropy solution $u(x,t)$ of (1.1)-(1.2), satisfying *only* (1.3)-(1.4), is in fact the unique Kruzkov solution, provided that the flux function has weakly genuine nonlinearity.

The outline of the paper is as follows. The main result is stated and proved in Section 4. In particular, it covers the case of the above-mentioned zero relaxation limit, which is recalled more precisely in Section 2. Our proof uses the theory of normal traces and the Gauss-Green formula for divergence-measure fields, which are discussed in Section 3.

Naturally, quite similar questions arise in studying the initial-boundary value problems. See e.g. [2, 4, 11, 17, 18, 34], also [26, 38, 40] for the zero relaxation approximation and related results in [28, 36, 37] for kinetic approximations. It would be also interesting to apply the techniques and ideas developed here to the boundary layer problems.

2. ZERO RELAXATION LIMIT

As a prototype, consider the semilinear relaxing system of (1.1)-(1.2):

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + a^2\, \partial_x u = \frac{1}{\varepsilon}\,(f(u) - v), \end{cases} \tag{2.1}$$

$$(u,v)(x,0) = (u_0,v_0)(x), \tag{2.2}$$

where $\varepsilon > 0$ is the relaxation time and $a > 0$ a constant (see [16]). We assume that a satisfies the classical *subcharacteristic condition* (see [39, 23]):

$$\sup_u\{f'(u)\} < a. \tag{2.3}$$

Now we list the following facts obtained by using either the compensated compactness method (see [6, 7, 9]), or BV estimates uniform in ε, see [25, 19], which are used in Section 4.

(i). For all $\varepsilon > 0$, there exists a unique globally defined solution $(u^\varepsilon, v^\varepsilon)$ to (2.1)-(2.2), uniformly bounded in L^∞.

(ii). This sequence $(u^\varepsilon, v^\varepsilon)$ is uniformly bounded in the space $L^\infty((0,\ \infty); BV(I\!R))$ if the initial data (2.2) are in the space of functions of x with bounded variation.

(iii). For any strictly convex entropy pair $(\eta(u), q(u))$ of equation (1.1), there exists a strictly convex entropy pair $(\eta_*, q_*)(u,v)$ such that

$$(\eta_*,\ q_*)(u, f(u)) \equiv (\eta, q)(u), \tag{2.4}$$

and

$$\eta_*(u, f(u)) \equiv \min_v\ \{\eta_*(u,v)\}. \tag{2.5}$$

Note that there is no difficulty in extending the convexity properties far from the equilibrium, since the relaxing system is semilinear (see [6, 7, 25]). Therefore, constructing these extended entropy pairs amounts to solving the linear wave equation with data on a curve which is noncharacteristic, in view of (2.3) (see [9, 15]).

(iv). For any $\varepsilon > 0$ and any function $\phi \in C_0^\infty(I\!R_+^2)$, we have

$$\int_0^\infty \int_{-\infty}^\infty (u\partial_t\phi + v\,\partial_x\phi)dxdt + \int_{-\infty}^\infty u_0(x)\ \phi(x,\ 0)dx = 0, \tag{2.6}$$

and a similar relation holds, with the obvious right-hand side, for the second equation.

(v). For all fixed $\varepsilon > 0$ and any convex entropy η, the entropy *equality* holds:

$$\begin{aligned} &\textstyle\int_0^\infty\int_{-\infty}^\infty (\eta(u,v)\partial_t\phi + q(u,v)\,\partial_x\phi)dxdt + \int_{-\infty}^\infty \eta(u_0,v_0)(x)\ \phi(x,0)dx \\ &= \tfrac{1}{\varepsilon}\textstyle\int_0^\infty\int_{-\infty}^\infty \partial_v\eta(u,v)\ (f(u) - v)\phi\ dxdt. \end{aligned} \tag{2.7}$$

In particular, the right-hand side is nonpositive if ϕ is nonnegative, in view of (iii).

(vi). Now we assume that $f(u)$ has weakly genuine nonlinearity (1.7). Then, using only two entropy pairs (see [8], also [6, 7, 9]), we conclude the following result, with the aid of the classical Cauchy-Schwarz inequality (see [8, 9, 29, 30, 31]).

Proposition 2.1. *Assume that* (1.7) *holds. Then there exists a subsequence (still denoted by)* $(u^\varepsilon, v^\varepsilon)$ *of solutions of (2.1)-(2.2) such that*

(a). $(u^\varepsilon,\, v^\varepsilon) \to (u,\, v)$ *boundedly a.e..*

(b). $v(x,t) = f(u(x,t)),\ t > 0$. *More precisely,*

$$\|v^\varepsilon - \ f(u^\varepsilon)\|_{L^2(Q)} = \ O(\varepsilon^{1/2}), \quad \textit{for any } Q \Subset I\!\!R^2_+. \tag{2.8}$$

(c). $u(x,t)$ *is a weak solution of (1.1)-(1.2).*

(d). For any convex entropy pair $(\eta, q)(u)$ *and any function* $\phi \in C_0^\infty(I\!\!R^2_+), \phi \geq 0$,

$$\int_0^\infty \int_{-\infty}^\infty (\eta(u)\partial_t\phi + q(u)\,\partial_x\phi)dx\,dt + \int_{-\infty}^\infty \eta_*(u_0(x), v_0(x))\phi(x,0)dx \ \geq\ 0, \tag{2.9}$$

where $\eta_*(u,v)$ *is the above-mentioned convex extension of* $\eta(u)$ *satisfying*

$$\eta(u_0(x)) \leq \eta_*(u_0(x), v_0(x)), \quad a.e. \tag{2.10}$$

(e). If the initial data are at the equilibrium

$$v_0(x) = \ f(u_0(x)) \qquad a.e.$$

then $u(x,t)$ *satisfies (1.6) and hence is the unique Kruzkov solution of (1.1)-(1.2).*

In Section 3, we give more details on the last point in Proposition 2.1 with the convexity argument of Section 1 in mind. In Section 4, we deal with the case where $v_0(\cdot) \neq f(u_0(\cdot))$.

3. Divergence-Measure Fields in L^∞ and Normal Traces

In this section, we discuss some properties of divergence-measure fields, especially their normal traces defined in [4] (also see [1] for a related notion from a different point of view). These properties are important in the proof of our main theorem in Section 4.

Definition 3.1. Let $D \subset I\!\!R^N$ be an open set. We say $F \in L^\infty(D; I\!\!R^N)$ is a divergence-measure field if

$$|\mathrm{div} F|(D) = \sup\Big\{ \int_D F\cdot\nabla\phi\, dy \ \mid\ \phi \in C_0^1(D; I\!\!R),\, |\phi(y)| \leq 1,\, y \in D \Big\} < \infty. \tag{3.1}$$

Then $\mathrm{div}F$ is a Radon measure over D. We define $\mathcal{DM}(D)$ as the space of divergence-measure fields over D and, under the norm

$$\|F\|_{\mathcal{DM}} = \|F\|_{L^\infty} + |\mathrm{div}F|(D),$$

$\mathcal{DM}(D)$ is a Banach space.

One of the main points for divergence-measure fields is to understand the normal traces on a deformable Lipschitz boundary. We recall the notion of normal traces introduced in [4]. The notion has the advantage of providing essential information about the normal trace on certain hypersurfaces from the knowledge of the normal traces in its neighboring hypersurfaces, as we will see in Theorem 3.1. This advantage is made possible by introducing Lipschitz deformations, which are important not only for the definition of the normal traces, but also for its applications. Note that a related notion of normal traces was also introduced with a different point of view in [1], in which a normal trace was defined as a representation function of a linear functional, in a more abstract fashion. However, the Gauss-Green formula (3.6) coincides, which means - fortunately ! - that both notions are consistent.

It is important to observe that, in general, one cannot define the trace for each component of a $\mathcal{DM}$ field over any Lipschitz boundary, as opposed to the case of BV fields. This fact can be easily seen through the example of the $\mathcal{DM}$ field $F(y_1, y_2) = (\cos(\frac{1}{y_1-y_2}), \cos(\frac{1}{y_1-y_2}))$. It is indeed impossible to define any reasonable notion of trace over the line $y_1 = y_2$ for its components. Nevertheless, the unit normal ν_τ to the line $y_1 - y_2 = \tau$ is the vector $(-1/\sqrt{2}, 1/\sqrt{2})$ so that the scalar product $F(y_1, y_1 - \tau) \cdot \nu_\tau$ is identically zero over this line. Hence, we note that $F \cdot \nu \equiv 0$ over the line $y_1 = y_2$ and the Gauss-Green formula in this case implies

$$0 = \langle \, divF|_{y_1>y_2}, \phi \rangle = \, - \int_{y_1>y_2} F \cdot \nabla\phi \, dy_1 dy_2,$$

for any $\phi \in C_0^1(I\!R^2)$.

Let Ω be an open subset in $I\!R^N$. Following [4], we say that $\partial\Omega$ is a *deformable Lipschitz boundary* provided that the following hold.

(i): For each $x \in \partial\Omega$, there exist $r > 0$ and a Lipschitz mapping $\gamma : I\!R^{N-1} \to I\!R$ such that, upon rotating and relabeling the coordinate axes if necessary,

$$\Omega \cap Q(x, r) = \{ \, y \in I\!R^N \mid \gamma(y_1, \cdots, y_{N-1}) < y_N \, \} \cap Q(x, r),$$

where $Q(x, r) = \{ \, y \in I\!R^N \mid |y_i - x_i| \le r, \; i = 1, \cdots, N \, \}$. We denote by $\tilde{\gamma}$ the map $(y_1, \cdots, y_{N-1}) \mapsto (y_1, \cdots, y_{N-1}, \gamma(y_1, \cdots, y_{N-1}))$.

(ii): There exists a map $\Psi : \partial\Omega \times [0, 1] \to \bar{\Omega}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(\cdot, 0) \equiv I$, where I is the identity map over $\partial\Omega$.

Denote $\partial\Omega_\tau \equiv \Psi(\partial\Omega \times \{\tau\})$, $\tau \in [0, 1]$, and denote Ω_τ the open subset of Ω whose boundary is $\partial\Omega_\tau$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

We say that the Lipschitz deformation is *regular* if

$$\lim_{\tau\to 0+} \nabla\Psi_\tau \circ \tilde{\gamma} = \nabla\tilde{\gamma} \qquad \text{in } L^1_{\mathrm{loc}}(B), \tag{3.2}$$

where $\tilde{\gamma}$ is a map as in condition (i), and Ψ_τ denotes the map of $\partial\Omega$ into Ω, given by $\Psi_\tau(x) = \Psi(x, \tau)$. Here B denotes the largest open set such that $\tilde{\gamma}(B) \subset \partial\Omega$.

Remark 3.1. Conditions (i)-(ii) above can be verified even for the star-shaped domains and the domains whose boundaries satisfy the cone property. If Ω is the image through a bi-Lipschitz map of a domain $\bar{\Omega}$ with a (regular) Lipschitz deformable boundary, then Ω itself possesses a (regular) Lipschitz deformable boundary.

Let $F \in \mathcal{DM}(D)$. Let $\Omega \subset D$ be an open set with deformable Lipschitz boundary. Let Ψ be a Lipschitz deformation of $\partial\Omega$. Then, following [4], there exists a $\mathcal{T} \subset (0, 1)$ with $meas([0, 1] - \mathcal{T}) = 0$ such that, for every $\tau \in \mathcal{T}$ and all $\phi \in C_0^1(I\!R^N)$,

$$\langle \mathrm{div} F|_{\Omega_\tau}, \phi \rangle = \int_{\partial\Omega_\tau} \phi(\omega) F(\omega) \cdot \nu_\tau(\omega) \, d\mathcal{H}^{N-1}(\omega) - \int_{\Omega_\tau} F(y) \cdot \nabla\phi(y) \, dy, \tag{3.3}$$

where ν_τ is a unit outward normal field defined $\mathcal{H}^{N-1}$-almost everywhere in $\partial\Omega_\tau$.

Since $F \cdot \nu_\tau$ is defined $\mathcal{H}^{N-1}$-almost everywhere on $\partial\Omega_\tau$ for $\tau \in \mathcal{T}$, we may regard $F \cdot \nu_\tau$ as either a Radon measure over $\partial\Omega_\tau$ or a Radon measure over $\partial\Omega$. We then define

$$F \cdot \nu|_{\partial\Omega} = \text{w-} \lim_{\tau\to 0 \;\, \tau\in\mathcal{T}} F \cdot \nu_\tau, \qquad \text{in } \mathcal{M}(\partial\Omega). \tag{3.4}$$

We justify (3.4) in the following theorem.

Theorem 3.1 ([4]). (1). Let $F \in \mathcal{DM}(D)$ and $\Omega \subset D$ be an open set with regular deformable Lipschitz boundary. The limit in (3.4) exists when $F \cdot \nu_\tau$ are regarded as Radon measures on $\partial\Omega$ through the formula:

$$\langle F \cdot \nu_\tau, \phi \rangle \equiv \int_{\partial\Omega_\tau} \phi(\Psi_\tau^{-1}(\omega)) F(\omega) \cdot \nu_\tau(\omega)\, d\mathcal{H}^{N-1}(\omega), \tag{3.5}$$

where $\Psi_\tau : \partial\Omega \to \partial\Omega_\tau$ is given by $\Psi_\tau(\omega) = \Psi(\omega, \tau)$.

(2). This definition for $F \cdot \nu$ over $\partial\Omega$ yields the following Gauss-Green formula:

$$\langle \mathrm{div} F|_\Omega, \phi \rangle = \int_{\partial\Omega} \phi(\omega) F(\omega) \cdot \nu(\omega)\, d\mathcal{H}^{N-1}(\omega) - \int_\Omega F(y) \cdot \nabla\phi(y)\, dy \tag{3.6}$$

for any $\phi \in C_0^1(\mathbb{R}^N)$, where, in the first integral in the right-hand side, the formal notation $F(\omega)\cdot\nu(\omega)\, d\mathcal{H}^{N-1}(\omega) \equiv F\cdot\nu$ is used for the normal trace measure justified in (i) below.

(3). The normal trace measure $F \cdot \nu$ has the following properties:

(i): $F \cdot \nu$ does not depend on the particular Lipschitz deformation for $\partial\Omega$ and is absolutely continuous with respect to $\mathcal{H}^{N-1}\big|_{\partial\Omega}$;

(ii): If $\partial\Omega \subset D$ and $|\mathrm{div} F|(\partial\Omega) = 0$, the density of $F \cdot \nu$ coincides with the function $F \cdot \nu$, $\mathcal{H}^{N-1}$- a.e. in $\partial\Omega$, whenever $\mathcal{H}^{N-1}(\partial\Omega \cap \mathcal{N}) = 0$, where $\mathcal{N}$ is the null set in the definition of the precise representative of F;

(iii): Denote also $F \cdot \nu$ the corresponding density. Then $F \cdot \nu \in L^\infty(\partial\Omega)$ and,

$$\|F \cdot \nu\|_{L^\infty(\partial\Omega)} \le \|F\|_{L^\infty(\Omega)}, \tag{3.7}$$

$$F \cdot \nu = w^* - ess \lim_{\tau \to 0+} (F \cdot \nu_\tau) \circ \Psi_\tau, \qquad \text{in } L^\infty(\partial\Omega). \tag{3.8}$$

The relations between divergence-measure fields and hyperbolic conservation laws can be seen via the Lax entropy inequality (1.4) and the Schwartz lemma [33]. Let u be an L^∞ entropy solution of (1.1)-(1.2). First, we deduce from (1.4) that, for any convex entropy pair (η, q),

$$\mu_{\eta,u} \equiv \partial_x q(u) + \partial_t \eta(u) \in \mathcal{M}(I\!R \times (0,\infty)), \tag{3.9}$$

using the Schwartz lemma, where $\mathcal{M}(I\!R \times (0,\infty))$ denotes the space of Radon measure defined on $I\!R \times (0,\infty)$. Therefore, we have

Lemma 3.1. Let $u \in L^\infty(I\!R^2_+)$ be an entropy solution to (1.1), which is endowed with a strictly convex entropy $\tilde\eta$. Then, for any C^2 entropy pair (η, q) and for any $T \in (0,\infty)$ and $-\infty < a < b < \infty$,

$$|\mu_{\eta,u}|((a,b) \times (0,T)) < \infty,$$

and

$$(q(u), \eta(u)) \in \mathcal{D}M((a,b) \times (0,T)).$$

Proof. We divide the proof into two steps.

Step 1. We first consider the case where $\eta(u)$ is convex. Then the measure $-\mu_{\eta,u}$ in $I\!R \times (0,\infty)$ is positive, that is,

$$-\mu_{\eta,u} \equiv -(\partial_x q(u) + \partial_t \eta(u)) \ge 0$$

in the sense of measures.

Set

$$Q_\delta = (a,b) \times (\delta, T), \qquad 0 < \delta < T < \infty,\ -\infty < a < b < \infty.$$

Then $Q_0 := Q \equiv (a,b)\times(0,T)$. We notice that

$$(q(u),\eta(u)) \in (L^\infty(Q))^2, \tag{3.10}$$

which implies that $(q(u),\eta(u))$ is a divergence-measure field over any Q_δ, $\delta>0$, for any C^2 convex entropy pair (η,q).

For any $\delta>0$, we use the Gauss-Green formula (3.6) with $\phi_\delta \in C_0^1(I\!\!R_+^2), \phi_\delta \geq 0, \phi_\delta|_{Q_\delta} \equiv 1$, and (3.7) to obtain

$$\begin{aligned}
&< -\mu_{\eta,u}, \phi_\delta >= | < \mu_{\eta,u}, \phi_\delta > | \\
&= | \textstyle\int_a^b \eta(u)(x,T_-)dx - \int_a^b \eta(u)(x,\delta_+)dx + \int_\delta^T q(u)(a_+,t)dt - \int_\delta^T q(u)(b_-,t)dt| \\
&\leq C\|(q(u),\eta(u))\|_{L^\infty} \equiv M < \infty,
\end{aligned}$$

where $M = M(a,b,T)$ is independent of $\delta>0$.

In particular, for any compact set $K \Subset Q$, there exists $\delta>0$ such that $K\subset Q_\delta$. Then, since $-\mu_{\eta,u}$ is a positive Radon measure, we have

$$|\mu_{\eta,u}|(K) = -\mu_{\eta,u}(K) \leq -\mu_{\eta,u}(Q_\delta) =< -\mu_{\eta,u}|_{Q_\delta}, 1 >\leq< -\mu_{\eta,u}, \phi_\delta >\leq M < \infty,$$

where $M>0$ is independent of K. This implies $|\mu_{\eta,u}|(Q)<\infty$.

Step 2. The extension to an arbitrary C^2 entropy η follows then from the same argument as in [3], by adding and subtracting if necessary the strictly convex entropy $C\,\tilde\eta$, with a suitable positive constant C so that both $C\,\tilde\eta$ and $C\,\tilde\eta - \eta$ are convex entropies. Then

$$|\mu_{\eta,u}|(Q) \leq |\mu_{C\tilde\eta,u}|(Q) + |\mu_{C\tilde\eta-\eta,u}|(Q) < \infty.$$

This fact, together with (3.10), implies

$$(q(u),\eta(u)) \in \mathcal{D}M(Q),$$

for any C^2 entropy-entropy flux pair (η,q). This completes the proof.

We will also need the following Lemma.

Lemma 3.2. (1). Let α be a bounded measure of one variable on $(0,1]$. Then $\alpha((0,t])$ tends to zero as t goes to 0_+.

(2). Similarly, let α be a bounded measure of two variables on $(a,b)\times(0,1]$. Then $\alpha((a,b)\times(0,t])$ tends to zero as t goes to 0_+, for any fixed $-\infty<a<b<\infty$.

Proof: *Step 1.* Fact (1) can be proved as follows:

$$\lim_{n\to\infty} \alpha((0,\frac{1}{n}]) = \alpha(\cap_{n=1}^\infty(0,\frac{1}{n}]) = 0,$$

since the set $\cap_{n=1}^\infty(0,\frac{1}{n}]$ is empty (see e.g. page 2 of [13]). More intuitively, we can decompose α into three parts: the part α_{ac} which is absolutely continuous with respect to the Lebesgue measure (i.e. the L^1 part), the Cantor part α_c, and the atomic part α_{at}:

$$\alpha = \alpha_{ac} + \alpha_c + \alpha_{at}.$$

The first two parts do not charge $t=0$. The last part is a convergent series of Delta functions, located on a countable set of points $\{t_n\}, t_n>0$, and their mass is a convergent series. Therefore, one concludes that $\alpha(\{t_n\})\to 0$ as t_n converges to 0_+.

Step 2. The extension to the two-dimensional case, or even to the multidimensional case is trivial by Fubini's Theorem.

Lemma 3.3. Let $u \in L^\infty(I\!R^2_+)$ be an entropy solution of hyperbolic conservation laws with a strictly convex entropy. Then, for any C^2 entropy pair (η, q), the difference between the normal traces of the divergence-measure field (q, η) at $t_- > 0$, denoted by $\eta(u)(\cdot, t_-)$, and at 0_+, by $\eta(u)(\cdot, 0_+)$, tends to zero in the L^∞ weak-star topology as $t \to 0_+$. That is,

$$\eta(u)(\cdot, t_-) - \eta(u)(\cdot, 0_+) \overset{*}{\rightharpoonup} 0, \qquad t \to 0_+.$$

Proof. For any $\phi \in C_0^1(I\!R^2_+)$, with $\phi \equiv 1$ on $\Omega_t := (a, b) \times (0, t)$, $-\infty < a < b < \infty$, we apply the Gauss-Green formula to the divergence-measure field $F = (q(u), \eta(u))$ with $\eta \in C^2$, by Lemma 3.1. We obtain

$$< \mathrm{div} F|_{\Omega_t}, 1 >= \int_{\partial\Omega_t} (q, \eta) \cdot \nu \, d\mathcal{H}^1(\omega).$$

Notice that

$$|\int_0^t q(u)(a_+, \tau)\, d\tau\,| + \,|\int_0^t q(u)(b_-, \tau)\, d\tau| \le Ct,$$

for some constant C. Using Lemma 3.2, we conclude that, for some new constant C,

$$\begin{aligned}
&|\int_a^b \eta(u)(x, t_-)dx - \int_a^b \eta(u)(x, 0_+)dx| \\
\le& | < -\mathrm{div} F|_{\Omega_t}, 1 > | + \,Ct \\
=& |\mathrm{div} F|\,((a, b) \times (0, t)) + \,Ct \to 0, \quad \text{when } t \to 0.
\end{aligned}$$

Remark. We emphasize that Lemmas 3.1 and 3.3 are also true for nonlinear hyperbolic systems of conservation laws in one space variable. They can be directly reformulated for the systems in several space variables.

4. Main Theorem and Its Applications

Main Theorem. *Assume that (1.7) is satisfied. Let $u(x, t)$ be an L^∞ weak entropy solution of the Cauchy problem (1.1)-(1.2). Then $u(x, t)$ satisfies (1.6), which implies that $u(x, t)$ is the unique Kruzkov solution.*

Proof. We divide the proof into eight steps.
Step 1. Let $(\tilde{\eta}, \tilde{q})$ be a strictly convex entropy pair of (1.1). Since $u(x, t)$ satisfies (1.3)-(1.4), Lemma 3.1 indicates that, for any C^2 entropy pair (η, q),

$$\partial_t \eta(u) + \,\partial_x q(u) := \mu_{\eta,u} \tag{4.1}$$

is a bounded measure on any bounded subset of $I\!R \times\, (0,\, \infty)$. Then, for *any* positive time t, $\eta(u)$ has two normal traces $\eta(u)(t_\pm) := \eta(u)(\cdot,\, t_\pm)$. These two traces are L^∞ functions, defined (at least) as weak-star limits in $L^\infty(I\!R)$, and coincide for *almost all* time t (those which are *not* charged by the bounded measure $\mu_{\eta,u}$ in (4.1)).

Moreover, the Gauss-Green formula (3.6) holds as stated in Section 3. In particular, the right-hand side of (1.1) is 0, so that u is continuous in time with values in the space of distributions in x. Therefore, the traces $u(t_-)$ and $u(t_+)$ coincide for *all* time $t > 0$. Furthermore, since u is a weak solution to (1.1)-(1.2),

$$u(0_+) = u_0. \tag{4.2}$$

In other words, there is no initial layer for u, at least in the L^∞ weak-star topology. On the other hand, there might be an initial layer for $\eta(u)$. Similarly, the traces of the flux $f(u)$ (resp. $q(u)$) at $x_\pm$ are defined for *all* (resp. almost all) $x \in I\!R$.

Now we have seen in Section 3 that the Gauss-Green formula (3.6) holds especially on any rectangle $(a,\ b) \times (0,\ t)$ for the divergence-measure field $F := (q, \eta)$ and any Lipschitz function ϕ, and naturally involves the traces of $\eta(u)$ at 0_+ and t_-, and the traces of $q(u)$ at a_+ and b_-. In particular, if

$$l := \tilde{\eta}(u)(0_+)\ , \tag{4.3}$$

we note that

$$\tilde{\eta}(u)(t_-) \ \stackrel{*}{\rightharpoonup}\ l, \qquad \text{as } \ t\ \to\ 0, \tag{4.4}$$

and similarly

$$u(t) =\ u(t_-) \ \stackrel{*}{\rightharpoonup}\ u_0, \qquad \text{as}\, t\ \to\ 0. \tag{4.5}$$

Step 2. The issue is now to show that, at least for a particular strictly convex entropy $\tilde{\eta}$, the weak limit (4.4) satisfies

$$l =\ \tilde{\eta}(u_0). \tag{4.6}$$

Indeed, if this is true, then, by Jensen's inequality, the convergence in (4.5) is in fact *strong*, in L^1, which implies the strong entropy inequality (1.6), as well as the desired (weak) property (1.5) of L^1 continuity in time.

Step 3. In order to show (4.6), we consider the (unique) Kruzkov solution U to the Cauchy problem (1.1)-(1.2), which satisfies (1.5). Let $(a,\ b)$ be an arbitrary interval. We are going to show that, at least for a subsequence $\{t_n\}$,

$$(u -\ U)(\cdot, t_n) \to\ 0, \quad \text{a.e. on (a, b) when } \ t_n \to\ 0, \tag{4.7}$$

and then the result follows.

Step 4. Let us first show (4.7). First, subtract the equations (1.1) for solutions u and U. Similarly, consider the entropy $\eta(u) := f(u)$ and subtract (4.1) for solutions u and U, we obtain

$$\begin{aligned} &\partial_t(u -\ U)\ +\ \partial_x(f(u) -\ f(U))\ =\ 0, \\ &\partial_t(f(u) -\ f(U))\ +\ \partial_x(g(u) -\ g(U)) :=\ \mu, \end{aligned} \tag{4.8}$$

where

$$g(u) = \int_0^u (f'(\xi))^2\, d\xi, \tag{4.9}$$

and the bounded measure μ is the difference of the corresponding right hand sides in (4.1). By the first equation, there exists a Lipschitz function ϕ such that

$$\partial_x\phi =\ u -\ U\ , \quad \partial_t\phi = -(f(u) -\ f(U)). \tag{4.10}$$

In fact, we choose

$$\phi(x,\ t) := \int_a^x (u - \ U)(y,\ t)dy - \int_0^t (f(u) - f(U))(a,\tau)d\tau, \tag{4.11}$$

where a is (arbitrarily) chosen below. From the above results, we observe that the trace of $u - \ U$ at $t = 0$ is zero. Therefore, since ϕ is Lipschitz continuous,

$$\phi(x,\ 0) \equiv\ 0, \quad \text{and} \quad |\phi(x,\ t)| \leq\ C\ t. \tag{4.12}$$

Here and below C denotes some constant independent of $a,\ b$, and t. We now multiply the second equation (4.8) by $-\phi$, and apply the divergence formula (3.6) on $\Omega_t :=\ (a,\ b) \times (0,\ t)$ to obtain

$$\begin{aligned} &\textstyle\int_0^t \int_a^b [-\ (f(u) -\ f(U))^2\ +\ (u -\ U)\ (g(u) -\ g(U))]\ dx\ d\tau \\ &-[\textstyle\int_a^b (f(u) -\ f(U))\ \phi\ dx]_0^t\ -[\textstyle\int_0^t (g(u) -\ g(U))\ \phi\ d\tau]_a^b \\ &:=\ I(t) +\ J(t) +\ K(t)\ =\ L(t) :=\ \textstyle\int_0^t \int_a^b \mu\ \phi. \end{aligned} \tag{4.13}$$

Define

$$h(u,\ U) =\ (u -\ U)\ (g(u) -\ g(U)) -\ (f(u) -\ f(U))^2. \tag{4.14}$$

Step 5. By the Cauchy-Schwarz inequality (see [29, 30, 31] for the p-system and [8, 7] for the scalar case), we see that $h(u,\ U) \geq\ 0$. More precisely, if assumption (1.7) is satisfied, then

$$h(u,U) \geq\ 0, \qquad \text{and} \qquad h(u,\ U) =\ 0 \quad \text{iff} \quad u =\ U. \tag{4.15}$$

On the other hand, by (4.12),

$$|K(t)| \leq\ C\ t^2\ , \tag{4.16}$$

and, by Lemma 3.2,

$$|L(t)| \leq\ C\ t\mu((a,\ b) \times (0,\ t)) \leq\ C\ t\ \varepsilon(t) =\ o(t), \quad \text{as}\ t \to\ 0. \tag{4.17}$$

In contrast, *a priori* we only have $|J(t)| \leq C\ t$, but observe that

$$J(t) =\ [\int_a^b \phi\ \partial_t\phi\ dx]_0^t.$$

Now, we integrate equation (4.13) in time again, and we use (4.12) and (4.15) :

$$\begin{aligned} &\textstyle\int_0^T \left(\int_0^t \int_a^b h(u,\ U)(x,\tau)\ dx\ d\tau\right) dt +\ \frac{1}{2}\ \int_a^b \phi(x,\ T)^2\ dx \\ &=\ -\ \textstyle\int_0^T K(t)\ dt\ +\ \int_0^T L(t)\ dt. \end{aligned} \tag{4.18}$$

Now, the two terms in the left-hand side of (4.18) are nonnegative and, by (4.16) and (4.17),

$$\int_0^T (|K(t)| +\ |L(t)|)\ dt \leq\ C\ T^3 +\ o(T^2)\ =\ o(T^2), \quad \text{as}\ T \to\ 0. \tag{4.19}$$

Using (4.18)-(4.19), we could conclude the expected result from (4.14), if the flux f were strictly convex. However, we want the result to be valid under the only assumption (1.7). Therefore we proceed as follows.

Step 6. First, by (4.18) and (4.19),

$$\begin{aligned} &\tfrac{1}{T^2} \int_0^T \int_0^t \int_a^b h(u,\, U)(x,\tau)\, dx\, d\tau\, dt \\ &:= \tfrac{1}{T^2} \int_0^T H(t)\, dt \;\to\; 0, \quad \text{as } T \to 0. \end{aligned} \tag{4.20}$$

Setting $t := Ts$, we have

$$\frac{1}{T}\int_0^1 H(Ts)\, ds \;\to\; 0, \quad \text{as } T \to 0. \tag{4.21}$$

Since h and H are nonnegative, there exists at least a subsequence $\{T_n\}$ converging to 0 such that, when $n \to \infty$,

$$\frac{1}{T_n}\, H(T_n s) = \frac{1}{T_n} \int_0^{T_n s} \int_a^b h(u,U)(x,\tau)\, dx\, d\tau \;\to\; 0, \quad \text{a.e. in } (0,\, 1). \tag{4.22}$$

Now, fixing an arbitrary non-exceptional positive s, dividing both sides of the above equation by s, and setting $\tau = T_n s\sigma$ yield

$$\int_0^1 \int_a^b h^n(x,\sigma)\, dx\, d\sigma \;\to\; 0, \quad \text{a.e. in } (0,\, 1) \text{ as } n \to \infty, \tag{4.23}$$

where

$$h^n(x,\sigma) := h(u,\, U)(x,\, T_n s\sigma).$$

Therefore, extracting if necessary a new subsequence (still denoted by) $\{T_n\}$, we deduce that

$$h^n(x,\sigma) \;\to\; 0, \qquad \text{a.e.} \quad \text{as} \quad n \to \infty.$$

Therefore, by assumption (1.7) and Lebesgue's Theorem, for an arbitrary given strictly convex entropy η, the subsequence

$$v^n:\ (x,\sigma) \;\mapsto\; (\eta(u) - \eta(U))(x, T_n s\sigma)$$

converges to 0 boundedly almost everywhere, and hence in $L^1((a,\ b) \times (0,\ 1))$.

Step 7. Since the Kruzkov solution U is continuous in time with values in $L^1_{loc}(I\!R)$, we conclude that

$$u(\cdot, T_n s\sigma) \;-\; u_0(\cdot) \;\to\; 0, \qquad \text{as} \quad n \;\to\; \infty. \tag{4.24}$$

Consequently, at least for this subsequence, the corresponding trace of $\tilde{\eta}(u)$ satisfies

$$\tilde{\eta}(u)(T_n s\sigma) \;\to\; \tilde{\eta}(u_0) \quad \text{in } L^1(a,\, b), \qquad \text{as } T_n \to 0. \tag{4.25}$$

Comparing with (4.4), which by Lemma 3.3 holds true for *any* time $t \to 0$, we deduce that

$$\tilde{\eta}(u_0) = l, \quad \text{a.e. in } (a,\, b). \tag{4.26}$$

Since the interval $(a,\ b)$ is arbitrary, this equality is valid *a.e.* in $I\!R$. Since $\tilde{\eta}$ is strictly convex, the convergence in (4.5) is strong for *any* time t converging to 0, which implies (1.5) and (1.6).

This completes the proof of the main theorem.

Remark. A striking observation is that even a weak solution which is *not* an entropy solution, but whose entropy production is controlled, is also *strongly* continuous in L^1 at time $t = 0$. Indeed, replace the entropy condition (1.4) in Main Theorem by: For any C^2 entropy-entropy flux pair (η, q),

$$(q(u(x,t)), \eta(u(x,t))) \in \mathcal{D}M(R \times (0,\infty)). \tag{4.27}$$

Then (1.5) still holds, but of course that does not imply that u is the Krushkov solution, if u does *not* satisfy (1.4)!

An application of the main theorem is the study of the zero relaxation limit described in Section 2, where $u(x,t)$ *a priori* only satisfied the weaker inequality (2.9), and where the initial data were *not* at the equilibrium. Then at the limit, the *initial layer* has entirely collapsed to the line $t = 0$. More precisely, we have

Corollary 4.1. *Consider the relaxing system* (2.1). *Assume that (1.7) is satisfied. Let $u(x,t) \in L^\infty$ be the zero relaxation limit of an arbitrary subsequence $\{u^\varepsilon\}$, as in Proposition 2.1, even if the initial data (2.2) for (2.1) are* not *at the equilibrium. Then the* whole *sequence $(u^\varepsilon,\ v^\varepsilon)$ converges strongly to $(u,\ f(u))$, where u is the unique Kruzkov solution of the local equation (1.1), with the initial data (1.2): at the limit, the initial layer has collapsed to the line $t = 0$. :*

$$v(\cdot,\ 0_+) = f(u)(\cdot,\ 0_+) \ = \ f(u_0) \ \neq v(\cdot,\ 0_-) \ = \ v_0.$$

Similarly, the main theorem can be applied to clarify the initial layers and uniqueness of the relaxation limits for various physical relaxation systems whose initial data are not at the equilibrium.

Acknowledgments. Gui-Qiang Chen's research was supported in part by the National Science Foundation grants INT-9726215 and DMS-9971793, and by an Alfred P. Sloan Foundation Fellowship. Michel Rascle's research was supported in part by the National Science Foundation grant DMS 9708261, by the EC TMR network HCL: n° ERB FMRX CT 96 0033, and by the (NSF)-CNRS contract n° 5909.

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