Hyperbolic Conservation Laws with Umbilic Degeneracy II

Gui-Qiang Chen[1]
Department of Mathematics, Northwestern University
2033 Sheridan Road, Evanston, IL 60208-2730, USA
Pui Tak Kan[2]
Department of Mathematical Sciences, Indiana University-Purdue University
Indianapolis, IN 46202, USA

Abstract

We continue to study hyperbolic systems of conservation laws with umbilic degeneracy. We further extend our compactness framework established earlier to other canonical classes of quadratic flux systems with an isolated umbilic point. With the aid of this compactness framework, we establish the compactness of solution operators and the long-time behavior of entropy solutions in L^∞ with large initial data, and we prove the convergence of the viscosity method, as well as the Lax-Friedrichs scheme and the Godnuov scheme, for a canonical class of nonlinear hyperbolic systems with umbilic degeneracy.

Key Words. Conservation laws, nonstrict hyperbolicity, umbilic points, compactness framework, viscosity method, global entropy solutions, compactness of solution operators, large-time behavior

AMS(MOS) Subject Classifications. Primary: 35L65, 35L80, 35D05; Secondary: 65M12, 35Q05

1. Introduction

We are concerned with the quadratic gradient systems

$$\partial_t U + \partial_x(\nabla C(U)) = 0, \tag{1.1}$$

where

$$C(U) = \frac{1}{2}(\frac{1}{3}au^3 + bu^2v + uv^2), \tag{1.2}$$

and a and b are real parameters with $a \neq 1 + b^2$.

These systems are nonstrictly hyperbolic, and the two eigenvalues of the systems coincide at the isolated point $U = 0$ in the state space. Such points are referred to as umbilic (degenerate) points. An umbilic point is hyperbolic if the Hessian of $C(U), \nabla^2 C(U)$, at the point is diagonalizable. This class of systems is generic in the sense that all quadratic systems with umbilic hyperbolic degeneracy can be transformed into systems in this class, while such quadratic flux functions determine the local behavior of hyperbolic singularity near the umbilic point. Moreover, for any general nonlinear system with umbilic hyperbolic degeneracy, at each

[1]Email address: gqchen@math.northwestern.edu
[2]Email address: kan@math.iupui.edu, ptkan@math.northwestern.edu

umbilic degenerate point, the wave curves of each family of elementary waves are tangential to the corresponding curves of one of the systems in (1.1) under a suitable linear transformation. In fact, the quadratic flux system resulting from truncating the Taylor expansion of the nonlinear flux function at any hyperbolic umbilic point can be transformed into a corresponding system in (1.1) by a linear coordinate transformation.

Systems with umbilic degeneracy arise in magnetohydrodynamics, multiphase flows in porous media, and elasticity. For systems in three space dimension, Lax [La1] showed that systems with $2\,(mod\,4)$ equations cannot be strictly hyperbolic in all spatial directions. The same conclusion holds for the systems with $\pm 2, \pm 3, \pm 4\,(mod\,8)$ equations (see [FR]). The study of plane waves for such systems along degenerate directions then reduces to the study of degenerate systems in one space dimension for which (1.1) is one of the simplest generic examples. The Riemann problem for (1.1) has been of intense interests in recent years. Solutions of the Riemann problem require the use of nonstandard shock waves, and the geometry of the elementary wave curves are extremely complex. This differs markedly from the classical strictly hyperbolic case. The nonlinear stability of these nonstandard waves, their admissibility via the viscous profile criterion, and the associated analytical issues such as the bifurcation of wave curves are also of obvious interests. For detailed survey and references, we refer the readers to [CK1,GM,IMPT,IT,KSZ1,LZ,SSMP]. The initial boundary value problems for some of these quadratic systems are also discussed in [KSZ1-2].

In [CK1], we developed a general compactness framework for the systems with general fluxes with isolated umbilic points in the state space satisfying structural assumptions (H1) and (H2) (see Theorem 2.1 of Section 2). Under this compactness theorem, any sequence of L^∞ bounded approximate solutions, generated by the viscosity method, the Godunov scheme [Go], or the Lax-Friedrichs scheme [La2], converges to a corresponding entropy solution of the system when the viscosity or the mesh size goes to zero. A prototype of such systems is the quadratic flux systems (1.1). In [CK1], the compactness framework was applied to a certain parameter range (Region IV) of (1.1), and the strong convergence of approximate solutions and the global existence of entropy solutions were obtained for large initial data in L^∞.

In this paper, we continue our investigation of systems (1.1) in the other parameter ranges (Regions I–IV). We study the analytical and geometrical properties of the elementary wave curves. These provide a basic understanding of the classification of systems (1.1) and their different behavior in the four regions of parameters a and b. Among other consequences, we also obtain invariant domains for viscous and finite-difference approximate solutions to (1.1). We then present a careful construction of the Riemann invariants, taking into account their singular behavior and the fact that they are not uniquely defined, and make an optimal choice maximizing certain analytical regularities of the coefficients of the entropy equation. Next, we investigate the monotonicity of the wave speeds in Riemann coordinates. In contrast to the strictly hyperbolic case, this monotonicity is not equivalent to the genuine nonlinearity of the systems but depends upon subtle cancellations which are implicit in the choice of the Riemann invariants. In fact, systems (1.1) are not always genuinely nonlinear. We then turn to the study of the analytical regularity of the coefficients of the entropy equation. This involves very detailed analyses and is crucial in verifying the structural assumptions (H1) and (H2) in the L^∞ Compactness Framework established in [CK1]. Finally, using this compactness framework theorem, we establish the existence and qualitative behavior of entropy solutions. In particular, we obtain the compactness of solution operators and large-time behavior of entropy solutions in L^∞ in one of the invariant domains of (1.1) for L^∞ periodic initial data. To our best knowledge, this seems to be the first demonstration of such compactness for systems with *hyperbolic* umbilic degeneracy. We also show the convergence of the viscosity method for canonical subclasses of the quadratic flux systems (1.1).

This paper is organized as follows. In Section 2, we recall the L^∞ Compactness Framework in [CK1] for general systems with isolated umbilic points. We also present several theorems characterizing the qualitative and structural behavior of the quadratic gradient systems (1.1) in order to verify the requirements of the Compactness Framework. In Section 3, we study the unsymmetric case in Region III. We analyze the rarefaction curves, construct Riemann invariants, and study their singularities. We clarify the relationship between the choice of the construction and analytical properties of the functions Γ_j's which are the ratios of Riemann invariants. With these, we give detailed analyses of the monotonicity of the wave speeds in Riemann coordinates and the analytic regularities of the coefficients of the entropy equation. In Section 4, we give corresponding results for the symmetric case in Regions III and IV. The statements of the results are similar to those in Section 3 but the technical proofs are different. We indicate the essential differences in the technical details. In Section 5, we apply the results in Sections 3 and 4 to verify the conditions in the L^∞ Compactness Framework for general Region III and the symmetric case in Region IV. We also obtain invariant domains for viscous and finite-difference approximate solutions to (1.1) in Regions I–IV. We then proceed to obtain the compactness of viscous and finite-difference approximate solutions, and the existence of global entropy solutions in Regions III and IV. We also establish the compactness of solution operators and the large-time asymptotics of entropy solutions in L^∞ with periodic initial data. In Section 6, we summarize briefly analyses of the structure of systems (1.1) in Regions I and II, highlighting the differences in the technical proofs when compared to those in Sections 3 and 4. Finally, we conclude Section 6 by giving the asymptotic decay of periodic entropy solutions in L^∞ in these regions.

2. Quadratic Systems, Entropy, and Compactness Framework

2.1. Quadratic Systems

Consider a quadratic gradient system in (1.1). The eigenvalues of the system are

$$\lambda_i = \frac{1}{2}\left((a+1)u + bv + (-1)^i\sqrt{((a-1)u+bv)^2 + 4(bu+v)^2}\right), \quad i = 1, 2, \tag{2.1}$$

and the corresponding eigenvectors are

$$\mathbf{r}_i \equiv (\mathbf{r}_{i1}, \mathbf{r}_{i2})^\top = ((a-1)u + bv + (-1)^i\sqrt{((a-1)u+bv)^2 + 4(bu+v)^2}\ ,\ 2(bu+v))^\top. \tag{2.2}$$

It is then obvious that, as long as $a \neq 1 + b^2$, $\lambda_1 = \lambda_2 \iff (u, v) = (0, 0)$, so that $(0, 0)$ is the unique umbilic point.

The j^{th} family of rarefaction curves $\mathbf{R}_j$ is defined as the family of integral curves of the vector field given by $\mathbf{r}_j$. Therefore, $\mathbf{R}_j$ is defined by the following ordinary differential equation:

$$\frac{du}{dv} = \frac{(a-1)u + bv + (-1)^j\sqrt{((a-1)u+bv)^2 + 4(bu+v)^2}}{2(bu+v)}. \tag{2.3}$$

The change in convexity of the rarefaction wave curves in the (u, v)-plane depends on the location and number of zeros of the cubic polynomial

$$h(\alpha) = -b\alpha^3 + (a-2)\alpha^2 + 2b\alpha + 1, \qquad \alpha = \frac{u}{v}.$$

The discriminant of $h(\alpha)$ is given by $\Delta = -32b^4 + b^2(27 + 36(a-2) - 4(a-2)^2) + 4(a-2)^3$. Thus, $\Delta = 0$ gives a new boundary in the (a, b)-plane which distinguishes different wave-curve geometries. This corresponds to the division between Region III and Region IV in [SS1]. When $\Delta < 0$, $h(\alpha)$ has three real zeros α_0, α_1, and α_2. When $\Delta > 0$, $h(\alpha)$ has only one real zero, which corresponds to Region IV. We refer the readers to [CK1] for the details of these analyses.

In this paper, for concreteness, we shall mainly restrict ourselves to the case $b > 0$, and

$$(u, v) \in \mathcal{I} \equiv \{(u, v) \,|\, v - \frac{1}{\alpha_0} u \geq 0\}.$$

The symmetric case $b = 0$ for Regions I–IV is analyzed briefly in subsections by following the discussion for the unsymmetric case of the corresponding regions.

2.2. Entropy

Consider a general 2×2 hyperbolic system for $U = (u, v)$ with a general flux function $F(U)$:

$$\partial_t U + \partial_x F(U) = 0, \qquad F(U) = (f(U), g(U)). \tag{2.4}$$

A pair of scalar functions $(\eta(u, v),\ q(u, v))$ is called an entropy-entropy flux pair of (2.4) if

$$\partial_t \eta(u(x,t), v(x,t)) + \partial_x q(u(x,t), v(x,t)) = 0,$$

for any smooth solution $(u(x,t), v(x,t))$ of (1.1). It is easy to check that this happens if and only if η and q satisfy the compatibility condition

$$\nabla\eta \nabla F = \nabla q. \tag{2.5}$$

Eliminating q, we get a second-order equation, the entropy equation,

$$g_u \eta_{uu} + (g_v - f_u)\eta_{uv} - f_v \eta_{vv} + (g_{uu} - f_{uv})\eta_u + (g_{uv} - f_{vv})\eta_v = 0. \tag{2.6}$$

Equation (2.6) is a linear hyperbolic equation whose characteristic variables turn out to be the Riemann invariants. A simple calculation gives its characteristic form:

$$\eta_{w_1 w_2} + \frac{\lambda_{2w_1}}{\lambda_2 - \lambda_1}\eta_{w_2} - \frac{\lambda_{1w_2}}{\lambda_2 - \lambda_1}\eta_{w_1} = 0. \tag{2.7}$$

The quadratic flux systems (1.1) admit entropy functions that are homogeneous polynomials in the physical variables u and v of arbitrarily high degrees. We remark that the simple function $\eta_*(u, v) = u^2 + v^2$ is a strictly convex entropy of (1.1) for all a and b with corresponding entropy flux q_*. This function plays a special role in obtaining the H^{-1} compactness of the entropy dissipation measures for approximate solution sequences.

Assume that the system under consideration has the following structure:

$(H1)$ There exist Riemann coordinates (w_1, w_2) such that $w_1 \leq 0 \leq w_2$ and the umbilic point $P = (0, 0)$ is in the coordinates;

$(H2)$ $$\frac{\lambda_{i w_j}}{\lambda_2 - \lambda_1} = \frac{\mathcal{A}_i(\frac{w_2}{w_1})}{w_2 - w_1} = \frac{\tilde{\mathcal{A}}_i(\theta)}{w_2 - w_1}, \quad \theta \in [\frac{\pi}{2}, \pi], \quad i \neq j,\ i, j = 1, 2,$$

where $tan\theta = \dfrac{w_2}{w_1}$, and $\tilde{\mathcal{A}}_i(\theta)$ is real analytic in $\theta \in (0, \dfrac{3\pi}{2})$.

Then, under the assumptions (H1) and (H2), the entropy equation for the quadratic flux systems (1.1) takes the form

$$\eta_{w_1w_2}+\frac{\mathcal{A}_2(\frac{w_2}{w_1})}{w_2-w_1}\eta_{w_2}-\frac{\mathcal{A}_1(\frac{w_2}{w_1})}{w_2-w_1}\eta_{w_1}=0, \tag{2.8}$$

in the domain $w_1 \le 0 \le w_2$, where $\tilde{\mathcal{A}}_i(\theta)=\mathcal{A}_i(tan\theta)$ are defined for $\theta\in[\frac{\pi}{2},\pi]$ and can be analytically extended in $\theta\in(0,\frac{3\pi}{2})$.

2.3. L^∞ Compactness Framework

Here, we recall the following L^∞ Compactness Framework for general systems (2.4) satisfying the structural assumptions $(H1)$ and $(H2)$ (see Theorem 6.2 in [CK1]).

Theorem 2.1. *[CK1]. Let (2.4) satisfy the assumptions $(H1)$ and $(H2)$, and*

$$\lambda_{iw_i}\neq 0. \tag{2.9}$$

Suppose that, given any nonnegative integers m,n,j,l with $m\ge j, n\ge l$, $m+n\le 2$, any entropy functions satisfying the estimates

$$\partial_{w_1}^j\partial_{w_2}^l\eta=\mathcal{O}(w_1^{m-j}w_2^{n-l}) \tag{2.10}$$

near the umbilic point, are C^2 in (u,v). Assume that a sequence of measurable functions $U^\epsilon(x,t)$ satisfy that

$$||U^\epsilon||_{L^\infty}\le C, \tag{2.11}$$

and

$$\partial_t\eta(U^\epsilon)+\partial_x q(U^\epsilon) \quad \textit{is compact in} \quad H^{-1}_{loc}, \tag{2.12}$$

for any entropy pair (η,q) satisfying (2.10). Then

$$(w_1(U^\epsilon(x,t)),w_2(U^\epsilon(x,t)))\to(w_1(x,t),w_2(x,t)), \quad a.e.\ (x,t), \quad as\ \epsilon\to 0.$$

In the following sections, we will prove that the quadratic systems and the viscous (or finite-difference) approximate solutions satisfy the assumptions of the framework with large L^∞ initial data. The fact that system (1.1) in different parameter regions satisfy the compactness framework in Theorem 2.1 is checked via the following sequence of theorems characterizing the qualitative and structural behavior of (1.1). Various analyses and estimates presented in subsequent sections will provide the ingredients for proving these theorems. Theorems 2.2, 2.4, and 2.5 hold for both the unsymmetric case ($b\neq 0$) and the symmetric case ($b=0$), although they require different analyses. Theorem 2.3 is concerned with the unsymmetric case ($b\neq 0$) of (1.1); and similar results hold for the symmetric case (see Sections 4 and 6.2).

Theorem 2.2. *Consider systems (1.1) in Regions I–III. Let $\mathcal{J}:(u,v)\to(w_1,w_2)$ denote the map from the state space to the plane of Riemann invariants. Suppose that η is an entropy function satisfying (2.10) near the umbilic point in Theorem 2.1. Then*

$$\eta\circ\mathcal{J}\in C^2(\mathbf{R}^2). \tag{2.13}$$

Theorem 2.3. *Consider systems (1.1) in the case $b \neq 0$ (the unsymmetric case).*

(a) For Region III, i.e., $a > 1 + b^2$ and $\Delta < 0$,

$$\lambda_{iw_i} \neq 0, \tag{2.14}$$

whenever $(u, v) \in \mathcal{I} - \{(-1)^i v > 0, \frac{u}{v} = \alpha_0\}$, or, equivalently, when $(w_1, w_2) \in \mathcal{J}(\mathcal{I}) - \{(w_1, w_2) \,|\, w_i = (-1)^i \infty\}$.

(b) For Region II, i.e. $\frac{3}{4}b^2 < a < 1 + b^2$,

$$\lambda_{iw_i} \neq 0, \tag{2.15}$$

whenever $\tilde{v} \geq 0$ except possibly at the umbilic point $(u, v) = (0, 0)$.

(c) For Region I, i.e. $a < \frac{3}{4}b^2$,

$$\lambda_{iw_i} \neq 0, \tag{2.16}$$

whenever $(u, v) \in \mathcal{I} - \{(a-1)u + bv = \frac{3b(a-3) \pm \sqrt{D}}{6a}(bu + v)\}$ with $D = 12(\frac{3}{4}b^2 - a)(a+3)^2$.

Theorem 2.4. *Consider the quadratic flux systems (1.1) in Regions I–III in the case $b \neq 0$. Then the coefficients $\mathcal{A}_i = \tilde{\mathcal{A}}_i(\theta), i = 1, 2$, of the entropy equation, which are well defined for $\theta \in [\frac{\pi}{2}, \pi]$, can be extended to be real analytic in $\theta \in (0, \frac{3\pi}{2})$.*

Theorem 2.5. *Consider the quadratic flux systems* (1.1)*. The following domains for the corresponding regions are convex invariant domains for the Riemann solutions and the parabolic approximate solutions of* (1.1)*.*

(1) *Region IV ($\Delta > 0$): $\Omega_{c_1 c_2} \equiv \{(u, v) \,|\, c_1 \leq w_1 \leq 0 \leq w_2 \leq c_2\}$, and $\Omega_{c_1 c_2} \cap \mathcal{I}$, for any pair of constants, $c_1 < 0 < c_2$.*

(2) *Region III: For any pair of constants, $c_3 < 0 < c_4$,*

$$\{(u, v) \,|\, c_3 \leq w_1(u, v) \leq 0 \leq w_2(u, v) \leq c_4, \alpha_0 v \leq u \leq \alpha_2 v, u \geq 0\},$$
$$\{(u, v) \,|\, c_3 \leq w_1(u, v) \leq 0 \leq w_2(u, v) \leq c_4, \alpha_0 v \leq u \leq \alpha_1 v, u \leq 0\},$$

and

$$\{(u, v) \,|\, c_3 \leq w_1(u, v), w_2(u, v) \leq c_4, \alpha_1 v \leq u \leq \alpha_2 v, v \geq 0\}.$$

(3) *Regions I and II: Domains*

$$\{(u, v) \,|\, \alpha_0 v \leq u \leq \alpha_2 v, u \geq 0\} \cap \mathcal{I}, \quad \{(u, v) \,|\, \alpha_0 v \leq u \leq \alpha_1 v, u \leq 0\} \cap \mathcal{I},$$

and

$$\{(u, v) \,|\, \alpha_1 v \leq u \leq \alpha_2 v, v \geq 0\} \cap \mathcal{I}.$$

3. Unsymmetric Case for Region III

Consider a hyperbolic system of conservation laws with quadratic flux in (1.1). In this section, we focus on Region III where $\Delta < 0, b \neq 0$. We study the Riemann invariants of the system in (1.1). We also study the genuine nonlinearity and the monotonicity of $\lambda_i, i = 1, 2$, as a function of Riemann invariants $w_j, j = 1, 2$, since the map $\mathcal{J} : (u, v) \to (w_1, w_2)$ is neither C^1 nor globally invertible in general.

3.1. Riemann Invariants, C^2 Entropy, and Verification of (H1)

The Riemann invariants $w_j = w_j(u, v), j = 1, 2$, are defined as the functions that are constants along any rarefaction wave curves of the i^{th} family $\mathbf{R}_i$ where $i \neq j$. On the domains where w_j is differentiable, it is easy to check that $\mathbf{r}_i \cdot \nabla w_j = 0, i \neq j$, since $\mathbf{R}_i$ curves are integral curves of the vector field $\mathbf{r}_i$.

The j^{th} family of rarefaction curves $\mathbf{R}_j$ is defined as the family of integral curves of the vector field given by $\mathbf{r}_j$. Therefore, $\mathbf{R}_j$ defined by (2.3) can be rewritten in the form

$$\frac{du}{dv} = \frac{(a-1)\alpha + b + (-1)^j sign(v)\sqrt{((a-1)+b\alpha)^2 + 4(b+\alpha)^2}}{2(b\alpha+1)}, \quad v \neq 0, \tag{3.1}$$

where $\alpha = \frac{u}{v}$. Set $\tilde{u} = u + \frac{1}{\alpha_0}v, \quad \tilde{v} = v - \frac{1}{\alpha_0}u, \quad \tilde{\alpha} = \frac{\tilde{u}}{\tilde{v}}$, with α_0 the smallest zero of $h(\alpha)$. Then (3.1) becomes

$$\frac{du}{dv} = \frac{(a-1)(\alpha_0\tilde{\alpha}-1) + b(\tilde{\alpha}+\alpha_0) + (-1)^j sign(\alpha_0\tilde{v})\sqrt{Q(\tilde{\alpha})}}{2(b(\alpha_0\tilde{\alpha}-1)+\tilde{\alpha}+\alpha_0)},$$

where $Q(\tilde{\alpha}) = ((a-1)(\alpha_0\tilde{\alpha}-1) + b(\alpha_0+\tilde{\alpha}))^2 + 4\left(b(\alpha_0\tilde{\alpha}-1)+\alpha_0+\tilde{\alpha}\right)^2.$

For $(u, v) \in \mathcal{I}$, by a simple calculation, (3.1) becomes

$$\frac{d\tilde{\alpha}}{d\tilde{v}} = \frac{1}{\tilde{v}}\left(-\frac{1}{\alpha_0+\tilde{\alpha}} + H_j(\tilde{\alpha}, sign(\alpha_0))\right)^{-1} \equiv \bar{H}_j(\tilde{\alpha}, sign(\alpha_0))^{-1} \tag{3.2}$$

with

$$H_j(\tilde{\alpha}, sign(\alpha_0)) = -\frac{(1+\alpha_0^2)E_j(\tilde{\alpha})}{2(\alpha_0+\tilde{\alpha})D(\tilde{\alpha})}, \tag{3.3}$$

where

$$\begin{aligned} E_j(\tilde{\alpha}) =& -2b(\alpha_0\tilde{\alpha}-1)^2 + (a-3)(\alpha_0+\tilde{\alpha})(\alpha_0\tilde{\alpha}-1) + b(\alpha_0+\tilde{\alpha})^2 \\ & + (-1)^i sign(\alpha_0)(\alpha_0+\tilde{\alpha})\sqrt{Q(\tilde{\alpha})}, \\ D(\tilde{\alpha}) =& -b(\alpha_0\tilde{\alpha}-1)^3 + (a-2)(\alpha_0+\tilde{\alpha})(\alpha_0\tilde{\alpha}-1)^2 \\ & + 2b(\alpha_0+\tilde{\alpha})^2(\alpha_0\tilde{\alpha}-1) + (\alpha_0+\tilde{\alpha})^3. \end{aligned} \tag{3.4}$$

The Riemann invariants are of the form

$$\begin{aligned} w_i(u,v) &= (-1)^i\tilde{v}^\beta \exp\{-\beta\int_0^{\tilde{\alpha}} \bar{H}_j(\tilde{\alpha}, sign(\alpha_0))d\tilde{\alpha}\} \\ &= (-1)^i\tilde{v}^\beta|\alpha_0+\tilde{\alpha}|^\beta exp\{-\beta\int_0^{\tilde{\alpha}} H_j(\tilde{\alpha}, sign(\alpha_0))d\tilde{\alpha}\}, \quad i \neq j, i = 1, 2, \end{aligned} \tag{3.5}$$

for any constant $\beta \neq 0$.

We remark that the $w_j{}'s$ are not uniquely defined and that care must be taken in the definition to make sure that we have a single-valued function globally. We will also make the choice of β that guarantees the maximal regularity of certain quantities. We elaborate on this below. The advantage of this choice becomes clear when we study the entropy functions.

Consider the polynomial $h(\alpha)$. When $\Delta > 0$ (Region IV), $h(\alpha)$ has one unique real zero denoted as α_0, which has been analyzed in detail in [CK1]. We now focus on the case of Region III, where $\Delta < 0$ and $a > 1 + b^2$. Since $\Delta < 0$, $h(\alpha)$ has three real zeros. Denote by $\alpha_0 \le \alpha_1 \le \alpha_2$ the three real zeroes of $h(\alpha)$. We consider the local behavior of $\frac{w_j}{w_i}$ near the rays $\alpha = \alpha_l, l = 0, 1, 2$. For definiteness, we first focus on the ray $\alpha = \alpha_0$ and $\{(u, v)\,|\,\alpha = \alpha_0, u > 0\} = \{(u, v)\,|\,\tilde{v} = 0, \tilde{u} > 0\}$ from the side $\{(u, v)\,|\,\tilde{v} \ge 0\}$. The discussions for the other cases are similar. Then

$$\frac{w_j}{w_i} = \Gamma_j(\tilde{\alpha}, sign(\alpha_0)) = -\exp\{\beta \int^{\tilde{\alpha}} \left(\bar{H}_j(\tilde{\alpha}, sign(\alpha_0)) - \bar{H}_i(\tilde{\alpha}, sign(\alpha_0))\right) d\tilde{\alpha}\}. \tag{3.6}$$

Notice that, for $(u, v) \in \mathcal{I}$,

$$\begin{aligned} &\lim_{\substack{|\tilde{\alpha}|\to+\infty \\ (-1)^k\tilde{u}>0}} \tilde{\alpha}(\bar{H}_i(\tilde{\alpha}, sign(\alpha_0)) - \bar{H}_j(\tilde{\alpha}, sign(\alpha_0))) \\ &= (-1)^{j+k} sign(\alpha_0) \frac{\sqrt{((a-1)\alpha_0 + b)^2 + 4(b\alpha_0 + 1)^2}}{h'(\alpha_0)}. \end{aligned} \tag{3.7}$$

Therefore, the local behavior of $\frac{w_j}{w_i}$ near $\{(u, v)\,|\,\alpha = \alpha_0, (-1)^k u > 0\}$ is determined by

$$|\alpha - \alpha_0|^{(-1)^{j+k} sign(\alpha_0) \frac{\sqrt{((a-1)\alpha_0+b)^2+4(b\alpha_0+1)^2}}{h'(\alpha_0)} \beta}. \tag{3.8}$$

We conclude by similar analyses that the local behavior of $\frac{w_j}{w_i}$ near $\{(u, v)\,|\,\alpha = \alpha_l\}, l = 1, 2$, is determined by

$$|\alpha - \alpha_l|^{(-1)^j sign(\alpha_0) \frac{\sqrt{((a-1)\alpha_l+b)^2+4(b\alpha_l+1)^2}}{h'(\alpha_l)} \beta}, \quad i \ne j. \tag{3.9}$$

We have, in Region III,

$$-\frac{1}{b} < \alpha_0 < \alpha_1 < 0 < \alpha_2.$$

Set

$$\beta_l = \frac{sign(\alpha_0) h'(\alpha_l)}{\sqrt{((a-1)\alpha_l + b)^2 + 4(b\alpha_l + 1)^2}}, \quad l = 0, 1, 2.$$

Noting $-\frac{1}{b} < \alpha_0 < \alpha_1 < 0 < \alpha_2$ and $h(\pm\infty) = \mp\infty$, we have $sign(h'(\alpha_l)) = (-1)^l sign(h'(\alpha_0)), l = 1, 2$, and, therefore, $sign(\beta_l) = (-1)^l sign(\beta_0)$.

Then, if we choose

$$\beta = \beta_2,$$

the functions $\Gamma_j(\tilde{\alpha}, -1)$ are continuous in the domain $\mathcal{I}$ and real analytic on the subdomain $\{(u, v)|\tilde{v} \ge 0, tan^{-1}\alpha_0 < \frac{u}{v} \le tan^{-1}\alpha_2\}$ except on the rays $\alpha = \alpha_1, \alpha_2$; if we choose

$$\beta = sign\beta_2 |\beta_1|,$$

the functions $\Gamma_j(\tilde{\alpha},-1)$ are continuous in the domain $\mathcal{I}$ and real analytic in the subdomain $\{(u,v)\,|\,\tilde{v}>0, \pi+tan^{-1}\alpha_1 \le \frac{u}{v} \le \pi+tan^{-1}\alpha_0\}$ or $\{(u,v)\,|\,\tilde{v}>0, tan^{-1}\alpha_2 < \frac{u}{v} \le \pi+tan^{-1}\alpha_1\}$.

To investigate the regularity of entropies as functions of the state variables u and v, we need to estimate the regularity of the Riemann invariants as functions of the state variables u and v for Theorem 2.1. In many strictly hyperbolic systems, such as the system of elasticity, the map $\mathcal{J}:(u,v)\to(w_1,w_2)$ is C^2. Therefore, C^2 regularity of entropies in the (u,v)-coordinates is a direct consequence of that in the w-coordinates. For nonstrictly hyperbolic systems, the coincidence of eigenvalues λ_1 and λ_2 usually means that the geometry of the wave curves are very singular at the umbilic point. Thus, $\mathcal{J}$ is usually not C^2, and additional analysis is necessary to get C^2 regularity for $\eta=\eta(u,v)$.

We are now ready to present a proof of Theorem 2.2 for Region III. To this end, we first prove some estimates for the Riemann invariants and their derivatives.

Proposition 3.1. *Consider* (1.1) *in Region III, i.e.,* $\Delta<0$ *and* $a>1+b^2$. *Near the umbilic point* $(u,v)=(0,0)$, *the derivatives of the Riemann invariants satisfy the following estimates:*

$$
\begin{aligned}
&|w_i| \le C\left|\frac{|\alpha|+1}{\alpha-\alpha_0}\right|^{|\frac{\beta}{\beta_0}|}, \quad i=1,2,\\
&|\partial_u^m\partial_v^n w_i| \le C\frac{|w_i|}{|\tilde{v}|^{m+n+\delta}|w_1w_2|^{(m+n)\delta}}\left|\frac{\alpha-\alpha_0}{1+|\alpha|}\right|^{m+(m+n)\bar{\delta}}, \quad 1\le m+n\le 2,
\end{aligned}
\tag{3.10}
$$

where $\delta = max(|\frac{\beta_1}{\beta}|,|\frac{\beta_2}{\beta}|,|\frac{1}{2\beta}|)+1$ *and* $\bar{\delta}=|\frac{\beta}{\beta_0}|\delta$.

Proof. The proof is more complicated than the case in Region IV in [CK1] because, in Region III, $|w_1w_2|$ vanishes on $\alpha=\alpha_1,\alpha_2$ and $|w_1w_2|=\infty$ on $\alpha=\alpha_0$.

We have, from the local analyses of Γ_j following (3.7),

$$
C^{-1}\tilde{v}^{\beta}\left|\frac{\alpha-\alpha_0}{1+|\alpha|}\right|^{|\frac{\beta}{\beta_0}|}\left|\frac{\alpha-\alpha_1}{1+|\alpha|}\right|^{|\frac{\beta}{\beta_1}|}\left|\frac{\alpha-\alpha_2}{1+|\alpha|}\right|^{|\frac{\beta}{\beta_2}|} \le |w_i| \le C\left|\frac{1+|\alpha|}{\alpha-\alpha_0}\right|^{|\frac{\beta}{\beta_0}|}.
$$

Now

$$
\begin{aligned}
&\partial_{\tilde{u}}w_i = \frac{w_i}{\tilde{v}}\mathcal{M}_{10}(\alpha,\alpha_0), \quad \partial_{\tilde{v}}w_i = \frac{w_i}{\tilde{v}}(1+\tilde{\alpha}\mathcal{M}_{10}(\alpha,\alpha_0)),\\
&\mathcal{M}_{10} = \frac{2\alpha_0(b\alpha+1)-(a-1)\alpha-b-(-1)^j sign(\alpha_0(\alpha_0+\tilde{\alpha}))\sqrt{Q(\tilde{\alpha})}}{-2\alpha(b\alpha+1)+(a-1)\alpha-b+(-1)^j sign(\alpha_0(\alpha_0+\tilde{\alpha}))\sqrt{Q(\tilde{\alpha})}}\frac{\alpha_0-\alpha}{\alpha_0^2+1}.
\end{aligned}
$$

Since

$$
\begin{aligned}
|w_1w_2| &\le \tilde{v}^{2\beta}\left|\frac{\alpha-\alpha_0}{1+|\alpha|}\right|^{-|\frac{\beta}{\beta_0}|}\left|\frac{\alpha-\alpha_1}{1+|\alpha|}\right|^{|\frac{\beta}{\beta_1}|}\left|\frac{\alpha-\alpha_2}{1+|\alpha|}\right|^{|\frac{\beta}{\beta_2}|}\\
&\le C\left[\tilde{v}\left|\frac{\alpha-\alpha_1}{1+|\alpha|}\right|\left|\frac{\alpha-\alpha_2}{1+|\alpha|}\right|\right]^{\frac{1}{\delta}}\left|\frac{\alpha-\alpha_0}{1+|\alpha|}\right|^{-|\frac{\beta}{\beta_0}|},
\end{aligned}
$$

we obtain

$$
|\mathcal{M}_{10}| \le C\frac{(1+|\alpha|)|\alpha-\alpha_0|}{|\alpha-\alpha_1||\alpha-\alpha_2|} \le C\tilde{v}^{-\delta}\left|\frac{\alpha-\alpha_0}{1+|\alpha|}\right|^{1-\bar{\delta}}|w_1w_2|^{-\delta}.
$$

Then we have

$$|\partial_{\tilde{u}} w_i| \leq C \frac{|w_i|}{\tilde{v}^{1+\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-\bar{\delta}}, \quad |\partial_{\tilde{v}} w_i| \leq C \frac{|w_i|}{\tilde{v}^{1+\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+\alpha\alpha_0} \right|^{-\bar{\delta}}.$$

Further differentiation yields

$$\begin{aligned}
|\partial_{\tilde{u}}^2 w_i| &= |\frac{\mathcal{M}_{10}}{\tilde{v}} \partial_{\tilde{u}} w_i + \frac{w_i}{\tilde{v}^2} \partial_{\tilde{\alpha}} \mathcal{M}_{10}| \\
&\leq C\left(\frac{|w_i|}{\tilde{v}^{1+\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-\bar{\delta}}\right)\left(\frac{1}{\tilde{v}^{1+\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-\bar{\delta}}\right) + C\frac{|w_i|}{\tilde{v}^2} \mathcal{O}\left(\frac{(\alpha-\alpha_0)^2}{|\alpha-\alpha_1||\alpha-\alpha_2|}\right) \\
&\leq C\left(\frac{|w_i|}{\tilde{v}^{2+2\delta}|w_1 w_2|^{2\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{2-2\bar{\delta}}\right.
\end{aligned}$$

We also have

$$\begin{aligned}
|\partial_{\tilde{u}} \partial_{\tilde{v}} w_i| =& |\frac{-w_i}{\tilde{v}^2} \frac{\mathcal{M}_{10}}{\tilde{v}} \partial_{\tilde{v}} w_i + \frac{w_i}{\tilde{v}} \frac{-\tilde{\alpha}}{\tilde{v}} \partial_{\tilde{\alpha}} \mathcal{M}_{10}|, \\
\leq & C\frac{|w_i|}{\tilde{v}^2} \frac{1}{\tilde{v}^{\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-\bar{\delta}}) + C\frac{|w_i|}{\tilde{v}^{2+2\delta}|w_1 w_2|^{2\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-2\bar{\delta}}) \\
&+ C|\tilde{\alpha}| \frac{|w_i|}{\tilde{v}^{2+2\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{2-\bar{\delta}} \\
\leq & C\frac{|w_i|}{\tilde{v}^{2+2\delta}|w_1 w_2|^{2\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-2\bar{\delta}}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
|\partial_{\tilde{v}}^2 w_i| =& |-\frac{w_i}{\tilde{v}^2}(1+\tilde{\alpha}\mathcal{M}_{10}) + \frac{1}{\tilde{v}}(1+\tilde{\alpha}\mathcal{M}_{10})(\frac{w_i}{\tilde{v}})(1+\tilde{\alpha}\mathcal{M}_{10}) + \frac{w_i}{\tilde{v}}(\frac{-\tilde{\alpha}}{\tilde{v}})(\mathcal{M}_{10} + \tilde{\alpha}\partial_{\tilde{\alpha}}\mathcal{M}_{10})| \\
\leq & C\frac{|w_i|}{\tilde{v}^2} \frac{\tilde{\alpha}}{\tilde{v}^{\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-\bar{\delta}} + C\frac{|\tilde{\alpha}|^2 w_i}{\tilde{v}^{2+2\delta}|w_1 w_2|^{2\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{2-2\bar{\delta}} \\
&+ C\frac{|\tilde{\alpha}| w_i}{\tilde{v}^2} \left[\frac{1}{\tilde{v}^{\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{1-\bar{\delta}} + \frac{|\tilde{\alpha}|}{\tilde{v}^{2\delta}|w_1 w_2|^{\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{2-\bar{\delta}} \right] \\
\leq & C\frac{|w_i|}{\tilde{v}^{2+2\delta}|w_1 w_2|^{2\delta}} \left| \frac{\alpha - \alpha_0}{1+|\alpha|} \right|^{-2\bar{\delta}}.
\end{aligned}$$

The analysis is similar for higher values of m and n. We omit the details. This completes the proof of Proposition 3.1.

Now, using Proposition 3.1 and the chain rule, we easily prove (2.13). This completes the proof of Theorem 2.2.

3.2. Genuine Nonlinearity

We now turn to the investigation of the genuine nonlinearity in the sense of Lax [La3] for the quadratic systems (1.1). It turns out that genuine nonlinearity allows a breakdown of the case $a < 1 + b^2$ into two subcases. By definition, a system in (1.1) is genuinely nonlinear in the

j^{th} characteristic field at a point (u,v) if $\mathbf{r}_j \cdot \nabla\lambda_j \neq 0, i \neq j$, at (u,v). A calculation by using (2.1) and (2.2) shows that

$$\mathbf{r}_j \cdot \nabla\lambda_j = h_j(\zeta, y) \equiv a\zeta + 3by + (-1)^j \frac{a\zeta^2 + 3by\zeta + 2(a+3)y^2}{\sqrt{\zeta^2 + 4y^2}}, \tag{3.11}$$

where $\zeta = (a-1)u + bv$ and $y = bu + v$.

By careful analyses, we find that all possible points of $\mathbf{r}_j \cdot \nabla\lambda_j = 0$ are on the lines

$$\begin{cases} y = 0, & a > \frac{3}{4}b^2, \\ y = 0, \zeta = \frac{b(a-3)}{2a}y, & a = \frac{3}{4}b^2, \\ y = 0, \zeta = \frac{3b(a-3)\pm\sqrt{D}}{6a}y, & a < \frac{3}{4}b^2. \end{cases} \tag{3.12}$$

This shows that the curve $a = \frac{3}{4}b^2$ divides the region $\{(a,b) \,|\, a < 1 + b^2\}$ into two subregions according to a global change in loci of loss of genuine nonlinearity. This corresponds to the division between Region I and Region II in [SS1]. For the details of these analyses, see [CK1].

We now investigate the monotonicity of the wave speed λ_i in the variable w_i, which is the content of Theorem 2.3, part (a). This is important in the verification of condition (2.9) in the Compactness Framework (Theorem 2.1).

Using $\frac{w_{iv}}{w_{iu}} = -\frac{r_{j1}}{r_{j2}}$, we compute to get

$$\lambda_{iw_i} = \frac{(-1)^i}{w_{1u}w_{2v} - w_{2v}w_{1u}}(\lambda_{iu}w_{jv} - \lambda_{iv}w_{ju}) \quad = \left(\frac{\mathbf{r}_i \cdot \nabla\lambda_i}{r_{i2}}\right) v_{w_i}. \tag{3.13}$$

Now we carefully compute v_{w_i} in terms of $(\tilde{u}, \tilde{v})$ to obtain

$$v_{w_i} = -\frac{\alpha_0}{1+\alpha_0^2} \frac{1 + (\alpha_0 + \tilde{\alpha})\bar{H}_i}{\beta\tilde{v}^{\beta-1}(\bar{H}_1 - \bar{H}_2)} \exp\{\beta \int_0^{\tilde{\alpha}} \bar{H}_j(\tilde{\alpha}, sign(\alpha_0)) d\tilde{\alpha}\}. \tag{3.14}$$

Then further detailed computations give the important formula:

$$\begin{aligned} \lambda_{iw_i} =& (-1)^i \alpha_0 \frac{\tilde{v}}{w_i} \times \frac{g(\alpha)}{\sqrt{((a-1)\alpha + b)^2 + 4(b\alpha+1)^2}} \times \\ & \times \frac{\frac{\mathbf{r}_i \cdot \nabla\lambda_i}{v}}{-2\alpha(b\alpha+1) + (a-1)\alpha + b + (-1)^i sign(v)\sqrt{((a-1)\alpha+b)^2 + 4(b\alpha+1)^2}}. \end{aligned} \tag{3.15}$$

We are now ready to prove part (a) of Theorem 2.3.

The proof is similar to that of the case in Region IV by using (3.15). The essential new feature is that λ_{iw_i} can vanish on $\alpha = \alpha_0$, because there is a factor of w_i in the denominator in (3.15), and w_i becomes unbounded on a half-line of $\alpha = \alpha_0$ in Region III. Consider

$$k_i(\alpha, sign(v)) = -2\alpha(b\alpha+1) + (a-1)\alpha + b + (-1)^i sign(v)\sqrt{((a-1)\alpha+b)^2 + 4(b\alpha+1)^2}.$$

Since $(a-1)\alpha_k + b = (\alpha_k^2 - 1)\frac{b\alpha_k+1}{\alpha_k}$, from $h(\alpha_k) = 0, k = 0,1,2$, we have

$$-2\alpha_k(b\alpha_k+1) + (a-1)\alpha_k + b = -(1+\alpha_k^2)\frac{b\alpha_k+1}{\alpha_k} \begin{cases} > 0, & k = 0,1, \\ < 0, & k = 2, \end{cases}$$

and

$$((a-1)\alpha+b)|_{\alpha=-\frac{1}{b}} = -\frac{a-1-b^2}{b} < 0.$$

Therefore, $k_i(\alpha, sign(v))$ has zeros

$$\begin{cases} \alpha_0, \alpha_1, & \text{when } (-1)^i sign(v) < 0, \\ -\frac{1}{b}, \alpha_2, & \text{when } (-1)^i sign(v) > 0. \end{cases}$$

The function $g(\alpha)$ has no zero. The only zero of $\mathbf{r}_i \cdot \nabla\lambda_i$ is $\alpha = -\frac{1}{b}$ and is of first-order. The denominator

$$-2\alpha(b\alpha+1) + (a-1)\alpha + b + (-1)^i sign(v)\sqrt{((a-1)\alpha+b)^2 + 4(b\alpha+1)^2}$$

also has the first-order zero $\alpha = -\frac{1}{b}$ and therefore they cancel. Finally, when $\tilde{v} = 0$ (i.e. $\alpha - \alpha_0 = 0$), either w_i or

$$-2\alpha(b\alpha+1) + (a-1)\alpha + b + (-1)^i sign(v)\sqrt{((a-1)\alpha+b)^2 + 4(b\alpha+1)^2}$$

vanishes to the same order. Thus, we conclude that λ_{iw_i} never vanishes except possibly at the umbilic point $(0,0)$. This completes the proof of part (a) of Theorem 2.3.

3.3. Coefficients of the Entropy Equation and Verification of (H2)

In this section, we study the analytical properties of the coefficients in the entropy equation (2.7). A direct consequence of these analyses will be the verification of the structural assumptions $(H2)$ in the Compactness Framework (Theorem 2.1). We treat Regions I–IV together in this section.

Notice that

$$\lambda_{jw_i} = \frac{\mathbf{r}_i \cdot \nabla\lambda_j}{r_{i2}} v_{w_i}, \tag{3.16}$$

and

$$\mathbf{r}_i \cdot \nabla\lambda_j = \zeta - by + + \frac{(-1)^i}{\sqrt{\zeta^2+4y^2}}(\zeta^2 - 3by\zeta + 2(a-1)y^2), \quad i \neq j. \tag{3.17}$$

Therefore,

$$\frac{\lambda_{jw_i}}{\lambda_2-\lambda_1} = \frac{\mathbf{r}_i \cdot \nabla\lambda_j}{2r_{i2}\sqrt{\zeta^2+4y^2}} v_{w_i} \equiv \frac{T_j(\tilde{\alpha})}{w_i}, \tag{3.18}$$

where $T_j(\tilde{\alpha}) = \frac{\alpha-\alpha_0}{2\beta}\frac{\mathbf{r}_i\cdot\nabla\lambda_j}{\zeta^2+4y^2}\frac{g(\alpha)}{k_j(\alpha,sign(\alpha-\alpha_0))}$.

Proposition 3.2. *The coefficients of the entropy equation of systems (1.1) defined by*

$$\mathcal{A}_i = T_i(\tilde{\alpha})\frac{w_2-w_1}{w_j}, \quad i \neq j, i = 1, 2,$$

are real analytic in $\tilde{\alpha} \in \mathbf{R} \cup \{\pm\infty\}$.

Proof: Notice that

$$\mathcal{A}_j = T_j(\tilde{\alpha})\frac{w_2-w_1}{w_i} = (-1)^i T_j(\tilde{\alpha})(1-\Gamma_j(\tilde{\alpha})).$$

Therefore, it suffices to prove that T_j and $T_j\Gamma_j$ are both real analytic in $\tilde{\alpha} \in \mathbf{R} \cup \{\pm\infty\}$.

To show the desired real analyticity, we need only check that the functions $T_j(\tilde{\alpha})$ has no singularities in any finite domain, nor at infinity.

In Regions I and II, the factor $k_j(\alpha, sign(v))$ has zeros

$$\begin{cases} -\frac{1}{b}, \alpha_1, & \text{when } (-1)^j sign(v) < 0, \\ \alpha_0, \alpha_2, & \text{when } (-1)^j sign(v) > 0. \end{cases}$$

On the other hand, $\frac{(\alpha-\alpha_0)(\mathbf{r}_i\cdot\nabla\lambda_j)}{\zeta^2+4y^2}$ has at least two zeros:

$$\begin{cases} -\frac{1}{b}, & \text{when } (-1)^i sign(v)((a-1)\alpha+b)|_{\alpha=-\frac{1}{b}} < 0, \\ \alpha_0, & \text{always}, \end{cases}$$

and $g(\alpha)$ has always two zeros α_1 and α_2. Thus, since $a < 1 + b^2$ in Regions I-II, the zeros in the denominator are cancelled by those in the numerator and T_j has no singularities in α in any finite domain.

In Region III, $k_j(\alpha, sign(v))$ has zeros

$$\begin{cases} -\frac{1}{b}, \alpha_2, & \text{when } (-1)^j sign(v) > 0, \\ \alpha_0, \alpha_1, & \text{when } (-1)^j sign(v) < 0. \end{cases}$$

Since now $a > 1 + b^2$ in Region III, we still obtain the same cancellations and $T_j(\alpha)$ has no singularity in α in any finite domain.

In Region IV, the zeros α_1, α_2 do not exist, and $k_j(\alpha, sign(v))$ has zeros

$$\begin{cases} -\frac{1}{b}, & \text{when } (-1)^j sign(v) > 0, \\ \alpha_0, & \text{when } (-1)^j sign(v) < 0. \end{cases}$$

The same cancellations occur.

Finally, we observed from the formula for T_j that it is easy to check that T_j remains bounded as $\alpha \to \pm\infty$. We also note that T_j is a ratio of sums and products of polynomials and radicals in α. We conclude that T_j is real analytic in α for all $\alpha \in \mathbf{R} \cup \{\pm\infty\}$.

It remains to consider $T_j\Gamma_j$. Since $\Gamma_j\Gamma_i \equiv 1$ whenever $i \neq j$, we have

$$T_j(\tilde{\alpha})\Gamma_j(\tilde{\alpha}) = \frac{\alpha-\alpha_0}{2\beta}\frac{\mathbf{r}_i\cdot\nabla\lambda_j}{\zeta^2+4y^2}\frac{g(\alpha)}{k_j(\alpha, sign(\alpha-\alpha_0))}\Gamma_j(\tilde{\alpha}).$$

Using an analysis similar to the above, we conclude that $T_j\Gamma_j$ is real analytic in α for all $\alpha \in \mathbf{R} \cup \{\pm\infty\}$. This completes the proof of Proposition 3.2.

We will now use Proposition 3.2 to prove Theorem 2.4 in the unsymmetric case in Regions I–IV.

Using Proposition 3.2, we know that $\mathcal{A}_j = (-1)^i T_j(1-\Gamma_j), i \neq j$, are real analytic in $\alpha \in \mathbf{R} \cup \{\pm\infty\}$.

We first assume that (u, v) are restricted to one of the invariant domains, described in Theorem 2.5. The first step of the proof of Theorem 2.4 in this case is to obtain an analytic inversion of Γ_2. We have

$$\partial_\alpha\Gamma_2(\alpha, sign\alpha_0) = \beta\Gamma_2\frac{(1+\alpha_0^2)\sqrt{Q(\tilde{\alpha})}}{D(\tilde{\alpha})} \neq 0,$$

for all $\tilde{\alpha}$ corresponding to the interior of the invariant domain under consideration as long as all the $|\beta_i|'s$ in the invariant domain have the same value. The coincidence of the $|\beta_i|'s$ ensure that Γ_2 has possibly only first-order zeros at $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, which are cancelled by the zeros of $D(\tilde{\alpha})$. Therefore, we can invert to get $\tilde{\alpha}$ as an analytic function in $\frac{w_2}{w_1}$ in the appropriate range and then substitute into $\mathcal{A}_i$, which is analytic in $\tilde{\alpha} \in \mathbf{R}$.

It remains to show that $\mathcal{A}_i$ can be extended as an analytic function in θ across the boundary of the invariant domains. The argument is similar to the proof of Theorem 4.1 in [CK1]. The extension across $\tilde{\alpha} = \pm\infty$ is identical to that in [CK1] and we omit the details. To extend across $\tilde{\alpha}_l, l = 1, 2$, we use the fact that β is chosen such that all $|\beta_i|'s$ are identical to write

$$\Gamma_2 = \frac{w_2}{w_1} = tan\theta = |\tilde{\alpha} - \tilde{\alpha}_l| f(\tilde{\alpha} - \tilde{\alpha}_l),$$

where $f(s)$ is real analytic near $s = 0$. We then extend $tan\theta$ analytically as a function of $\tilde{\alpha} - \tilde{\alpha}_l$ across the point $\tilde{\alpha} = \tilde{\alpha}_l$ as follows:

$$tan\theta = (\tilde{\alpha} - \tilde{\alpha}_l) f(\tilde{\alpha} - \tilde{\alpha}_l) = \bar{\Gamma}_2(\tilde{\alpha} - \tilde{\alpha}_l) \equiv \begin{cases} \Gamma_2(\tilde{\alpha} - \tilde{\alpha}_l), & \tilde{\alpha} - \tilde{\alpha}_l < 0, \\ -\Gamma_2(\tilde{\alpha}_l - \tilde{\alpha}), & \tilde{\alpha} - \tilde{\alpha}_l \geq 0. \end{cases}$$

Then $tan\theta$ is analytic near $\tilde{\alpha} - \tilde{\alpha}_l = 0$. Finally, we check explicitly that $\partial_{\tilde{\alpha} - \tilde{\alpha}_l} \bar{\Gamma}_2 \neq 0$ near $\tilde{\alpha} - \tilde{\alpha}_l = 0$. Thus, $\tilde{\mathcal{A}}_i(\theta)$ can be analytically extended across $\theta = \frac{\pi}{2}, \pi$. We refer the readers to [CK1] for more details. This completes the proof of Theorem 2.4.

We now discuss the possibility of the equality of the three quantities $|\beta_0|, |\beta_1|, |\beta_2|$. Such a consideration is important in determining the maximal domain of analyticity of Γ_j given any particular choice of β. We recall that

$$|\beta_l| = \frac{|h'(\alpha_l)|}{\sqrt{[(a-1)\alpha_l + b]^2 + 4(b\alpha_l + 1)^2}}, \quad h(\alpha) = -b\alpha^3 + (a-2)\alpha^2 + 2b\alpha + 1.$$

For definiteness, we concentrate on $|\beta_1|$ and $|\beta_2|$. After careful algebraic manipulations, we find the following compatibility condition for the equality $|\beta_1| = |\beta_2|$:

$$0 = (a+1)^3(5-a) - (a+1)(a^2 + 11a + 20)b^2 + (2a^2 + a - 5)b^4; \; b \neq 0, a \neq -1.$$

For $b = 0$, $|\beta_1| = |\beta_2|$ but the above argument does not apply (see the next section for details).

Finally, we consider the example $a = \frac{3}{2}$. In this case, an explicit computation gives

$$h(\alpha) = (2b\alpha + 1)(1 - \frac{1}{2}\alpha^2), \quad \beta_0 = \frac{1 + 8b^2}{1 + b^2}, \quad \beta_{1,2} = \frac{2(\pm 2\sqrt{2}b - 1)}{3(1 \mp \sqrt{2}b)}.$$

Therefore, in general, $|\beta_0| \neq |\beta_1| \neq |\beta_2|$ for $b \neq 0$, and $\beta_1 = \beta_2$ for $b = 0$.

4. Symmetric Case in Regions III and IV

Consider the symmetric case $b = 0$ of (1.1), i.e. the flux takes the form:

$$F(U) = \nabla C(U) = \frac{1}{2}(au^2 + v^2, 2uv)^\top. \tag{4.1}$$

The unsymmetric case in Region IV ($\Delta > 0, b \neq 0$) was treated in [CK1]. In this section we analyze the symmetric case in Regions III and IV. We will study the Riemann invariants of (1.1) for this case. We will also study the genuine nonlinearity and the monotonicity of $\lambda_i, i = 1, 2$, as a function of Riemann invariants $w_j, j = 1, 2$. For simplicity, we restrict ourselves to the half-plane domain

$$\mathcal{I} \equiv \{(u, v) \,|\, v \geq 0\}.$$

This domain is also an invariant domain for the viscous and finite-difference approximate solutions of (1.1) (see Section 5). For the other half-plane, the situation is very similar.

4.1. Riemann Invariants, C^2 Entropy, and Verification of (H1)

In this case, the Riemann invariants are of the form

$$w_i(u, v) = (-1)^i v^\beta \exp\{-2\beta \int_0^\alpha \frac{d\alpha}{(a-3)\alpha + (-1)^j \sqrt{(a-1)^2\alpha^2 + 4}}\}, \quad i = 1, 2,\ i \neq j, \tag{4.3}$$

for any constant $\beta \neq 0$.

We consider and summarize below the local behavior of $\frac{w_j}{w_i},\ i \neq j,\ i, j = 1, 2$. Using (4.3),

$$\frac{w_j}{w_i} = \Gamma_j(\alpha) = -\exp\{(-1)^j \beta \int_0^\alpha \frac{\sqrt{(a-1)^2\alpha^2 + 4}}{(a-2)\alpha^2 + 1} d\alpha\}. \tag{4.4}$$

Region IV $(a > 2)$: $\frac{w_j}{w_i}$ are analytic in $|\alpha| \in [0, \infty]$, and $|\alpha|$ is analytic in $\frac{w_j}{w_i}$ in their defined domains.

It is easy to check that $\Gamma_j(\alpha)$ are analytic in $\alpha \in (-\infty, \infty)$. Now we check that $\Gamma_j(\alpha)$ are locally analytic near $\alpha \sim \pm\infty$, or $\sigma = \frac{1}{\alpha} \sim 0$. Notice that

$$\Gamma_j(\alpha) \sim |\sigma|^{(-1)^{i+k}\beta\frac{a-1}{a-2}}, \quad \text{as} \quad (-1)^k u > 0,$$

near $\sigma \sim 0$. Choose $\beta = \frac{a-2}{a-1}$. We have

$$\frac{w_j}{w_i} = Cexp\{(-1)^i \beta \int_0^\sigma \frac{sign(\sigma)}{\sigma} \frac{\sqrt{(a-1)^2 + 4\sigma^2}}{a - 2 + \sigma^2} d\sigma\} \equiv |\sigma|^{sign(\sigma) i} G(|\sigma|; sign(\sigma)),$$

near $\sigma \sim 0$ and $v \geq 0$, where C is some constant. Here the functions $G(\tau; \pm 1)$ are real analytic near $\tau = 0$.

Furthermoe, we have

$$\partial_\alpha \Gamma_j(\alpha) = (-1)^j \frac{a-2}{a-1} \frac{\sqrt{(a-1)^2\alpha^2 + 4}}{(a-2)\alpha^2 + 1} \Gamma_j(\alpha) \neq 0, \quad \text{for all } \alpha \in \mathbf{R}.$$

Therefore, we have $\alpha = \Lambda_j(\Gamma_j) = \Lambda_j(\frac{w_j}{w_i})$, where $\Lambda_j(\tau)$ is real analytic in $\tau \in \Gamma_j(\mathbf{R}) = (-\infty, 0)$.

Region III $(1 < a < 2)$: In this case,

$$\frac{w_j}{w_i} = -\exp\{(-1)^j \beta \int^\alpha \frac{\sqrt{(a-1)^2\alpha^2 + 4}}{(1 + \sqrt{2-a}\alpha)(1 - \sqrt{2-a}\alpha)} d\alpha\}.$$

Near $\alpha = \frac{1}{\sqrt{2-a}}$,

$$\frac{w_j}{w_i} \sim |\alpha - \frac{1}{\sqrt{2-a}}|^{(-1)^{j+1}\beta\frac{3-a}{2(2-a)}}.$$

If we choose $\beta = \beta_1 = \frac{2(2-a)}{3-a}$, then the ray $\{\alpha = \frac{1}{\sqrt{2-a}}, v > 0\}$ corresponds to $w_1 = 0$.

Similarly, near $\alpha = -\frac{1}{\sqrt{2-a}}$,

$$\frac{w_j}{w_i} \sim |\alpha - \frac{1}{\sqrt{2-a}}|^{(-1)^{j}\beta\frac{3-a}{2(2-a)}}.$$

If we choose $\beta = \beta_2 = \frac{2(2-a)}{3-a} \equiv \beta_1$, then the ray $\{\alpha = -\frac{1}{\sqrt{2-a}}, v > 0\}$ corresponds to $w_2 = 0$.

Near $\alpha = \pm\infty$, that is, $\sigma = \frac{1}{\alpha} = 0$,

$$\frac{w_j}{w_i} = -\exp\{(-1)^j\beta\int^{\sigma}(-\frac{1}{\sigma^2})\frac{\sqrt{(a-1)^2\frac{1}{\sigma^2}+4}}{1-(2-a)\frac{1}{\sigma^2}}d\sigma\} \sim |\sigma|^{(-1)^{i+k}\beta\frac{a-1}{a-2}}, \quad \text{when} \quad (-1)^k\sigma > 0.$$

If we choose $\beta = \beta_0 \equiv \frac{2-a}{a-1}$, then the ray $\{v = 0, u > 0\}$ corresponds to $w_2 = 0$. This is so because

$$\begin{aligned} w_i &\sim (-1)^i v^\beta |\sigma|^{\frac{\beta}{2-a}}, \quad \text{when } (-1)^j u > 0,\\ &\sim (-1)^i |u|^{\frac{\beta}{2-a}} v^{-1}, \quad \text{when } (-1)^j u > 0. \end{aligned}$$

We now check that $w_j = (-1)^j\infty$ on the half-ray $\{v = 0, (-1)^{j+1}u > 0\}$ with the choice $\beta = \beta_1 = \beta_2 = \frac{2(2-a)}{3-a}$. Indeed,

$$w_i(u,v) \sim \begin{cases} (-1)^i v^\beta |\sigma|^{-\frac{\beta}{2-a}}, \quad (-1)^j sign(\sigma) > 0,\\ (-1)^i |u|^\beta, \quad (-1)^j sign(\sigma) < 0. \end{cases}$$

A careful analysis using (4.3) now yields

Proposition 4.1. *Consider a symmetric system in* (1.1) *in Regions III and IV. Suppose that* $(u,v) \in \mathcal{I}$. *Near the umbilic point* $(u,v) = (0,0)$, *the derivatives of the Riemann invariants satisfy the following estimates:*

$$\begin{aligned} w_i =& \mathcal{O}(1), \quad i = 1,2,\\ |\partial_u^m \partial_v^n w_i| \le& \begin{cases} C(\frac{|w_i|}{|\tilde{v}|^{m+n}|\alpha - \frac{(-1)^j}{\sqrt{2-a}})}|, \quad a < 2,\\ C(\frac{|w_i|}{|\tilde{v}|^{m+n}}), \quad a > 2, \end{cases} \qquad 1 \le m+n \le 2. \end{aligned}$$

Combining Proposition 4.1 with the chain rule, we conclude Theorem 2.2 for the symmetric case.

4.2. Genuine Nonlinearity

We now turn to the investigation of the genuine nonlinearity in the sense of Lax [La3] for the symmetric quadratic systems (1.1).

We find that all possible points of $\mathbf{r}_j \cdot \nabla\lambda_j = 0$ are

$$\begin{cases} v = 0, & a \geq 0, \\ v = 0, v = \pm\frac{(1-a)\sqrt{-3a}}{|3+a|}, & a < 0. \end{cases} \tag{4.5}$$

On the line $v = 0$, $\mathbf{r}_j \cdot \nabla\lambda_j = 0$ if and only if $(-1)^j(a-1)u < 0$; on the line $v = \pm\frac{(1-a)\sqrt{-3a}}{|3+a|}$ for the case $a < 0, \mathbf{r}_j \cdot \nabla\lambda_j = 0$ if and only if $\pm(-1)^j a(a-1)(a^2-9)u < 0$. The point $a = 0$ divides the interval $-\infty < a < 1$ into two subregions according to a global change in loci of loss of genuine nonlinearity. This corresponds to the division between Region I and Region II in [SS1].

We are now concerned with the monotonicity of the wave speed λ_i in the variable w_i. This is important in the verification of the structural assumptions $(H1)$ and $(H2)$. Using (4.3) and (3.13), we obtain

$$\begin{aligned} \lambda_{iw_i} =& \frac{-2vw_j}{w_1 w_2} \frac{(-1)^j((a-2)\alpha^2+1)}{(a-3)\alpha + (-1)^i\sqrt{(a-1)^2\alpha^2+4}} \\ &\times \frac{a(a-1)\alpha\sqrt{(a-1)^2\alpha^2+4} + (-1)^i(a(a-1)^2\alpha^2 + 2(a+3))}{\beta(\sqrt{(a-1)^2\alpha^2+4})^2}. \end{aligned}$$

Using this formula, it can be checked that

For Region IV $(a > 2)$, we have

$$\lambda_{iw_i} \begin{cases} \neq 0, & \text{for all } |\alpha| < \infty, \text{and } \alpha = (-1)^i\infty, \\ = \mathcal{O}(\frac{1}{\alpha}), & \alpha \to (-1)^j\infty. \end{cases}$$

Therefore,

$$\lambda_{iw_i} \neq 0, \quad \text{for all} \quad (w_1, w_2) \in \mathcal{J}(I) - \{w_i = 0\}.$$

For Region III $(1 < a < 2)$, we have

$$\frac{1 + \sqrt{2-a}\alpha}{(a-3)\alpha - \sqrt{(a-1)^2\alpha^2+4}} \neq 0,$$

$$\frac{a(a-1)\alpha\sqrt{(a-1)^2\alpha^2+4} + (-1)^i(a(a-1)^2\alpha^2 + 2(a+3))}{\beta(\sqrt{(a-1)^2\alpha^2+4})^2} = \begin{cases} \mathcal{O}(1), & \alpha \to (-1)^i\infty, \\ \mathcal{O}(\frac{1}{\alpha^2}), & \alpha \to (-1)^j\infty, \end{cases}$$

and

$$\frac{v(1 - \sqrt{2-a}\alpha)}{w_i} \neq 0,$$

for all $(w_1, w_2) \in \mathcal{J}(I) - \{w_i = (-1)^i\infty\}$, or $(u, v) \in \mathcal{I} - \{v = 0, (-1)^i u < 0\}$. Therefore,

$$\lambda_{iw_i} \neq 0, \quad \text{for all } (w_1, w_2) \in \mathcal{J}(\mathcal{I}) - \{w_i = (-1)^i\infty, \text{or } w_i = 0\}.$$

4.3. Coefficients of the Entropy Equation and Verification of $(H2)$

Notice that

$$\mathbf{r}_i \cdot \nabla\lambda_j = (a-1)u + \frac{(-1)^j}{\sqrt{(a-1)^2\alpha^2+4}}\left((a-1)^2u^2 + 2(a-1)v^2\right), \quad i \neq j. \tag{4.16}$$

Therefore, from (4.6) and (3.6), we have

$$
\begin{aligned}
\mathcal{A}_i(\frac{w_2}{w_1}) &= \frac{\lambda_{jw_i}}{\lambda_2-\lambda_1}(w_2-w_1) = \frac{\mathbf{r}_i\cdot\nabla\lambda_j}{2\sqrt{(a-1)^2u^2+4v^2}}\frac{v_{w_i}(w_2-w_1)}{v} \\
&= (-1)^{i+j}(1-\Gamma_j(\alpha))\frac{(a-1)\alpha+\frac{(-1)^i(a-1)((a-1)\alpha^2+2)}{\sqrt{(a-1)^2\alpha^2+4}}}{2\beta\left((a-3)\alpha+(-1)^i\sqrt{(a-1)^2\alpha^2+4}\right)} \\
&\equiv T_i(\alpha)(1-\Gamma_j(\alpha)), \quad i\neq j.
\end{aligned}
$$

We know that $\Gamma_j(\alpha)$ is real analytic in $\alpha\in\mathbf{R}$. Moreover, the denominator of T_i has no zero in the range $|\alpha|<\infty$. Therefore, $\mathcal{A}_i$ is real analytic in $\alpha\in\mathbf{R}$.

Now we have

$$
\Gamma_j(\alpha)=\mathcal{O}(|\alpha|^{-sign(\alpha)i}), \qquad i\neq j, \text{ as } \alpha\to\infty,
$$

and

$$
\begin{aligned}
T_i(\alpha) &\sim \frac{-(a-1)\left((a-1)\alpha|\alpha|(1+\frac{2}{(a-1)^2\alpha^2}+\mathcal{O}(\frac{1}{\alpha^4}))+(-1)^i(a-1)\alpha^2\right)}{2\beta(a-1)|\alpha|\left((a-3)\alpha+(-1)^i(a-1)|\alpha|\right)}, \quad \text{as } |\alpha|\to\infty \\
&\sim \begin{cases} \mathcal{O}(1), & \alpha\to(-1)^i\infty, \\ \mathcal{O}(\frac{1}{\alpha^2}), & \alpha\to(-1)^j\infty, \ i\neq j. \end{cases}
\end{aligned}
$$

Therefore, for $i\neq j$, $T_i\Gamma_j$ is real analytic in $\alpha\in\mathbf{R}\cup\{\pm\infty\}$. Thus, we have shown that $\mathcal{A}_i$ is real analytic in $\alpha\in\mathbf{R}\cup\{\pm\infty\}$.

After arriving at this conclusion, an argument similar to the unsymmetric case yields Theorem 2.4 for the symmetric case.

5. Existence and Qualitative Behavior of Entropy Solutions

We first consider the behavior of entropy solutions in L^∞.

5.1. Compactness and Large-Time Asymptotics of Entropy Solutions

First we have

Theorem 5.1. *Any bounded entropy solution operator $S_tU_0(\cdot)=U(\cdot,t)$ with $U_0(x)$ in any invariant domain, described in Theorem 2.5, is compact in L^1_{loc} for $t>0$. In other words, the initial oscillations instantaneously cancel as time evolves.*

Given any uniformly bounded oscillatory initial data sequence $U_0^\epsilon(x)$ in any invariant domain described in Theorem 2.5, we denote $U^\epsilon(x,t)$ as the corresponding bounded entropy solution sequence, which is also in the same invariant domain. It suffices to show that the sequence $U^\epsilon(x,t)$ is compact in L^1_{loc} for $t>0$.

By the definition of entropy solutions, $U^\epsilon(x,t)$ satisfies that

$$
\mu^\epsilon\equiv\partial_t\eta(U^\epsilon)+\partial_x q(U^\epsilon)\leq 0,
$$

in the sense of distributions for any convex entropy pair (η,q). Also

$$
\mu^\epsilon \qquad \text{is a bounded subset of } W^{-1,\infty}_{loc}(\mathbf{R}^2_+).
$$

Murat's lemma [Mu] indicates that

$$\mu^\epsilon \qquad \text{is compact in } H^{-1}_{loc}(\mathbf{R}^2_+).$$

In particular,

$$\mu^\epsilon_* \equiv \partial_t \eta_*(U^\epsilon) + \partial_x q_*(U^\epsilon) \qquad \text{is compact in } H^{-1}_{loc}(\mathbf{R}^2_+),$$

with $\eta_*(U) = u^2 + v^2$.

Following an idea in [Ch], for any (not necessarily convex) C^2 entropy pair (η, q), we use the fact that $(\eta + C_\eta \eta_*, q + C_\eta q_*)$ is a convex entropy for some $C_\eta > 0$. Then we find that $\mu^\epsilon_\eta + C_\eta \mu^\epsilon_*$ is compact in $H^{-1}_{loc}(\mathbf{R}^2_+)$. By linearity, we conclude that

$$\mu^\epsilon \qquad \text{is compact in } H^{-1}_{loc}(\mathbf{R}^2_+).$$

Then, combining our analyses in Sections 2-4 with Theorem 2.1, we conclude the compactness of $U^\epsilon(x,t)$ in L^1_{loc} for $t > 0$. This concludes Theorem 5.1.

Remark 5.1. Theorem 5.1 shows that the nonstrictly hyperbolic degeneracy of the systems does not affect the compactness of solution operators.

Furthermore, we have

Theorem 5.2. *Let $U(x,t)$ be any periodic entropy solution with period $P = [0,a]$. Then $U(x,t)$ asymptotically decays to the average of the initial data over the period P:*

$$ess \lim_{t\to\infty} \int_0^a |U(x,t) - \bar{U}| dx = 0, \quad \text{with } \bar{U} = \frac{1}{a}\int_0^a U_0(x) dx.$$

This can be achieved by combining Theorem 5.1 above with the arguments in Chen-Frid [CF] as follows.

1. Set $U^\epsilon(x,t) = U(x/\epsilon, t/\epsilon)$. Then $U^\epsilon(x,t)$ is a sequence of entropy solutions with oscillating initial data. Theorem 5.1 implies the compactness of $U^\epsilon(x,t)$ in $L^1_{loc}(\mathbf{R}^2_+)$ (see the arguments for Theorem 5.1 above). Therefore there exists a subsequence (still denoted) $U^\epsilon(x,t)$ converging to some function $\overline{U}(x,t) \in L^\infty(\mathbf{R}^2_+)$ in $L^1_{loc}(\mathbf{R}^2_+)$. We conclude that $\overline{U}(x,t) = \overline{U}(t)$ from the periodicity of $U^\epsilon(x,t)$.

Now, writing the equation of $U^\epsilon(x,t)$ in the weak integral form and setting $\epsilon \to 0$, we can check that $\partial_t \overline{U}(t) = 0$ in the weak sense. This implies that $\overline{U}(t) = \overline{U} = \frac{1}{a}\int_0^a U_0(x)dx = w^* - lim\, U_0(x/\epsilon)$, since $U_0(x)$ is periodic.

Since the limit is unique, the whole sequence $U^\epsilon(x,t)$ strongly converges to $\overline{U}$ in $L^1_{loc}(\mathbf{R}^2_+)$ when $\epsilon \to 0$. Therefore, we have

$$\int_0^1 \int_{|x|\le rt} |U^\epsilon(x,t) - \overline{U}| dx dt \to 0, \qquad \text{when } \epsilon \to 0. \tag{5.1}$$

2. Denote $U = (u,v)$ and $\bar{U} = (\bar{u}, \bar{v})$. Using the special strictly convex entropy η_*, we find that the periodic entropy solution $U(x,t)$ satisfies the entropy inequality

$$\partial_t \eta_\sharp(U) + \partial_x q_\sharp(U) \le 0, \tag{5.2}$$

where

$$\eta_\sharp \equiv |U(x,t)-\overline{U}|^2, \quad q_\sharp \equiv 2(\frac{au^3}{3}+bu^2v+uv^2)-\overline{u}(au^2+2buv+v^2)-\overline{v}(bu^2+2uv).$$

Then we can deduce that, for any $\alpha \in \mathbf{R}$,

$$\int_\alpha^{\alpha+a} |U(x,t_2)-\overline{U}|^2 dx \le \int_\alpha^{\alpha+a} |U(x,t_1)-\overline{U}|^2 dx, \tag{5.3}$$

for all $0 \le t_1 < t_2, t_1, t_2 \in (0,\infty) - \mathcal{T}$, where $meas(\mathcal{T}) = 0$.

3. Given $T > 0, T \in (0,\infty) - \mathcal{T}$, we take all the rectangles given by $x \in [\alpha, \alpha+a]$, for α integer, and $t \in [[rT]/(2r), T]$, in the interior of the cone $\{|x| \le rt \,|\, 0 \le t \le T\}$, where $[\alpha]$ denotes the largest integer less than or equal α. The number of such rectangles is larger than $[rT]$. Using the periodicity of $U(x,t)$, inequality (5.3) with $t_2 = T$, which holds for a.e. $t_1 = t \in (0,T)$ over the period P, and the strict convexity of the entropy $\eta_\sharp$, we find that there exists $c_0 > 0, C > 0$, independent of T, such that

$$\begin{aligned}
\int_0^a |U(x,T)-\bar{U}|^2 dx &\le c_0 \frac{[rT]}{T^2} \int_{\frac{[rT]}{2r}}^T \int_0^a |U(x,T)-\bar{U}|^2 dxdt \\
&\le c_0 \frac{[rT]}{T^2} \int_{\frac{[rT]}{2r}}^T \int_0^a |U(x,t)-\bar{U}|^2 dxdt \\
&\le c_0 \frac{1}{T^2} \int_0^T \int_{|x|\le rt} |U(x,t)-\bar{U}|^2 dxdt \\
&\le c_0(|U|_{L^\infty}+|\bar{U}|_{L^\infty}) \frac{1}{T^2} \int_0^T \int_{|x|\le rt} |U(x,t)-\bar{U}| dxdt \\
&\le C \int_0^1 \int_{|x|\le rt} |U^\epsilon(x,t)-\bar{U}| dxdt \to 0, \qquad \epsilon = 1/T \to 0.
\end{aligned} \tag{5.4}$$

Then (5.4) implies the result of Theorem 5.2.

Remark 5.2. In Theorem 5.2, the assumption $U(x,t) \in L^\infty(\mathbf{R}^2_+)$ can be replaced by $U(x,t) \in L^p(\mathbf{R}^2_+), p \ge 3$. This can been easily seen in the arguments above for Theorem 5.2.

Remark 5.3. Although the uniqueness of entropy solutions is still unknown, we show in Theorem 5.2 that periodic solutions asymptotically decay to the unique constant state, determined solely by the initial data.

5.2. Convergence of Approximate Solutions and Existence of Entropy Solutions

To establish the existence of solution operators, i.e. entropy solutions in Theorems 5.1-5.2, we use the vanishing viscosity method. The analysis also works for the convergence of the Lax-Friedrichs scheme [La3] and the Godunov scheme [Go].

Consider the parabolic approximate solutions $U^\epsilon(x,t)$ governed by the associated parabolic system

$$\begin{cases} \partial_t U^\epsilon + \partial_x(\nabla C(U^\epsilon)) = \epsilon \Delta U^\epsilon, \\ U^\epsilon(x,0) = U_0(x). \end{cases} \tag{5.6}$$

We are concerned with the convergence of U^ϵ to an entropy solution U of (1.1) with the same initial data in any invariant domain described in Theorem 2.5 as $\epsilon \to 0^+$. Since $\eta_* = u^2 + v^2$ is a C^2 strictly convex entropy for (1.1), and $U_0(x)$ tends to a constant state $\bar{U}$ as $|x| \to \infty$ and $U_0 - \bar{U} \in L^2 \cap L^\infty$, we multiply (5.6) by $\nabla\eta_*(U^\epsilon) - \nabla\eta_*(\bar{U})$, a standard argument of integration by parts gives the estimate

$$\epsilon \int_0^\infty \int_{-\infty}^\infty |U_x^\epsilon|^2 \, dx \, dt \le C,$$

where C depends only upon U_0.

Consider any C^2 entropy-entropy flux pair (η, q) constructed in Theorem 2.2 for the system. Multiplying (5.6) by $\nabla\eta(U^\epsilon)$, and integrating by parts, we find, after using the L^∞ bound on U^ϵ, the boundedness of $\nabla^2\eta$ on compact sets, a standard application of Murat's Lemma [Mu], and a weak compactness estimate for the dissipation measures, that

$$\partial_t \eta(U^\epsilon) + \partial_x q(U^\epsilon) \qquad \text{is compact in } H^{-1}_{loc}. \tag{5.7}$$

Applying the compactness framework (Theorem 2.1) to (1.1) with $\Delta < 0$, we can show the strong convergence of viscosity approximate solutions to entropy solutions of (1.1) with large initial data for all positive time. As a corollary, we obtain the global existence of entropy solutions of the Cauchy problem of (1.1) with arbitrary large data in any invariant domain, described in Theorem 2.5.

We first verify L^∞ apriori estimates for these approximate solutions. We then apply the compactness framework (Theorem 2.1) together with Theorems 2.3 and 2.4 to achieve our results. To this end, invariant domains will be used for the viscosity approximate solutions.

Using the convexity properties of the $\mathbf{R}_j$ curves or the explicit form of w_i, we immediately conclude part (1) and (2) of Theorem 2.5, which yields an L^∞ apriori bound for viscosity solutions U^ϵ for (1.1) in Regions III and IV.

Theorem 5.3. *Consider the Cauchy problem* (1.1) *in Regions III and IV with the Cauchy data in one of the invariant domains. Then, for any $\epsilon > 0$, $U^\epsilon(x,t)$ is well defined for all (x,t) and, moreover, $|U^\epsilon(x,t)| \le C||U_0||_{L^\infty}$, for some $C < \infty$.*

Combining Theorems 2.2–2.5 and 5.3 with the Compactness Framework (Theorem 2.1), we obtain

Theorem 5.4. *Consider the viscosity approximation $\{U^\epsilon\}_{\epsilon>0}$ of the Cauchy problem* (1.1) *in Regions III and IV with the Cauchy data U_0 in one of the invariant domains, described in Theorem 2.5. Suppose that there is a constant $\bar{U}$ such that*

$$U_0 - \bar{U} \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R}). \tag{5.8}$$

Then, as $\epsilon \to 0^+$, there exists a subsequence of $U^\epsilon(x,t)$ that converges a.e. (x,t) to $U(x,t)$, a global entropy solution of (1.1).

The strong convergence of a subsequence of the viscosity approximations $\{U^\epsilon\}_{\epsilon>0}$ follows directly from Theorem 5.3 and the verification of the assumptions of Theorem 2.1 in Sections 3 and 4, and the one-to-one correspondence between one of the invariant domains in the U-plane and $\{w \mid w_1 \leq 0 \leq w_2\}$ in the w-plane. It is standard to show that $U(x,t)$ is an entropy solution, that is, $U(x,t)$ satisfies the entropy inequality.

Remark 5.4. Similarly, following the arguments in [DCL,Di,CL] for the Lax-Friedrichs or Godunov approximate solutions $\{U^\epsilon\}_{\epsilon>0}$ of the Cauchy problem (1.1) in Regions III and IV with the Cauchy data $U_0(x)$ in one of the invariant domains, described in Theorem 2.5, we can show that, as $\epsilon \to 0^+$, there exists a subsequence of $U^\epsilon(x,t)$ that converges a.e. (x,t) to a global entropy solution of (1.1).

Remark 5.5. Condition (5.8) can be removed. In particular, if the Cauchy data are periodic, there exists a global periodic entropy solution. For the finite-difference approximate solutions, the requirement that the data be in L^2 in condition (5.8) is not needed because finite-difference approximate solutions keep the main feature of hyperbolic equations: the finiteness of propagation speeds, which is the main advantage of the finite-difference methods over the viscosity method.

Theorem 5.5. *Given any bounded Cauchy data $U_0(x)$ in one of the invariant domains, described in Theorem 2.5, there always exists a global entropy solution for the Cauchy problem for* (1.1) *with the Cauchy data $U_0(x)$, which defines a solution operator for the Cauchy problem. Moreover, if the Cauchy data $U_0(x)$ are periodic in x with period P, then there exists a global entropy solution that is periodic in x with the same period.*

6. Regions I and II

We now summarize analytical results about the structure of the quadratic systems (1.1) in the case $a < 1 + b^2$, which include Regions I and II. Many of the results will be similar to those in Regions III and IV. Their proofs, however, are not the same. We will briefly highlight the essential differences.

6.1. Structure of the Systems: Unsymmetric Case

First, we discuss the unsymmetric case when $b \neq 0$. Recall that, for $(u,v) \in \mathcal{I}$, the Riemann invariants are of the form

$$w_i(u,v) = (-1)^i \tilde{v}^\beta \exp\{-\beta \int_0^{\tilde{\alpha}} \bar{H}_j(\tilde{\alpha}, sign(\alpha_0)) d\tilde{\alpha}\}, \quad i \neq j, \tag{6.1}$$

for any constant $\beta \neq 0$.

Consider the polynomial $h(\alpha)$. Denote by $\alpha_0 \leq \alpha_1 \leq \alpha_2$, the three real zeros of $h(\alpha)$ when $\Delta \leq 0$ (Regions I–III). We consider the local behavior of $\frac{w_j}{w_i}$ near the rays $\alpha = \alpha_l, l = 0,1,2$. For definiteness, we focus on the ray $\alpha = \alpha_0$ and $\{(u,v) \mid \alpha = \alpha_0, u > 0\} = \{(u,v) \mid \tilde{v} = 0, \tilde{u} > 0\}$ from the side $\{(u,v) \mid \tilde{v} \geq 0\}$. The discussions for the other cases are much the same.

When $(u,v) \in \mathcal{I}$, for Regions I and II, similar to Region III, near $\alpha = \alpha_l, \tilde{v} > 0$,

$$\frac{w_j}{w_i} = \Gamma_j(\tilde{\alpha}, sign(\alpha_0)) = -\exp\{\beta \int^{\tilde{\alpha}} \left(\bar{H}_j(\tilde{\alpha}, sign(\alpha_0)) - \bar{H}_i(\tilde{\alpha}, sign(\alpha_0))\right) d\tilde{\alpha}\} \sim |\tilde{\alpha} - \tilde{\alpha}_l|^{\mathcal{E}},$$

where

$$\mathcal{E} = \lim_{\tilde{\alpha} \to \tilde{\alpha}_l} \{(-1)^j \beta sign(\alpha_0)(1+\alpha_0^2) \frac{\sqrt{Q(\tilde{\alpha})}}{(\frac{D(\tilde{\alpha})}{\tilde{\alpha} - \tilde{\alpha}_l})}\} = (-1)^j \beta sign(\alpha_0) \frac{\sqrt{((a-1)\alpha_l + b)^2 + 4(b\alpha_l + 1)^2}}{h'(\alpha_l)}.$$

Set

$$\beta_l = \frac{sign(\alpha_0)h'(\alpha_l)}{\sqrt{((a-1)\alpha_l + b)^2 + 4(b\alpha_l + 1)^2}}, \quad l = 0, 1, 2.$$

Noting $\alpha_0 < -\frac{1}{b} < \alpha_1 < 0 \leq \alpha_2$ and $h(\alpha = \pm\infty) = \mp\infty$, we have $sign(h'(\alpha_l)) = (-1)^l sign(h'(\alpha_0))$, $l = 1, 2$, and, therefore, $sign(\beta_l) = (-1)^l sign(\beta_0)$. Then, if we choose

$$\beta = \beta_2,$$

the function $\Gamma_j(\tilde{\alpha}, -1)$ are continuous in the domain $\mathcal{I}$ and real analytical on the subdomain $\{(u,v)|\tilde{v} \geq 0, tan^{-1}\alpha_0 < \frac{u}{v} \leq tan^{-1}\alpha_2\}$ except on the ray $\alpha = \alpha_1, \alpha_2$; if we choose

$$\beta = sign\beta_2|\beta_1|,$$

the function $\Gamma_j(\tilde{\alpha}, -1)$ are continuous in the domain $\mathcal{I}$ and real analytical in the subdomain $\{(u,v)|\tilde{v} > 0, \pi + tan^{-1}\alpha_1 \leq \frac{u}{v} \leq \pi + tan^{-1}\alpha_0\}$ or $\{(u,v)|\tilde{v} > 0, tan^{-1}\alpha_2 < \frac{u}{v} \leq \pi + tan^{-1}\alpha_1\}$.

We now collect a number of Propositions describing analytical properties of the Riemann invariants, the coefficients of the entropy equations, and the monotonicity of wave speeds in the nonsymmetric Regions I-II. Most of the proofs are omitted. They can be carried out similar to the cases in Regions III and IV although the technical details are different.

Proposition 6.1. *Consider Regions I and II, i.e.,* $\Delta < 0$ *and* $a < 1 + b^2, b \neq 0$. *Near the umbilic point* $(u,v) = (0,0)$, *the derivatives of the Riemann invariants satisfy the following estimates:*

$$\begin{aligned} &w_i = \mathcal{O}(1), \quad i = 1, 2, \\ &|\partial_u^m \partial_v^n w_i| \leq C\left(\frac{|w_i|}{|\tilde{v}|^{m+n+2\gamma}|w_1 w_2|^{2\gamma}}\right), \quad 1 \leq m + n \leq 2, \end{aligned} \tag{6.3}$$

where $\gamma = max(|\frac{\beta}{\beta_1}|, |\frac{\beta}{\beta_2}|, |\frac{1}{2\beta}|) + 1$.

The proof is similar to the case in Region IV. We remark that the estimates here are different because, in Regions I and II, $|w_1 w_2|$ vanishes on $\alpha = \alpha_1, \alpha_2$. In fact,

$$|w_1 w_2| \leq C\tilde{v}^{2\beta} \left|\frac{\alpha - \alpha_0}{1 + |\alpha|}\right|^{|\frac{\beta}{\beta_0}|} \left|\frac{\alpha - \alpha_1}{1 + |\alpha|}\right|^{|\frac{\beta}{\beta_1}|} \left|\frac{\alpha - \alpha_2}{1 + |\alpha|}\right|^{|\frac{\beta}{\beta_2}|}. \tag{6.4}$$

We are now ready to briefly discuss the proof of Theorem 2.3, parts (b) and (c) which are concerned with the monotonicity of the wave speeds. The proof is similar to those for Regions III and IV and uses (3.13). In Regions I and II, w_i is bounded on the line $\alpha = \alpha_0$. Therefore, λ_{iw_i} remains nonzero there. In Region I, there is an additional zero in the term $\mathbf{r}_i \cdot \nabla\lambda_i$ in the numerator in (3.13). Therefore, λ_{iw_i} vanishes on the half of the line $\{(u,v)\,|\,(a-1)u + bv = \frac{3b(a-3)\pm\sqrt{D}}{6a}(bu + v)\}$.

6.2. Structure of the Systems: Symmetric Case

Next, we consider the symmetric case in Regions I and II, i.e., $b = 0$, $a < 1$. For simplicity, we restrict ourselves to the following half-plane domain $\mathcal{I} \equiv \{(u,v)\,|\,v \geq 0\}$. This domain is also an invariant domain for the parabolic, the Lax-Friedrichs, and the Godunov approximate solutions for (1.1). For the other half-plane, the situation is very similar.

Now we have

$$w_i(u,v) = (-1)^i v^\beta \exp\{-\beta \int_0^\alpha \frac{2d\alpha}{(a-3)\alpha + (-1)^j\sqrt{(a-1)^2\alpha^2+4}}\}, \quad i=1,2, i\neq j, \tag{6.5}$$

for any constant $\beta \neq 0$, and

$$\Gamma_j(\alpha) = \frac{w_j}{w_i} = -\exp\{(-1)^j\beta \int_0^\alpha \frac{\sqrt{(a-1)^2\alpha^2+4}}{(a-2)\alpha^2+1}d\alpha\}. \tag{6.6}$$

Similar to the case of Region III, we choose $\beta_1 = \beta_2 = \frac{2(2-a)}{3-a}$ to guarantee the one side analyticity of $\frac{w_j}{w_i}$ near the rays $\alpha = \pm\frac{1}{\sqrt{2-a}}$.

Now we check that the correspondence between the rays $\{(u,v)|v=0, u>0, \text{or } v = -\sqrt{2-a}\}$ and $w_2 = 0$ and between the rays $\{(u,v)|v=0, u<0, \text{ or } v=\sqrt{2-a}\}$ and $w_1 = 0$; and the boundedness of w_j near these rays. In fact, near $\sigma = 0$,

$$\begin{aligned} w_i(u,v) &= (-1)^i v^\beta \exp\{2\beta \int^\sigma \frac{1}{a-3+(-1)^j sign(\sigma)\sqrt{(a-1)^2+4\sigma^2}} \frac{d\sigma}{\sigma}\} \\ &\sim \begin{cases} (-1)^i v^\beta |\sigma|^{-\beta} = (-1)^i |u|^\beta, & (-1)^j sign(\sigma) > 0, \\ (-1)^i v^\beta |\sigma|^{\frac{\beta}{a-2}} = (-1)^i v^{\beta\frac{1-a}{2-a}} |u|^{\frac{\beta}{2-a}}, & (-1)^j sign(\sigma) < 0. \end{cases} \end{aligned}$$

It can be similarly confirmed near the rays $\{(u,v)\,|\,v = \pm\sqrt{2-a}\}$. We omit the details.

Recall that all possible points of $\mathbf{r}_j \cdot \nabla\lambda_j = 0$ are

$$\begin{cases} v = 0, & a \geq 0, \\ v = 0, v = \pm\frac{(1-a)\sqrt{-3a}}{|3+a|}, & a < 0. \end{cases} \tag{6.7}$$

On the line $v=0$, $\mathbf{r}_j \cdot \nabla\lambda_j = 0$ if and only if $(-1)^j(a-1)u < 0$; on the line $v = \pm\frac{(1-a)\sqrt{-3a}}{|3+a|}$ for the case $a<0, \mathbf{r}_j \cdot \nabla\lambda_j = 0$ if and only if $\pm(-1)^j a(a-1)(a^2-9)u < 0$. Recall that, for $i \neq j$,

$$\begin{aligned} \lambda_{iw_i} =& \frac{-2vw_j}{w_1w_2} \frac{(-1)^j((a-2)\alpha^2+1)}{(a-3)\alpha + (-1)^i\sqrt{(a-1)^2\alpha^2+4}} \\ &\times \frac{a(a-1)\alpha\sqrt{(a-1)^2\alpha^2+4} + (-1)^i(a(a-1)^2\alpha^2 + 2(a+3))}{\beta(\sqrt{(a-1)^2\alpha^2+4})^2}, \end{aligned}$$

Proposition 6.2. *(a) Consider the symmetric systems in* (1.1) *($b=0, a<1$). Suppose that $(u,v) \in \mathcal{I}$. Near the umbilic point $(u,v) = (0,0)$, the derivatives of the Riemann invariants satisfy the following estimates:*

$$\begin{aligned} w_i =& \mathcal{O}(1), \quad i = 1,2, \\ |\partial_u^m \partial_v^n w_i| \leq& \begin{cases} C\frac{|w_i|}{|\tilde{v}|^{m+n}|\alpha - \frac{(-1)^j}{\sqrt{2-a}}|}|, & a<2, \\ C\frac{|w_i|}{|\tilde{v}|^{m+n}}, & a>2, \end{cases} \qquad 1 \leq m+n \leq 2. \end{aligned}$$

(b) For Region II ($0<a<1$),

$$\lambda_{iw_i} \neq 0, \qquad \text{for all } (w_1,w_2) \in \mathcal{J}(\mathcal{I}) - \{(0,0)\};$$

For Region I ($a < 0$),

$$\lambda_{iw_i} \neq 0, \quad \textit{for all } (w_1, w_2) \in \mathcal{J}(\mathcal{I} - \{v = (-1)^i \frac{(a-1)\sqrt{-3a}}{a+3}\}).$$

6.3. Qualitative Behaviors of L^∞ Weak Entropy Solutions

Using the properties of the systems discussed above, the arguments as in Section 5.1, and the Compactness Framework (Theorem 2.1), we have

Theorem 6.1. *Let $U(x,t)$ be a bounded entropy solution of (1.1) in Regions I-II with the Cauchy data in one of the invariant domains (see Theorem 2.5). Set $U^T(x,t) = U(Tx, Tt)$ its self similar scaling sequence. Then $U^T(x,t)$ is compact in L^1_{loc} for $t > 0$.*

Combining Theorem 6.1 with Theorems 3.1 and 3.2 of Chen-Frid [CF], we conclude

Theorem 6.2. *Let $U(x,t)$ be any bounded periodic entropy solution in one of the invariant domains with period $P = [0, a]$. Then $U(x,t)$ asymptotically decays to the average of the initial data over the period P:*

$$ess \lim_{t\to\infty} \int_0^a |U(x,t) - \bar{U}| dx = 0, \qquad \textit{with } \bar{U} = \frac{1}{a}\int_0^a U_0(x) dx.$$

Acknowledgments. Gui-Qiang Chen's research was supported in part by NSF grants DMS-9971793, INT-9987378, and INT-9726215. Pui Tak Kan's research was supported in part by NSF grants DMS-9100383 and DMS-9404341.

References

[Ch] Chen, G.-Q.: *Hyperbolic systems of conservation laws with a symmetry*, Comm. Partial Diff. Eqs. 16, 1461-1487 (1991).

[CF] Chen, G.-Q. and Frid, H.: *Decay of entropy solutions of nonlinear conservation laws*, Arch. Rational Mech. Anal. 1998 (to appear).

[CK1] Chen, G.-Q., and Kan, P.-T.: *Hyperbolic conservation laws with umbilic degeneracy I*, Arch. Rational Mech. Anal. 130, 231-276 (1995).

[CL] Chen, G.-Q., and LeFloch, P.: *Compressible Euler equations with general pressure law*, Arch. Rational Mech. Anal. 153 (2000), 221-259.

[CCS] Chueh, K.N., Conley, C.C., and Smoller, J.: *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J. 26, 372–411 (1977).

[Daf] Dafermos, C. M.: *Hyperbolic Systems of Conservation Laws in Continuum Physics,* Springer-Verlag: Berlin, 2000,

[Da] Darboux, G.: *Lecons Sur La Theorie Generale des Surfaces*, T. II, Paris, (1914).

[DCL] Ding, X., Chen, G.-Q., and Luo, P.: *Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics*, Commun. Math. Phys. 121 (1989), 63-84.

[Di] DiPerna, R.: *Convergence of approximate solutions to conservation laws*, Arch. Rat. Mech. Anal. 82, 27–70 (1983).

[FR] Friedlands, S., Robbin, J. W., and Sylvester, J: *On the crossing rule*, Comm. Pure Appl. Math. 37, 19-38 (1984).

[G1] Glimm, J.: *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. 18, 697-715 (1965),

[GM] Glimm, J. and Majda, A.: *Multidimensional Hyperbolic Problems and Computations*, IMA Vol. 29, Springer-Verlag: New York, 1991.

[Go] Godunov, S.: *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics.* Mat. Sb. 47 (89), 271-360 (1959).

[IMPT] Isaacson, E., Marchesin, D., Plohr, B., and Temple, B.: *The Riemann problem near a hyperbolic singularity: the classification of solutions of quadratic Riemann problem (I)*, SIAM J. Appl. Math. 48, 1-24 (1988).

[IT] Isaacson, E., and Temple, B.: *The classification of solutions of quadratic Riemann problem (II)-(III)*, SIAM J. Appl. Math. 48, 1287-1301, 1302-1318 (1988).

[KSZ1] Kan, P.T., Santos, M., Xin, Z.: *Initial boundary value problems for a class of quadratic systems of conservation laws*, Contemp. Mat. 11, 1-33 (1996).

[KSZ2] Kan, P.T., Santos, M., Xin, Z.: *Initial boundary value problems for conservation laws*, Commun. Math. Phys. 186, 701-730 (1997).

[La1] Lax, P.D.: *The multiplicity of eigenvalues*, Bull. AMS. 6, 213-214 (1982).

[La2] Lax, P.D.: *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure Appl. Math. 7, 159-193 (1954).

[La3] Lax, P.D.: *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, Philadelphia (1973).

[LZ] Liu, T.-P. and Zumbriun, K., : *On nonlinear stability of general undercompressive viscous shock waves*, Commun. Math. Phys. 174 (1995), 319-345.

[Mu] Murat, F.: *L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$*, J. Math. Pures Appl. 60, 309–322 (1981).

[SS1] Schaeffer, D., and Shearer, M.: *The classification of 2×2 systems of nonstrictly hyperbolic conservation laws, with application to oil recovery*, Comm. Pure Appl. Math. 40, 141-178 (1987).

[SSMP] Shearer, M., Schaeffer, D., Marchesin, D., and Paes-Leme, P. J.: *Solution of the Riemann problem for a prototype 2×2 system of nonstrictly hyperbolic conservation laws*, Arch. Rat. Mech. Anal. 97, 299-320 (1987).

[Ta] Tartar, L.: *Compensated compactness and applications to partial differential equations*, In: Research Notes in Mathematics, Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. 4, ed. R. J. Knops, pp 136–212, Pitman Press, New York, 1979.