Hyperbolic Conservation Laws with Umbilic Degeneracy I

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Abstract

In this paper a compactness framework for approximate solutions to nonlinear hyperbolic systems with umbilic degeneracy is established by combining compensated compactness ideas with some classical methods, and by a detailed analysis of a highly singular Euler-Poisson-Darboux-type equation. Then this framework is successfully applied to prove the convergence of the viscosity method, and the existence of global entropy solutions for the Cauchy problem with large initial data for a canonical class of the systems with quadratic flux form.

Key Words. Conservation laws, nonstrict hyperbolicity, umbilic points, compactness framework, viscosity method, global entropy solutions

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1. Introduction

We are concerned with hyperbolic systems of conservation laws with umbilic degenerate points. A point in the state space $U_0 \in \mathbf{R}^n$ is called an umbilic degenerate point for a system of hyperbolic conservation laws

$$\partial_t U + \partial_x F(U) = 0, \qquad U \in \mathbf{R}^n, \tag{1.1}$$

if certain wave speeds coincide at this point U_0, that is, at least two eigenvalues $\lambda_i(U)$ and $\lambda_j(U), i \neq j$, among the n real eigenvalues of the matrix $\nabla F(U)$, are such that $\lambda_i(U_0) = \lambda_j(U_0)$. Such umbilic degeneracy allows a degree of interaction, or nonlinear resonance, between distinct modes, and leads to high singularities, a phenomenon missing in the strictly hyperbolic case.

The study of degenerate hyperbolic equations has an extensive history dating back at least to the work of Euler [Eu] two hundred years ago, when he proposed a degenerate equation — the Euler-Poisson-Darboux equation. Its close relation with wave theory, fluid dynamics as well as geometry has attracted great attention from mathematicians for two centuries, including Poisson (1823), Darboux (1914), Riemann (1860), Volterra (1892) and also Weinstein, Erdélyi, Lions, and others in the 60's (see [Ya]). Recently, a study of the behavior of its solutions led to a solution of the nonlinear system of isentropic gas dynamics [Ch1, DCL] (also see [Di2, Ch2]). On the other hand, the theory of linear equations with multiple characteristics is also well developed. An important feature of such equations is the loss of differentiability [DeG], which leads to ill-posedness in Sobolev spaces but well-posedness in the Gevrey classes [Ge]. Another feature is that sign conditions on the subprincipal symbol play an important role (cf. [Fo, Oh, H1]).

Recently, nonlinear hyperbolic systems with such degeneracy have arisen from such disparate areas as multiphase flow in porous media, elasticity, water wave problems, and maganetohydrodynamics. Such umbilic points appear naturally in multidimensional systems of conservation laws (cf. [FR, H2, Jo, La1]). In particular, Lax [La1] showed that in three space variables there must be degenerate points if the number of equations n is $2(mod\ 4)$. More generally, the same result holds if $n = \pm 2, \pm 3, \pm 4 (mod\ 8)$ (see [FR]). This means that, in general, plane wave solutions for such multi-dimensional systems are governed by one-dimensional hyperbolic systems with umbilic degeneracy.

The theory of local solutions for such systems is well developed because many tools for linear equations can still be used. However, since these systems are nonlinear, even if the initial data are smooth, the solutions of the Cauchy problem generally develop singularities and become discontinuous in finite time. This is a reflection of the physical phenomena of breaking of waves and the development of shock waves. An effort has been made to understand the Riemann solutions for such systems (e.g. [G1, IMPT, IT, SS2, SSMP]). Two kinds of degeneracy are classified, which govern different behavior of solutions near umbilic points. A typical example of parabolic degeneracy is provided by the system of isentropic gas dynamics (e.g. [Ch2]). The simplest example of hyperbolic degeneracy is the system with rotational symmetry, exhibiting one linear degenerate characteristic field and one contact characteristic field (cf. [KK, LW, Fr, Ch3]).

This is the first in a series of papers in which we focus on isolated umbilic points with hyperbolic degeneracy. Near such an isolated umbilic point one can scale and blow up singularities to reduce generically the flux to a homogeneous polynomial form, determined by the lowest-order nontrivial terms, through a Galilean transformation. For the 2×2 case, this process generically leads to a homogeneous quadratic polynomial flux. Such a polynomial flux contains some inessential scaling parameters and the selection of a unique flux from each equivalence class is the problem of normal forms solved by Isaacson, Plohr, and Temple and in a more satisfactory form by Schaeffer and Shearer [SS1]. The classification of the geometry of rarefaction curves and some preliminary tools for the analysis of Riemann problems for the systems with quadratic flux are also presented in [SS1]. The Riemann solutions for such systems were constructed by Isaacson, Marchesin, Paes-Leme, Plohr, Schaeffer, Shearer, Temple, and others (e.g. [IMPT, IT, SS2, SSMP]).

The global existence of weak solutions to the Cauchy problem for a special case of systems with quadratic flux form was solved in [Ka]. He employs the method of compensated compactness in the context of vanishing viscosity. The analysis involves detailed estimates and characterization of the singularities of solutions to the associated entropy equation which is of Euler-Poisson-Darboux type. Classes of Goursat data are then carefully chosen to cancel these singularities in the construction of general classes of regular entropies. In [FS, Rb], the analysis in [Ka] is applied to a slightly different system. And in [Lu], a different independent proof is given to the problem studied in [Ka].

In this paper we establish an L^∞ compactness framework for sequences of approximate solutions to general nonstrictly hyperbolic systems with umbilic points. Techniques from the theory of compensated compactness [Ta, Mu, Di1, Se, Ch2, Mo] are used and the analysis of singularities in [Ka] is generalized to attain our objective. Under this framework approximate solution sequences, which are a priori bounded in L^∞ and which produce correct entropy dissipations, lead to compactness of the resulting sequences of Riemann invariants. One of the principal difficulties associated with degenerate systems is how to generate enough entropy function which satisfy the conditions in the div-curl lemma of Tartar [Ta] and Murat [Mu]. This is due to possible singularities of entropy functions near the regions of nonstrict hyperbolicity. The analysis leading to the compactness involves two major steps:

In the first step, we construct regular entropy functions governed by a highly singular entropy equation. We have to overcome two difficulties. The first is that, in general, near the umbilic points the coefficients of the entropy equation are multiple-valued functions in the Riemann invariant coordinates, which is not the case in the special cases [Ka, FS, Rb]. This difficulty is overcome by a detailed analysis of the singularities of the Riemann function of the entropy equation in Section 3 and Section 4 involving a study of a corresponding Euler-Poisson-Darboux equation and requiring very complicated estimates. An appropriate choice of Goursat data leads to a cancellation of singularities and yields regular entropies in the Riemann invariant coordinates. The second difficulty is that the nonlinear correspondence between the U-coordinates and the Riemann invariant coordinates is, in general, irregular. A regular entropy function in the Riemann invariant coordinates is usually no longer regular in the physical coordinates U. We overcome this by a detailed analysis of the correspondence between these two coordinates.

In the second step, we study the structure of the Young measure associated with the approximating sequences, and prove that its support lies in finite isolated points or separate lines in the Riemann invariant space. This is achieved through the usual Tartar-Murat commutation equation [Ta] for the Young measure by a delicate use of Serre's technique

[Se] (also see [Mo]) together with the regular entropy functions, constructed in the first step.

This compactness framework is successfully applied to prove the convergence of the viscosity method for a canonical class of the systems with quadratic flux form in Section 7. This leads to an existence theorem of global entropy solutions for such systems. The compactness is achieved by reducing the support of the corresponding Young measure to a Dirac mass in the physical space.

In a forthcoming paper [CK] we will develop this framework to solve the other three remaining canonical classes of the systems with quadratic flux form and related systems. Several convergence theorems of L^∞ approximate solutions and existence theorems of global solutions will be presented. This theory will also be applied to the convergence analysis of shock capturing schemes including the Lax-Friedrichs scheme [La3] and the Godunov scheme [Go].

In connection with earlier work on compactness frameworks on approximate solutions to scalar and 2×2 systems of hyperbolic conservation laws, we refer the reader to the works [Ch1-2, Da2, DCL, Di1-2, FS, Ka, Lu, NRT, Mo, Rb, Se, Ta] and references cited therein. The requirement of strict hyperbolicity of the 2×2 systems is crucial in the earlier work [Di1, Se, M0]. The analysis in [Di1] is based on a study of the Lax progressing entropy waves in the state space [La2], and in particular, on relationships between their structure and the structure of the Young measure with the aid of the conditions of strict hyperbolicity and convexity of the systems. The work in [Se] and in [Mo] provides alternative approaches to establishing compactness frameworks for general systems by using certain kinds of Goursat entropies whose regularities are ensured by the strict hyperbolicity of the systems.

For other related topics on hyperbolic conservation laws with umbilic points, we refer the reader to [Fr, G1-2, IMP, IMPP, IMPT, IT, Ke, KK, SS1-2, SSMP] and references cited therein for various disscusions on the Riemann problem, the classification of degenerate systems, and the structure and admissibility of overcompressive and transitional shock waves. Recent studies on the asymptotic stability of overcompressive and transitional shock waves can be found in [LX, LZ, ZPM]. We also refer the reader to [G1-2] for a review of applications of bifurcation theory and geometry to the analysis of Riemann solutions.

2. The Classification of Quadratic Fluxes and Wave Curves

Consider a hyperbolic system of conservation laws

$$\partial_t U + \partial_x F(U) = 0, \qquad U \in \mathbf{R}^2, \tag{2.1}$$

with an isolated umbilic point U_0, namely, $\nabla F(U_0)$ is diagonalizable, and there is a neighborhood $\mathbf{N}$ of U_0 such that $\nabla F_T(U)$ has distinct eigenvalues for all $U \in \mathbf{N} - U_0$, where

$$\nabla F_T(U) = F(U_0) + \nabla F(U_0)(U - U_0) + \frac{1}{2}(U - U_0)^\top \nabla^2 F(U_0)(U - U_0). \tag{2.2}$$

Take the Taylor expansion for $F(U)$ about $U = U_0$:

$$F(U) = \nabla F_T(U) + h.o.t. \tag{2.3}$$

where $h.o.t.$ represents the remainder. The flux function $\nabla F_T(U)$ determines the local behavior of the hyperbolic singularity near the umbilic point U_0. Since $\nabla F(U_0)$ is diagonalizable, we can make a coordinate transformation to eliminate the linear term from (2.2) and relabel $U - U_0$ as U to obtain

$$\partial_t U + \partial_x Q(U) = 0, \tag{2.4}$$

where $Q(U) = \frac{1}{2}U^\top \nabla^2 F(U_0)U$. From the normal form theorem in [SS1], there is a nonsingular linear coordinate transformation to transform the system (2.4) into

$$\partial_t U + \partial_x (dC(U)) = 0, \tag{2.5}$$

where

$$C(U) = \frac{1}{2}(\frac{1}{3}au^3 + bu^2v + uv^2),$$

and a and b are real parameters with $a \neq 1 + b^2$.

We now analyze the geometry of rarefaction wave curves and the genuine nonlinearity of the system with quadratic flux function (2.5) in the sense of Lax [La4]. We will make use of this information to understand the classification of the canonical form into four regions in the (a, b)-plane thereby rederiving some of the results in [SS1] from a different viewpoint.

This analysis is also essential in obtaining an L^∞ a priori estimate for the sequences of viscous approximate solutions via invariant region techniques. We establish such estimates in Section 7. For the symmetric case of (2.5) ($b = 0$), the existence of invariant regions and the geometry of wave curves were analyzed in [Ka]. We remark here that our analysis will be consistent with Darboux's local analysis near an isolated umbilic point (see [Dar]).

Recall from (2.5) that the flux vector and matrix take the form

$$dC(U) = \frac{1}{2}(au^2 + 2buv + v^2, bu^2 + 2uv)^\top, \tag{2.6}$$

and

$$\nabla(dC(U)) = \begin{pmatrix} au + bv & bu + v \\ bu + v & u \end{pmatrix}. \tag{2.7}$$

The eigenvalues and eigenvectors are from (2.7),

$$\lambda_i = \frac{1}{2}\left((a+1)u + bv + (-1)^i\sqrt{((a-1)u + bv)^2 + 4(bu + v)^2}\right), \quad i = 1, 2, \tag{2.8}$$

and

$$\begin{aligned} \mathbf{r}_i &\equiv (\mathbf{r}_{i1}, \mathbf{r}_{i2})^\top \\ &= ((a-1)u + bv + (-1)^i\sqrt{((a-1)u + bv)^2 + 4(bu + v)^2}\ ,\ 2(bu + v))^\top, \end{aligned} \tag{2.9}$$

respectively. It is then immediate that as long as $a \neq 1 + b^2$, $\lambda_1 = \lambda_2$ if and only if $(u, v) = (0, 0)$, so that $(0, 0)$ is the unique umbilic point for (2.5).

The j^{th} family of rarefaction curves $\mathbf{R}_j$ is defined as the family of integral curves of the vector field given by $\mathbf{r}_j$. Upon setting $\alpha = \frac{u}{v}$ and

$$g_j(\alpha) \equiv \frac{(a-1)\alpha + b + (-1)^j \sqrt{((a-1)\alpha + b)^2 + 4(b\alpha + 1)^2}}{2(b\alpha + 1)}, \tag{2.10}$$

$\mathbf{R}_j$ is defined by the following ordinary differential equation:

$$\begin{aligned} \frac{du}{dv} &= \frac{(a-1)u + bv + (-1)^j \sqrt{((a-1)u + bv)^2 + 4(bu + v)^2}}{2(bu + v)} \\ &\equiv F_j(u, v) \equiv \begin{cases} g_j(\alpha), & v > 0, \\ g_i(\alpha), & v < 0, \quad i \neq j. \end{cases} \end{aligned} \tag{2.11}$$

To analyze the geometry of the $\mathbf{R}_j$ curves, we start by noting that

$$\frac{du}{dv} = F_j(u, v), \tag{2.12}$$

$$\frac{d^2u}{d^2v} = \frac{1}{v}\partial_\alpha F_j(u, v)(F_j - \alpha), \quad \text{when} \quad v \neq 0. \tag{2.13}$$

We collect, in the following lemma, some basic properties of g_j.

Lemma 2.1. *The function $g_j(\alpha), j = 1, 2,$ satisfies*

(1) $g_1(\alpha)g_2(\alpha) = -1$,

(2) $g_j(0) = \dfrac{b + (-1)^j\sqrt{b^2 + 4}}{2}$,

(3) $g_j(\pm\infty) = \dfrac{a - 1 \pm (-1)^j\sqrt{(a-1)^2 + 4b^2}}{2b}$,

(4) $\lim_{\alpha \to -\frac{1}{b} \pm 0} g_j(\alpha) = \begin{cases} 0, & (-1)^j sign((a - 1 - b^2)b) > 0, \\ \pm sign(b)(-1)^j\infty, & (-1)^j sign((a - 1 - b^2)b) < 0, \end{cases}$

(5) $sign(g_j'(\alpha)) = sign(a - 1 - b^2)$.

Next, we make use of (2.10) – (2.13) and Lemma 2.1 to obtain a qualitative picture of the $\mathbf{R}_j$ curves. We distinguish several cases. We remark that from (4)-(5) in Lemma 2.1, $a = 1 + b^2$ defines a boundary curve in the (a, b)-plane separating qualitatively different wave curve geometries. We shall elaborate on this below.

For simplicity, we first consider the case $a > 1 + b^2, b > 0$. From (2.13), the change in convexity (as a function of the slope) of the wave curves depends on the location and number of roots of $g_j(\alpha) - \alpha$. A computation using the formulae for the $g_j(\alpha)'s$ shows that, in the present case, these locations are given by the roots of the cubic polynomial

$$h(\alpha) = -b\alpha^3 + (a-2)\alpha^2 + 2b\alpha + 1.$$

The discriminant of $h(\alpha)$ is given by $\Delta = -32b^4 + b^2(27 + 36(a-2) - 4(a-2)^2) + 4(a-2)^3$. Thus, $\Delta = 0$ gives a new boundary in the (a,b)-plane which distinguishes different wave curve geometries. This corresponds to the partition between regions III and IV in [SS1]. When $\Delta < 0$, $h(\alpha)$ has three real roots α_0, α_1, and α_2. The behavior of the $\mathbf{R}_j$ curves in the case $b > 0, a > 1 + b^2, \Delta < 0$, can be described in Fig. 2.1. When $\Delta > 0$, $h(\alpha)$ has only one real root and the wedge-shaped regions in Fig. 2.1 collapse into a line. This corresponds to region IV in [SS1] (see Fig. 2.2). The case $b < 0, a > 1 + b^2$, is completely similar.

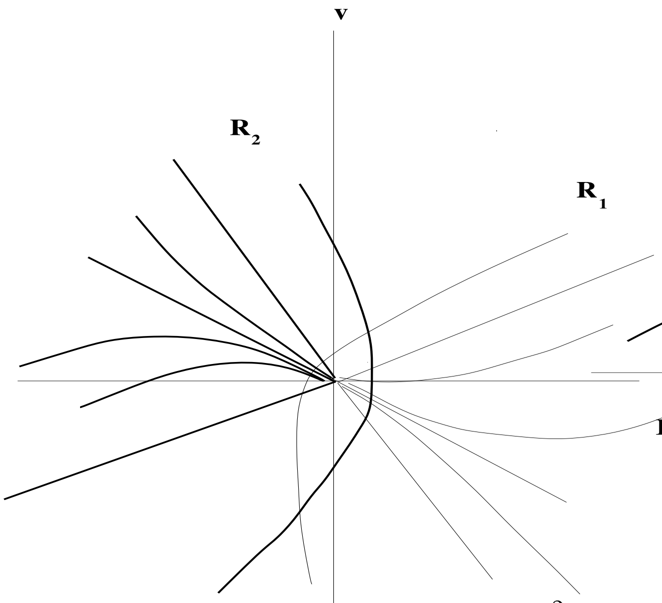

Fig. 2.1

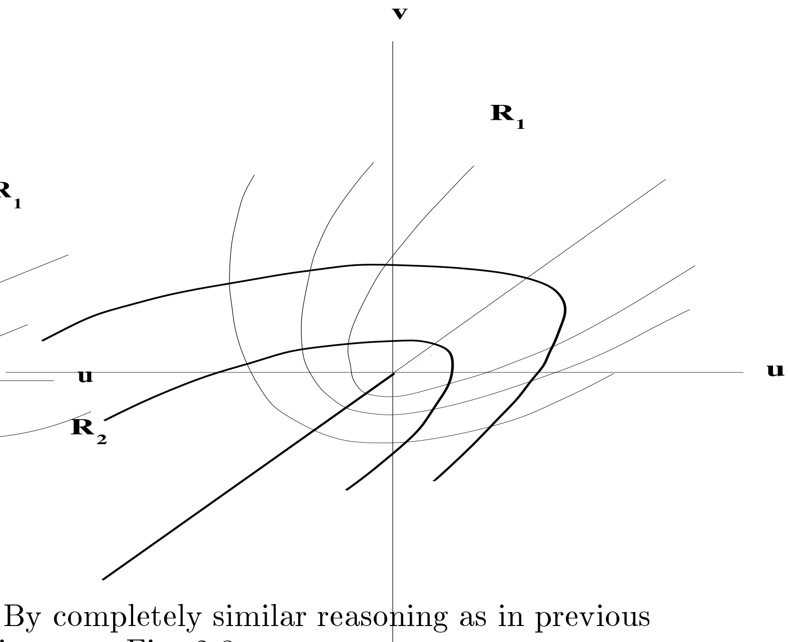

Fig. 2.2

Next, we consider the case $a < 1 + b^2$. By completely similar reasoning as in previous cases, we obtain the following $\mathbf{R}_j$ curves diagrams, Fig. 2.3:

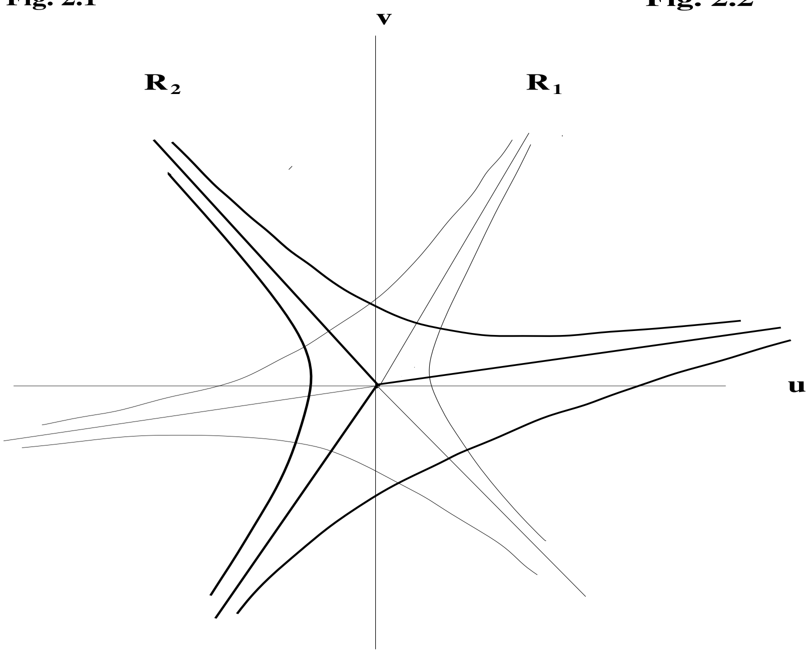

Fig. 2.3

We now turn to investigate the genuine nonlinearity in the sense of Lax [La4] for the quadratic system (1.1) – (1.2). It will turn out that genuine nonlinearity allows a breakdown of the last case above ($a < 1 + b^2$) into two subcases. By definition, (2.5) is genuinely nonlinear in the j^{th} characteristic field at a point (u, v) if $\mathbf{r}_j \cdot \nabla\lambda_j \neq 0$ at (u, v). A calculation using (2.8) – (2.9) shows that

$$\mathbf{r}_j \cdot \nabla\lambda_j = a\zeta + 3by + (-1)^j \frac{a\zeta^2 + 3by\zeta + 2(a+3)y^2}{\sqrt{\zeta^2 + 4y^2}} \equiv h_j(\zeta, y), \tag{2.14}$$

where

$$\zeta = (a-1)u + bv, \quad y = bu + v.$$

Then the roots of $h_j(\zeta, y)$ satisfy

$$y = 0, \quad \text{and} \quad (-1)^j\zeta < 0,$$

or

$$\begin{cases} 3a\zeta^2 + 3b(3-a)\zeta y + ((a+3)^2 - 9b^2)y^2 = 0, \\ (-1)^j sign[(a\zeta + 3by)(a\zeta^2 + 3by\zeta + 2(a+3)y^2)] < 0. \end{cases}$$

The discriminant of the latter quadratic form is

$$D = -12(a - \frac{3}{4}b^2)(a+3)^2,$$

and also

$$h_j(\zeta, 0) = 0, \quad (-1)^j\zeta < 0.$$

Therefore, we have three cases:

(1) $a > \frac{3}{4}b^2$. Then there exists only one root $y = 0$ of $h_j(\zeta, y)$ such that

$$\mathbf{r}_j \cdot \nabla\lambda_j = 0, \quad \text{on} \quad \{bu + v = 0, (-1)^j((a-1)u + bv) < 0\}; \tag{2.15}$$

(2) $a = \frac{3}{4}b^2$. Then there exist two roots $y = 0$ and $\zeta = \frac{b(a-3)}{2a}y$ such that

$$\mathbf{r}_j \cdot \nabla\lambda_j = 0, \quad \text{on} \quad \{bu + v = 0, (-1)^j((a-1)u + bv) < 0\}; \tag{2.16}$$

and

$$\mathbf{r}_j \cdot \nabla\lambda_j = 0, \quad \text{on} \quad \{(a-1)u + bv = \frac{b(a-3)}{2a}(bu + v), (-1)^j b(bu + v) < 0\}; \tag{2.17}$$

(3) $a < \frac{3}{4}b^2$. Then there exist three roots $y = 0$ and $\zeta = \frac{3b(a-3) \pm \sqrt{D}}{6a}y$ such that

$$\mathbf{r}_j \cdot \nabla\lambda_j = 0, \quad \text{on} \quad \{bu + v = 0, (-1)^j((a-1)u + bv) < 0\}; \tag{2.18}$$

and
(2.19)

$$\mathbf{r}_j \cdot \nabla\lambda_j = 0, \quad \text{on} \;\; \{(a-1)u + bv = \frac{3b(a-3) \pm \sqrt{D}}{6a}(bu+v), (-1)^j a(a+3)(bu+v)\sigma < 0\},$$

where $\sigma = 3b(a+3)^2 \pm (a-3)\sqrt{D}$.

Therefore, we find that all possible roots of $\mathbf{r}_j \cdot \nabla\lambda_j$ are located on the lines

$$\tag{2.20} \begin{cases} y = 0, & a > \frac{3}{4}b^2, \\ y = 0, \zeta = \frac{b(a-3)}{2a}y, & a = \frac{3}{4}b^2, \\ y = 0, \zeta = \frac{3b(a-3)\pm\sqrt{D}}{6a}y, & a < \frac{3}{4}b^2. \end{cases}$$

On the line $y = 0$, $\mathbf{r}_j \cdot \nabla\lambda_j = 0$ if and only if $(-1)^j\zeta < 0$; on the line $\zeta = \frac{3b(a-3)\pm\sqrt{D}}{6a}y$ for the case $a < \frac{3}{4}b^2$, $\mathbf{r}_j \cdot \nabla\lambda_j = 0$ if and only if $(-1)^j a(a+3)(3b(a+3)^2 \pm (a-3)\sqrt{D}(bu+v)) < 0$ and $(u,v) \neq (0,0)$. (2.20) shows that the curve $a = \frac{3}{4}b^2$ divides the region $\{(a,b)|a < 1 + b^2\}$ into two subregions according to a global change in the curve on which genuine nonlinearity fails. This corresponds to the partition between region I and II in [SS1].

We summarize the three boundaries in the (a,b)-plane separating four different types of behavior of the wave curves and affecting the qualitative behavior in the system (2.5) : $a = \frac{3}{4}b^2, a = 1 + b^2$, and $\Delta = 0$. These three boundaries divide the (a,b)-plane into four regions. These regions are labeled I, II, III, and IV from left to right on the (a,b)-plane.

3. Riemann Invariants and Genuinely Nonlinearity for the Quadratic System

In this section we study the Riemann invariants of the system with quadratic flux form (2.5). We also study the monotonicity of $\lambda_i, i = 1,2$, as a function of Riemann invariants $w_j, j = 1,2$. We remark that this does not follow from our knowledge of genuine nonlinearity (Section 2) as the map $\mathcal{J} : (u,v) \to (w_1, w_2)$ is neither C^1 nor globally invertible, in general.

Riemann invariants $w_i = w_i(u,v), i = 1,2$, are defined as functions that are constant along any rarefaction wave curves of the j^{th} family $\mathbf{R}_j$ where $i \neq j$. On regions where w_i is differentiable, it is easy to check that $\mathbf{r}_j \cdot \nabla w_i = 0, i \neq j$, since $\mathbf{R}_j$ curves are integral curves of the vector field $\mathbf{r}_j$.

We will use the ordinary differential equation (2.11) to define w_j taking care that it comes out as a single-valued function globally. For simplicity, we assume $b > 0$ and restrict ourselves from now on to a half plane domain

$$\mathcal{I} \equiv \{ \ (u,v) \mid v - \frac{1}{\alpha_0}u \geq 0 \ \}.$$

The case $b < 0$ is completely similar and the case $b = 0$ will be discussed in details in [CK]. This domain $\mathcal{I}$ is also an invariant region for the system with viscosity associated with (2.5) (see Section 7). We remark that the $w_j's$ are not uniquely defined. We shall make the choice that guarantees maximal regularity of certain quantities. The advantage of this choice will become clear in Section 4 when we study the entropy functions.

Theorem 3.1. *Consider* (2.5) *with* $\Delta > 0$. *Then the cubic polynomial* $h(\alpha)$ *has only one real root* $\alpha_0 > 0$. *Denote*

$$\beta = \frac{-h'(\alpha_0)}{\sqrt{((a-1)\alpha_0 + b)^2 + 4(\alpha_0+1)^2}}.$$

Then

(1) The following formulae define a pair of Riemann invariants for (2.5) *on* $\mathcal{I}$:

$$w_i(\tilde{u},\tilde{v}) = (-1)^i |\tilde{v}|^\beta exp\{-\beta \int_0^{\tilde{\alpha}} H_j(\tilde{\alpha})\ d\tilde{\alpha}\}, \quad i \neq j, \tag{3.1}$$

with

$$H_j(\tilde{\alpha}) = -\frac{(1+\alpha_0^2)}{2(\alpha_0+\tilde{\alpha})}\frac{E_j(\tilde{\alpha})}{D(\tilde{\alpha})} - \frac{1}{\alpha_0+\tilde{\alpha}}; \tag{3.2}$$

(2) The ratio of the pair of Riemann invariants (w_1, w_2) *is of the form*

$$\begin{aligned}\frac{w_j}{w_i} &= -exp\{(-1)^j \beta \int_0^{\tilde{\alpha}} \frac{(1+\alpha_0^2)\sqrt{Q(\tilde{\alpha})}}{D(\tilde{\alpha})} d\tilde{\alpha}\} \\ &\equiv \Gamma_j(\tilde{\alpha}), \quad i \neq j,\ i,j = 1,2,\end{aligned} \tag{3.3}$$

where the function $\Gamma_j(\tilde{\alpha})$ *is real analytic in* $\tilde{\alpha} \in \mathbf{R}$ *and*

$$\Gamma_j(\frac{1}{\tilde{\sigma}}) = |\tilde{\sigma}|^{(-1)^i sign(\tilde{\sigma})} f_j(\tilde{\sigma}), i \neq j,$$

near $\tilde{\sigma} = 0$, *where* $f_j(\tilde{\sigma})$ *is real analytic near* $\tilde{\sigma} = 0$.

Here

$$\begin{aligned}
E_j(\tilde{\alpha}) =& -2b(\alpha_0\tilde{\alpha}-1)^2 + (a-3)(\alpha_0+\tilde{\alpha})(\alpha_0\tilde{\alpha}-1) + b(\alpha_0+\tilde{\alpha})^2 \\
&+ (-1)^i(\alpha_0+\tilde{\alpha})\sqrt{Q(\tilde{\alpha})}, \\
Q(\tilde{\alpha}) =& \left((a-1)(\alpha_0\tilde{\alpha}-1) + b(\alpha_0+\tilde{\alpha})\right)^2 + 4\left(b(\alpha_0\tilde{\alpha}-1) + \alpha_0 + \tilde{\alpha}\right)^2, \\
D(\tilde{\alpha}) =& -b(\alpha_0\tilde{\alpha}-1)^3 + (a-2)(\alpha_0+\tilde{\alpha})(\alpha_0\tilde{\alpha}-1)^2 \\
&+ 2b(\alpha_0+\tilde{\alpha})^2(\alpha_0\tilde{\alpha}-1) + (\alpha_0+\tilde{\alpha})^3, \\
\tilde{u} = u + \frac{1}{\alpha_0}v,& \quad \tilde{v} = v - \frac{1}{\alpha_0}u, \quad \tilde{\alpha} = \frac{\tilde{u}}{\tilde{v}}, \quad \tilde{\sigma} = \frac{1}{\tilde{\alpha}}.
\end{aligned}$$

Proof. We know that the j^{th} family of rarefaction curves $\mathbf{R}_j$ is defined by the ordinary differential equation (2.11). Notice that

$$sign(v)sign(\tilde{\alpha}+\alpha_0) = sign(\alpha_0\tilde{v}) > 0.$$

Performing a simple calculation and using (2.11), we get

$$\frac{d\tilde{\alpha}}{d\tilde{v}} = \frac{1}{\tilde{v}} H_j^{-1}(\tilde{\alpha}), \tag{3.4}$$

where H_j is determined by (3.2). Therefore, the Riemann invariants are of the form (3.1) for any constant $\beta \neq 0$.

Now we consider the local behavior of $\frac{w_j}{w_i}$ near the rays $\tilde{\alpha} = (-1)^k \infty$, which corresponds to the ray $\{(u,v)|\alpha = \alpha_0, (-1)^k u > 0\} = \{(u,v)|\tilde{v} = 0, (-1)^k \tilde{u} > 0\}, k = 1, 2$, from the side $\{(u,v)|\tilde{v} \geq 0\}$. Using (3.1), we have

$$\frac{w_j}{w_i} = -\exp\{\beta \int_{-\infty}^{\tilde{\sigma}} \frac{1}{\tilde{\sigma}} \left(\frac{H_i(\frac{1}{\tilde{\sigma}}) - H_j(\frac{1}{\tilde{\sigma}})}{\tilde{\sigma}} \right) d\tilde{\sigma}\}, \quad i \neq j, \tag{3.5}$$

where $\tilde{\sigma} = \frac{1}{\tilde{\alpha}}$. Notice that, for $(u,v) \in \mathcal{I}$,

$$H_i(\tilde{\alpha}) - H_j(\tilde{\alpha}) = (-1)^i \frac{(1+\alpha_0^2)\sqrt{Q(\tilde{\alpha})}}{D(\tilde{\alpha})},$$

and

$$\lim_{\substack{\tilde{\sigma} \to 0 \\ (-1)^k \tilde{u} > 0}} \frac{H_i(\frac{1}{\tilde{\sigma}}) - H_j(\frac{1}{\tilde{\sigma}})}{\tilde{\sigma}} = (-1)^{j+k} \frac{\sqrt{((a-1)\alpha_0 + b)^2 + 4(b\alpha_0 + 1)^2}}{h'(\alpha_0)}. \tag{3.6}$$

Therefore, the local behavior of $\frac{w_j}{w_i}$ near $\tilde{\alpha} = (-1)^k \infty, k = 1, 2$, is determined by

$$|\tilde{\sigma}|^{(-1)^{j+k} \frac{\sqrt{((a-1)\alpha_0+b)^2 + 4(b\alpha_0+1)^2}}{h'(\alpha_0)} \beta}.$$

Choose

$$\beta \equiv \beta_0 = \frac{-h'(\alpha_0)}{\sqrt{((a-1)\alpha_0 + b)^2 + 4(b\alpha_0 + 1)^2}} > 0$$

to ensure that the functions $\frac{w_j}{w_i} \equiv \Gamma_j(\tilde{\sigma}) = |\tilde{\sigma}|^{(-1)^i sign(\tilde{\sigma})} f_j(\tilde{\sigma}), i \neq j$, near $\tilde{\sigma} = 0$, where $f_j(\tilde{\sigma})$ is real analytic near $\tilde{\sigma} = 0$. Furthermore,

$$\frac{w_j}{w_i} = -\exp\{\beta \int_{-\infty}^{\tilde{\alpha}} (H_j(\tilde{\alpha}) - H_i(\tilde{\alpha}))\, d\tilde{\alpha}\}$$

are analytic in $\tilde{\alpha}$ on any finite interval. Therefore, with this choice of β the functions $\Gamma_j(\tilde{\alpha})$ are real analytic in $\tilde{\alpha} \in \mathbf{R}$ which corresponds to the set $\mathcal{I} - \{(u,v)|u = \alpha_0 v, (-1)^j u > 0\}$ in the (u,v)-plane. This completes the proof of Theorem 3.1.

Remark 1. The sign of β in the theorem is consistent with the conventional choice as in the case $b = 0$ [SS1, Ka]. We can choose the opposite sign in a completely similar approach.

Remark 2. In Theorem 3.1, we consider the half plane domain $\mathcal{I} \equiv \{ (u,v) \mid v - \frac{1}{\alpha_0}u \geq 0 \}$ only for simplicity. It is completely similar to deal with the complementary half plane domain $\{ (u,v) \mid v - \frac{1}{\alpha_0}u \leq 0 \}$.

The next theorem concerns the monotonicity of the wave speed λ_i in the variable w_i. This is important in the reduction of the Young measure for approximate solution sequences in Section 6.

Theorem 3.2. *Suppose that $\Delta > 0$. Given w_1 and w_2 as defined in Theorem 3.1, the eigenvalues $\lambda_i, i = 1,2$, are well-defined functions in $(w_1, w_2) \in \mathcal{J}(\mathcal{I})$. Moreover, we have*

$$\frac{\partial \lambda_i}{\partial w_i} \neq 0, \ \textit{for all } (w_1,w_2) \in \mathcal{J}(\mathcal{I}) - \{(w_1,w_2)|w_i = 0\}, \ i = 1,2. \tag{3.7}$$

Proof. Notice that

$$\frac{w_{iv}}{w_{iu}} = -\frac{r_{j1}}{r_{j2}}$$

from $\mathbf{r}_j \cdot \nabla \lambda_i = 0$ and

$$u_{w_i} = (-1)^i \frac{w_{jv}}{J}, \quad v_{w_i} = (-1)^j \frac{w_{ju}}{J}, \quad i \neq j,$$

where

$$J = w_{2u}w_{1v} - w_{2v}w_{1u}.$$

We have, from $\lambda_{iw_i} = \lambda_{iu}u_{w_i} + \lambda_{iv}v_{w_i}$,

$$\lambda_{iw_i} = \left(\frac{\mathbf{r}_i \cdot \nabla \lambda_i}{r_{i2}} \right) v_{w_i}. \tag{3.8}$$

Furthermore,

$$v_{w_i} = -\frac{\alpha_0}{1+\alpha_0^2} \frac{1 + (\alpha_0 + \tilde{\alpha})H_i(\tilde{\alpha})}{\beta \tilde{v}^{\beta-1}(H_1(\tilde{\alpha}) - H_2(\tilde{\alpha}))} \exp\{\beta \int_0^{\tilde{\alpha}} H_j(\tilde{\alpha})d\tilde{\alpha}\}. \tag{3.9}$$

Therefore, from (3.8) and (3.9),

$$\lambda_{iw_i} = \frac{\alpha_0 \tilde{v}^{1-\beta} \exp\{\beta \int_0^{\tilde{\alpha}} H_j(\tilde{\alpha})d\tilde{\alpha}\} \frac{\mathbf{r}_i \cdot \nabla \lambda_i}{v}}{k_i(\alpha, sign(v))} \frac{g(\alpha)}{\sqrt{((a-1)\alpha + b)^2 + 4(b\alpha+1)^2}}, \tag{3.10}$$

where

$$\begin{aligned} &g(\alpha) = \frac{\alpha_0 + \tilde{\alpha}}{1+\alpha_0^2} h(\alpha), \\ &k_i(\alpha, s) = -2\alpha(b\alpha+1) + (a-1)\alpha + b + (-1)^i s \sqrt{((a-1)\alpha+b)^2 + 4(b\alpha+1)^2}. \end{aligned} \tag{3.11}$$

Notice that $\beta > 0$ and the only bounded root, $\alpha = -\frac{1}{b}$, of the numerator is canceled by that of the denominator from (3.10). Moreover, near the line $\tilde{v} = 0, (-1)^i\tilde{u} > 0$ (i.e. $\tilde{\alpha} = (-1)^i\infty$),

$$\tilde{v}^{1-\beta}\exp\{\beta\int_0^{\tilde{\alpha}} H_j(\tilde{\alpha})d\tilde{\alpha}\}\frac{\frac{\mathbf{r}_i\cdot\nabla\lambda_i}{v}}{k_i(\alpha, sign(v))} \neq 0,$$
$$\frac{g(\alpha)}{\sqrt{((a-1)\alpha+b)^2+4(b\alpha+1)^2}} \neq 0$$

for any $i = 1, 2$, hence we conclude (3.7). This completes the proof.

4. Entropy Functions

4.1. The Entropy Equation

Following the standard definition in [La2], we call a pair of scalar functions $(\eta(u,v),\ q(u,v))$ an entropy-entropy flux pair for (2.1) if, for smooth solutions $(u(x,t), v(x,t))$ of (2.1), we have the additional conservation law

$$\partial_t\eta(u(x,t), v(x,t)) + \partial_x q(u(x,t), v(x,t)) = 0.$$

It is easy to check that this happens if and only η and q satisfy the compatibility condition

$$\nabla\eta\nabla F = \nabla q. \tag{4.1}$$

Eliminating q, we get a second order partial differential equation, the entropy equation,

$$g_u\eta_{uu} + (g_v - f_u)\eta_{uv} - f_v\eta_{vv} + (g_{uu} - f_{uv})\eta_u + (g_{uv} - f_{vv})\eta_v = 0, \tag{4.2}$$

where we denote $F(u,v) = (f(u,v), g(u,v))^\top$.

(4.2) is a linear hyperbolic equation whose characteristic variables turn out to be the Riemann invariants. A simple calculation gives the characteristic form as

$$\eta_{w_1w_2} + \frac{\lambda_{2w_1}}{\lambda_2-\lambda_1}\eta_{w_2} - \frac{\lambda_{1w_2}}{\lambda_2-\lambda_1}\eta_{w_1} = 0. \tag{4.3}$$

4.2. The Entropy Equation for the Quadratic System

We now investigate the properties of the coefficients of (4.3) in the case of the system with quadratic flux form (2.5). It will turn out that, in general, these coefficients cannot be written down as functions of w_1 and w_2 in closed form. Instead, we will compute them as functions of $\tilde{\alpha} = \frac{\tilde{u}}{\tilde{v}}$ and study their properties using information from Section 3. We summarize the main results in the following proposition:

Theorem 4.1. *Consider the system with quadratic flux form* (2.5) *in the case* $\Delta > 0$. *Then the coefficients of* (4.3) *satisfy, for* $i \neq j$,

(1) $\frac{\lambda_{jw_i}}{\lambda_2-\lambda_1} = \frac{T_j(\tilde{\alpha})}{w_i} = \frac{\mathcal{A}_j(\frac{w_2}{w_1})}{w_2-w_1}$,

where $T_j(\tilde{\alpha})$ *and* $T_j(\tilde{\alpha})\Gamma_j(\tilde{\alpha})$ *are real analytic in* $\tilde{\alpha} \in \mathbf{R} \cup \{\pm\infty\}$ *and* $\mathcal{A}_j(\tau)$ *are real analytic in* $\tau \in \mathbf{R}$;

(2)
$$\frac{\lambda_{jw_i}}{\lambda_2 - \lambda_1} = \frac{\tilde{\mathcal{A}}_j(\theta)}{w_2 - w_1}, \quad \theta \in [\frac{\pi}{2}, \pi],$$

where $\tilde{\mathcal{A}}_j(\theta), j = 1, 2,$ *are well defined and real analytic in* $\theta \in (0, \frac{3\pi}{2})$.

Proof. Since
$$\lambda_{jw_i} = \frac{\mathbf{r}_i \cdot \nabla \lambda_j}{r_{i2}} v_{w_i},$$

and, for $i \neq j$,
$$\mathbf{r}_i \cdot \nabla \lambda_j = \zeta - by + (-1)^i \frac{\zeta^2 - 3by\zeta + 2(a-1)y^2}{\sqrt{\zeta^2 + 4y^2}},$$

we have

$$\frac{\lambda_{jw_i}}{\lambda_2 - \lambda_1} = \frac{(\alpha - \alpha_0)g(\alpha)}{2\beta k_i(\alpha, sign(v))} \frac{\mathbf{r}_i \cdot \nabla \lambda_j}{\zeta^2 + 4y^2} \frac{1}{w_i} \equiv \frac{T_j(\tilde{\alpha})}{w_i}, \tag{4.4}$$

where $\zeta = (a-1)u + bv, y = bu + v$.

Note that $k_i(\alpha, sign(v))$ has roots

$$\begin{cases} -\frac{1}{b}, & \text{when}(-1)^i sign(v)((a-1)\alpha + b) < 0, \\ \alpha_0, & \text{when}(-1)^j sign(v) < 0. \end{cases}$$

At the same time, $\frac{\tilde{v}(\mathbf{r}_i \cdot \nabla \lambda_j)}{v\sqrt{\zeta^2+4y^2}}$ as a function of α has two roots:

$$\begin{cases} -\frac{1}{b}, & \text{when}(-1)^i sign(v)((a-1)\alpha + b) < 0, \\ \alpha_0, & \text{always}, \end{cases}$$

and $g(\alpha)$ has no root.

Therefore, $T_j(\tilde{\alpha})$ is real analytic in $\tilde{\alpha}$ for all $\tilde{\alpha} \in \mathcal{R} \cup \{\pm\infty\}$ and $T_j(\tilde{\alpha}) = \tilde{\alpha}^{-1}K_j(\tilde{\alpha})$ near $\tilde{\alpha} = (-1)^j\infty$, where $K_j(\tilde{\alpha})$ is real analytic in $\tilde{\alpha} \in \mathcal{R} \cup \{\pm\infty\}$. Moreover,

$$T_j(\tilde{\alpha}) = \mathcal{O}\left(|\tilde{\alpha}|^{-\frac{1+(-1)^j sign(\tilde{\alpha})}{2}}\right)$$

as $|\tilde{\alpha}| \to \infty$.

Furthermore, using Theorem 3.1 (2), we have

$$\Gamma_j(\tilde{\alpha}) = \mathcal{O}(|\tilde{\alpha}|^{(-1)^j sign(\tilde{\alpha})}), \quad \text{as } |\tilde{\alpha}| \to \infty.$$

We conclude that

$$\mathcal{A}_j(\frac{w_2}{w_1}) = (-1)^i T_j(\tilde{\alpha})(1 - \Gamma_j(\tilde{\alpha}))$$

are real analytic in $\tilde{\alpha} \in \mathcal{R} \cup \{\pm\infty\}$.

On the other hand, we have that

$$\frac{w_2}{w_1} = \Gamma_2(\tilde{\alpha})$$

is real analytic in $\tilde{\alpha}$ except at values of $\tilde{\alpha}$ where w_1 vanishes. To complete the proof of the theorem, we need only obtain an analytic inversion of Γ_2. To this end, we consider

$$\partial_{\tilde{\alpha}}\Gamma_2(\tilde{\alpha}) = \beta\Gamma_2(\tilde{\alpha})\frac{(1+\alpha_0^2)\sqrt{Q(\tilde{\alpha})}}{D(\tilde{\alpha})}.$$

Consider $\tilde{\alpha}$ near any finite point $\tilde{\alpha}_0$. We have $\partial_{\tilde{\alpha}}\Gamma_2(\tilde{\alpha}) \neq 0$. Therefore, by the standard theorem on inversion of analytic functions, $\tilde{\alpha}$ is an analytic function of $\frac{w_2}{w_1}$ near $\frac{w_2}{w_1} = \Gamma_2(\tilde{\alpha}_0)$. Then, by substituting into the expression $\mathcal{A}_j = (-1)^i T_j(\tilde{\alpha})(1-\Gamma_j(\tilde{\alpha}))$, we conclude that $\mathcal{A}_i$ is an analytic function of $\frac{w_2}{w_1}$ in the range $\Gamma_2(\mathbf{R}) = (-\infty, 0)$.

That $\mathcal{A}_j(\frac{w_2}{w_1})$ can be extended analytically to all nonnegative values of its argument will now be established together with the second result in Theorem 4.1 as follows.

First, we have $\frac{w_2}{w_1} = tan\theta$. Therefore, $\frac{w_2}{w_1}$ is real analytic in $\theta \in (\frac{\pi}{2}, \pi)$. Using the result we just proved above, we conclude that $\mathcal{A}_i$ is a real analytic function of θ on $\theta \in (\frac{\pi}{2}, \pi)$.

Next, we prove that $\tilde{\mathcal{A}}_i(\theta)$ can be extended analytically across the point $\theta = \pi$. Let $\tilde{\sigma} = \frac{1}{\tilde{\alpha}}$. We notice that $\theta \lesssim \pi$ corresponds to $\tilde{\sigma} \lesssim 0$. In particular, near $\theta \lesssim \pi$, we have

$$\Gamma_2(\frac{1}{\tilde{\sigma}}) = \frac{w_2}{w_1} = tan\theta = -|\tilde{\sigma}|f(\tilde{\sigma}),$$

where $f(\tilde{\sigma})$ is analytic in the vicinity of $\tilde{\sigma} = 0$.

Then we extend $tan\theta$ analytically as a function of $\tilde{\sigma}$ across the point $\tilde{\sigma} = 0$ as follows:

$$tan\theta = \tilde{\sigma}f(\tilde{\sigma}) \equiv \bar{\Gamma}_2(\frac{1}{\tilde{\sigma}}) = \begin{cases} \Gamma_2(\frac{1}{\tilde{\sigma}}), & \tilde{\sigma} < 0, \\ -\Gamma_2(-\frac{1}{\tilde{\sigma}}), & \tilde{\sigma} \geq 0. \end{cases}$$

Then $tan\theta$ is analytic in $\tilde{\sigma}$ near $\tilde{\sigma} = 0$.

Now, we show that $\partial_{\tilde{\sigma}}\bar{\Gamma}_2 \neq 0$ near $\tilde{\sigma} = 0$,

$$\frac{w_2}{w_1} = \bar{\Gamma}_2(\frac{1}{\tilde{\sigma}}).$$

We have

$$\partial_{\tilde{\sigma}}\bar{\Gamma}_2 = \tilde{\alpha}_{\tilde{\sigma}}\partial_{\tilde{\alpha}}\bar{\Gamma}_2 = \begin{cases} -\frac{\Gamma_2(\frac{1}{\tilde{\sigma}})}{\tilde{\sigma}^2}\frac{(1+\alpha_0^2)\sqrt{Q(\frac{1}{\tilde{\sigma}})}}{D(\frac{1}{\tilde{\sigma}})}, & \tilde{\sigma} \lesssim 0, \\ \frac{\Gamma_2(-\frac{1}{\tilde{\sigma}})}{\tilde{\sigma}^2}\frac{(1+\alpha_0^2)\sqrt{Q(-\frac{1}{\tilde{\sigma}})}}{D(-\frac{1}{\tilde{\sigma}})}, & \tilde{\sigma} \gtrsim 0. \end{cases}$$

Notice that

$$\bar{\Gamma}_2(\frac{1}{\tilde{\sigma}}) = \mathcal{O}(|\tilde{\sigma}|), \quad \text{near} \quad \tilde{\sigma} \simeq 0.$$

We also observe that $D(\tilde{\alpha})$ is quadratic in $\tilde{\alpha}$, and $\sqrt{Q(\tilde{\alpha})}$ is of first order in $\tilde{\alpha}$.

Combining these, we conclude that, at $\tilde{\sigma} = 0\pm$ (i.e. $\tilde{\alpha} = \pm\infty$), $\partial_{\tilde{\sigma}}\bar{\Gamma}_2 \neq 0$. It follows that $\tilde{\sigma}$ is an analytic function of θ near $\theta = \pi$. Actually, it is easy to check that $\tilde{\sigma}$ can be extended to become analytic in $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$.

Similarly, working with $cotan\theta$ near $\theta = \frac{\pi}{2}$, we find that $\tilde{\sigma}$ can be extended analytically on the range $(0, \frac{\pi}{2})$.

Combining these two analyses, we conclude that $\tilde{\mathcal{A}}_i(\theta)$ is analytic in $\theta \in (0, \frac{3\pi}{2})$. This completes the proof of Theorem 4.1.

Remark 1. Theorem 4.1 implies the bounds $||\mathcal{A}_i||_{L^\infty} \leq C$. The analyticity of $\tilde{\mathcal{A}}_i$ in Theorem 4.1 will turn to be crucial in analyzing the singular behavior of η. We shall elaborate on this in Section 4.

Remark 2. The analyticity of $\tilde{\mathcal{A}}_i(\theta)$ on $(0, \frac{3\pi}{2})$ implies that $\mathcal{A}_i(s)$ can be extended analytically to $s \in \mathcal{R}$, which corresponds to $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, for example.

As a corollary to Theorem 4.1, we obtain

Corollary 4.2. *The entropy equation for the system with quadratic flux form* (2.5) *takes the form*

$$\eta_{w_1w_2} + \frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_2} - \frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_1} = 0, \tag{4.5}$$

in the region $w_1 \leq 0 \leq w_2$, *where* $\tilde{\mathcal{A}}_i(\theta) = \mathcal{A}_i(tan\theta)$ *is defined on* $\theta \in [\frac{\pi}{2}, \pi]$, *and can be analytically extended in* $\theta \in (0, \frac{3\pi}{2})$.

It is clear from (4.5) that the coefficients of the entropy equation become singular along the line $w_1 = w_2$. Since $w_1 \leq 0 \leq w_2$, the coefficients are singular, multi-valued, only at the umbilic point. The singular nature of the coefficients also suggest that in general solutions η cannot be expected to be C^2. This presents major difficulties in verifying H^{-1} compactness conditions in the div-curl lemma. This is reminiscent of similar difficulties for proving convergence of approximate solutions to the gas dynamics equations (see [Ch2, Di2]). However, the singularities of the entropies are of somewhat different character in the two cases and we use different approaches to resolve these difficulties. The method we shall present generalizes that in [Ka] and consists in constructing very general classes of nonsingular C^2 entropies. This analysis is not restricted to the systems with quadratic flux form and can be generalized to obtain a compactness framework theorem (Theorem 6.3) for convergence of approximate solutions to general nonstrictly hyperbolic systems with an isolated umbilic point.

We remark that, although (2) in Theorem 4.1 implies that $\tilde{\mathcal{A}}_i(\theta)$ is real analytic in $\theta \in [\frac{\pi}{2}, \pi]$, $\mathcal{A}_i(\frac{w_2}{w_1})$ is multi-valued at the umbilic point. We now interpret this by comparing (4.5) to the classical Euler-Poisson-Darboux equation. The classical EPD equation in characteristic form is

$$\eta_{w_1 w_2} + \frac{\beta_1}{w_2 - w_1}\eta_{w_2} - \frac{\beta_2}{w_2 - w_1}\eta_{w_1} = 0, . \tag{4.6}$$

where β_1 and β_2 are constants. The significance of these constants lies in the fact that they completely determine the singular behavior of solutions to (4.6). The EPD equation arises as the entropy equation in isentropic gas dynamics (see [Di2, Ch2]). A comparison between (4.5) and (4.6) indicates that, heuristically, we should expect that the singularity of the entropy near the umbilic point depends on the angle of approach and the “size” of $\mathcal{A}_i$. This turns out to be the correct picture and motivates much of the work on the analysis and cancellation of singularities in Subsection 4.5.

4.3 Polynomial Entropies for the Quadratic System

The quadratic system (2.5) admits entropy functions that are homogeneous polynomials in the physical variables u and v of arbitrary high degree. This was observed in [Ka] for the symmetric case $b = 0$ of (2.5). We now generalize this result for arbitrary a and b. We remark that $u^2 + v^2$ is a strictly convex entropy function for (2.5) for all a and b. This function plays a special role in obtaining H^{-1} compactness estimates for the dissipation of entropy for approximate solution sequences.

Using (4.1) and the form of the flux matrix in (2.5), we obtain the entropy equation for (2.5) as follows:

$$((a-1)u + bv)\eta_{uv} + (bu + v)(\eta_{vv} - \eta_{uu}) = 0. \tag{4.7}$$

Proposition 4.1. *Given any a and b, there exists an infinite sequence of solutions to (4.7), η_k ($k = 1, 2, 3, \cdots$) that is a homogeneous polynomial in u and v of degree k.*

This is achieved by looking for special entropy functions of the form

$$\eta_k = v^k \phi(\alpha), \quad \alpha = \frac{u}{v}.$$

Then (4.2) reduces to an ordinary differential equation for $\phi(\alpha)$. It is then easy to check that for each positive integer k, this equation admits a polynomial solution in α of degree k. Thus, η_k is a homogeneous polynomial in u and v of degree k.

4.4. The Riemann Function for the Entropy Equation

Next, we study the general properties of entropy in the (w_1, w_2)-coordinates. In particular, we are interested in understanding the possible singularities of η and the construction of C^2 Goursat entropies. Goursat entropies are solutions of the entropy equation satisfying characteristic boundary conditions. To this end, we first study the Riemann function $\mathcal{R}$ for the hyperbolic equation (4.3). Recall that the Riemann function contains all information about the general solution and is defined as the solution to the following Goursat problem (characteristic boundary value problem for the adjoint equation of (4.3)) with special boundary data:

$$\begin{aligned}
&\mathcal{R} = \mathcal{R}(w_1, w_2; \sigma, \tau), \\
&\mathcal{R}_{w_1 w_2} - \frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1}\mathcal{R}_{w_2} + \frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2 - w_1}\mathcal{R}_{w_1} + \\
&\quad + \left(\frac{\mathcal{A}_1(\frac{w_2}{w_1}) + \mathcal{A}_2(\frac{w_2}{w_1})}{(w_2 - w_1)^2} + \frac{\partial_{w_1}\mathcal{A}_1(\frac{w_2}{w_1}) - \partial_{w_2}\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1} \right) \mathcal{R} = 0, \\
&\mathcal{R}(w_1, \tau; \sigma, \tau) = exp\left\{ - \int_\sigma^{w_1} \frac{\mathcal{A}_2(\frac{\tau}{s})}{s - \tau}\, ds \right\}, \\
&\mathcal{R}(\sigma, w_2; \sigma, \tau) = exp\left\{ \int_\tau^{w_2} \frac{\mathcal{A}_1(\frac{y}{\sigma})}{\sigma - y}\, dy \right\}.
\end{aligned} \tag{4.8}$$

Consider the general Goursat problem for the entropy equation (4.5)

$$\begin{aligned}
&\eta_{w_1 w_2} + \frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_2} - \frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2 - w_1}\eta_{w_1} = 0, \\
&\eta(\xi, w_2) = \phi(w_2), \\
&\eta(w_1, \kappa) = \psi(w_1),
\end{aligned} \tag{4.9}$$

where ϕ and ψ are prescribed functions, ξ and κ are fixed constants, and $\phi(\kappa) = \psi(\xi)$. It is well known that the solution η to (4.9) can be expressed in integral form in terms of $\mathcal{R}, \phi$, and ψ (at least in regions where η is C^2) as follows:

$$\begin{aligned}
\eta(w_1, w_2) = \psi(\xi)\mathcal{R}(w_1, w_2; \xi, \kappa) &+ \int_\kappa^{w_2} \mathcal{R}(w_1, w_2; \xi, s) \left(\phi'(s) - \frac{\mathcal{A}_1(\frac{s}{\xi})}{s - \xi}\phi(s) \right) ds \\
&+ \int_\xi^{w_1} \mathcal{R}(w_1, w_2; y, \kappa) \left(\psi'(y) + \frac{\mathcal{A}_2(\frac{\kappa}{y})}{\kappa - y}\psi(y) \right) dy.
\end{aligned} \tag{4.10}$$

Our strategy is first to analyze the existence and singular behavior of $\mathcal{R}$ and then make use of (4.10) to show how to choose general classes of data ϕ and ψ to cancel these singularities. The existence of $\mathcal{R}$ is nontrivial as the coefficients of (4.8) are singular. In the case of strictly hyperbolic systems, the corresponding $\mathcal{R}$ and η equations will have regular coefficients only and existence of general solutions follows from standard iteration methods. This is not the case here and is one of the key difficulties in treating nonstrictly hyperbolic systems.

For the conservation laws (2.5) under consideration, the Riemann invariants satisfy $w_1 \leq 0 \leq w_2$. In this quadrant of the Riemann invariant plane, the entropy function is as regular as its boundary data when restricted to a region bounded away from the umbilic point (the origin) and the line $\{w_1 = 0\}$. This follows from standard iteration methods (cf. [CH]). We state this as the following lemma.

Lemma 4.2. *Given $\epsilon > 0$, suppose that the boundary data ϕ and ψ in (4.10) belong to the class C^k. Assume also that $\xi < 0$. Then on the domain $\{(w_1, w_2)|w_2 \geq 0, -\epsilon \geq w_1\}$, there exists a unique solution η to (4.9) belonging to the class C^k.*

Granting Lemma 4.2, we now concentrate on investigating the singular structure of the Riemann function in a neighborhood of the umbilic point. To this end, we demonstrate an explicit construction of $\mathcal{R}$ in such a neighborhood in power series in polar coordinates. For simplicity, we consider the case when $\kappa = 0$ and $\phi \equiv 0$ in (4.9). We observe then that, in the integral representation (4.10) of η, we need only consider $\mathcal{R}(w_1, w_2; \sigma, 0)$. Such $\mathcal{R}$'s are sufficient for our purpose.

We rewrite the equation for $\mathcal{R}$ in (4.8) in polar coordinates:

$$\begin{cases} w_1 &= -\sigma r \ cos\theta, \\ w_2 &= -\sigma r \ sin\theta, \end{cases} \tag{4.11}$$

for $\theta \in [\frac{\pi}{2}, \pi]$.

A simple computation shows that the $\mathcal{R}$ equation becomes

$$\begin{aligned} & sin\theta cos\theta \ \mathcal{R}_{rr} + \frac{cos^2\theta - sin^2\theta}{r} \ \mathcal{R}_{r\theta} - \frac{sin\theta cos\theta}{r^2} \ \mathcal{R}_{\theta\theta} \\ & + \frac{cos\theta \tilde{\mathcal{A}}_1(\theta) - sin\theta \tilde{\mathcal{A}}_2(\theta) - sin\theta cos\theta (sin\theta - cos\theta)}{r(sin\theta - cos\theta)} \ \mathcal{R}_r \\ & - \frac{cos\theta \tilde{\mathcal{A}}_2(\theta) + sin\theta \tilde{\mathcal{A}}_1(\theta) + (cos^2\theta - sin^2\theta)(sin\theta - cos\theta)}{r^2(sin\theta - cos\theta)} \ \mathcal{R}_\theta \\ & + \frac{\tilde{\mathcal{A}}_1(\theta) + \tilde{\mathcal{A}}_2(\theta) - (sin\theta - cos\theta)(sin\theta \tilde{\mathcal{A}}_1'(\theta) + cos\theta \tilde{\mathcal{A}}_2'(\theta))}{r^2(sin\theta - cos\theta)^2} \ \mathcal{R} = 0, \end{aligned} \tag{4.12}$$

where

$$\tilde{\mathcal{A}}_i(\theta) = \mathcal{A}_i(\tan\theta) = \mathcal{A}_i(\frac{w_2}{w_1}), \quad i = 1, 2.$$

Abusing notation, we write $\mathcal{R} = \mathcal{R}(r, \theta)$ for $\mathcal{R}(w_1, w_2; \sigma, 0)$ written in polar coordinates in (4.11). Then the boundary conditions in (4.8) are translated into

$$\begin{aligned} & \mathcal{R}(r, \theta)|_{\theta=\pi} = r^{-\mathcal{A}_2(0)} = r^{-\tilde{\mathcal{A}}_2(\pi)}, \\ & \mathcal{R}(r, \theta)|_{r=\frac{-1}{cos\theta}} = exp\{\int_0^{tan\theta} \frac{\mathcal{A}_1(y)}{1 - y} \ dy\}, \qquad \frac{\pi}{2} < \theta \leq \pi. \end{aligned} \tag{4.13}$$

We look for solutions to (4.12) of the form

$$\mathcal{R}(r,\theta) = r^{-\tilde{\mathcal{A}}_2(\pi)}\alpha(\theta)\sum_{n=-\infty}^{\infty} a_n(\theta)r^n,$$

$$\alpha(\theta) = exp\{\int_0^{tan\theta}\frac{\mathcal{A}_1(y)}{1-y}\,dy\}(\frac{-1}{cos\theta})^{\tilde{\mathcal{A}}_2(\pi)}. \tag{4.14}$$

The boundary conditions (4.13) are satisfied if a_n satisfies

$$a_n(\pi) = \delta_{n,0}, \qquad n \text{ integers},$$

$$\sum_{n=-\infty}^{\infty} a_n(\theta)(\frac{-1}{cos\theta})^n = 1, \qquad \frac{\pi}{2} < \theta \leq \pi. \tag{4.15}$$

We look for solution to (4.12) and (4.13) of the form (4.14). We will first treat (4.14) as a formal solution and compute the equation satisfied by a_n. Then we analyze the behavior of a_n and show that the series in (4.14) converges and determines a well-defined solution to (4.12) – (4.13) in the range of (r,θ) stated in the theorem.

Substituting (4.14) into the equation (4.12), and equating coefficients of r^n, we conclude that $a_n(\theta)$ satisfies the following ordinary differential equation:

$$a_n''(\theta) + B_n(\theta)a_n'(\theta) + C_n(\theta)a_n(\theta) = 0. \tag{4.16}$$

Here

$$B_n(\theta) = n\frac{sin^2\theta - cos^2\theta}{sin\theta cos\theta} + \tilde{B}(\theta),$$

$$C_n(\theta) = -n^2 + n\left(2\tilde{\mathcal{A}}_2(\pi) + 2 + \frac{\alpha'(\theta)}{\alpha(\theta)}\frac{sin^2\theta - cos^2\theta}{sin\theta cos\theta} + \frac{cos\theta\tilde{\mathcal{A}}_1(\theta) - sin\theta\tilde{\mathcal{A}}_2(\theta)}{sin\theta cos\theta(cos\theta - sin\theta)}\right) + \tilde{C}(\theta),$$

where

$$\tilde{B}(\theta) = 2\frac{\alpha'(\theta)}{\alpha(\theta)} + (\tilde{\mathcal{A}}_2(\pi)+1)\frac{cos^2\theta - sin^2\theta}{sin\theta cos\theta} + \frac{cos\theta\tilde{\mathcal{A}}_2(\theta) + sin\theta\tilde{\mathcal{A}}_1(\theta)}{sin\theta cos\theta(sin\theta - cos)}$$

and $\tilde{C}(\theta)$ depends on $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2$, and α has the following properties:

$$\lim_{\theta\to\pi} \tilde{B}_n(\theta)(\theta-\pi) = 1,$$

$$\lim_{\theta\to\pi} \tilde{C}(\theta)(\theta-\pi)^2 = 0; \tag{4.17}$$

and

$$\lim_{\theta\to\frac{\pi}{2}} \tilde{B}(\theta)(\theta - \frac{\pi}{2}) = 1 + \tilde{\mathcal{A}}_1(\frac{\pi}{2}) - \tilde{\mathcal{A}}(\pi),$$

$$\lim_{\theta\to\frac{\pi}{2}} \left(\tilde{C}(\theta) - \frac{\alpha''(\theta)}{\alpha(\theta)} - \frac{\alpha'(\theta)}{\alpha(\theta)}(\tilde{B}(\theta) - 2\frac{\alpha'(\theta)}{\alpha(\theta)})\right)(\theta - \frac{\pi}{2})$$

$$= \tilde{\mathcal{A}}_2(\frac{\pi}{2})(\tilde{\mathcal{A}}_2(\pi)+1) - \tilde{\mathcal{A}}_1'(\frac{\pi}{2}) + \tilde{\mathcal{A}}_1(\frac{\pi}{2}),$$

$$\lim_{\theta\to\frac{\pi}{2}} \tilde{C}(\theta)(\theta - \frac{\pi}{2})^2 = 0. \tag{4.18}$$

Furthermore, we have

Lemma 4.3. *$\tilde{B}(\theta)$ and $\tilde{C}(\theta)$ are analytic on $(\frac{\pi}{4},\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)\cup(\pi,\frac{5\pi}{4})$ and have first order poles at $\theta=\frac{\pi}{2}$ and $\theta=\pi$. Moreover, let $\tau=tan\theta$, $\bar{B}(\tau)\equiv\tilde{B}(\theta)$, and $\bar{C}(\tau)\equiv\tilde{C}(\theta)$. Then $\tau\bar{B}(\tau)$ and $\tau\bar{C}(\tau)$ are analytic near $\tau=0$ and their k^{th} Taylor coefficients at $\tau=0$ grow at most exponentially in k.*

Remark. The second boundary condition in (4.15) is nonlocal and couples the system of infinite differential equations for the $a_n's$.

Theorem 4.3. *There exist solutions a_n to* (4.16) *together with the boundary conditions* (4.15) *satisfying the following properties:*

(1) $a_n\equiv 0$, *for* $n\le -1$;

(2) $a_n(\theta)$ *is analytic in* $\theta\in(\frac{\pi}{2},\pi]$;

(3) *Given* $k>0$, *there exists* $n_0>0$ *such that*

$\frac{d^j}{d\theta^j}(\alpha(\theta)a_n(\theta))|_{\theta=\frac{\pi}{2}}$ *are bounded for all* $n\ge n_0,\ 0\le j\le k$.

Proof. It is easy to check that the coefficients of (4.16) are analytic in θ except at isolated points and that $\theta=\frac{\pi}{2},\pi$ are regular singular points of the o.d.e.. Thus, any general solution a_n will be analytic in $\theta\in(\frac{\pi}{2},\pi)$.

Using (4.17), we obtain that the indicial equation of (4.6) at $\theta=\pi$ is

$$\gamma^2-n\gamma=0,$$

which has roots $\gamma=0,n$.

First, we set $a_n\equiv 0$, for all $n\le -1$. Then, for $n\ge 0$ and θ close to π, using the theory for regular singular points of ordinary differential equations, there are two linear independent solutions to (4.16) of the form

$$(4.19)\qquad \begin{aligned}&(\theta-\pi)^n\sum_{k=0}^{\infty}\alpha_{nk}(\theta-\pi)^k;\\ &(\theta-\pi)^{\delta_{n,0}}\sum_{k=0}^{\infty}\beta_{nk}(\theta-\pi)^k+d_n\ log|\theta-\pi|(\theta-\pi)^n\sum_{k=0}^{\infty}\alpha_{nk}(\theta-\pi)^k.\end{aligned}$$

Here, α_{nk},β_{nk}, and d_n are constants determined by the coefficients in (4.16). We also use the normalization $\alpha_{n0}=\beta_{n0}=1$, for all $n\ge 0$.

We will only make use of the first solution in (4.18). That is, for $n\ge 0$, we set

$$a_n(\theta)=c_n(\theta-\pi)^n\sum_{k=0}^{\infty}\alpha_{nk}(\theta-\pi)^k, \qquad (4.20)$$

where c_n are constants to be adjusted to fit the boundary conditions in (4.15). Moreover, the first boundary condition in (4.15) is satisfied automatically.

Recall that $a_n \equiv 0$, for all $n \leq -1$. The second condition in (4.15) becomes

$$\sum_{n=0}^{\infty} a_n(\theta)(\frac{-1}{cos\theta})^n = 1, \qquad \frac{\pi}{2} < \theta \leq \pi. \tag{4.21}$$

We expand the cosine factors into power series in $\theta - \pi$:

$$(\frac{-1}{cos\theta})^n = \sum_{l=0}^{\infty} b_{nl}(\theta - \pi)^l.$$

Using (4.20), (4.21) becomes

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} a_n(\theta)(\frac{-1}{cos\theta})^n \\ &= \sum_{\tau=0}^{\infty}(\theta - \pi)^\tau \sum_{n=0}^{\tau} c_n \sum_{k=0}^{\tau-n} \alpha_{nk} b_{n\tau-k-n}. \end{aligned} \tag{4.22}$$

Therefore, c_n should solve the system of infinite linear equations

$$\delta_{\tau,0} = \sum_{n=0}^{\tau} c_n \sum_{k=0}^{\tau-n} \alpha_{nk} b_{n\tau-k-n}, \quad \tau = 0, 1, 2, \cdots. \tag{4.23}$$

This system is lower triangular and the diagonal elements are of the form $\alpha_{\tau 0} b_{\tau 0} = 1 \neq 0$. Hence, the c_n's are uniquely determined. This completes the constructions of the a_n's and proves statements (1) and (2) in the theorem.

It remains to verify Theorem 4.3(3). Using (4.18), we have that the indicial equation of (4.6) at $\theta = \frac{\pi}{2}$ is

$$\gamma^2 - \gamma(n + \tilde{\mathcal{A}}_1(\frac{\pi}{2}) - \tilde{\mathcal{A}}_2(\pi)) + n(\tilde{\mathcal{A}}_2(\pi) - \tilde{\mathcal{A}}_1(\frac{\pi}{2})) = 0,$$

which has roots $\gamma = n, -\tilde{\mathcal{A}}_1(\frac{\pi}{2}) + \tilde{\mathcal{A}}_2(\pi)$.

Therefore, near $\theta = \frac{\pi}{2}$, we have

$$a_n(\theta) = E_n(\theta - \frac{\pi}{2})^n \sum_{k=0}^{\infty} \sigma_{nk}(\theta - \frac{\pi}{2})^k + F_n|\theta - \frac{\pi}{2}|^{\tilde{\mathcal{A}}_2(\pi) - \tilde{\mathcal{A}}_1(\frac{\pi}{2})} \sum_{k=0}^{\infty} \tau_{nk}(\theta - \frac{\pi}{2})^k. \tag{4.24}$$

Here the constants σ_{nk} and τ_{nk} are determined by the coefficients of (4.16). Since $a_n(\theta)$ is analytic in $\theta \in (\frac{\pi}{2}, \pi)$, the constants E_n and F_n are determined uniquely by c_n in (4.22).

We remark here that, in the formula (4.24), we have assumed that $n - \tilde{\mathcal{A}}_2(\pi) + \tilde{\mathcal{A}}_1(\frac{\pi}{2})$ is not an integer. If this quantity is an integer, then (4.24) will involve logarithmic terms. In that case, a slightly more complicated analysis than the one we will present below will lead to the same result. We omit the details.

Notice that

$$\begin{aligned} \alpha(\theta) &= (\frac{-1}{cos\theta})^{-\tilde{\mathcal{A}}_2(\pi)} exp\{\int_0^{tan\theta} \frac{\mathcal{A}_1(y)}{1-y}\, dy\} \\ &= |\theta - \frac{\pi}{2}|^{-\tilde{\mathcal{A}}_2(\pi)+\tilde{\mathcal{A}}_1(\frac{\pi}{2})}\ \mathfrak{G}(\theta - \frac{\pi}{2}), \end{aligned} \tag{4.25}$$

where $\mathfrak{G}$ is an analytic function.

Combining (4.24) and (4.25), we obtain

$$\alpha(\theta)a_n(\theta) = E_n(\theta - \frac{\pi}{2})^{n+\tilde{\mathcal{A}}_1(\frac{\pi}{2})-\tilde{\mathcal{A}}_2(\pi)} \sum_{k=0}^{\infty} \tilde{\sigma}_{nk}(\theta - \frac{\pi}{2})^k + F_n \sum_{k=0}^{\infty} \tilde{\tau}_{nk}(\theta - \frac{\pi}{2})^k \tag{4.26}$$

with appropriate constants $\tilde{\sigma}_{nk}$ and $\tilde{\tau}_{nk}$. Statement (3) of the theorem now follows from the formula in (4.26). This completes the proof of the theorem.

To complete the construction of $\mathcal{R}(r,\theta)$ satisfying (4.12) and (4.13), we prove that the tail end of the series in (4.14) is convergent. That is, for N large enough and $r \geq 0$ small enough, $\sum_{n=N}^{\infty} \alpha(\theta)a_n(\theta)r^n$ converges for all $\frac{\pi}{2} \leq \theta \leq \pi$.

To this end, we estimate $\alpha(\theta)a_n(\theta)$ and show that it grows at most exponentially in n.

Theorem 4.4. *There exists constants $N, C > 0$ such that for all $n > N$ and $\theta \in [\frac{\pi}{2}, \pi]$,*

$$|\alpha(\theta)a_n(\theta)| \leq Cexp(Cn).$$

Proof. The proof will consist of three steps. In step 1, we estimate $a_n(\theta)$ when θ is close to π by estimating the constants c_n and α_{nk} in (4.20). In step 2, restricting ourselves to a region of θ strictly bounded between $\frac{\pi}{2}$ and π, we derive a similar estimate using the o.d.e. (4.16). Finally, we show that a_n is nonsingular and satisfies the same exponential bound (for n large) in a neighborhood of $\frac{\pi}{2}$ using an iteration procedure involving the Green's function of the principal singular part of (4.16).

In a neighborhood of $\theta = \pi$, consider the function

$$f_n(\theta) = (-cos\theta)^{-n}a_n(\theta).$$

Then, from (4.16), f_n satisfies

$$\begin{aligned} f_n'' &= p_n f_n' + q_n f_n, \\ p_n &= 2n\, tan\theta - B_n, \\ q_n &= -n(n-1)tan^2\theta + n + n\, tan\theta B_n - C_n, \end{aligned}$$

and f_n has the form

$$f_n(\theta) = c_n(\theta-\pi)^n \sum_{k=0}^{\infty} \tilde{f}_{nk}(\theta-\pi)^k,$$

from (4.21).

We will bound f_n by estimating c_n and $(\theta-\pi)^n \sum_{k=0}^{\infty} \tilde{f}_{nk}(\theta-\pi)^k$, separately. An essential observation is that, for each j fixed, p_{nj} and q_{nj} (the Taylor coefficients of p_n and q_n at π) are $\mathcal{O}(n)$ and $\mathcal{O}(1)$, respectively. The corresponding Taylor coefficients occurring in the recurrence formula for a_n are $\mathcal{O}(n^2)$. It turns out that the sum $(\theta-\pi)^n \sum_{k=0}^{\infty} \tilde{f}_{nk}(\theta-\pi)^k$ can be more conveniently estimated when it is rewritten in a different variable. The reason will become apparent below. Let

$$s = \beta(\theta), \quad \dot{} = \frac{d}{ds}.$$

Then we have

$$\ddot{f_n}(s) = J_n \dot{f_n}(s) + L_n f_n(s), \tag{4.27}$$

where

$$J_n = \frac{p_n}{\beta'(\theta)} - \frac{\beta''(\theta)}{(\beta'(\theta))^2}, \qquad L_n = \frac{q_n}{(\beta'(\theta))^2}.$$

Also, if $\beta(\pi) = 0$, then

$$f_n(s) = c_n s^n \sum_{k=0}^{\infty} f_{nk} s^k, \tag{4.28}$$

for appropriate constants f_{nk}. Moreover, we can write down the recurrence formula for f_{nk}:

$$(m+n+2)(m+n+1-J_{n,-1}) f_{nm+2} = \sum_{j=-1}^{m} ((m+n-j)J_{n,j+1} + L_{n,j}) f_{nm-j}, \tag{4.29}$$

with $J_n = \sum_{j=-1}^{\infty} J_{nj} s^j$ and $L_n = \sum_{j=-1}^{\infty} L_{nj} s^j$.

A computation using the expression for p_n and q_n shows that

$$\begin{aligned} J_n &= \frac{n}{sin\theta cos\theta \beta'(\theta)} + \tilde{J}, \\ L_n &= (\beta'(\theta))^{-2} \tilde{L}, \end{aligned} \tag{4.30}$$

where $\tilde{J}$ and $\tilde{L}$ are independent of n. Therefore, $J_{nj} = \mathcal{O}(n)$ for each j fixed. Now we make a choice of the new variable $s = \beta(\theta)$ to further reduce the order of J_{nj} in the recurrence formula (4.29).

We choose $s = \beta(\theta)$ to satisfy the relation $\beta(\theta) = (\frac{1}{2}sin2\theta)\beta'(\theta)$. That is, $\beta(\theta) = tan\theta$. Then we have, from (4.30),

$$J_n = \frac{n}{s} + \tilde{J}.$$

With this choice of variable, we found $J_{n,j+1}, j \geq -1$, in the recurrence formula (4.29) are independent of n.

Using (4.30) and $s = tan\theta$, a detailed computation shows that

$$\tilde{J} = -\frac{2s + \tilde{B}(\theta)}{1+s^2},$$
$$\tilde{L} = -\frac{\tilde{C}(\theta)}{(1+s^2)^2}. \tag{4.31}$$

Using Lemma 4.3, we conclude that the Taylor coefficients J_{nk} and L_{nk} at $s = 0$ satisfy the following estimates

$$|J_{nk}| + |L_{nk}| \leq Cexp(Ck), \tag{4.32}$$

for any $k \geq 0$, here and henceforth C denotes a universal, sufficiently large, constant independent of n. Using (4.29) and (4.32), we now show that

$$|f_{nm}| \leq Cexp(Cm). \tag{4.33}$$

We prove this estimate by induction. Assuming that $|f_{n,m-j}| \leq Cexp(C(m-j))$ for all $-1 \leq j \leq m$ and then, using (4.32) and (4.33), we have

$$\begin{aligned}|f_{nm+2}| &\leq \frac{1}{(m+n+2)(m+2)} \sum_{j=-1}^{m} ((m+n-j)Cexp(C(j+1)) + C\exp(Cj))\, Cexp(C(m-j)) \\ &\leq Cexp(C(m+2)),\end{aligned}$$

if C is large enough. This proves the estimate (4.33).

It remains to estimate the constants c_n in (4.28). We have

$$c_n = -\sum_{j=0}^{n-1} f_{j,n-j}\ c_j$$

from the boundary condition (4.15). Then using the estimates on f_{nm} just obtained in (4.33), we will show that

$$|c_n| \leq Cexp(Cn). \tag{4.34}$$

We have, by (4.33),

$$|c_n| \leq C \sum_{j=0}^{n-1} exp(C(n-j))\ |c_j|.$$

Let
$$G(n) \equiv \sum_{j=0}^{n-1} exp(-Cj)|c_j|.$$
Then
$$\begin{aligned} G(n) - G(n-1) &= exp(1-n)|c_n| \\ &\leq exp(-1) \sum_{j=0}^{n-2} exp(-Cj)|c_j| \\ &= exp(-1)G(n-1). \end{aligned}$$

Therefore,
$$\begin{aligned} G(n) &\leq (1 + exp(-1))G(n-1) \\ &\leq \cdots \leq C(1 + exp(-1))^n. \end{aligned}$$

Then we have
$$|c_n| \leq Cexp(n)G(n) \leq C^2(exp(1)+1)^n.$$
This proves (4.34).

Combining these two estimates, we conclude that for sufficiently small $\delta > 0$, we have

$$|\alpha(\theta)a_n(\theta)| \leq Cexp(Cn), \qquad \forall \theta \in [\pi - \delta, \pi]. \tag{4.35}$$

Next, we consider θ in the range $[\frac{\pi}{2} + \delta, \pi - \delta]$. In this range, the differential equation (4.16) has no singular points. It is then straightforward to show that αa_n satisfies a bound similar to (4.35) with a slightly larger constant C. We omit the details.

Finally, we show that αa_n satisfies a similar bound in a neighborhood of $\frac{\pi}{2}$. Let $b_n = \alpha a_n$, then b_n satisfies

$$b_n''(\theta) + D_n(\theta)b_n'(\theta) + E_n(\theta)b_n(\theta) = 0, \tag{4.36}$$

where
$$\begin{aligned} D_n(\theta) =& n\frac{sin^2\theta - cos^2\theta}{sin\theta cos\theta} + \tilde{B}(\theta) - \frac{2\alpha'(\theta)}{\alpha(\theta)}, \\ E_n(\theta) =& -n^2 + n\left(2\tilde{\mathcal{A}}_2(\pi) + 2 - \frac{sin\theta\tilde{\mathcal{A}}_2(\theta) - cos\theta\tilde{\mathcal{A}}_1(\theta)}{sin\theta cos\theta(cos\theta - sin\theta)}\right) \\ &- \frac{\alpha''(\theta)}{\alpha(\theta)} + \frac{\alpha'(\theta)}{\alpha(\theta)}\left(2\frac{\alpha'(\theta)}{\alpha(\theta)} - \tilde{B}(\theta)\right) + \tilde{C}(\theta). \end{aligned}$$

We rewrite this equation as

$$b_n''(\theta) + \frac{D_{n,-1}}{\theta - \frac{\pi}{2}} b_n'(\theta) + \frac{E_{n,-1}}{\theta - \frac{\pi}{2}} b_n(\theta) = g_n(\theta) b_n'(\theta) + h_n(\theta) b_n(\theta), \tag{4.37}$$

where

$$\begin{aligned}
D_{n,-1} &= -n + 1 + \tilde{\mathcal{A}}_2(\pi) - \tilde{\mathcal{A}}_1(\frac{\pi}{2}), \\
E_{n,-1} &= -n\tilde{\mathcal{A}}_2(\frac{\pi}{2}) + \tilde{\mathcal{A}}_2(\frac{\pi}{2})(\tilde{\mathcal{A}}_2(\pi) + 1) - \tilde{\mathcal{A}}_1'(\frac{\pi}{2}) + \tilde{\mathcal{A}}_1(\frac{\pi}{2}), \\
g_n(\theta) &= D_n(\theta) - \frac{D_{n,-1}}{\theta - \frac{\pi}{2}}, \quad h_n(\theta) = E_n(\theta) - \frac{E_{n,-1}}{\theta - \frac{\pi}{2}}.
\end{aligned}$$

Notice that g_n and h_n are analytic functions near $\theta = \frac{\pi}{2}$. We note that the right hand side of (4.37) is the principal singular part of the differential equation. Furthermore, from the formulae for D_n and E_n, we have the estimates

$$|g_n(\theta)| + |\frac{h_n(\theta)}{n}| \le Cn, \quad \forall \theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \delta]. \tag{4.38}$$

We will use the Green's function of the right hand side of (4.37) to set up an iteration scheme. Consider the homogeneous equation

$$e_n''(\theta) + \frac{D_{n,-1}}{\theta - \frac{\pi}{2}} e_n'(\theta) + \frac{E_{n,-1}}{\theta - \frac{\pi}{2}} e_n(\theta) = 0. \tag{4.39}$$

The two elementary solutions of (4.39) have the form

$$\begin{aligned}
\beta_n(\theta) &= \sum_{k=0}^{\infty} \tau_{nk} (\theta - \frac{\pi}{2})^k, \\
\mu_n(\theta) &= (\theta - \frac{\pi}{2})^{n - \tilde{\mathcal{A}}_2(\pi) + \tilde{\mathcal{A}}_1(\frac{\pi}{2})} \sum_{k=0}^{\infty} \sigma_{nk} (\theta - \frac{\pi}{2})^k.
\end{aligned} \tag{4.40}$$

Here we have assumed that $n - \tilde{\mathcal{A}}_2(\pi) + \tilde{\mathcal{A}}_1(\frac{\pi}{2})$ is not an integer. If this quantity is an integer then the elementary solutions will involve logarithmic terms. The analysis will then be slightly more complicated but similar to what we present below. Let $W(\beta_n, \mu_n)$ be the Wronskian of the two elementary solutions in (4.40). Then, it is not difficult to obtain the following estimates:

$$\begin{aligned}
|\tau_{nk}| + |\sigma_{nk}| &\le \mathcal{O}(1) \frac{1}{k!}, \\
W(\beta_n(\theta), \mu_n(\theta)) &= n(\theta - \frac{\pi}{2})^{n-1+\tilde{\mathcal{A}}_1(\frac{\pi}{2}) - \tilde{\mathcal{A}}_2(\pi)} \{-1 + \mathcal{O}(\frac{1}{n}) + h.o.t.\},
\end{aligned} \tag{4.41}$$

where $\mathcal{O}(1)$ and $h.o.t.$ denote, respectively, a constant and higher order terms that are bounded independent of n.

Consider the Green's function of (4.39) and its derivative in θ:

$$G_1(\theta,\tau) = \frac{\beta_n(\theta)\mu_n(\tau) - \beta_n(\tau)\mu_n(\theta)}{W(\beta_n(\tau),\mu_n(\tau))},$$
$$G_2(\theta,\tau) = \frac{\beta_n'(\theta)\mu_n(\tau) - \beta_n(\tau)\mu_n'(\theta)}{W(\beta_n(\tau),\mu_n(\tau))}.$$

Using (4.41), we derive the following estimates:

$$n|G_1(\theta,\tau)| + |G_2(\theta,\tau)| \le C, \tag{4.42}$$

for all $\frac{\pi}{2} \le \theta \le \tau \le \frac{\pi}{2} + \delta < \frac{3\pi}{4}$.

Denote $\mu_n(\theta) = (\theta - \frac{\pi}{2})^{n+\tilde{A}_1(\frac{\pi}{2})-\tilde{A}_2(\pi)}\tilde{\mu}_n(\theta)$. Then

$$\begin{aligned}
|G_1(\theta,\tau)| &= \left| \frac{\beta_n(\theta)\mu_n(\tau) - \beta_n(\tau)\mu_n(\theta)}{n(\tau-\frac{\pi}{2})^{n-1+\tilde{A}_1(\frac{\pi}{2})-\tilde{A}_2(\pi)}\{-1+\mathcal{O}(\frac{1}{n})+h.o.t.\}} \right| \\
&= \left| \frac{\tau - \frac{\pi}{2}}{n\{-1+\mathcal{O}(\frac{1}{n})+h.o.t.\}} \{\beta_n(\theta)\tilde{\mu}_n(\tau) - (\frac{\theta-\frac{\pi}{2}}{\tau-\frac{\pi}{2}})^{n+\tilde{A}_1(\frac{\pi}{2})-\tilde{A}_2(\pi)}\beta_n(\tau)\tilde{\mu}_n(\theta)\} \right| \\
&\le \frac{C}{n}, \quad \forall\ \frac{\pi}{2} \le \theta \le \tau \le \frac{\pi}{2} + \delta.
\end{aligned}$$

Similarly, we can obtain

$$|G_2(\theta,\tau)| \le C, \quad \forall \frac{\pi}{2} \le \theta \le \tau \le \frac{\pi}{2}.$$

This proves (4.42).

We now use G_1 and G_2 to set up an iteration scheme to solve (4.37) and therefore (4.36) near $\frac{\pi}{2}$. Let $y_n(\theta) = n^2 b_n(\theta)$ and $z_n(\theta) = nb_n'(\theta)$. Then the o.d.e. (4.37) is equivalent to the system of integral equations

$$y_n(\theta) = y_{n0} + \int_{\frac{\pi}{2}+\delta}^{\theta} nG_1(\theta,\tau)\left(g_n(\tau)z_n(\tau) + \frac{h_n(\tau)}{n}y_n(\tau)\right)\,d\tau,$$
$$z_n(\theta) = z_{n0} + \int_{\frac{\pi}{2}+\delta}^{\theta} G_2(\theta,\tau)\left(g_n(\tau)z_n(\tau) + \frac{h_n(\tau)}{n}y_n(\tau)\right)\,d\tau.$$

Here $y_{n0} = y_n(\frac{\pi}{2}+\delta)$ and $z_{n0} = z_n(\frac{\pi}{2}+\delta)$ satisfy the estimates $|y_{n0}|+|z_{n0}| \le Cn^2 exp(Cn)$ by the exponential bounds on αa_n near the point $\frac{\pi}{2}+\delta$. Consider an iteration scheme

$$y_n^{m+1}(\theta) = y_{n0} + \int_{\frac{\pi}{2}+\delta}^{\theta} nG_1(\theta,\tau)\left(g_n(\tau)z_n^m(\tau) + \frac{h_n(\tau)}{n}y_n^m(\tau)\right)\,d\tau,$$
$$z_n^{m+1}(\theta) = z_{n0} + \int_{\frac{\pi}{2}+\delta}^{\theta} G_2(\theta,\tau)\left(g_n(\tau)z_n^m(\tau) + \frac{h_n(\tau)}{n}y_n^m(\tau)\right)\,d\tau.$$

Using the estimates (4.38) and (4.42), we show by induction that

$$|y_n^{m+1}(\theta) - y_n^m(\theta)| + |z_n^{m+1}(\theta) - z_n^m(\theta)| \leq n^2 \frac{(Cn(\theta - \frac{\pi}{2} - \delta))^{m+1} exp(Cn)}{m!}.$$

Therefore, the sequence y_n^m converges to a limit $y_n(\theta)$ as $m \to +\infty$ satisfying

$$\begin{aligned} |y_n(\theta)| &\leq n^2 exp(Cn)\ exp\left\{Cn(\theta - \frac{\pi}{2} - \delta)\right\} \\ &\leq n^2 exp(Cn), \quad \forall\ \theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \delta]. \end{aligned}$$

This immediately implies that

$$|\alpha(\theta) a_n(\theta)| \leq exp(Cn), \quad \text{for n large enough and for all} \quad \theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \delta].$$

This completes the proof of the theorem.

Using Theorem 4.3 and Theorem 4.4, we conclude that the formal solution (4.14) is indeed a well-defined solution and the construction of $\mathcal{R}(r, \theta) = \mathcal{R}(w_1, w_2; \sigma, 0)$ is complete.

Theorem 4.5. *There exists a constant $r_0 > 0$, such that, for all $0 < r \leq r_0$ and $\frac{\pi}{2} \leq \theta \leq \pi$,* (4.14) *represents a well-defined solution to* (4.12) *with boundary conditions* (4.13).

4.5. C^2 Goursat Entropies

In this subsection, we characterize the singularities of entropies and we show how to construct general classes of Goursat entropies that are C^k in w_1 and w_2. We also obtain decay estimates for such Goursat entropies and their derivatives near the umbilic point.

The domain of interest for our problem is $\{w_1 \leq 0 \leq w_2\}$. It is enough to focus our attention on a compact set containing the umbilic point $\{w_1^- \leq w_1 \leq 0 \leq w_2 \leq w_2^+\}$ where $w_2^+ > 0$ and $w_1^- < 0$ are fixed constants. Fix a constant $\delta > 0$ satisfying $|\delta| \ll w_2^+,\ -w_1^-$.

Without loss of generality, we consider (4.9) with the special choice of Goursat data $\phi(w_2) \equiv 0$:

$$\begin{aligned} &\eta_{w_1 w_2} + \frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2 - w_1} \eta_{w_2} - \frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2 - w_1} \eta_{w_1} = 0, \\ &\eta(\xi, w_2) \equiv 0, \\ &\eta(w_1, \kappa) = \psi(w_1). \end{aligned} \tag{4.43}$$

There is no loss of generality since an analysis similar to what is shown below will allow us to construct C^2 entropies with Goursat data $\psi \equiv 0$ and ϕ arbitrary (up to a finite number of conditions, see Theorem 4.7). The general case as in (4.11) then follows from a simple linear superposition of these two special cases.

For simplicity, we assume, as in Section 4.4, that $\kappa = 0$ in (4.43) and construct regular solutions η. Such entropies are sufficient for reducing the Young measure and proving convergence of approximate solutions to (2.5). Recall from (4.10) that we have an integral representation for the solution to (4.43) ($\kappa = 0$, $\phi \equiv 0$):

$$\eta(w_1, w_2) = \int_{\xi}^{w_1} \mathcal{R}(w_1, w_2; y, 0) \left(\psi'(y) - \frac{\tilde{\mathcal{A}}_2(\pi)}{y} \psi(y) \right) dy. \tag{4.44}$$

Our first task is to decompose $\mathcal{R}$ into the sum of its singular and regular parts using Theorem 4.3. It will turn out that the singular part only loses its regularity as $r \to 0+$ or as $\theta \to \frac{\pi}{2}+$. The interaction of special classes of Goursat boundary data with the singular part of $\mathcal{R}$ (via (4.44)) will induce regularity for the entropy function.

Theorem 4.6. *Given $k > 0$, $\mathcal{R}$ admits a decomposition*

$$\mathcal{R}(r, \theta) = \mathcal{R}_{sing}(r, \theta) + \mathcal{R}_{regular}(r, \theta),$$

where $\mathcal{R}_{regular} \in C^k([0, r_0] \times [\frac{\pi}{2}, \pi]) \cap \mathfrak{A}((0, r_0] \times (\frac{\pi}{2}, \pi])$ and $\mathcal{R}_{sing} \in \mathfrak{A}((0, r_0] \times (\frac{\pi}{2}, \pi])$. Here $\mathfrak{A}$ denotes the real analytic class.

Proof. Using (4.14) and the fact that $a_n \equiv 0, \quad \forall n \leq -1$, we have

$$\mathcal{R}(r, \theta) = r^{-\tilde{\mathcal{A}}_2(\pi)} \alpha(\theta) \sum_{n=0}^{\infty} a_n(\theta) r^n.$$

Let $r_0 > 0$ be as defined in Theorem 4.5. Given $k > 0$, (2) and (3) of Theorem 4.5 imply that there exists $n_0 = n_0(k) > 0$ such that $\alpha(\theta) a_n(\theta) \in C^k([\frac{\pi}{2}, \pi]) \cap \mathfrak{A}((\frac{\pi}{2}, \pi])$, for all $n \geq n_0$.

Now, we pick $N_0 = n_0(k) + |\tilde{\mathcal{A}}_2(\pi)| + k + 1$ and write

$$\mathcal{R}(r, \theta) = \mathcal{R}_{sing}(r, \theta) + \mathcal{R}_{regular}(r, \theta),$$

$$\mathcal{R}_{sing}(r, \theta) = r^{-\tilde{\mathcal{A}}_2(\pi)} \sum_{n=0}^{N_0} \alpha(\theta) a_n(\theta) r^n,$$

$$\mathcal{R}_{regular}(r, \theta) = \sum_{n=N_0+1}^{\infty} \alpha(\theta) a_n(\theta) r^{n - \tilde{\mathcal{A}}_2(\pi)}. \tag{4.45}$$

Using Theorem 4.5 again, we conclude that $\mathcal{R}_{sing}$ and $\mathcal{R}_{regular}$ belong to the regularity classes stated in the theorem. This completes the proof.

Remark. If $\tilde{\mathcal{A}}_1(\frac{\pi}{2}) - \tilde{\mathcal{A}}_2(\pi)$ is not an integer, then $\mathcal{R}_{regular}(r, \theta) \in \mathfrak{A}((0, r_0] \times [\frac{\pi}{2}, \pi])$.

Using the above theorem and the integral representation (4.44), we show how to choose Goursat data ψ to cancel the singular effects of $\mathcal{R}_{sing}$ by requiring ψ to satisfy certain zero moment conditions. These conditions define a subspace $\mathcal{M}$ with finite codimension in L^2 such that $\psi \in \mathcal{M}$ implies that the entropy function η is regular.

We now state our main result in this section.

Theorem 4.7. *Given a positive integer m and a compact set K containing $(0,0)$ in $\mathbf{R}^2$, consider the Goursat problem (4.43) with $\kappa = 0$. There exists a subspace $\mathcal{M} \subset L^2[w_1^-, -\delta]$ with codim$\mathcal{M} < \infty$ such that if*

(1) $\psi \in \mathcal{M} \cap C^m$,

(2) $\psi(w_1) = 0, \quad$ *for all* $w_1 \geq -\delta$,

then there exists a solution $\eta \in C^m([w_1^-, 0] \times [0, w_2^+])$ *to (4.43) satisfying*

$$\max_{0 \leq j+l \leq m} \sup_{(w_1,w_2) \in K} |w_1^{-j} w_2^{-l} \partial_{w_1}^j \partial_{w_2}^l \eta| \leq C.$$

Proof. We start with the representation formula

$$\eta(w_1, w_2) = \int_\xi^{w_1} \mathcal{R}(w_1, w_2; y, 0) \left(\psi'(y) - \frac{\tilde{\mathcal{A}}_2(\pi)}{y} \psi(y) \right) dy.$$

Let $m \geq 1$ be the positive integer in the statement of the theorem. We will assume that the Goursat data $\psi \in C^m(\mathbf{R})$ satisfying $\psi(w_1) = 0$, for all $w_1 \geq -\delta$. Using Lemma 4.2, it suffices to consider the regularity of η in a neighborhood of the umbilic point $(0,0)$. Assuming that $|w_1| + |w_2| \ll \delta$,

$$\eta(w_1, w_2) = \int_\xi^{-\delta} \mathcal{R}(w_1, w_2; y, 0) \left(\psi'(y) - \frac{\tilde{\mathcal{A}}_2(\pi)}{y} \psi(y) \right) dy. \tag{4.46}$$

Let $k > 0$ be a large integer to be fixed below. Using (4.46), we write

$$\begin{aligned} \eta(w_1, w_2) &= I_1 + I_2, \\ I_1(w_1, w_2) &= \int_\xi^{-\delta} \mathcal{R}_{sing}(w_1, w_2; y, 0) \left(\psi'(y) - \frac{\tilde{\mathcal{A}}_2(\pi)}{y} \psi(y) \right) dy, \\ I_2(w_1, w_2) &= \int_\xi^{-\delta} \mathcal{R}_{regular}(w_1, w_2; y, 0) \left(\psi'(y) - \frac{\tilde{\mathcal{A}}_2(\pi)}{y} \psi(y) \right) dy. \end{aligned}$$

Using (4.11) and (4.45), we have

$$\mathcal{R}_{regular}(w_1, w_2; y, 0) = \sum_{n=N_0+1}^{\infty} \alpha(\theta) a_n(\theta) \left(\frac{w_1^2 + w_2^2}{y^2}\right)^{\frac{n - \tilde{\mathcal{A}}_2(\pi)}{2}}.$$

Here $N_0 = n_0(k) + |\tilde{\mathcal{A}}_2(\pi)| + k + 1$ is defined as in (4.45). Therefore, we obtain

$$I_2(w_1, w_2) = \sum_{n=N_0+1}^{\infty} \left(\int_\xi^{-\delta} (-y)^{\tilde{\mathcal{A}}_2(\pi) - n} \left(\psi'(y) - \frac{\tilde{\mathcal{A}}_2(\pi)}{y} \psi(y) \right) dy \right) \alpha(\theta) a_n(\theta) (w_1^2 + w_2^2)^{\frac{n - \tilde{\mathcal{A}}_2(\pi)}{2}}.$$

Then it is easy to check that, for $k>0$ large enough, $I_2 \in C^m([w_1^-,0]\times[0,w_2^+])$ and its derivatives satisfy

$$\max_{0\le j+l\le m}\sup_{(w_1,w_2)\in K}|w_1^{-j}w_2^{-l}\partial_{w_1}^j\partial_{w_2}^l I_2|\le C.$$

It remains to consider I_1. We require ψ to satisfy certain zero moment conditions to make $I_1\equiv 0$. Using (4.11) and (4.45) again, and denoting $\tilde r^2=w_1^2+w_2^2$,

$$\begin{aligned}I_1(w_1,w_2)&=\int_\xi^{-\delta}\mathcal{R}_{sing}(w_1,w_2;y,0)\ (-y)^{\tilde{\mathcal{A}}_2(\pi)}\left((-y)^{\tilde{\mathcal{A}}_2(\pi)}\psi(y)\right)'\ dy,\\&=\tilde r^{-\tilde{\mathcal{A}}_2(\pi)}\alpha(\theta)\sum_{n=0}^{N_0}a_n(\theta)\tilde r^n\Phi_n(\xi,\delta;\psi),\end{aligned}$$

where $\Phi_n(\xi,\delta;\psi)=(n-2\tilde{\mathcal{A}}_2(\pi))\int_\xi^{-\delta}(-y)^{-n-1+\tilde{\mathcal{A}}_2(\pi)}\psi(y)dy$.

Therefore,

$$I_1\equiv 0$$

provided $\psi\in\mathcal{M}$, where

$$\mathcal{M}=\left\{\psi\in C^m(\mathbf{R})\ \middle|\ \begin{array}{l}\psi(y)=0\qquad\forall y\ge-\delta,\\ \displaystyle\int_\xi^{-\delta}(-y)^{-n-1+\tilde{\mathcal{A}}_2(\pi)}\psi(y)dy=0,\ n\in\{0,1,\cdots,N_0\}\end{array}\right\}.$$

This defines the subspace $\mathcal{M}$ and completes the proof of the theorem.

We now turn to investigate the regularity of entropies constructed in Theorem 4.7 as functions of the state variables u and v. This regularity is needed in our use of compensated compactness to prove the convergence of approximate solutions to (2.1).

In many strictly hyperbolic systems, such as the system of elasticity, the map $\mathcal{J}:(u,v)\to(w_1,w_2)$ is C^2. Therefore, C^2 regularity of entropies in (u,v)-coordinates is a direct consequence of Theorem 4.7. For nonstrictly hyperbolic systems, the coincidence of eigenvalues λ_1 and λ_2 usually means that the geometry of the wave curves is very singular at the umbilic point. Thus, $\mathcal{J}$ is usually not C^2 and additional work is necessary to get C^2 regularity for $\eta=\eta(u,v)$.

We now restrict ourselves to region IV of the system with quadratic flux form (2.5), i.e., where a and b satisfy $\Delta=-32b^4+b^2(27+36(a-2)-4(a-2)^2)+4(a-2)^3>0$. Using Proposition 3.1, we have

Proposition 4.4. *Consider the system with quadratic flux form* (1.1) – (1.2) *with* $\Delta>0$. *Near the umbilic point* $(u,v)=(0,0)$, *the derivatives of the Riemann invariants satisfy the following estimates:*

$$w_i=\mathcal{O}(1),\quad i=1,2,$$

$$\partial_u^m\partial_v^n w_i=\mathcal{O}\Big(\frac{w_i}{|\tilde v|^{m+n}|w_1w_2|^n}\Big),\quad 1\le m+n\le 2. \tag{4.47}$$

Proof. The Riemann invariants are of the form

$$\begin{aligned} w_i(u,v) &= (-1)^i \tilde{v}^\beta exp\{-\beta \int_0^{\tilde{\alpha}} H_j(\tilde{\alpha}) d\tilde{\alpha}\} \\ &= \mathcal{O}(1). \end{aligned}$$

We have

$$\begin{aligned} \partial_{\tilde{u}} w_i &= -\beta \frac{w_i}{\tilde{v}} H_j(\tilde{\alpha}) \equiv \frac{w_i}{\tilde{v}} \mathcal{M}_{10}(\alpha, \alpha_0), \\ \partial_{\tilde{v}} w_i &= \beta \frac{w_i}{\tilde{v}} (1 + \tilde{\alpha} H_j(\tilde{\alpha})) \equiv \frac{w_i}{\tilde{v}} \mathcal{M}_{01}(\alpha, \alpha_0), \end{aligned}$$

where

$$\begin{aligned} \frac{-\mathcal{M}_{10}}{\beta} &= \frac{2\alpha_0(b\alpha+1) - (a-1)\alpha - b - (-1)^j sign(\alpha_0(\alpha_0+\tilde{\alpha}))\sqrt{Q(\tilde{\alpha})}}{-2\alpha(b\alpha+1) + (a-1)\alpha - b + (-1)^j sign(\alpha_0(\alpha_0+\tilde{\alpha}))\sqrt{Q(\tilde{\alpha})}} \frac{\alpha_0 - \alpha}{\alpha_0^2 + 1} \\ &= \mathcal{O}(\frac{|\alpha - \alpha_0|}{|\alpha| + 1}), \end{aligned}$$

and

$$\frac{M_{01}}{\beta} = 1 + \tilde{\alpha} M_{10} = \mathcal{O}(\frac{|\alpha - \alpha_0|}{|\alpha| + 1}).$$

Therefore,

$$\begin{aligned} \partial_{\tilde{u}} w_i &= \mathcal{O}(\frac{w_i}{\tilde{v}}), \\ \partial_{\tilde{v}} w_i &= \mathcal{O}(\frac{w_i}{\tilde{v}}). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \partial^2_{\tilde{u}} w_i &= \frac{w_i}{\tilde{v}^2} \mathcal{M}^2_{10} + \frac{w_i}{\tilde{v}^2} \partial_{\tilde{\alpha}} \mathcal{M}_{10} \\ &= \mathcal{O}(\frac{w_i}{\tilde{v}^2}). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \partial_{\tilde{u}} \partial_{\tilde{v}} w_i &= \frac{w_i}{\tilde{v}^2} (\mathcal{M}_{10} \mathcal{M}_{01} - \mathcal{M}_{10} - \tilde{\alpha} \partial_{\tilde{\alpha}} \mathcal{M}_{10}) \\ &= \mathcal{O}(\frac{w_i(|\alpha| + 1)}{\tilde{v}^2 |\alpha - \alpha_0|}) \\ &= \mathcal{O}(\frac{w_i}{\tilde{v}^2 |w_1 w_2|}), \end{aligned}$$

and

$$
\begin{aligned}
\partial_{\tilde{v}}^2 w_i &= \frac{w_i}{\tilde{v}^2}(\mathcal{M}_{01}^2 - \mathcal{M}_{01} - \tilde{\alpha}\mathcal{M}_{10} - \tilde{\alpha}^2\partial_{\tilde{\alpha}}\mathcal{M}_{10}) \\
&= \mathcal{O}(\frac{w_i(|\alpha|^2+1)}{\tilde{v}^2|\alpha-\alpha_0|^2}) \\
&= \mathcal{O}(\frac{w_i}{\tilde{v}^2|w_1w_2|^2}).
\end{aligned}
$$

Since derivatives in the (u,v)-coordinates can be expressed in terms of linear combinations of derivatives in the $(\tilde{u},\tilde{v})$-coordinates, we obtained the desired estimates in the Proposition. This completes the proof.

Next, we recover from the proof of Theorem 4.7 certain fine estimates on the derivatives η in w_1 and w_2. These derivatives vanish to high order as they approach the umbilic point.

Proposition 4.5. *Consider* (2.5) *with* $\Delta > 0$. *Given any nonnegative integers* m,n,j,l *with* $m \geq j, n \geq l$, *there exist entropy functions constructed using the method in Theorem 4.7 which satisfy the following estimates near the umbilic point:*

$$\partial_{w_1}^j\partial_{w_2}^l\eta = \mathcal{O}(w_1^{m-j}w_2^{n-l}).$$

Proposition 4.5 is a direct corollary of Theorem 4.7. Now using the chain rule and combining the estimates in Propositions 4.4 and 4.5, we obtain

Theorem 4.8. *Consider* (2.5) *with* $\Delta > 0$. *Let* $\mathcal{J} : (u,v) \to (w_1,w_2)$ *denote the map from the state space to the Riemann invariants plane. Suppose that* η *is the solution to* (4.9) *as constructed in Theorem 4.7. Then we have*

$$\eta \circ \mathcal{J} \in C^2(\mathbf{R}^2).$$

Proof. The map $\mathcal{J}$ is C^2 outside a neighborhood of the origin (the umbilic point) in the (u,v)-plane. Therefore, it suffices to prove that η is regular near $(u,v) = (0,0)$, i.e., $(w_1,w_2) = (0,0)$.

We consider the derivative η_{vv}. Other derivatives of η with respect to (u,v) can be treated in a similar fashion. By the chain rule, we have

$$\eta_{vv} = \eta_{w_1w_1}(w_{1v})^2 + 2\eta_{w_1w_2}w_{1v}w_{2v} + \eta_{w_2w_2}(w_{2v})^2 + \eta_{w_1}w_{1vv} + \eta_{w_2}w_{2vv}.$$

We now take η to be an entropy function constructed in Theorem 4.7 which satisfies the estimates on derivatives in Proposition 4.5 with sufficiently large m and n. Furthermore, we have

$$w_j = (-1)^j\tilde{v}^\beta \ exp\{-\beta\int_0^{\tilde{\alpha}} \bar{H}_i(\tilde{\alpha})d\tilde{\alpha}\}, \quad j = 1,2,$$

where $(\tilde{u}, \tilde{v})$ are the rotated coordinates in Proposition 3.1.

Now, the chain rule and Proposition 4.3 give

$$
\begin{aligned}
|\eta_{vv}| &= \mathcal{O}(\frac{w_1^{m-2} w_2^{n-2}}{\tilde{v}^2}) \\
&\leq \mathcal{O}(w_1^{m-2-\frac{1}{2\beta}} w_2^{n-2-\frac{1}{2\beta}} exp\{\frac{1}{2\beta}\int_0^{\tilde{\alpha}} (H_1(\tilde{\alpha}) + H_2(\tilde{\alpha}))d\tilde{\alpha}\}) \\
&= \mathcal{O}(1), \quad \text{for } m, n \text{ large enough.}
\end{aligned}
$$

This completes the proof of the Theorem.

5. Parabolic Approximate Solutions and Young Measure

5.1. Parabolic Approximation

We consider parabolic approximations to the general system (1.1) by adding artificial viscosity. Recall that (1.1) together with initial data $U_0(x)$ takes the form

$$
\begin{aligned}
&\partial_t U + \partial_x F(U) = 0, \qquad x \in \mathbf{R}, t > 0, \\
&U(x,0) = U_0(x).
\end{aligned} \tag{5.1}
$$

Consider a sequence of approximate solutions U^ϵ of the associated parabolic system

$$
\begin{aligned}
&\partial_t U^\epsilon + \partial_x F(U^\epsilon) = \epsilon \partial_x (D \partial_x U^\epsilon), \qquad x \in \mathbf{R}, t > 0, \\
&U^\epsilon(x,0) = U_0(x),
\end{aligned} \tag{5.2}
$$

where $D \geq 0$ is a non-negative matrix (viscosity matrix) and $\epsilon > 0$ measures the amount of artificial viscosity in (5.2).

We are concerned with the convergence of U^ϵ to a weak solution U of (5.1) as $\epsilon \to 0^+$. A compactness framework theorem (Theorem 6.3) will be established in Section 6 to attain this goal. It will then be applied to the system with quadratic flux form (2.5). This is made possible by a reduction analysis of the Young measure (Theorem 6.2) using a large class of regular entropy functions (Theorem 4.7 and Proposition 4.4) constructed in Section 4. In this section, we dispense with certain preliminaries concerning existence and L^∞ a priori estimates on solutions to (5.2). For simplicity, we take $D = I$, the identity matrix.

Theorem 5.1. *Let c_1 and c_2 be constants. Let $\Omega_{c_1c_2} \equiv \{(u,v) \mid c_1 \leq w_1 \leq 0 \leq w_2 \leq c_2\}$. Suppose that the Riemann invariants w_1 and w_2 are quasi-convex on $\partial\Omega_{c_1c_2}$. That is, on $\partial\Omega_{c_1c_2}$,*

$$
\begin{aligned}
&\mathbf{r}_1^\top \cdot \nabla^2 w_2 \cdot \mathbf{r}_1 \geq 0, \\
&-\mathbf{r}_2^\top \cdot \nabla^2 w_1 \cdot \mathbf{r}_2 \geq 0.
\end{aligned}
$$

Then $\Omega_{c_1c_2}$ is an invariant region for (5.2), i.e., $U_0(x) \in \Omega_{c_1c_2}$, for all $x \in \mathbf{R}$, implies $U^\epsilon(x,t) \in \Omega_{c_1c_2}$, for all $x \in \mathbf{R}$, $t > 0$. Moreover, if (5.2) admits a family of such invariant regions which spans $\mathbf{R}^2$, then we obtain an a priori L^∞ bound for solutions U^ϵ to (5.2).

Theorem 5.1 is a straightforward consequence of the invariant region theorem (see [CCS]). Granting such an L^∞ a priori estimate, a local solution in t to (5.2) obtained by standard iteration arguments can be extended globally in $t > 0$.

5.2. Young Measure and Compensated Compactness

The Young measure representation (see [Ta]) for sequences of bounded functions in an appropriate space is an efficient tool for studying the limit behavior of the approximate solutions of nonlinear problems, especially for conservation laws because of the lack of regularity of the limit problems. By combining the Young measure representation with compensated compactness first introduced by Tartar and Murat [Ta, Mu], one can transfer the singular limit problem to the problem of solving some functional equations for the corresponding Young measure, that is, to studying the structure of the Young measure satisfying the functional equations. If one can solve these functional equations to clarify the structure of the Young measure, the limit behavior of corresponding sequences can be well understood. Therefore, the essential difficulty is how to solve these functional equations for the Young measure. This difficulty is overcome for some important systems of conservation laws (cf. [Di, Ch1, Ch2, DCL, Ka, Mo, Se]). In this section we review some results on Young measure and compensated compactness for subsequent development to solve general nonstrictly hyperbolic systems.

Theorem 5.2 ([Ta]). Suppose that $U^\epsilon : \mathbf{R}^2_+ \to \mathbf{R}^n$ is a sequence of bounded measurable functions

$$U^\epsilon(x,t) \in K, \qquad \text{a.e.} \tag{5.3}$$

for a bounded set K in $\mathbf{R}^n$. Then there exists a subsequence (still labeled U^ϵ) and a family of Young measures

$$\nu_{x,t}(\lambda) \in \text{Prob.}(\mathbf{R}^n),$$

such that

(1) for any continuous function g,

$$w^* - lim\ g(U^\epsilon(x,t)) =< \nu_{x,t}(\lambda), g(\lambda) >\equiv \int_{\mathbf{R}^n} g(\lambda) d\nu_{x,t}(\lambda);$$

(2) $U^\epsilon(x,t) \to U(x,t)$ a.e. if and only if $\nu_{x,t}$ is a Dirac mass

$$\nu_{x,t} = \delta_{U(x,t)}, \quad \text{for almost all } (x,t);$$

(3) $\nu_{x,t}$ satisfies

$$< \nu_{x,t}, \begin{vmatrix} \eta_1 & q_1 \\ \eta_2 & q_2 \end{vmatrix} >= \begin{vmatrix} < \nu_{x,t}, \eta_1 > & < \nu_{x,t}, q_1 > \\ < \nu_{x,t}, \eta_1 > & < \nu_{x,t}, q_2 > \end{vmatrix}, \qquad \text{a.e.} \tag{5.4}$$

provided that

$$\eta_i(U^\epsilon)_t + q_i(U^\epsilon)_x \subset \text{compact set in } H^{-1}_{\text{loc}}, \tag{5.5}$$

for continuous function pairs $(\eta_i, q_i), i = 1, 2$.

This theorem ensures the existence of the Young measure determined by the sequence of bounded functions. The second result indicates that the strong convergence of the sequence is equivalent to the one point structure of the support of the Young measure. Furthermore, the boundedness of U^ϵ automatically ensures that

$$\eta_i(U^\epsilon)_t + q_i(U^\epsilon)_x \in H^{-1}_{loc}, \tag{5.6}$$

for continuous function pairs $(\eta_i, q_i), i = 1, 2$. This theorem indicates that an extra condition of weak composite compactness for $\eta_i(U^\epsilon)_t + q_i(U^\epsilon)_x$ can give us very useful information for the Young measure. These theorems provide a framework by which one can prove strong convergence of the sequence $U^\epsilon(x, t)$ satisfying (5.3) and (5.4) by deducing

$$\nu_{x,t}(\lambda) = \delta_{w^*-lim U^\epsilon(x,t)}(\lambda)$$

from the functional equations (5.5) for some continuous function pairs.

The following compactness embedding theorems are useful for obtaining the condition (5.4) for conservation laws.

Theorem 5.3. *Let $1 < q \le p < r < \infty$. Then*

$$(\textit{compact set of } W^{-1,q}_{loc}) \cap (\textit{bounded set of } W^{-1,r}_{loc}) \subset (\textit{compact set of } W^{-1,p}_{loc}).$$

The proof of Theorem 5.3 can be found in [DCL, Ch2].

Theorem 5.4. *([Mu]). The embedding of the positive cone of $W^{-1,p}$ in $W^{-1,q}$ is completely continuous for $q < p$.*

Theorem 5.3 indicates that compactness in $W^{-1,q}_{loc}$ coupled with boundedness in $W^{-1,r}_{loc}$ yields compactness in $W^{-1,p}_{loc}$. Theorem 5.4 says that the uniformly lower (or upper) bound in the dual sense in $W^{-1,p}$ of the sequence in $W^{-1,q}$ leads to compactness in $W^{-1,q}$, $q < p$.

5.3 The Dissipation Measures $\eta(U^\epsilon)_t + q(U^\epsilon)_x$

Consider a sequence of viscosity approximate solutions $\{U^\epsilon\}_{\epsilon>0}$ to (5.2). Now suppose that η_* is a C^2 strictly convex entropy for (5.1), and that $U_0(x)$ tends to a constant state $\bar{U}$ as $|x| \to \infty$ and $U_0 - \bar{U} \in L^2 \cap L^\infty$. In the case of (2.5) we may choose η_* to be $u^2 + v^2$. Multiplying (5.2) by $\nabla\eta_*(U^\epsilon) - \nabla\eta_*(\bar{U})$, a standard integration by parts argument gives the estimate

$$\epsilon \int_0^\infty \int_{-\infty}^\infty |U^\epsilon{}_x|^2 \, dx \, dt \le C,$$

where C depends on U_0.

Consider any C^2 entropy-entropy flux pair (η, q) for the system (5.1). In the case of (2.5), such pairs were constructed in Section 4. Multiplying (5.2) by $\nabla\eta(U^\epsilon)$, and integrating by parts, we get, after using the L^∞ bound on U^ϵ, the boundedness of $\nabla^2\eta$ on compact sets and a standard application of Theorem 5.4 [Mu], a weak compactness estimate for the dissipation measure

$$\eta(U^\epsilon)_t + q(U^\epsilon)_x \subset \text{compact set in } H^{-1}_{loc}. \tag{5.7}$$

By the L^∞ bound of U^ϵ, as $\epsilon \to 0^+$, U^ϵ (or at least some subsequence) converges in the weak star topology in L^∞. Then, by Theorem 5.2 of [Ta], there exists a family of probability measures $\nu_{x,t}$, the Young measures, that describe weak convergence in the following way. For any continuous function g, we have

$$w^* - lim_{\epsilon\to 0^+} g(U^\epsilon(x,t)) = \int g(\lambda)\ d\nu_{x,t}(\lambda) \equiv < \nu_{x,t}, g > .$$

Let (η, q) and $(\bar\eta, \bar q)$ be two pairs of C^2 entropy-entropy flux. They satisfy compactness conditions in H^{-1} as stated above. Then, by Theorem 5.2 of [Ta] again, we get the commutation relation:

$$< \nu, \eta\bar q - \bar\eta q > = < \nu, \eta >< \nu, \bar q > - < \nu, \bar\eta >< \nu, q >, \tag{5.8}$$

where we have dropped the subscript (x,t) on $\nu_{x,t}$.

6. Compactness Framework for Approximate Solutions

Now we establish a structure framework for the Young measure, which are determined by the approximate solutions to general nonstrictly hyperbolic system (1.1) with an isolated umbilic point $P = (\bar w_1, \bar w_2)$:

$$\lambda_1(P) = \lambda_2(P),$$

in Riemann coordinates. Then we conclude a corresponding compactness framework for approximate solution sequences to the system (1.1).

First, we list some basic assumptions on the structure of (1.1) for the framework theorems that we will state:

$(H1)$ There exist Riemann coordinates (w_1, w_2) such that

$w_1 \le 0 \le w_2$ and the umbilic point $P = (0,0)$;

$(H2)$ $$\frac{\lambda_{i w_j}}{\lambda_2 - \lambda_1} = \frac{\mathcal{A}_i(\frac{w_2}{w_1})}{w_2 - w_1} = \frac{\tilde{\mathcal{A}}_i(\theta)}{w_2 - w_1}, \quad \theta \in [\frac{\pi}{2}, \pi],$$

where $tan\theta = \dfrac{w_2}{w_1}$, $\tilde{\mathcal{A}}_i(\theta)$ is real analytic in $\theta \in (0, \dfrac{3\pi}{2})$, and $i \ne j,\ i,j = 1,2.$

Remark. The assumption (H1) is natural for many such systems. The condition (H2) is satisfied by the system (2.5) (see Section 3). Moreover, the analyticity conditions on $\tilde{\mathcal{A}}_i$ in (H2) implies that $||\mathcal{A}_i||_{L^\infty(\mathbf{R}^1)} < +\infty$, which is a natural assumption.

Theorem 6.1. *Assume that (1.1) satisfies conditions (H1)-(H2). Suppose that a Young measure $\nu \in Prob.(\mathbf{R}^2)$ satisfies*

$$\text{supp } \nu(w_1, w_2) \subset [w_1^-, w_1^+] \times [w_2^-, w_2^+], \tag{6.1}$$

$$< \nu, \eta_1 q_2 - \eta_2 q_1 >=< \nu, \eta_1 >< \nu, q_2 > - < \nu, \eta_2 >< \nu, q_1 >, \tag{6.2}$$

for all C^2 Goursat entropy-entropy flux pairs $(\eta_i, q_i), i = 1, 2,$ as constructed in Section 4 for the systems (1.1) with an isolated umbilic point P. Then

$$\text{supp } \nu(w_1, w_2) \subset \{ \ (w_1, w_2) \mid \lambda_{i w_i} = 0, \ \ i = 1, 2 \ \}.$$

Combining Theorem 6.1 with Theorem 5.2, we have

Theorem 6.2. *Suppose that (1.1) satisfies (H1) and (H2) and that*

$$\lambda_{i w_i} \neq 0, \ \ i = 1, 2, \ \ on \ \ (-\infty, 0] \times [0, \infty) - \{(w_1, w_2) | w_i = 0\}. \tag{6.3}$$

Suppose that $U^\epsilon(x, t)$ are measurable functions satisfying

$$\|U^\epsilon\|_{L^\infty} \le M < +\infty, \tag{6.4}$$

$$\eta(U^\epsilon)_t + q(U^\epsilon)_x \subset \text{compact set in } H^{-1}_{loc}, \tag{6.5}$$

for any C^2 Goursat entropy-entropy flux pair (η, q) as constructed in Section 4 for the systems (1.1) with an isolated umbilic point. Then

$$(w_1(U^\epsilon), w_2(U^\epsilon)) \longrightarrow (w_1(x, t), w_2(x, t)), \quad a.e. \tag{6.6}$$

where $w_1(U)$ and $w_2(U)$ are the Riemann invariants.

Remark 1. Theorem 6.1 and Theorem 6.2 provide a framework for the compactness of the sequence of the Riemann invariants, determined by the sequence of approximate solutions satisfying the conditions (6.3) and (6.4).

Remark 2. If $U^\epsilon(x, t)$ are generated by the viscosity method, the Lax-Friedrichs scheme, or the Godunov scheme, then the condition (6.5) for $U^\epsilon(x, t)$ can be established by a standard argument (see [Ch2, Di1]).

Now we prove Theorem 6.1 in several steps. Suppose that R is the minimal rectangle in the (w_1, w_2)-space containing the support of ν. There are two cases:

1. R does not contain the umbilic point $(0, 0)$;
2. R contains the umbilic point $(0, 0)$.

We will treat the case 2. The case 1 is similar and less complicated. We also assume that R is not a line segment parallel to any axis (this case can be dealt with using the method developed below), thus, $R = [w_1^-, 0] \times [0, w_2^+]$ for some $w_1^- < 0 < w_2^+$.

Let $\delta > 0$ denote the constant used in the construction of entropies in Section 4. Assume that δ is chosen a priori to satisfy $w_1^- < -\delta < 0$.

The proof will rely on the following Propositions and Lemmas.

Proposition 6.1. *Consider the entropy function constructed in Theorem 4.7. Assume that $w_1^* \in (w_1^-, -\delta)$ is a fixed constant and that the Goursat data ψ are supported either on the interval (w_1^-, w_1^*) (west type entropy with limit w_1^*) or on $(w_1^*, -\delta)$ (east type entropy with limit w_1^*). Then the entropy η and its flux q admit integral representations of the form*

$$\eta(w_1, w_2) = I(w_1, w_2)\psi(w_1) + \int_{w_1^*}^{w_1} J(w_1, w_2; y)\psi(y)\ dy,$$

$$q(w_1, w_2) = K(w_1, w_2)\psi(w_1) + \int_{w_1^*}^{w_1} L(w_1, w_2; y)\psi(y)\ dy,$$

where I, J, K, L are smooth as long as $w_1 < 0$ and $y < 0$. Moreover, $I \geq const. > 0$ on any domain bounded away from the origin in the w-plane.

Proof. For simplicity, we only derive the representation formula for η. The corresponding representation for q can be established in a similar way.

We start with the representation (4.44) and perform an integration by parts to shift the derivative onto the Riemann function $\mathcal{R}$:

$$\eta(w_1, w_2) = \int_{w_1^*}^{w_1} \mathcal{R}(w_1, w_2; y, 0)(-y)^{\tilde{\mathcal{A}}_2(\pi)} \left((-y)^{-\tilde{\mathcal{A}}_2(\pi)}\psi(y)\right)' dy$$

$$= [\mathcal{R}(w_1, w_2; y, 0)\psi(y)]_{y=w_1^*}^{w_1} - \int_{w_1^*}^{w_1} \left(\partial_y \mathcal{R}(w_1, w_2; y, 0) + \frac{\tilde{\mathcal{A}}_2(\pi)}{y}\mathcal{R}(w_1, w_2; y, 0)\right)\psi(y)dy$$

$$= I(w_1, w_2)\psi(w_1) + \int_{w_1^*}^{w_1} J(w_1, w_2; y)\psi(y)dy.$$

Therefore, using (4.8), we obtain

$$I(w_1, w_2) = \mathcal{R}(w_1, w_2; w_1, 0) = exp\{\int_0^{w_2} \frac{\mathcal{A}_1(\frac{\sigma}{w_1})}{w_1 - \sigma} d\sigma\},$$

$$J(w_1, w_2; y) = -\partial_y \mathcal{R}(w_1, w_2; y, 0) - \frac{\tilde{\mathcal{A}}_2(\pi)}{y}\mathcal{R}(w_1, w_2; y, 0).$$

Now, from the above expressions, since $w_2 \geq 0$, I is C^∞ in (w_1, w_2) whenever $w_1 < 0$. Moreover, from Theorem 4.6, $\mathcal{R}(w_1, w_2; y, 0)$ is C^∞ in (w_1, w_2, y) as long as $w_1, y < 0$. Thus, J shares the same smoothness property.

Finally, there exists a constant c_δ such that $I \geq c_\delta > 0$ on any compact domain with $w_1 \leq -\delta < 0$.

Lemma 6.2. *Let $[w_1^-, 0] \times [0, w_2^+]$ be the minimal rectangle containing $supp\nu$ and $w_1^- \leq \alpha < -\delta$. If there exists an east type entropy η_{ϵ_0} with limit $\alpha + \epsilon_0 < -\delta$ for some small $\epsilon_0 > 0$ such that $< \nu, \eta_{\epsilon_0} >> 0$, then*

$$supp\ \nu \subset \{\ (w_1, w_2)\ |\ \frac{\partial \lambda_1}{\partial w_1}(\alpha, w_2) = 0\ \}.$$

If we assume also that $\frac{\partial\lambda_1}{\partial w_1}(\alpha, w_2)$ *does not vanish, then supp* $\nu \cap \{(w_1, w_2)|w_2 \geq 0, w_1 = \alpha\}$ *is empty.*

The proof of Lemma 6.2 relies on constructing the trace of ν on the line $w_1 = \alpha$ using the integral representations of η and q in Proposition 6.1. The argument is quite standard and we refer the reader to [Se].

For simplicity, we assume from now on that $\frac{\partial\lambda_i}{\partial w_i} \neq 0$, whenever $w_i \neq 0$, $i = 1, 2$. The general case is only slightly more tedious. We remark that the quadratic system (2.5) with $\Delta > 0$ satisfies this assumption (see Theorem 3.2). With this assumption, ν can be reduced to a point mass in the Riemann plane below.

Next, using Proposition 6.1 and some properties of the kernel functions I, J, K, L, we can show

Proposition 6.3. *Fix* $\delta > 0$. *Let* w_1^* *satisfy* $w_1^- < w_1^* < -\delta < 0$. *If for all east type entropies* η *with limit* w_1^* *we have* $<\nu, \eta> = 0$, *then supp* $\nu \cap \{(w_1, w_2)|w_2 \geq 0, w_1^* \leq w_1 \leq -\delta\}$ *is empty.*

Proof. We sketch the main ideas. The arguments needed here are variations of the proof of a similar proposition in [Ka].

We begin by assuming that there exists α satisfying $w_1^* < \alpha < -\delta$ and such that $supp\nu \cap \{(w_1, w_2) \mid w_1 = \alpha\}$ is nonempty. We then derive a contradiction by constructing an east type entropy η with limit w_1^* such that $<\nu, \eta> > 0$.

Recall from Proposition 6.1 that an east type entropy η can be written in the form

$$\eta(w_1, w_2) = I(w_1, w_2)\psi(w_1) + \int_{w_1^*}^{w_1} J(w_1, w_2; y)\psi(y)\, dy.$$

Here ψ is the Goursat data in (4.43).

Let $N > 0$ be a sufficiently large number to be fixed later. Denote $S_l = \{(w_1, w_2) \mid w_2 \geq 0, \alpha - l \leq w_1 \leq \alpha + l\}$ the vertical strip of width l centered on the line $w_1 = \alpha$. Let ψ satisfy

$$\psi(y) = \begin{cases} N, & \text{for } y \in S_{\frac{1}{N^2}}\ , \\ O(1), & \text{for } y \notin S_{\frac{1}{N^2}+\frac{1}{N^3}}. \end{cases}$$

Moreover, we require that $\psi < N$ in $S_{\frac{1}{N^2}+\frac{1}{N^3}} - S_{\frac{1}{N^2}}$. Finally, we demand that $\psi \in \mathcal{M}$ (with $\mathcal{M}$ as defined in Theorem 4.7). It is not difficult to check that such ψ with high amplitude concentrated near α can be constructed.

Then, using the smoothness properties of I and J, and the fact that $I > c_\delta > 0$ when $(w_1, w_2) \in [w_1^-, -\delta] \times [0, w_2^+]$ (Proposition 6.3), we deduce

$$\eta(w_1, w_2) \begin{cases} \geq c'N, & \text{for } w_1 \in S_{\frac{1}{N^2}}\ , \\ = O(1), & \text{otherwise}\ , \end{cases}$$

where c' is a constant depending on c_δ.

However, since $supp\nu \cap \{(w_1, w_2) \mid w_1 = \alpha\}$ is nonempty, for N large enough, this implies that $< \nu, \eta > > 0$. This gives a contradiction.

Now, we will make use of Proposition 6.1 and Lemmas $6.2 - 6.3$ to reduce ν to a point mass. The reduction process will take several steps. First, we show that the support of ν must concentrate only at the four corners of R, i.e., ν is the sum of four delta functions. Then we further reduce the number of delta functions from four to one.

It turns out that after some detailed analysis, Lemmas 6.2- 6.3 allow us to concentrate the support of ν on the extreme left vertical edge of R and an arbitrarily narrow vertical strip containing the umbilic point.

Proposition 6.4. *supp* $\nu \subseteq \{(w_1, w_2)|0 \le w_2 \le w_2^+, \quad \text{and } w_1 = w_1^- \text{ or } -\delta \le w_1 \le 0\}$.

Proof. Let $\alpha \in (w_1^-, -\delta)$. Let $\epsilon > 0$ be sufficiently small such that $w_1^- < \alpha - \epsilon < \alpha + \epsilon < -\delta$. Assume that there exists an east type entropy η_e with limit $\alpha + \epsilon$ satisfying $< \nu, \eta_e > > 0$. Then $supp\nu \cap \{(w_1, w_2) \mid w_1 = \alpha\}$ is empty by virtue of Lemma 6.2.

If no such east type entropies η_e exist then, by Lemma 6.3, *supp* $\nu \cap \{(w_1, w_2)|w_2 \ge 0, \alpha + \epsilon \le w_1 \le -\delta\}$ is empty for all ϵ sufficiently small. This implies that *supp* $\nu \cap \{(w_1, w_2)|w_2 \ge 0, \alpha < w_1 \le -\delta\}$ is empty. Now suppose that $supp\nu \cap \{(w_1, w_2) \mid w_1 = \alpha\}$ is nonempty. Then since ν also has nonzero mass on the line $w_1 = w_1^-$, Lemma 6.2 indicates that, for any $\beta \in (w_1^-, \alpha)$, $supp\nu \cap \{(w_1, w_2) \mid w_1 = \beta\}$ is empty. So ν concentrates on the lines $w_1 = \alpha, w_1^-$ and the vertical strip of width δ between $w_1 = -\delta$ and $w_1 = 0$. If $supp\nu \cap \{(w_1, w_2) \mid w_1 = \alpha\}$ is nonempty, then we can once again use Lemma 6.2 to conclude that $supp\nu \cap \{(w_1, w_2) \mid w_1 = w_1^-\}$ is empty. But this contradicts the minimality of R. Therefore, we obtain the result we expected.

Combining these two cases, we obtain *supp* $\nu \subseteq \{(w_1, w_2)|0 \le w_2 \le w_2^+, \quad \text{and } w_1 = w_1^- \text{ or } -\delta \le w_1 \le 0\}$. This completes the proof of the proposition.

By using entropies of type north and south defined in a similar way and by performing a similar reduction process on horizontal line segments interior to the rectangle R, we conclude that *suppν* is concentrated on a horizontal strip of width δ and on the line $w_2 = w_2^+$. Since the proof is completely similar, we omit this argument but summarize the result as follows:

Proposition 6.5. *supp* $\nu \subseteq \{(w_1, w_2)| \ 0 \le w_1 \le w_1^-, \quad \text{and } w_2 = w_2^+ \text{ or } \delta \ge w_2 \ge 0\}$.

Propositions 6.4 and 6.5 imply that the support of ν is in fact concentrated on the square $\{(w_1, w_2)|0 \le w_2 \le \delta, -\delta \le w_1 \le 0\}$, and the points $(w_1^-, 0)$, $(0, w_2^+)$, and (w_1^-, w_2^+). Now, $\delta > 0$ is small but arbitrarily fixed. Let $\delta \to 0$. We obtain

Proposition 6.6. *The support of* ν *is concentrated at the four corners of* R*, i.e.* $(0,0)$, $(0, w_2^+)$, $(w_1^-, 0)$, *and* (w_1^-, w_2^+).

Knowing that the Young measure ν is the sum of four weighted delta functions at the corners of R, we reduce ν further. Let P_i, $i = 1, 2, 3, 4$, denote the corners of R, $P_1 = (0, w_2^+), P_2 = (0, 0), P_3 = (w_1^-, 0), P_4 = (w_1^-, w_2^+)$.

By Proposition 6.6,

$$\nu = \sum_{1 \le i \le 4} \beta_i \delta_{P_i},$$

where

$$\beta_i \ge 0, \quad i = 1, 2, 3, 4, \quad \text{and} \sum_{1 \le i \le 4} \beta_i = 1.$$

To reduce ν further, we apply a variant of Theorem 6.1 in [Se]:

Lemma 6.7. *Suppose that $\beta_i > 0$ for all $i = 1, 2, 3, 4$. Then*

$$\begin{aligned} &\partial_{w_1}\lambda_1 = 0, \ \textit{on the line segments } P_1P_4,\ P_2P_3; \\ &\partial_{w_2}\lambda_2 = 0, \ \textit{on the line segments } P_1P_2,\ P_3P_4. \end{aligned}$$

Now, by assumption, $\partial_{w_1}\lambda_1$ and $\partial_{w_2}\lambda_2$ are nonzero in the interior of the segments P_1P_4 and P_2P_3, and P_1P_2 and P_3P_4, respectively. This contradicts the conclusion of the above proposition. Thus we have

Lemma 6.8. *There exists an i, $1 \le i \le 4$, such that $\beta_i = 0$.*

Therefore, ν is only a sum of at most three Dirac masses. Further reduction of ν can be achieved by using Goursat entropies with appropriate support. Finally, we conclude that the Young measure ν is a point mass on the (w_1, w_2)-plane. This completes the proof of Theorem 6.1.

7. Convergence of the Viscosity Method

In this section, we apply the compactness framework theorem to the system with quadratic flux form with $\Delta > 0$. For any (even large) initial data, we prove the strong convergence of approximate solution sequences constructed by the viscosity method to weak entropy solutions of (2.5) for all positive time. As a corollary, we obtain the global existence of weak solutions to the Cauchy problem of (2.5) with large data in L^∞.

We first verify L^∞ a priori estimates for these approximate solution sequences. We then apply the compactness framework (Theorem 6.2) together with Theorems 4.7-4.8 on the existence of regular entropies. To this end, the principle of invariant regions will be used for the viscosity approximate sequence. We remark that the Riemann problem for (2.5) has not been solved before by purely analytical methods. Using the result we obtained here, one can deduce existence of solutions to the Riemann problem on arbitrary compact sets in the (x, t)-plane.

Consider the Cauchy problem

$$
\begin{aligned}
&\partial_t U + \partial_x (dC(U)) = 0, \\
&U(x,0) = U_0(x), \\
&C(U) = \frac{1}{2}(\frac{1}{3}au^3 + bu^2 v + uv^2), \\
&\Delta > 0.
\end{aligned} \tag{7.1}
$$

Using the convexity properties of the $\mathbf{R}_j$ curves or the explicit form of w_i, we obtain

Theorem 7.1. *Consider the system with quadratic flux form* (2.5) *with* $\Delta > 0$. *Then for any pair of constants,* $c_1 < 0 < c_2$, $\Omega_{c_1 c_2}$ *and* $\Omega_{c_1 c_2} \cap \mathcal{I}$, *with* $\mathcal{I}$ *as defined in Theorem 5.1, are convex invariant regions for the parabolic systems with viscosity associated with* (2.5).

Using Theorem 7.1, we obtain an L^∞ a priori bound for viscosity approximate solutions U^ϵ.

Theorem 7.2. *Consider the Cauchy problem* (7.1) *with Cauchy data* $U_0 \in \mathcal{I}$, *and its associated viscosity approximation* $\{U^\epsilon\}_{\epsilon>0}$. *Suppose that there is a constant* $\bar{U}$ *such that* $U_0 - \bar{U} \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$. *Then, for any* $\epsilon > 0$, $U^\epsilon(x,t)$ *is well-defined for all* (x,t) *and moreover,* $|U^\epsilon(x,t)| \le C||U_0||_{L^\infty}$, *for some* $C < \infty$.

Combining Theorem 7.2 and Theorem 6.2 with Theorem 4.8, we obtain

Theorem 7.3. *Consider the viscosity approximation* $\{U^\epsilon\}_{\epsilon>0}$ *of the Cauchy problem* (7.1) *with Cauchy data* $U_0 \in \mathcal{I}$. *Suppose that* $\Delta > 0$ *and that there is a constant* $\bar{U}$ *such that* $U_0 - \bar{U} \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$. *Then, as* $\epsilon \to 0^+$, *there exists a subsequence of* $U^\epsilon(x,t)$ *that converges a.e.* (x,t) *to a global entropy weak solution of the hyperbolic system* (7.1).

The strong convergence of a subsequence of the viscosity approximations $\{U^\epsilon\}_{\epsilon>0}$ follows directly from (5.7), Theorem 7.1, Theorem 6.2, and the one-to-one correspondence between the region $\mathcal{I}$ in the U-plane and $\{w : w_1 \le 0 \le w_2\}$ in the w-plane. For any convex entropy η, we multiply (5.2) by $\nabla\eta(U^\epsilon)$ and obtain

$$
\eta(U^\epsilon)_t + q(U^\epsilon)_x = \epsilon\eta(U^\epsilon)_{xx} - (U^\epsilon_x)^\top \nabla^2\eta U^\epsilon_x \le \epsilon\eta(U^\epsilon)_{xx}. \tag{7.2}
$$

Then we multiply (7.2) by any nonnegative test function $\phi(x,t) \in C^1_0(\mathbf{R}^2_+)$ and integrate by part

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty (\eta(U^\epsilon)\phi_t + q(U^\epsilon)\phi_x)\, dxdt + \int_{-\infty}^\infty \eta(U_0(x))\phi(x,0)dx \\
\ge -\int_0^\infty \int_{-\infty}^\infty \eta(U^\epsilon)\phi_{xx} dxdt.
\end{aligned} \tag{7.3}
$$

Taking a convergent subsequence of U^{ϵ_k} in (7.3) and letting $k \to +\infty$, we obtain that the limit function $U(x,t)$ satisfies

$$\int_0^\infty \int_{-\infty}^\infty (\eta(U)\phi_t + q(U)\phi_x)\, dxdt + \int_{-\infty}^\infty \eta(U_0(x))\phi(x,0)dx \geq 0,$$

for any convex entropy η and any nonnegative test function $\phi(x,t) \in C_0^1(\mathbf{R}_+^2)$. This means that $U(x,t)$ is an entropy solution.

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