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Relaxation Limit for Conservation Laws

We are concerned with the limit behavior of hyperbolic systems of conservation laws with stiff relaxation terms to the local systems of conservation laws as the relaxation time tends to zero. The connections of this limit problem with many important challenging problems in related areas are discussed. Some recent developments in this direction are reviewed and analyzed.

1. Introduction

There are two basic theories to describe the nonequilibrium phenomena in mechanics: kinetic theory from microscopic level and continuum theory from macroscopic level. Since the pioneering work of Hilbert [16] and Chapman-Enskog (cf. [5]), there have been many activities in studying the kinetic limits from the kinetic nonequilibrium processes to the continuum (equilibrium or nonequilibrium) processes with the aid of the moment closure techniques from kinetic theory and the kinetic formulation techniques from continuum theory (cf. [1,3,4,20,21,26,27,29,40,41]).

we are concerned with the relaxation limit of hyperbolic systems of conservation laws with stiff relaxation terms to the local systems, which models dynamic limit from the continuum and kinetic nonequilibrium processes to the equilibrium processes, as the relaxation time tends to zero. Typical examples for the limit include gas flow near thermo-equilibrium, viscoelasticity with vanishing memory, kinetic theory with small Knudsen number, and phase transition with small transition time. An important case is that the relaxation depends only on the local values of the basic dependent variables and can be modeled by the following hyperbolic system in the form:

$$\partial_t U + \nabla_x \cdot F(U) + \frac{1}{\epsilon} R(U) = 0, \qquad x \in \mathbf{R}^D, \tag{1}$$

where $U = U(x,t) \in \mathbf{R}^N$ represents the density vector of basic physical variables. The relaxation term is endowed with an $n \times N$ constant matrix $\mathcal{Q}$ with rank $n < N$ such that $\mathcal{Q}R(U) = 0$. There are two basic types of relaxation terms: (I) the manifold of local equilibria is uniquely determined by n independent conserved quantities $u = \mathcal{Q}U$: $U = \mathcal{E}(u)$; (II) the dimension of the manifold of local equilibria equals the number, N, of equations in (1). The local equilibrium limit turns out to be highly singular because of shock and initial layers and to involve many challenging problems in nonlinear analysis and applied sciences. Roughly speaking, the relaxation time measures how far the nonequilibrium states are away from the corresponding equilibrium states; understanding its limit behavior is equivalent to understanding the stability of the equilibrium states. It connects nonlinear integral partial differential equations with nonlinear partial differential equations. This limit also involves the singular limit problem from nonlinear strictly hyperbolic systems to mixed hyperbolic-elliptic ones, even purely elliptic ones in some cases (see [8]). The basic issue for such a limit problem is the stability theory. In this article we focus on Type (I) relaxation terms in Sections 2-4. We remark Type (II) and related topics in Section 5.

2. Stability of Zero Relaxation Limit

In general, the zero relaxation limit is not stable even for the linear case: the characteristic speeds of the local system must be interlaced with the characteristic speeds of the relaxing system to ensure the stability of the limit. The same condition is true (see [28]) for the 2×2 quasilinear case to ensure that the local relaxation approximation is dissipative. This condition is referred to as the subcharacteristic condition by Liu [28]. This can be understood by improving upon the local relaxation approximation with the aid of the idea of Chapman-Enskog expansion for the kinetic theory. For $N \times N$ system (1), the use of the same spirit of Chapman-Enskog expansion leads to the following first order correction to the local relaxation approximation, which is the analogue of the compressible Navier-Stokes approximation in the kinetic theory:

$$\partial_t u + \nabla_x \cdot \mathcal{Q}F(\mathcal{E}(u)) = \varepsilon \, \nabla_x \Big[D(u) \nabla_x u \Big] , \tag{2}$$

where $D(u)$ is nonlinear 4-tensor in $\mathbf{R}^{N \times N} \otimes \mathbf{R}^{D \times D}$, which is very complicated. It is not generally clear that this first-order correction will be dissipative. It can be shown that this will be the case whenever the linear constant coefficient problem obtained by linearizing the original problem about any absolute equilibrium $\mathcal{E}(\bar{u})$ is stable as $\varepsilon \to 0$. However, this is a cumbersome criterion to check.

In Chen-Levermore-Liu [8] we introduced a simple alternative criterion, namely, the existence of a strictly convex entropy Φ with corresponding entropy flux Ψ: For the case $R(U) = 0$, (Φ, Ψ) is consistent with the classical entropy-entropy flux (see Lax [25]); in order to consistent with the relaxation terms, the entropy Φ is required to be locally dissipated and to characterize completely the local equilibria $\mathcal{E} = \mathcal{E}(u)$ in terms of Φ and $\mathcal{Q}$. This is a refinement of the notion of entropy introduced by Boltzmann into his kinetic theory to describe kinetic relaxation to fluid dynamics. His key observation was that his entropy characterizes the local equilibria of the kinetic equation, the celebrated H theorem (cf. [4]). We adopt the notion of entropy that shares all of above properties in the level of nonlinear hyperbolic systems of balance laws. In Chen-Levermore-Liu [8], we establish the following stability theory: The existence of strictly convex entropy Φ implies the followings: (a) The local equilibrium system is hyperbolic with a strictly convex entropy pair $(\phi(u), \psi(u)) = (\Phi, \Psi)|_{U=\mathcal{E}(u)}$; (b) The characteristic speeds of the local system are interlaced with the characteristic speeds of the original system (1); (c) The first-order correction is locally dissipative with nonnegative diffusion $D(u)$. For the 2×2 case, the pure dissipativity $D(u) > 0$ with the coupling condition (see [28]) is equivalent to the strict stability condition on the equilibrium curve. This leads to the converse of (a)-(c) as follows: Let (ϕ, ψ) be a strictly convex entropy pair for the local equilibrium equation. Assume that the subcharacteristic condition holds. Then there exists a strictly convex entropy pair (Φ, Ψ) for the 2×2 system (1) over an open set containing the local equilibrium curve, along which it takes $(\phi(u), \psi(u))$. For the converse, the strict stability criterion and the coupling condition imply that the local equilibrium curve is a noncharacteristic curve for the entropy equation, which is a second-order hyperbolic equation for which we pose the Cauchy data in the form $\Phi = \phi(u), \partial_v \Phi = 0$ along the local equilibrium curve. The classical local existence theory ensures that there is a solution Φ of this Cauchy problem over an open domain containing the local equilibrium curve. Since the strict stability condition is satisfied along the local equilibrium curve, then it will also be satisfied in some open domain containing the local equilibria curve by continuity. In the stability theory, the convexity of entropy Φ is essential. The most physical systems with stiff relaxation term are endowed with a convex entropy. In the context, we refer to Levermore [26], where such systems are systematically derived via the moment closures from kinetic theory.

3. Convergence of Relaxation Limit

This limit is the compressible Euler type one and the solutions of the relaxation systems tend to those of the local relaxation approximation, which are inviscid conservation laws. This limit is highly singular because of shock and initial layers. The main difficulty is that the solutions of the full systems are only the measurable functions with certain boundedness. The most basic class of the systems is the 2×2 one. Consider uniformly bounded solutions $U^\varepsilon = (u^\varepsilon, v^\varepsilon) \in L^\infty$ of the 2×2 systems satisfying the entropy inequality in the sense of distributions. Assume that the strict stability condition holds and the subcharacteristic speed is monotone almost everywhere for the local variable $u \in \mathbf{R}$. The stability theory ensures the existence of such a strictly convex entropy. Then it is proved in [7,8] that U^ε strongly converges to (u, v) and the limit functions $(u(x,t), v(x,t))$ are on the equilibrium curve for almost all $(x,t), t > 0$, where $u(x,t)$ is the entropy solution of the Cauchy problem for scalar conservation law with the Cauchy data $w^*\text{-}\lim u_0^\varepsilon(x)$ in L^∞. We remark: (a) Notice that the initial data may even be far from equilibrium. The convergence result indicates that the limit functions (u, v) indeed come into local equilibrium as soon as $t > 0$. This shows that the limit is highly singular. In fact, this limit consists of two processes simultaneously: one is the initial layer limit, and the other is the shock layer limit. (b) The compactness of the zero relaxation limit indicates that the sequence U^ε is compact no matter how oscillatory the initial data are. Note that the relaxation systems are allowed to be linearly degenerate; the initial oscillations can propagate along the linearly degenerate fields for the homogeneous systems (cf. [6]). This fact shows that the relaxation mechanism coupling with the nonlinearity of the equilibrium equations can kill the initial oscillations, just as the nonlinearity for the homogeneous system can kill the initial oscillations. (c) The above discussions are based on the L^∞ a priori estimate. In many physical systems, the estimate can be achieved. Such examples include the p-system and models in viscoelasticity, chromatography, and combustion (see [7,8,37,41,33,39,22,31]), which have natural invariant regions. For some special models, even uniform BV bound of relaxation solutions $(u^\varepsilon, v^\varepsilon)$ can be achieved [39], which ensures the convergence of zero relaxation limit via the Helly principle.

4. Convergence of Weakly Nonlinear Relaxation Limit

The weakly nonlinear relaxation limit is the incompressible Navier-Stokes type one just as the limit from the Boltzmann equations to the incompressible Navier-Stokes equations [1]. The main observation is that the linearization of the local relaxation approximation about an equilibrium gives a simple advection dynamics with the equilibrium characteristic speed. This can be understood in a formal fashion (see [8]). If one applies the same asymptotic scaling to the first correction to the local equilibrium approximation, one again arrives at the weakly nonlinear approximation. This shows that the latter is a distinguished limit of the former and makes clear why it inherits the good features of the former. Its advantage is that the solutions of the Burgers equation are smooth even for the case that the initial data are not smooth. Thus the solutions remain globally consistent with all the assumptions that were used to derive the weakly nonlinear approximation. In Chen-Levermore-Liu [8], this approximation is justified by using the stability theory and the energy estimate techniques. Linearized version of the limit is well understood which relates to the "random walk" in Brownian motion (cf. [12,35,24]).

5. Relaxation Limit for the Systems with Type (II) Relaxation Terms

When the dimension of local equilibrium manifold is equal to the number of equations of the relaxation systems, the situation is different and our stability theory established in [8] can not be directly applied. Such systems arise from many physical areas including elastoplasticity and combustion. In [9] we established a similar stability theory and applied this theory to study the limit behavior of the zero relaxation limit for such systems. A notion of admissible weak solutions for the rate-independent systems as the limit functions of the zero relaxation limit is formulated. More details can be found in [9].

Remark 1. *For the systems violating the strict stability criterion, the equilibrium speed may equal one of the frozen speeds. It is shown in Chen-Liu [7] for a model in phase transitions that no oscillation arises when the dissipation is present and goes to zero more slowly than the relaxation. It would be interesting to further investigate relaxation systems arising from various physical areas to understand how the feature of the failure of the strictly stability criterion affects the limiting behavior of zero relaxation and dissipation limits.*

Remark 2. *The relaxation systems with more than one local equilibrium manifold in the level of reaction-diffusion equations have been studied for the typical models (See [13,14,38] and references cited therein). It would be interesting to investigate the zero relaxation limit behavior of such relaxation systems in the level of reaction-convection equations, which are modeled by the hyperbolic conservation laws with stiff relaxation terms.*

Remark 3. *For the systems with stiff relaxation terms such that the corresponding dynamic systems* $\partial_t U + \frac{1}{\epsilon}R(U) = 0$ *are of limit circles, the situation is much more complicated. Such systems in the level of reaction-diffusion equations have been studied for some models (see [23,32] and references cited therein). One of interesting connections of such a limit with physics is the characterization of the Ginzburg-Landau vortices. Such a limit problem for stationary Ginzburg-Landau equations has been systematically studied in Bethuel-Brezis-Hélein [2]. It would be interesting to study the zero relaxation limit behavior for the hyperbolic systems with such stiff relaxation terms.*

Remark 4. *It is natural to use the relaxation methods to construct shock capturing schemes to calculate the numerical solutions to the local systems. Some efforts have been made in this direction with the aid of the stability theory of local relaxation limit (See [17,18,19,33,42]). Such ideas closely relate to those of the kinetic schemes.*

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