Existence and Stability of Supersonic Euler Flows past Lipschitz Wedges

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Abstract

It is well known that, when the vertex angle of a straight wedge is less than the critical angle, there exists a shock-front emanating from the wedge vertex so that the constant states on both sides of the shock-front are supersonic. Since the shock-front at the vertex is usually strong, especially when the vertex angle of the wedge is large, then such a global flow is physically required to be governed by the isentropic or adiabatic Euler equations. In this paper, we systematically study two-dimensional steady supersonic Euler (i.e. non-potential) flows past Lipschitz wedges and establish the existence and stability of supersonic Euler flows when the total variation of the tangent angle functions along the wedge boundaries is suitably small. We develop a modified Glimm difference scheme and identify a Glimm-type functional, by incorporating the Lipschitz wedge boundary and the strong shock-front naturally and by tracing the interaction not only between the boundary and weak waves but also between the strong shock front and weak waves, to obtain the required BV estimates. Then these estimates are employed to establish the convergence of both approximate solutions to a global entropy solution and corresponding approximate strong shock-fronts emanating from the vertex to the strong shock-front of the entropy solution. The regularity of strong shock-fronts emanating from the wedge vertex and the asymptotic stability of entropy solutions in the flow direction are also established.

1. Introduction

We are concerned with the existence and behavior of two-dimensional steady supersonic Euler flows past Lipschitz wedges with arbitrary vertex angles that are less than the critical angle so that there is a supersonic shock-front emanating from the wedge vertex. The two-dimensional steady supersonic Euler flows are generally governed by

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ (\rho u v)_x + (\rho v^2 + p)_y = 0, \\ (u(E+p))_x + (v(E+p))_y = 0, \end{cases}$$
(1)

where (u, v) is the velocity, ρ the density, p the scalar pressure, and

$$E = \frac{1}{2}\rho(u^{2} + v^{2}) + \rho e(\rho, p)$$

is the total energy with e the internal energy (a given function of (ρ, p) defined through thermodynamical relationships). The other two thermodynamic variables are the temperature T and the entropy S. If (ρ, S) are chosen as the independent variables, then we have the constitutive relations:

$$(e, p, T) = (e(\rho, S), p(\rho, S), T(\rho, S)),$$
(2)

governed by

$$TdS = de - \frac{p}{\rho^2} d\rho.$$
(3)

For an ideal gas,

$$p = R\rho T, \qquad e = c_v T, \qquad \gamma = 1 + \frac{R}{c_v} > 1,$$
(4)

and

$$p = p(\rho, S) = \kappa \rho^{\gamma} e^{S/c_v}, \qquad e = \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v} = \frac{RT}{\gamma - 1}, \tag{5}$$

where R, κ , and c_v are all positive constants.

If the flow is isentropic, i.e., S = const., then the pressure p is a function of the density ρ , $p = p(\rho)$, and the flow is governed by the following simpler isentropic Euler equations:

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ (\rho u v)_x + (\rho v^2 + p)_y = 0. \end{cases}$$
(6)

For polytropic isentropic gases, by scaling, the pressure-density relationship can be expressed as

$$p(\rho) = \rho^{\gamma} / \gamma, \qquad \gamma > 1. \tag{7}$$

For the isothermal flow, $\gamma = 1$. The quantity

$$c = \sqrt{p_{\rho}(\rho, S)}$$

is defined as the sonic speed and, for polytropic gases, $c = \sqrt{\gamma p/\rho}$.

System (1) or (6) governing a supersonic flow (i.e., $u^2 + v^2 > c^2$) has all real eigenvalues and is hyperbolic, while system (1) or (6) governing a subsonic flow (i.e., $u^2 + v^2 < c^2$) has complex eigenvalues and is elliptic-hyperbolic mixed and composite.

The study of two-dimensional steady supersonic flows past wedges can date back to the 1940s (cf. Courant-Friedrichs [8]). Local solutions around the wedge vertex were first constructed in Gu [12], Li [16], Schaeffer [22], and the references cited therein. Global potential solutions were constructed in [4–8,28,29] when the wedge has certain convexity or the wedge is a small perturbation of the straight wedge with fast decay in the flow direction, whose vertex angle is less than the critical angle. In particular, in Zhang [29], the existence of two-dimensional steady supersonic potential flows past piecewise smooth curved wedges, which are a small perturbation of the straight wedge, was established.

As is well-known, the potential flow equation is an excellent model for the flow containing only weak shocks since it approximates to the isentropic Euler equations up to third-order in shock strength. For the flow containing shocks of large strength, the isentropic or adiabatic Euler equations are required to govern the physical flow. For the wedge problem, when the vertex angle is large, the flow contains a strong shock-front emanating from the wedge vertex and, for this case, the Euler equations should take the position to describe the physical flow. Thus it is important to study the two-dimensional steady supersonic flows governed by the Euler equations, rather than the potential flow equation, for the wedge problem with a large vertex angle. When a wedge is straight and the wedge vertex angle is less than the critical angle, there exists a supersonic shock-front emanating from the wedge vertex so that the constant states on both sides of the shock-front are supersonic; the critical angle condition is necessary and sufficient for the existence of the supersonic shock. These facts can be seen through the shock polar in Fig. 3 in Section 2 (also cf. Courant-Friedrichs [8]).

In this paper, we analyze the two-dimensional steady supersonic Euler flows past two-dimensional Lipschitz wedges whose vertex angles are less than the critical angle, along which the total variation of the tangent angle function is suitably small; and we establish the existence and behavior of such global supersonic Euler flows, especially the nonlinear stability of the strong shock-front emanating from the wedge vertex under the BV perturbation.

For concreteness, we will analyze the problem in the region below the lower side Γ of the wedge for the Euler flows for $U = (u, v, p, \rho)$ governed by system (1) and $U = (u, v, \rho)$ by (6); the case above the wedge can be handled in the same fashion. Then we have

(i). There exists a Lipschitz function $g \in Lip(\mathbb{R}_+)$ with $g' \in BV(\mathbb{R}_+)$, g'(0+) = 0, and g(0) = 0 such that

$$\varOmega:=\{(x,y) \ : \ y < g(x), \ x \geq 0\}, \qquad \varGamma:=\{(x,y) \ : \ y=g(x), \ x \geq 0\},$$

and $\mathbf{n}(x\pm) = \frac{(-g'(x\pm),1)}{\sqrt{(g'(x\pm))^2+1}}$ is the outer normal vector to Γ at the point $x\pm$ (see Fig. 1);



Fig. 1. Supersonic flow past a curved wedge

(ii). The upstream flow is a constant state U_{-} satisfying

$$u_{-} > 0, \quad v_{-} > 0, \quad u_{-}^2 + v_{-}^2 > c_{-}^2 := \frac{\gamma p_{-}}{\rho_{-}}$$

and

$$0 < \arctan(v_-/u_-) < \omega_{crit},$$

so that there is a supersonic shock-front emanating from the wedge vertex, where ω_{crit} is the critical vertex angle (cf. Fig. 3).

With this setup, the wedge problem can be formulated into the following problem of initial-boundary value type for system (1) or (6):

Cauchy Condition:

$$U|_{x=0} = U_{-}; (8)$$

Boundary Condition:

$$(u,v) \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma. \tag{9}$$

The main theorem of this paper is the following.

Main Theorem (Existence and Stability). There exist $\varepsilon > 0$ and C > 0such that, if

$$TV(g'(\cdot)) < \varepsilon, \tag{10}$$

then there exists a pair of functions

$$U \in BV_{loc}(\mathbb{R}^2_+), \qquad \sigma \in BV(\mathbb{R}_+),$$

with $\chi = \int_0^x \sigma(t) dt \in Lip(\mathbb{R}_+)$ such that (i) U is a global entropy solution of problem (1), or (6), and (8)–(9) in Ω with

$$TV\{U(x,\cdot) : (-\infty, g(x))\} \le CTV(g'(\cdot))$$
 for every $x \in \mathbb{R}_+$, (11)

$$(u, v) \cdot \mathbf{n}|_{y=g(x)} = 0$$
 in the trace sense; (12)

(ii) The curve $y = \chi(x)$ is a strong shock-front with $\chi(x) < g(x)$ for any x > 0 and

$$U|_{\{y < \chi(x)\}} = U_{-}, \qquad \sqrt{u^2 + v^2}|_{\{\chi(x) < y < g(x)\}} < u_{-}, \tag{13}$$

that is, $y = \chi(x)$ is the strong shock next to the constant state U_{-} ; (iii) There exist constants p_{∞} and σ_{∞} such that

$$\lim_{x \to \infty} \sup\{|p(x, y) - p_{\infty}| : \chi(x) < y < g(x)\} = 0,$$
$$\lim_{x \to \infty} |\sigma(x) - \sigma_{\infty}| = 0,$$

and

$$\lim_{x \to \infty} \sup\{|\arctan(v(x, y)/u(x, y)) - \omega_{\infty}| : \chi(x) < y < g(x)\} = 0,$$

where $\omega_{\infty} = \lim_{x \to \infty} \arctan(g'(x+))$. This theorem indicates that the strong shock-front emanating from the wedge vertex is nonlinearly stable in structure, although there may be many weak waves and vortex sheets between the wedge boundary and the strong shock-front, under the BV perturbation of the wedge boundary so long as the wedge vertex angle is less than the critical angle. This asserts that any supersonic shock for the wedge problem is nonlinearly stable.

In order to establish this theorem, we develop a modified Glimm scheme and identify a Glimm-type functional by incorporating the curved wedge boundary and the strong shock-front naturally and by tracing the interactions not only between the wedge boundary and weak waves but also the interaction between the strong shock-front and weak waves. Some detailed interaction estimates are carefully made to ensure that the Glimm-type functional monotonically decreases in the flow direction. In particular, one of the essential estimates is on the strengths of the reflected 4-waves for (1), or 3-waves for (6), in the interaction between the strong shock-front and weak waves; and the second essential estimate is the interaction estimate between the wedge boundary and weak waves. Another essential estimate is on tracing the approximate strong shocks in order to establish the nonlinear stability and asymptotic behavior of the strong shock-front emanating from the wedge vertex under the wedge perturbation.

We remark that, in Lien-Liu [17], the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone with small vertex angle was established upon the perturbation of the obstacle. It would be interesting to combine the analysis in this paper with the argument in [17] to study the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone with arbitrary vertex angle. We also remark in passing that condition (10) can be relaxed by combining the analysis in this paper with the argument in [24, 25].

In this paper we first focus on the isentropic Euler flows in Sections 2–5 and then we extend to the adiabatic (full) Euler flows in Section 6.

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In Section 2, we study the lateral Riemann problem and the classical Riemann problem, and analyze the properties of the Riemann solutions to the isentropic Euler equations (6), which are essential for the interaction estimates among the nonlinear waves and the wedge boundary in Section 3 and for the existence and behavior of entropy solutions of the wedge problem in Sections 4–5. In Section 3, we make estimates on the wave interactions and reflections on the wedge and the strong shock, respectively. In Section 4, we develop a modified Glimm scheme to construct approximate solutions and establish necessary estimates for them in the approximate domains. In Section 5, we establish the convergence of approximate solutions to a global entropy solution and prove the nonlinear stability and asymptotic behavior of the strong shock-front emanating from the wedge vertex under the wedge perturbation. In Section 6, we extend the analysis and approach to establish the existence and behavior of two-dimensional steady supersonic flows past the Lipschitz wedges for the adiabatic Euler equations.

2. Riemann Problems and Riemann Solutions

In this section, we study the lateral Riemann problem and the classical Riemann problem, and analyze the properties of the Riemann solutions to the isentropic Euler equations (6), which are essential not only for the interaction estimates among the nonlinear waves and the wedge boundary but also for the existence and behavior of solutions for the wedge problem in Sections 3–5.

2.1. Euler Equations

The Euler system can be written in the following conservation form:

$$W(U)_x + H(U)_y = 0, (14)$$

where

$$U = (u, v, \rho), \ W(U) = (\rho u, \rho u^2 + p, \rho uv), \ H(U) = (\rho v, \rho uv, \rho v^2 + p).$$

For a smooth solution U(x, y), system (14) is equivalent to

$$\nabla_U W(U)U_x + \nabla_U H(U)U_y = 0. \tag{15}$$

Then the eigenvalues of (14) are the roots of the third order polynomial:

$$det(\lambda \nabla_U W(U) - \nabla_U H(U)), \tag{16}$$

and are thus the solutions of the cubic equation:

$$(v - \lambda)\left((v - \lambda u)^2 - c^2(1 + \lambda^2)\right) = 0,$$

where $c = \sqrt{p'(\rho)}$ is the sonic speed. If the flow is supersonic (i.e., $u^2 + v^2 > c^2$), then we have three eigenvalues $\lambda_j, j = 1, 2, 3$:

$$\lambda_2 = v/u, \qquad \lambda_j = \frac{uv + (-1)^{\frac{j+1}{2}} c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 3,$$
(17)

which implies that the system is always hyperbolic in the flow. When $u \neq 0$, the corresponding eigenvectors are

$$\mathbf{r}_{2} = (1, v/u, 0)^{\top}, \qquad \mathbf{r}_{j} = \kappa_{j} (-\lambda_{j}, 1, \rho(\lambda_{j}u - v)/c^{2})^{\top}, \quad j = 1, 3,$$
(18)

where κ_j are chosen so that $\mathbf{r}_j \cdot \nabla \lambda_j = 1$ because of genuine nonlinearity of the j^{th} -characteristic fields, j = 1, 3. Note that the second characteristic field is always linearly degenerate: $\mathbf{r}_2 \cdot \nabla \lambda_2 = 0$.

Definition 1 (Entropy Solutions). A function $U = U(x, y) \in BV_{loc}(\Omega)$ is called an entropy solution of problem (6) and (8)–(9) provided that

(i) U is a weak solution of (6) and satisfies

$$(u,v) \cdot \mathbf{n}|_{y=g(x)} = 0$$
 in the trace sense;

(ii) U satisfies the following entropy inequality:

$$(u(E+p(\rho)))_{x} + (v(E+p(\rho)))_{y} \le 0$$
(19)

in the sense of distributions in Ω including the boundary.

2.2. Basic Properties of Nonlinear Waves

In this subsection, we analyze some basic properties of nonlinear waves, especially the global behavior of shock curves and rarefaction wave curves in the phase space.

We first seek the self-similar solution to (6):

$$(u, v, \rho)(x, y) = (u(\xi), v(\xi), \rho(\xi)), \qquad \xi = y/x,$$

which connects to a state $U_0 = (u_0, v_0, \rho_0)$. Then we have

$$det\left(\xi\nabla_U W(U) - \nabla_U H(U)\right) = 0.$$
(20)

Hence

$$\xi = \lambda_2 = v/u, \quad \text{or} \quad \xi = \lambda_j = \frac{uv + (-1)^{\frac{j+1}{2}}c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 3.$$
(21)

Plugging $\xi = \lambda_2$ into (20), we obtain

$$d\rho = 0, \quad vdu - udv = 0.$$

Then the contact discontinuity curve $C_2(U_0)$ in the phase space is:

$$C_2(U_0): \quad \rho = \rho_0, \quad w = v/u = v_0/u_0,$$
(22)

which describes compressible vortex sheets.

Plugging $\xi = \lambda_j$ into (20), we get the j^{th} -rarefaction wave curve $R_j(U_0)$ in the phase space through U_0 :

$$R_j(U_0): \quad du = -\lambda_j dv, \quad \rho(\lambda_j u - v) dv = dp, \qquad j = 1, 3.$$
(23)

We now compute $\frac{d\lambda_j}{d\rho}$ along $R_j(U_0), j = 1, 3$. Since $(v - \xi u)^2 = c^2(1 + \xi^2)$ along $R_j(U_0)$, we differentiate the equation to have

$$\left(c^2\lambda_j + u(v - \lambda_j u)\right)\frac{d\lambda_j}{d\rho} = -(1 + \lambda_j^2)\left(\frac{1}{\rho}\frac{dp}{d\rho} + c\frac{dc}{d\rho}\right) < 0$$

Noting that $c^2 \lambda_j + u(v - \lambda_j u) = (-1)^{\frac{j+3}{2}} c \sqrt{u^2 + v^2 - c^2}$, we conclude

$$\frac{d\lambda_1}{d\rho}|_{R_1(U_0)} < 0, \quad \frac{d\lambda_4}{d\rho}|_{R_3(U_0)} > 0.$$
(24)

Now we consider discontinuous solutions so that (6) is satisfied in the distributional sense, which implies that the following Rankine-Hugoniot conditions hold along the discontinuity with speed σ , which connects to a state $U_0 = (u_0, v_0, \rho_0)$:

$$\sigma[\rho u] = [\rho v],\tag{25}$$

$$\sigma[\rho u^2 + p] = [\rho uv], \tag{26}$$

$$\sigma[\rho uv] = [\rho v^2 + p], \tag{27}$$

where the jump symbol $[\cdot]$ stands for the value of the quantity of the front state minus that of the back state. This can be rewritten as

$$\begin{pmatrix} -\sigma\rho & \rho & v_0 - \sigma u_0 \\ \rho(v_0 - \sigma u_0) & 0 & -\sigma \bar{c}_0^2 \\ 0 & \rho(v_0 - \sigma u_0) & \bar{c}_0^2 \end{pmatrix} \begin{pmatrix} [u] \\ [v] \\ [\rho] \end{pmatrix} = 0$$
(28)

with $\bar{c}_0^2 = \frac{\rho}{\rho_0} \frac{[p]}{[\rho]}$. Then we have

$$\sigma = \sigma_2 := v_0/u_0, \quad \sigma = \sigma_j := \frac{u_0 v_0 + (-1)^{\frac{j+1}{2}} \bar{c}_0 \sqrt{u_0^2 + v_0^2 - \bar{c}_0^2}}{u_0^2 - \bar{c}_0^2}, \quad j = 1, 3.$$
(29)

Plugging σ_2 into (28), we get the same $C_2(U_0)$ as defined in (22). Plugging σ_j into (28), we obtain the j^{th} -shock curve $S_j(U_0)$ in the phase space through U_0 :

$$S_j(U_0): \quad [u] = -\sigma[v], \quad \rho_0(\sigma_j u_0 - v_0)[v] = [p], \qquad j = 1, 3.$$
(30)

It is straightforward to see that the shock curve $S_j(U_0)$ contacts with $R_j(U_0)$ at U_0 up to second-order and, along $S_j(U_0), j = 1, 3$,

$$\frac{d\sigma_1}{d\rho}|_{S_1(U_0)} < 0, \qquad \frac{d\sigma_3}{d\rho}|_{S_3(U_0)} > 0.$$
(31)

Lemma 1. If U is a piecewise smooth solution, then, on the shock wave, the entropy inequality (19) in Definition 2.1 is equivalent to any of the following:
(i) The physical entropy condition: the density increases across the shock in the flow direction:

$$\rho_{front} < \rho_{back}; \tag{32}$$

(ii) The Lax entropy condition: on the j^{th} -shock with the shock speed σ_j :

$$\lambda_j(back) < \sigma_j < \lambda_j(front), \qquad j = 1, 3,$$
(33)

$$\sigma_1 < \lambda_2(back), \qquad \lambda_2(front) < \sigma_3.$$
 (34)

Proof. Since the system is Galilean invariant, we may assume that the back state of the shock is $U_+ = (u_+, 0, \rho_+)$ with $u_+ > 0$.

We first show the equivalence between (19) and (32). Along the discontinuity with speed σ and back state U_+ , (19) is equivalent to

$$\sigma[u(E+p)] \ge [v(E+p)]. \tag{35}$$

Plugging (25) into (35) and using $v_{+} = 0$, we have

$$\rho^{\gamma-1}/(\gamma-1) + (u^2 + v^2)/2 \ge \rho_+^{\gamma-1}/(\gamma-1) + u_+^2/2.$$
(36)

Using (25) and (26), we have

$$u = u_{+} - \frac{p - p_{+}}{\rho_{+} u_{+}}.$$
(37)

Using (25) and (27), we get

$$p - p_+ = \sigma \rho_+ u_+ v. \tag{38}$$

Combining (25) with (38) yields

$$v^{2} = \frac{(\rho u - \rho_{+} u_{+})(p - p_{+})}{\rho \rho_{+} u_{+}}.$$
(39)

Then plugging (37) and (39) into (36) yields that (19) is equivalent to

$$H(\rho) := 2\gamma(p\rho_{+} - p_{+}\rho) - (\gamma - 1)(p - p_{+})(\rho + \rho_{+}) \ge 0,$$

with $H(\rho_+) = 0$, which implies $\rho < \rho_+$ that is (32).

Now we show the equivalence between (32) and (33)-(34).

Case $(32) \Rightarrow (33)$ -(34): We now prove the 1-shock with the physical entropy condition satisfies the Lax entropy condition; the 3-shock can be proved in the same way.

First, since $\lambda_1(U_+) = \frac{-c_+}{\sqrt{u_+^2 - c_+^2}}$ and $\sigma_1 = \frac{-\bar{c}_+}{\sqrt{u_+^2 - \bar{c}_+^2}}$, then $\sigma_1 < \lambda_2(U_+)$ is direct, and $\lambda_1(U_+) < \sigma_1$ is equivalent to the inequality:

$$c_{+}/\sqrt{u_{+}^{2}-c_{+}^{2}} > \bar{c}_{+}/\sqrt{u_{+}^{2}-\bar{c}_{+}^{2}}.$$
 (40)

Since the function $f(x) = \frac{x}{\sqrt{u_+^2 - x^2}}$ is strictly increasing in $x \in [0, u_+)$, inequality (40) holds if and only if $c_+ > \bar{c}_+$, which is equivalent to

$$p'(\rho_{+}) > \frac{\rho_{-}}{\rho_{+}} \frac{p_{-} - p_{+}}{\rho_{-} - \rho_{+}} = \frac{\rho_{-}}{\rho_{+}} p'(\theta\rho_{+} + (1 - \theta)\rho_{-})$$
(41)

for some $\theta \in (0, 1)$, where $U_{-} = (u_{-}, v_{-}, \rho_{-})$ is the front state of the 1-shock.

By the entropy condition, $\rho_{-} < \rho_{+}$, so that $\theta \rho_{+} + (1-\theta)\rho_{-} < \rho_{+}$. Then the convexity of $p(\rho)$ implies $p'(\theta \rho_{+} + (1-\theta)\rho_{-}) \leq p'(\rho_{+})$. Then inequality (41) follows.

Second, set $\omega_{-} = \arctan(v_{-}/u_{-})$ and

$$\omega_{ma} = \arctan(\frac{c_-}{\sqrt{u_-^2 + v_-^2 - c_-^2}}), \quad \bar{\omega}_{ma} = \arctan(\frac{\bar{c}_-}{\sqrt{u_-^2 + v_-^2 - \bar{c}_-^2}}).$$

A direct calculation shows that $\sigma_1 < \lambda_1(U_-) \Leftrightarrow \omega_{ma} < \bar{\omega}_{ma} \Leftrightarrow c_- < \bar{c}_-$, while $c_- < \bar{c}_-$ is equivalent to

$$p'(\rho_{-}) < \frac{\rho_{+}}{\rho_{-}} \frac{p_{+} - p_{-}}{\rho_{+} - \rho_{-}} = \frac{\rho_{+}}{\rho_{-}} p'(\theta\rho_{+} + (1 - \theta)\rho_{-})$$

for some $\theta \in (0, 1)$, which is a corollary of the convexity of $p(\rho)$ and $\rho_+ > \rho_-$.

Case (33)–(34) \Rightarrow (32): We prove by contradiction. Instead, if $\rho_+ < \rho_-$, similarly to the previous case, we find that $\lambda_1(U_+) < \sigma_1$ is equivalent to $c_+ > \bar{c}_+$, i.e.,

$$p'(\rho_+) > \frac{\rho_-}{\rho_+} \frac{p_- - p_+}{\rho_- - \rho_+} = \frac{\rho_-}{\rho_+} p'(\theta \rho_+ + (1 - \theta)\rho_-)$$

for some $\theta \in (0, 1)$, which is impossible since $p''(\rho) > 0$ and $\rho_+ < \rho_-$.

In view of (29) and (32), for a shock wave, $u_0^2 + v_0^2 \ge \bar{c}_0^2 > c_0^2$, which indicates that the front state of a shock must be supersonic. Choosing a coordinate system so that $u_0 > 0$ and $v_0 = 0$, then we have $\frac{y}{x} = \sigma_j = (-1)^{\frac{j+1}{2}} \bar{c}_0 / \sqrt{u_0^2 - \bar{c}_0^2}$. Thus, 1-shocks and 3-shocks must be as shown in Fig. 2.



Fig. 2. Shock waves in the physical plane



Fig. 3. Shock polar and critical angle

Define the angle of the flow direction:

$$\omega = \arctan(v/u).$$

Then the shock curves $S_j(U_0), j = 1, 3$, in the (ω, ρ) -plane and (u, v)-plane form shock polars, respectively, as shown in Fig. 3. In general,

$$\omega_{crit} = \sup\{|\omega(u,v) - \omega(u_0,v_0)| : (u,v,\rho) \in S(U_0), c_*^2 < u^2 + v^2 < u_0^2 + v_0^2\}$$

where $S(U_0) = S_1(U_0) \cup S_3(U_0)$ is the shock polar associated with U_0 , similar to that shown above, and $c_* > 0$ is a constant such that $u^2 + v^2 \ge c^2(\rho)$ is equivalent to $u^2 + v^2 > c_*^2$ on $S(U_0)$.

2.3. Lateral Riemann Problem

The simplest case of problem (6) and (8)–(9) is $g \equiv 0$. It has been shown in [8] that, if $g \equiv 0$, then problem (6) admits an entropy solution that consists of the constant states U_- and U_+ , with $U_+ = (u_+, 0, \rho_+)$ and $u_+ > c_+ > 0$ in the subdomain of Ω , separated by a straight shockfront emanating from the vertex. That is to say that the state ahead of the shock-front is U_- , while the state behind the shock-front is U_+ (see Fig. 4). When the angle between the flow direction of the front state and the wedge boundary at a boundary vertex is larger than π , then an entropy solution contains a rarefaction wave that separates the front state from the back state (see Fig. 5 for the difference).

2.4. Riemann Problem Involving Only Weak Waves

Consider the Riemann problem for (14):

$$U|_{x=x_0} = \underline{U} = \begin{cases} U_a, \, y > y_0, \\ U_b, \, y < y_0, \end{cases}$$

$$\tag{42}$$

where U_a and U_b are constant states which are regarded as the *above* state and *below* state with respect to the line $y = y_0$, respectively.



Fig. 4. Unperturbed case when $g \equiv 0$



Fig. 5. Lateral Riemann solutions

Following Lax [14], we can parameterize any physically admissible wave curve in a neighborhood of a constant state U_+ , $O_{\varepsilon}(U_+)$, by $\alpha_j \mapsto \Phi_j(\alpha_j; U_b)$, with $\Phi \in C^2$, $\Phi_j|_{\alpha_j=0} = U_b$, and $\frac{\partial \Phi_j}{\partial \alpha_j}|_{\alpha_j=0} = \mathbf{r}_j(U_b)$. Set

$$\Phi(\alpha_3, \alpha_2, \alpha_1; U_b) := \Phi_3(\alpha_3; \Phi_2(\alpha_2; \Phi_1(\alpha_1; U_b))).$$

Here and in what follows, we denote $O_{\varepsilon}(W)$ is a universal ball with radius $M\varepsilon > 0$ and center W, where M > 0 is a universal constant depending only on the parameters in the system and possibly on the boundary function g(x) starting from Section 4.2, which may be different at each occurrence. Then we have

Lemma 2. There exits $\varepsilon > 0$ such that, for any states $U_a, U_b \in O_{\varepsilon}(U_+)$, the Riemann problem (42) admits a unique admissible solution consisting of three elementary waves. In addition, state U_a can be represented by

$$U_a = \Phi(\alpha_3, \alpha_2, \alpha_1; U_b)$$

with $\Phi|_{\alpha_1=\alpha_2=\alpha_3=0} = U_b$ and $\frac{\partial \Phi}{\partial \alpha_i}|_{\alpha_1=\alpha_2=\alpha_3=0} = \mathbf{r}_i(U_b), i = 1, 2, 3.$

Furthermore, we find that the renormalization factors $\kappa_j(U), j = 1, 3$, in (18) are positive in a neighborhood $O_{\varepsilon}(U_0)$ of any state $U_0 = (u_0, 0, \rho_0)$ with $u_0 > 0$.

Lemma 3. At any state $U_0 = (u_0, 0, \rho_0)$ with $u_0 > 0$,

$$\kappa_1(U_0) = \kappa_3(U_0) > 0,$$

which implies $\kappa_j(U) > 0$ for any $U \in O_{\varepsilon}(U_0)$ since $\kappa_j(U)$ are continuous for j = 1, 3.

At state $U_0 = (u_0, 0, \rho_0)$, it is straightforward to see that

$$\nabla_U \lambda_{10} \cdot (-\lambda_{10}, 1, \rho_0 u_0 \lambda_{10} / c_0^2) = \nabla_U \lambda_{30} \cdot (-\lambda_{30}, 1, \rho_0 u_0 \lambda_{30} / c_0^2) > 0.$$

Therefore, we have $\kappa_1(U_0) = \kappa_3(U_0) > 0$.

2.5. Riemann Problem Involving a Strong 1-Shock

For simplicity, we use the notation $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3)$ to denote that $U_a = \Phi(\alpha_3, \alpha_2, \alpha_1; U_b)$ throughout the paper. For any $U \in S_1(U_-)$, we also use $\{U_-, U\} = (\sigma, 0, 0)$ to denote the 1-shock that connects U_- and Uwith speed σ . Then we have

Lemma 4. Let $\{U_-, U_+\} = (\sigma_0, 0, 0)$ with $U_+ = (u_+, 0, \rho_+)$, $\rho_+ > \rho_-$, and $\gamma \ge 1$. Then

$$\sigma_0 < 0, \qquad u_+ < u_- < (1+1/\gamma)u_+.$$

Proof. From the Rankine-Hugoniot conditions (25)-(27), we have

$$\sigma_0(\rho_- u_- - \rho_+ u_+) = \rho_- v_-, \tag{43}$$

$$\sigma_0(\rho_- u_-^2 - \rho_+ u_+^2 + p_- - p_+) = \rho_- v_- u_-, \tag{44}$$

which implies $\sigma_0 = -\bar{c}_0/\sqrt{u_+^2 - \bar{c}_0^2} < 0$. Substituting (43) into (44), we have $p_+ - p_- = \rho_+ u_+ (u_- - u_+)$. By the entropy condition: $\rho_+ > \rho_-$, which implies $p_+ > p_-$. Then we have $u_- > u_+$. Furthermore, since $p_- > 0$, we have $p_+ > \rho_+ u_+ (u_- - u_+)$, which implies

$$\rho_+ c_+^2 / \gamma > \rho_+ u_+ (u_- - u_+).$$

Using $u_{+} > c_{+}$, we have $u_{+}/\gamma > u_{-} - u_{+}$ and thus $u_{-} < (1 + 1/\gamma)u_{+}$.

Moreover, it is direct to conclude

Lemma 5. There exists a neighborhood $O_{\varepsilon}(U_+)$ of U_+ such that the shock polar $S_1(U_-) \cap O_{\varepsilon}(U_+)$ can be parameterized by the shock speed σ as

$$\sigma \to G(\sigma)$$

with $G \in C^2$ near σ_0 and $G(\sigma_0) = U_+$.

The following lemma is essential to estimate the strengths of reflected weak waves in the interaction between the strong 1-shock and weak waves (see the proofs for Propositions 1–4).

Lemma 6. Set $A = \nabla_U H(U_+) - \sigma_0 \nabla_U W(U_+)$. Then

 $\det A < 0, \quad \det(A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+} < 0, \quad \det(A\mathbf{r}_3, A\mathbf{r}_2, AG_{\sigma}(\sigma_0))|_{U=U_+} < 0.$

Proof. A direct calculation shows that

$$A = \begin{pmatrix} -\sigma_0 \rho_+ & \rho_+ & -\sigma_0 u_+ \\ -2\sigma_0 \rho_+ u_+ & \rho_+ u_+ & -\sigma_0 (u_+^2 + c_+^2) \\ 0 & -\sigma_0 \rho_+ u_+ & c_+^2 \end{pmatrix},$$

and

$$\mathbf{r}_2(U_+) = (1,0,0)^{\top}, \quad \mathbf{r}_j(U_+) = \kappa_j(U_+)(-\lambda_{j+},1,\rho_+u_+\lambda_{j+}/c_+^2)^{\top}, \ j = 1,3.$$

Then we have

$$A\mathbf{r}_{2}(U_{+}) = -\sigma_{0}\rho_{+}(1, 2u_{+}, 0)^{\top},$$

$$A\mathbf{r}_{j}(U_{+}) = \frac{\kappa_{j}(U_{+})\rho_{+}(\lambda_{j+} - \sigma_{0})}{\lambda_{j+}}(1, u_{+}, u_{+}\lambda_{j+})^{\top}, \ j = 1, 3,$$

and

$$AG_{\sigma}(\sigma_0) = W(U_+) - W(U_-) = -\frac{\rho_- v_-}{\sigma_0} (1, u_-, \sigma_0 u_-)^{\top}.$$

Using Lemma 4, we can directly identify the signs of the following determinants:

$$\det A = \sigma_0 \rho_+^2 u_+ (\lambda_{1+}^2 - \sigma_0^2) (u_+^2 - c_+^2) < 0,$$

$$\det(A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+} = \frac{(\kappa_3(U_+))^2 \sigma_0 \rho_+^3 u_+^2}{\lambda_{1+} \lambda_{3+}} (\lambda_{3+} - \sigma_0)(\sigma_0 - \lambda_{1+})(\lambda_{1+} - \lambda_{3+}) < 0,$$

and

$$\begin{aligned} &\det(A\mathbf{r}_{3}, A\mathbf{r}_{2}, AG_{\sigma}(\sigma_{0}))|_{U=U_{+}} \\ &= \frac{\kappa_{3}(U_{+})\rho_{-}v_{-}\rho_{+}^{2}u_{+}}{\lambda_{3+}}(\lambda_{3+}-\sigma_{0})\left((2u_{+}-u_{-})\lambda_{3+}-\sigma_{0}u_{-}\right) < 0. \end{aligned}$$

3. Estimates on wave interactions and reflections

We now make estimates on wave interactions and reflections on the wedge and the strong 1-shock, respectively.

3.1. Estimates on Weak Wave Interactions

We first estimate the interactions among weak waves. We will use the following elementary identities, whose proofs are straightforward.

Lemma 7. (i) If $f \in C^1(\mathbb{R})$, then, for any $x \in \mathbb{R}$,

$$f(x) - f(0) = x \int_0^1 f_x(rx) dr;$$
(45)

(ii) If $f \in C^2(\mathbb{R}^2)$, then, for any $(x, y) \in \mathbb{R}^2$,

$$f(x,y) - f(x,0) - f(0,y) + f(0,0) = xy \int_0^1 \int_0^1 f_{xy}(rx,sy) drds.$$
(46)

Proposition 1. Suppose that U_b, U_m , and U_a are three states in a small neighborhood $O_{\varepsilon}(U_+)$ with

$$\{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3), \quad \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3), \quad \{U_b, U_a\} = (\gamma_1, \gamma_2, \gamma_3).$$

Then

$$\gamma_i = \alpha_i + \beta_i + O(1) \triangle(\alpha, \beta), \tag{47}$$

where $\triangle(\alpha,\beta) = |\alpha_3||\beta_1| + |\alpha_2||\beta_1| + |\alpha_3||\beta_2| + \sum_{j=1,3} \triangle_j(\alpha,\beta)$ with

$$\Delta_j(\alpha,\beta) = \begin{cases} 0, & \alpha_j \ge 0 \text{ and } \beta_j \ge 0, \\ |\alpha_j||\beta_j|, & otherwise. \end{cases}$$



Fig. 6. Weak wave interactions

Proof. First, Lemma 6 yields

$$\det\left(\frac{\partial \Phi(\gamma_3, \gamma_2, \gamma_1; U_b)}{\partial(\gamma_3, \gamma_2, \gamma_1)}\right)|_{\gamma_1 = \gamma_2 = \gamma_3 = 0} = \frac{1}{\det A} \det(A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U = U_+} > 0.$$

Then, by the implicit function theorem, there exists $(\gamma_3, \gamma_2, \gamma_1)$ as a C^2 -function of $(\beta_1, \beta_2, \beta_3, \alpha_3, \alpha_2, \alpha_1; U_b)$ so that

$$\Phi(\beta_3, \beta_2, \beta_1; \Phi(\alpha_3, \alpha_2, \alpha_1; U_b)) = \Phi(\gamma_3, \gamma_2, \gamma_1; U_b).$$

We omit U_b now for simplicity and will only compute γ_3 here since the estimates for γ_1 and γ_2 can be carried out in the same way. We can rewrite

$$\gamma_3 = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{split} I_1 &= \gamma_3(\beta_3, \beta_2, \beta_1, \alpha_3, \alpha_2, \alpha_1) - \gamma_3^a(\beta_3, \beta_2, \beta_1, 0, \alpha_2, \alpha_1) \\ &- \gamma_3^b(\beta_3, \beta_2, 0, \alpha_3, \alpha_2, \alpha_1) + \gamma_3(\beta_3, \beta_2, 0, 0, \alpha_2, \alpha_1), \\ I_2 &= \gamma_3^a(\beta_3, \beta_2, \beta_1, 0, \alpha_2, \alpha_3) - \gamma_3^c(\beta_3, \beta_2, \beta_1, 0, 0, \alpha_1) \\ &- \gamma_3(\beta_3, \beta_2, 0, 0, \alpha_2, \alpha_1) + \gamma_3(\beta_3, \beta_2, 0, 0, 0, \alpha_1), \\ I_3 &= \gamma_3^b(\beta_3, \beta_2, 0, \alpha_3, \alpha_2, \alpha_1) - \gamma_3(\beta_3, \beta_2, 0, 0, \alpha_2, \alpha_1) \\ &- \gamma_3^d(\beta_3, 0, 0, \alpha_3, \alpha_2, \alpha_1) + \gamma_3(\beta_3, 0, 0, 0, \alpha_2, \alpha_1), \\ I_4 &= \gamma_3^c(\beta_3, \beta_2, \beta_1, 0, 0, \alpha_1) + \gamma_3^d(\beta_3, 0, 0, \alpha_3, \alpha_2, \alpha_1) \\ &+ \gamma_3(\beta_3, \beta_2, 0, 0, \alpha_2, \alpha_1) - \gamma_3(\beta_3, \beta_2, 0, 0, 0, \alpha_1) \\ &- \gamma_3(\beta_3, 0, 0, 0, \alpha_2, \alpha_1). \end{split}$$

Note here we add the super-scriptions a, b, c, and d just to trace the terms. From (46), we have

$$I_1 = O(1)|\beta_1||\alpha_3|, \quad I_2 = O(1)|\beta_1||\alpha_2|, \quad I_3 = O(1)|\beta_2||\alpha_3|.$$

Now we estimate I_4 . First we have We rely on the implicit function theorem to obtain the uniqueness of the solution in a small neighborhood of U_+ , $O_{\varepsilon}(U_+)$, to get the following facts, which make the above decomposition possible:

$$\begin{split} &\gamma_3(\beta_3,\beta_2,\beta_1,0,0,0) = \gamma_3(\beta_3,\beta_2,0,0,0,0) = \beta_3, \\ &\gamma_3(0,0,0,\alpha_3,\alpha_2,\alpha_1) = \alpha_3, \gamma_3(0,0,0,0,\alpha_2,\alpha_1) = 0, \\ &\gamma_3(\beta_3,\beta_2,0,0,0,\alpha_1) = \gamma_3(\beta_3,0,0,0,\alpha_2,\alpha_1) = \beta_3. \end{split}$$

The implicit function theorem also implies $\gamma_3(\beta_3, \beta_2, 0, 0, \alpha_2, \alpha_1) = \beta_3$. Therefore,

$$I_4 = \beta_3 + \alpha_3 + O(1)(|\alpha_1||\beta_1| + |\alpha_3||\beta_3|).$$

Summing up I_1 , I_2 , I_3 , and I_4 , we get (47).

3.2. Estimates on the Weak Wave Reflections on the Boundary (see Fig. 7)

Denote $\{C_k(a_k, b_k)\}_{k=0}^{\infty}$ the points $\{(a_k, b_k)\}_{k=0}^{\infty}$ in the *xy*-plane with $a_{k+1} > a_k > 0$. Set

$$\omega_{k,k+1} = \arctan\left(\frac{b_{k+1} - b_k}{a_{k+1} - a_k}\right), \quad \omega_k = \omega_{k,k+1} - \omega_{k-1,k}, \quad \omega_{-1,0} = 0,
\Omega_{k+1} = \{(x,y) : x \in [a_k, a_{k+1}), \ y < b_k + (x - a_k) \tan(\omega_{k,k+1})\}, \quad (48)
\Gamma_{k+1} = \{(x,y) : x \in [a_k, a_{k+1}), \ y = b_k + (x - a_k) \tan(\omega_{k,k+1})\},$$

and the outer normal vector to Γ_k :

$$\mathbf{n}_{k+1} = \frac{-(b_{k+1} - b_k, a_{k+1} - a_k)}{\sqrt{(b_{k+1} - b_k)^2 + (a_{k+1} - a_k)^2}} = (-\sin(\omega_{k,k+1}), \cos(\omega_{k,k+1})).$$
(49)



Fig. 7. Weak wave reflections on the boundary

Then we consider the initial-boundary value problem:

$$\begin{cases} (14) & \text{in } \Omega_{k+1}, \\ U|_{x=a_k} = \underline{U}, \\ (u, v) \cdot \mathbf{n}_{k+1} = 0 \text{ on } \Gamma_{k+1}, \end{cases}$$

where \underline{U} is a constant state.

Proposition 2. Let $\{U_b, U_m\} = (\alpha_3, \alpha_2, 0)$ and $\{U_m, U_k\} = (0, 0, \beta_1)$ with

$$(u_k, v_k) \cdot \mathbf{n}_k = 0.$$

Then there exists U_{k+1} such that

$$\{U_b, U_{k+1}\} = (0, 0, \delta_1)$$
 with $(u_{k+1}, v_{k+1}) \cdot \mathbf{n}_{k+1} = 0.$

Furthermore,

$$\delta_1 = \beta_1 + K_{b3}\alpha_3 + K_{b2}\alpha_2 + K_{b0}\omega_k$$

where K_{b3}, K_{b2} , and K_{b0} are C^2 -functions of $(\alpha_3, \alpha_2, \beta_1, \omega_k; U_b)$ satisfying that

$$K_{b3}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 1, \quad K_{b2}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 0,$$

and K_{b0} is bounded.

Proof. Since

$$\frac{\partial}{\partial \delta_1} (\Phi(0,0,\delta_1;U_b) \cdot (\mathbf{n}_{k+1},0))|_{\{\delta_1=0,U_b=U_+,\omega_{k,k+1}=0\}}$$
(50)
= $\kappa_1(U_+)(-\lambda_{1+},1,\rho_+u_+\lambda_{1+}/c_+^2) \cdot (0,1,0) > 0,$

we know from the implicit function theorem that δ_1 can be solved as a C^2 -function of $(\alpha_3, \alpha_2, \beta_1, \omega_{k-1,k}, \omega_k; U_b)$ such that

$$\Phi(0, 0, \beta_1; \Phi(\alpha_3, \alpha_2, 0; U_b)) \cdot (\mathbf{n}_k, 0) = \Phi(0, 0, \delta_1; U_b) \cdot (\mathbf{n}_{k+1}, 0).$$
(51)

Since $\omega_{k-1,k}$ and U_b are constant, we will also omit U_b and $\omega_{k-1,k}$ and write $\delta_1 = \delta_1(\omega_k, \alpha_2, \alpha_3, \beta_1)$ for simplicity. Again, from (45), we can obtain

$$\delta_{1}(\omega_{k}, \alpha_{2}, \alpha_{3}, \beta_{1}) = \delta_{1}(\omega_{k}, \alpha_{2}, \alpha_{3}, \beta_{1}) - \delta_{1}(0, \alpha_{2}, \alpha_{3}, \beta_{1}) + \delta_{1}(0, \alpha_{2}, \alpha_{3}, \beta_{1}) - \delta_{1}(0, 0, \alpha_{3}, \beta_{1}) + \delta_{1}(0, 0, \alpha_{3}, \beta_{1}) - \delta_{1}(0, 0, 0, \beta_{1}) + \delta_{1}(0, 0, 0, \beta_{1}) = K_{b0}\omega_{k} + K_{b2}\alpha_{2} + K_{b3}\alpha_{3} + \beta_{1}.$$

Differentiating (51) with respect to α_3 and α_2 , respectively, and letting $\omega_k = \alpha_3 = \alpha_2 = \beta_1 = 0$ and $U_b = U_+$ yields

$$\mathbf{r}_{3}(U_{+})\cdot(0,1,0) = \frac{\partial\delta_{1}}{\partial\alpha_{3}}\mathbf{r}_{1}(U_{+})\cdot(0,1,0), \quad \mathbf{r}_{2}(U_{+})\cdot(0,1,0) = \frac{\partial\delta_{1}}{\partial\alpha_{2}}\mathbf{r}_{1}(U_{+})\cdot(0,1,0).$$

Hence, $K_{b3}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}}=1$ and $K_{b2}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}}=0$. It is clear that $K_{b0}=\frac{\partial \delta_1}{\partial \omega_k}$ is bounded. This completes the proof.

3.3. Estimate on the Boundary Perturbation of the Strong Shock

Proposition 3. For $\varepsilon > 0$, there exists $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon) < \varepsilon$ so that $G(O_{\hat{\varepsilon}}(\sigma_0)) \subset O_{\varepsilon}(U_+)$ and, when $|\omega_k| < \varepsilon$, the following equation

$$G(\sigma) \cdot (\mathbf{n}_k, 0) = 0 \tag{52}$$

admits a unique solution $\sigma_k \in O_{\hat{\varepsilon}}(\sigma_0)$. Moreover, we have

$$\sigma_{k+1} = \sigma_k + K_{bs}\omega_k + O(1)|\omega_k|^2, \tag{53}$$

where K_{bs} is bounded.

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Proof. We first show that there exists a solution $\sigma = \sigma(h)$ to the following equation:

$$G(\sigma) \cdot (-\sin h, \cos h, 0) = 0. \tag{54}$$

This can be seen as follows: Differentiating the quantity $G(\sigma) \cdot (-\sin h, \cos h, 0)$ with respect to σ , following the part of the calculations we had in Lemma 6, and denoting by (A_{ij}^*) the cofactor matrix of (a_{ij}) , we have

$$\begin{split} &\frac{\partial}{\partial \sigma} (G(\sigma) \cdot (-\sin h, \cos h, 0))|_{\{\sigma = \sigma_0, h = 0\}} \\ &= (A^{-1} \cdot AG_{\sigma}(\sigma_0)) \cdot (0, 1, 0) = \frac{1}{\det A} (A_{12}^*, A_{22}^*, A_{32}^*) \cdot AG_{\sigma}(\sigma_0) \\ &= \frac{\rho_{-} v_{-} \rho_{+}}{\det A} \left((u_{-} - 2u_{+}) c_{+}^2 + \sigma_0^2 u_{-} (c_{+}^2 - u_{+}^2) \right) > 0. \end{split}$$

Hence, by the implicit function theorem, there exists a unique C^2 -function $\sigma = \sigma(h)$ with $\sigma(0) = \sigma_0$, which solves (54) in some neighborhood of $(\sigma, h) =$ $(\sigma_0, 0)$. Then

$$\sigma(\Omega_j) = \sigma_j, \qquad j = k, k+1,$$

and, by the Taylor expansion formula, we have the desired estimates (53).

3.4. Estimates on the Interactions Between the Strong Shock and Weak Waves

Proposition 4. Let $U_m, U_a \in O_{\varepsilon}(U_+)$ with

$$\{G(\sigma), U_m\} = (0, \alpha_2, \alpha_3), \qquad \{U_m, U_a\} = (\beta_1, \beta_2, 0).$$

Then there exists a unique $(\sigma', \delta_2, \delta_3)$ such that the Riemann problem (42) with $U_b = U_-$ admits an admissible solution that consists of a strong 1-shock of strength σ' , a contact discontinuity of strength δ_2 , and a weak 3-wave of strength δ_3 :

$$\{U_-, U_a\} = (\sigma', \delta_2, \delta_3).$$

Moreover,

$$\delta_3 = \alpha_3 + K_{s3}\beta_1 + O(1)\Delta, \\ \delta_2 = \alpha_2 + \beta_2 + K_{s2}\beta_1 + O(1)\Delta, \\ \sigma' = \sigma + K_{s1}\beta_1 + O(1)\Delta,$$
(55)

where

$$|K_{s3}| < 1,$$
 (56)

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 $|K_{s2}|$ and $|K_{s3}|$ are bounded, and $\Delta = |\alpha_3||\beta_1| + |\alpha_2||\beta_1| + |\alpha_3||\beta_2|$. Furthermore, we can write the estimates in a more precise fashion:

$$\sigma' = \sigma + \widetilde{K_{s1}}\beta_1 + O(1)|\alpha_3||\beta_2|, \quad \delta_2 = \alpha_2 + \beta_2 + \widetilde{K_{s2}}\beta_1 + O(1)|\alpha_3||\beta_2|,$$

$$\delta_3 = \alpha_3 + \widetilde{K_{s3}}\beta_1 + O(1)|\alpha_3||\beta_2|, \tag{57}$$

where

$$|\widetilde{K_{s3}}| < 1, \qquad |\widetilde{K_{s1}}| + |\widetilde{K_{s2}}| \le M \qquad for some \ constant \ M > 0.$$
 (58)



Fig. 8. Interactions between the strong shock and weak waves

Proof. We first show that there exists a unique solution

$$(\sigma', \delta_2, \delta_3) = (\sigma'(\sigma, \alpha_2, \alpha_3, \beta_1, \beta_2), \delta_2(\sigma, \alpha_2, \alpha_3, \beta_1, \beta_2), \delta_3(\sigma, \alpha_2, \alpha_3, \beta_1, \beta_2))$$
to

$$\Phi(0,\beta_2,\beta_1;\Phi(\alpha_3,\alpha_2,0;G(\sigma))) = \Phi(\delta_3,\delta_2,0;G(\sigma')).$$
(59)

By Proposition 1, there exists $(\gamma_3, \gamma_2, \gamma_1)$ such that

$$\Phi(0,\beta_2,\beta_1;\Phi(\alpha_3,\alpha_2,0;G(\sigma))) = \Phi(\gamma_3,\gamma_2,\gamma_1;G(\sigma))$$
(60)

with $\gamma_1 = \beta_1 + O(1)\Delta$, $\gamma_2 = \beta_2 + \alpha_2 + O(1)\Delta$, and $\gamma_3 = \alpha_3 + O(1)\Delta$. Thus, (59) can be reduced to

$$\Phi(\gamma_3, \gamma_2, \gamma_1; G(\sigma)) = \Phi(\delta_3, \delta_2, 0; G(\sigma')).$$
(61)

Furthermore, Lemma 6 implies

$$\det\left(\frac{\partial\Phi(\delta_3,\delta_2,0;G(\sigma'))}{\partial(\delta_3,\delta_2,\sigma')}\right)|_{\{\delta_3=\delta_2=0,\sigma'=\sigma_0\}}$$
$$=\frac{1}{\det A}\det(A\mathbf{r}_3(U_+),A\mathbf{r}_2(U_+),AG_{\sigma}(\sigma_0))>0.$$

Therefore, the implicit function theorem implies that $(\delta_3, \delta_2, \sigma')$ can be solved as a C^2 -function of $(\gamma_1, \gamma_2, \gamma_3, \sigma)$ uniquely:

$$\delta_3 = \delta_3(\gamma_3, \gamma_2, \gamma_1, \sigma), \quad \delta_2 = \delta_2(\gamma_3, \gamma_2, \gamma_1, \sigma), \quad \sigma' = \sigma'(\gamma_3, \gamma_2, \gamma_1, \sigma).$$

Using identity (45), we find that, for i = 2, 3,

$$\delta_i = \delta_i(\gamma_3, \gamma_2, \gamma_1, \sigma) - \delta_i(\gamma_i, \gamma_2, 0, \sigma) + \delta_i(\gamma_3, \gamma_2, 0, \sigma) = K_{si}\gamma_1 + \gamma_i,$$

$$\sigma' = \sigma'(\gamma_3, \gamma_2, \gamma_1, \sigma) - \sigma'(\gamma_3, \gamma_2, 0, \sigma) + \sigma'(\gamma_3, \gamma_2, 0, \sigma) = K_{s1}\gamma_1 + \sigma,$$

where $K_{s1} = \int_0^1 \partial_{\gamma_1} \sigma'(\gamma_3, \gamma_2, \lambda \gamma_1, \sigma) d\lambda$ and $K_{si} = \int_0^1 \partial_{\gamma_1} \delta_i(\gamma_3, \gamma_2, \lambda \gamma_1, \sigma) d\lambda$. When $\gamma_3 = \gamma_2 = \gamma_1 = 0$, it is clear that $|\frac{\partial \delta_3}{\partial \gamma_1}|, |\frac{\partial \delta_2}{\partial \gamma_1}|$, and $|\frac{\partial \sigma'}{\partial \gamma_1}|$ are bounded. We can further claim the important feature that

$$|\frac{\partial \delta_3}{\partial \gamma_1}| < 1$$
 when $\gamma_3 = \gamma_2 = \gamma_1 = 0$

This can be shown by differentiating (61) with respect to γ_1 and let $\gamma_3 =$ $\gamma_2 = \gamma_1 = 0$ to obtain

$$\mathbf{r}_1(U_+) = \mathbf{r}_3(U_+)\frac{\partial\delta_3}{\partial\gamma_1} + \mathbf{r}_2(U_+)\frac{\partial\delta_2}{\partial\gamma_1} + G_\sigma(\sigma_0)\frac{\partial\sigma'}{\partial\gamma_1}.$$
 (62)

Multiplying both sides by A which is defined in Lemma 6, we have

$$\begin{aligned} \frac{\partial \delta_3}{\partial \gamma_1} | &= |\frac{\det(A\mathbf{r}_1(U_+), A\mathbf{r}_2(U_+), AG_{\sigma}(\sigma_0))}{\det(A\mathbf{r}_3(U_+), A\mathbf{r}_2(U_+), AG_{\sigma}(\sigma_0))}| \\ &= |\frac{\lambda_{3+} + \sigma_0}{\lambda_{3+} - \sigma_0}||\frac{(2u_+ - u_-)\lambda_{3+} + \sigma_0u_-}{(2u_+ - u_-)\lambda_{3+} - \sigma_0u_-}| < 1. \end{aligned}$$

Combining these with the estimates we had for γ_1, γ_2 , and γ_3 , we conclude the proof.

4. Approximate Solutions

In this section, we develop a modified Glimm difference scheme to construct a family of approximate solutions and establish their necessary estimates for the initial-boundary value problem (1) and (8)-(9) in the corresponding approximate domains $\Omega_{\Delta x}$.

4.1. A Modified Glimm Scheme

To define the scheme more clearly, we first use the fact that the boundary is a perturbation of the straight wedge:

$$\sup_{x \ge 0} |g'(x)| < \varepsilon \qquad \text{for sufficiently small } \varepsilon > 0.$$

For any $\Delta x \ge 0$, set $a_k := k \Delta x$ and $b_k := y_k = g(k \Delta x)$ in (48) and (49), and follow the notations in Subsection 3.2 (also see Fig. 7). Then

$$m := \sup_{k>0} \left\{ \frac{y_k - y_{k-1}}{\Delta x} \right\} < \varepsilon.$$
(63)

Define

$$\Omega_{\Delta x} = \bigcup_{k \ge 0} \Omega_{\Delta x, k},$$

where $\Omega_{\Delta x,k} = \{(x,y) : (k-1)\Delta x \leq x < k\Delta x, y \leq y_{k-1} + (x - (k - 1)\Delta x) \tan(\omega_{k-1,k})\}$. We also need the Courant-Friedrichs-Lewy type condition: $\frac{\Delta y - m\Delta x}{|x| + |x|} \leq |x_{k}| + \max(|x_{k}| + |x|)|$

$$\frac{y - m\Delta x}{\Delta x} < |\sigma_0| + \max_{j=1,3} (\sup_{U \in O_{\varepsilon}(U_+)} |\lambda_j(U)|).$$

Define

$$a_{k,n} = (2n+1+\theta_k)\Delta y + y_k$$

where θ_k is randomly chosen in (-1, 1). Then we choose

$$p_{k,n} = (k\Delta x, a_{k,n}), \qquad k \ge 0, n = 0, -1, -2, ...,$$

to be the mesh points and define the approximate solutions $U_{\Delta x,\theta}$ in $\Omega_{\Delta x}$ for any $\theta = (\theta_0, \theta_1, \theta_2, ...)$ in an inductive way:

For k = 0, we define $U_{\Delta x,\theta}$ in $\{0 \le x < \Delta x\} \cap \Omega_{\Delta x}$ to be the strong 1-shock solution starting from $U_{\Delta x,\theta}|_{\{x=0,y<0\}} = U_{-}$.

Assume that $U_{\Delta x,\theta}$ has been constructed for $\{0 \le x < k\Delta x\}$. Denoting, for $n \le -1$,

$$U_k^0 := U_{\Delta x,\theta}(k\Delta x - a_{k,n}) \quad \text{if } y \in (y_k + 2n\Delta y, y_k + (2n+2)\Delta y),$$

then we define $U_{\Delta x,\theta}$ in $\{k\Delta x \leq x < (k+1)\Delta x\}$ as follows: We first solve the following lateral Riemann problem in diamond $T_{k,0}$, whose vertices are $((k+1)\Delta x, y_{k+1}), ((k+1)\Delta x, -\Delta y + y_{k+1}), (k\Delta x, y_k), \text{ and } (k\Delta x, -\Delta y + y_k):$

$$\begin{cases} W(U_k)_x + H(U_k)_y = 0 & \text{in } T_{k,0} \\ U_k|_{x=k\Delta x} = U_k^0, \\ (u_k, v_k) \cdot \mathbf{n}_k = 0 & \text{on } \Gamma_k \end{cases}$$

to obtain the lateral Riemann solution U_k in $T_{k,0}$ as constructed in Section 2.3 and define

$$U_{\Delta x,\theta} = U_k \qquad \text{in } T_{k,0}.$$

Then we solve the following Riemann problem in each diamond $T_{k,n}$ for $n \leq -1$, whose vertices are $((k+1)\Delta x, (2n-1)\Delta y + y_{k+1}), ((k+1)\Delta x, (2n+1)\Delta y + y_{k+1}), (k\Delta x, (2n-1)\Delta x + y_k),$ and $(k\Delta x, (2n+1)\Delta y + y_k)$:

$$\begin{cases} W(U_k)_x + H(U_k)_y = 0 & \text{in } T_{k,n}, \\ U_k|_{x=k\Delta x} = U_k^0, \end{cases}$$

to obtain the Riemann solution $U_k(x, y)$ in $T_{k,n}$ as constructed in Sections 2.4–2.5, and define

$$U_{\Delta x,\theta} = U_k$$
 in $T_{k,n}, n \le -1$.

In this way, we have constructed the approximate solutions $U_{\Delta x,\theta}(x,y)$ globally provided that we can obtain a uniform bound of the approximate solutions.

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4.2. Glimm-Type Functional and Its Bounds

In this section, we prove that the approximate solutions can be well defined in $\Omega_{\Delta x}$ indeed via the steps in Section 4.1 by providing a uniform bound for them. First, we introduce the following lemma.

Lemma 8. (i) If $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3)$ with $U_b, U_a \in O_{\varepsilon}(U_+)$, then

$$|U_b - U_a| \le s_1(|\alpha_1| + |\alpha_2| + |\alpha_3|)$$

with $s_1 = \max_{1 \le i \le 3} \left(\sup_{U \in O_{\varepsilon}(U_+)} |\partial_{\alpha_i} \Phi(\alpha_3, \alpha_2, \alpha_1; U)| \right);$ (ii) For any $\sigma \in O_{\widehat{\varepsilon}}(\sigma_0) \subset O_{\varepsilon}(U_+),$

$$|G(\sigma) - G(\sigma_0)| \le s_2 |\sigma - \sigma_0|$$

with $s_2 = \max_{\sigma \in O_{\varepsilon}(\sigma_0)} \{ G'_{\sigma}(\sigma) \}.$

Next, we show that $U_{\Delta x,\theta}$ can be globally defined. Assume that $U_{\Delta x,\theta}$ has been defined in $\{x < k\Delta x\} \cap \Omega_{\Delta x}$ by the steps in Section 4.1 and assume that the following conditions are satisfied:

 $C_1(k-1): \begin{cases} \text{In each } \Omega_{\Delta x,j} \text{ for } 0 \leq j \leq k-1, \text{ there is a strong 1-shock } S_*(\sigma_{(j)}) \\ \text{in } U_{\Delta x,\theta} \text{ with speed } \sigma_{(j)} \in O_{\hat{\varepsilon}}(\sigma_0), \text{ which divides } \Omega_{\Delta x,j} \text{ into two} \\ \text{parts: } \Omega^+_{\Delta x,j} \text{ and } \Omega^-_{\Delta x,j}, \text{ where } \Omega^+_{\Delta x,j} \text{ is the part bounded by} \\ S_*(\sigma_{(j)}) \text{ and } \Gamma_j = \{y = g(x, j, \Delta x)\}; \end{cases}$

$$C_2(k-1): U_{\Delta x,\theta}|_{\Omega^+_{\Delta x,j}} \in O_{\varepsilon}(U_+) \text{ and } U_{\Delta x,\theta}|_{\Omega^-_{\Delta x,j}} = U_- \text{ for } 0 \le j \le k-1;$$

 $C_3(k-1): \begin{cases} \{S_*(\sigma_{(j)})\}_{j=0}^{k-1} \text{ forms an approximate 1-characteristic } \chi_{\Delta x,\theta}: y = \chi_{\Delta x,\theta}(x), \text{ which emanating from the origin.} \end{cases}$

Here and in what follows, we use $S_*(\sigma_{(j)})$ to denote the strong 1-shock with speed $\sigma_{(j)}$. Then we prove that $U_{\Delta x,\theta}$ can be defined in $\Omega_{\Delta x,k}$ and satisfies $C_1(k), C_2(k)$, and $C_3(k)$.

From the construction steps in Section 4.1, we first define $U_{\Delta x,\theta}$ and the strong 1-shock $S_*(\sigma_{(k)})$ in $\Omega_{\Delta x,k}$. Then there exists a diamond $\Lambda_{k,n(k)}$ such that $S_*(\sigma_{(k-1)})$ enters $\Lambda_{k,n(k)}$ and $S_*(\sigma_{(k)})$ emanates from the center of $\Lambda_{k,n(k)}$. We extend $\chi_{\Delta x,\theta}$ to $\Omega_{\Delta x,k}$ such that $\chi_{\Delta x,\theta} = S_*(\sigma_{(k)})$ in $\Omega_{\Delta x,k}$ and define $\Omega^-_{\Delta x,j}$ and $\Omega^+_{\Delta x,j}$ in the same way as in $C_1(k-1)$. Then it suffices to impose some conditions so that $C_2(k-1)$ holds and $\sigma_{(k)} \in O_{\hat{\varepsilon}}(\sigma_0)$.

To achieve this, we establish the bound on the total variation of $U_{\Delta x,\theta}$ on a class of space-like curves. Denote by

$$N(\theta_{k+1}, n) = \begin{cases} P_{k+1,n} & \text{if } \theta_{k+1} \le 0, \\ P_{k+1,n-1} & \text{if } \theta_{k+1} > 0, \end{cases} \quad S(\theta_k, n) = \begin{cases} P_{k-1,n-1} & \text{if } \theta_k \le 0, \\ P_{k-1,n} & \text{if } \theta_k > 0. \end{cases}$$

Then we introduce

Definition 2. A j-mesh curve J is defined to be an unbounded space-like curve lying in the strip $\{(j-1)\Delta x \leq x \leq (j+1)\Delta x\}$ and consisting of the segments of the form $P_{k,n-1}N(\theta_{k+1},n)$, $P_{k,n-1}S(\theta_k,n)$, $S(\theta_k,n)P_{k,n}$, and $N(\theta_{k+1},n)P_{k,n}$.

This means that we connect the mesh point $P_{k,n}$ by two line segments to the two mesh points $P_{k-1,n-1}$ and $P_{k-1,n}$ if $\theta_k \leq 0$, or we connect the mesh point $P_{k,n}$ by two line segments to the two mesh points $P_{k-1,n}$ and $P_{k-1,n+1}$ if $\theta_k > 0$ (see Fig. 9).



Fig. 9. Interaction diamond $\Lambda_{k,n}$ and orientation of the segments

Clearly, for any k > 0, each k-mesh curve I divides the plane \mathbb{R}^2 into I^+ part and I^- part, where I^- is the one containing the set $\{x < 0\}$. As in Glimm [10], we also partially order these mesh curves by saying J > I if every point of the mesh curve J is either on I or contained in I^+ , and we call J an immediate successor to I if J > I and every mesh point of J except one is on I.

With these mesh curves J, we associate the Glimm-type functional $F_s(J)$:

Definition 3. We define

$$F_s(J) = C^* |\sigma^J - \sigma_0| + F(J)$$

with

$$\begin{split} F(J) &= L(J) + KQ(J), \\ L(J) &= K_0^* L_0(J) + L_1(J) + K_2^* L_2(J) + K_3^* L_3(J), \\ Q(J) &= \sum \{ |\alpha_i| |\beta_j| : \text{both } \alpha_i \text{ and } \beta_j \text{ cross } J \text{ and approach} \} \end{split}$$

and

$$L_0(J) = \sum \{ |\omega(C_k)| : C_k \in \Omega_J \}, \quad L_j(J) = \sum \{ |\alpha_j| : \alpha_j \text{ crosses } J \}, \ j = 1, 2, 3,$$

where K and C^* will be defined later, while Ω_J is the set of the corner points C_k lying in J^+ :

$$\Omega_J = \{ C_k \in J^+ \cap \partial \Omega_{\Delta x} : k \ge 0 \},\$$

 σ^J stands for the speed of the strong shock crossing J, and K_0^*, K_2^* and K_3^* are the constants that satisfy the following conditions:

$$K_0^* > |K_{b0}|, \quad |K_{b2}| < K_2^* < \frac{1 - |K_{s3}|K_3^*}{|K_{s2}|}, \quad |K_{b3}| < K_3^* < \frac{1}{K_{s3}}$$

which can be achieved from our discussions of the properties of K_{bi} and K_{si} , i = 1, 2, 3, as in Propositions 3.2–3.4 in Section 3.

As indicated in Section 2.4, from now on, we denote M > 0 a universal constant, depending only the parameters in the system and the boundary function g(x), which may be different at each occurrence. We now prove the decreasing property of our functional F_s . We first have

Proposition 5. Suppose that the wedge function g(x) satisfies (63), and I and J are two k-mesh curves such that J is an immediate successor of I. Suppose that

$$|U_{\Delta x,\theta}|_{I\cap (\Omega^+_{\Delta x,k-1}\cup \Omega^+_{\Delta x,k})} - U_+| < \varepsilon, \qquad |\sigma^I - \sigma_0| < \hat{\varepsilon}(\varepsilon),$$

where $\hat{\varepsilon}(\varepsilon)$ is determined in Proposition 3 and Lemma 8. Then there exist constants $\tilde{\varepsilon} > 0, K > 0$, and $C^* > 1$, depending only on the system in (6) and states U_- and U_+ , such that, if $F_s(I) \leq \tilde{\varepsilon}$, then

$$F_s(J) \le F_s(I),$$

and hence

$$|U_{\Delta x,\theta}|_{J\cap(\Omega^+_{\Delta x,k-1}\cup\Omega^+_{\Delta x,k})} - U_+| < \varepsilon, \qquad |\sigma^J - \sigma_0| < \hat{\varepsilon}(\varepsilon).$$

Proof. Let Λ be the diamond that is formed by I and J. We can always assume that $I = I_0 \cup I'$ and $J = J_0 \cup J'$ such that $\partial \Lambda = I' \cup J'$. We divide our proof into four cases depending on the location of the diamond.

Case 1 (interior weak-weak interaction): Λ lies in the interior of $\Omega_{\Delta x}$ and does not touch $\chi_{\Delta x,\theta}$. Then only weak waves enter Λ . Denote $Q(\Lambda) = \Delta(\alpha,\beta)$ defined as in Proposition 1. Then, for some constant M > 0,

$$L(J) - L(I) \le (1 + K_2^* + K_3^*) MQ(\Lambda).$$

Since $L(I_0) < \tilde{\varepsilon}$ from $F_s(I) < \tilde{\varepsilon}$, we have

$$Q(J) - Q(I) = Q(I_0) + Q(\gamma_1, I_0) + Q(\gamma_2, I_0) + Q(\gamma_3, I_0) -(Q(I_0) + Q(\Lambda) + Q(\alpha_1, I_0) + Q(\alpha_2, I_0) + Q(\alpha_3, I_0) +Q(\beta_1, I_0) + Q(\beta_2, I_0) + Q(\beta_3, I_0)) \leq Q(MQ(\Lambda), I_0) - Q(\Lambda) = (ML(I_0) - 1)Q(\Lambda) \leq -\frac{1}{2}Q(\Lambda).$$

Hence, by choosing suitably large K, we obtain

$$F(J) - F(I) \le ((1 + K_2^* + K_3^*)M - K/2)Q(\Lambda) \le -\frac{1}{4}Q(\Lambda).$$



Fig. 10. Case 2: near the boundary

Case 2 (near the boundary): Λ touches the approximate boundary $\partial \Omega_{\Delta x}$ and is away from the strong shock $\chi_{\Delta x,\theta}$. Then $\Omega_J = \Omega_I \setminus \{C_k\}$ for certain k and $\sigma^I = \sigma^J$.

Let δ_1 be the weak 1-wave going out of Λ through J', and let β_1 , α_2 , and α_3 be the weak waves entering Λ through I', as shown in Fig. 10. Then

$$\begin{split} L_{0}(J) - L_{0}(I) &= -|\omega_{k}|, \\ L_{2}(J) - L_{2}(I) &= \sum_{\gamma_{2} \text{ crosses } I_{0}} |\gamma_{2}| - (|\alpha_{2}| + \sum_{\gamma_{2} \text{ crosses } I_{0}} |\gamma_{2}|) = -|\alpha_{2}|, \\ L_{3}(J) - L_{3}(I) &= \sum_{\gamma_{3} \text{ crosses } I_{0}} |\gamma_{3}| - (|\alpha_{3}| + \sum_{\gamma_{3} \text{ crosses } I_{0}} |\gamma_{3}|) = -|\alpha_{3}|, \\ L_{1}(J) - L_{1}(I) &= (|\delta_{1}| + \sum_{\gamma_{1} \text{ crosses } I_{0}} |\gamma_{1}|) - (|\beta_{1}| + \sum_{\gamma_{1} \text{ crosses } I_{0}} |\gamma_{1}|) \\ &= |\delta_{1}| - |\beta_{1}| \leq |K_{b3}| |\alpha_{3}| + |K_{b2}| |\alpha_{2}| + |K_{b0}| |\omega_{k}|, \end{split}$$

where the last step is from Proposition 2. Thus,

$$L(J) - L(I) \le (|K_{b0}| - K_0^*)|\omega_k| + (|K_{b2}| - K_2^*)|\alpha_2| + (|K_{b3}| - K_3^*)|\alpha_3|.$$

From our requirement in Definition 3, we find $L(J) - L(I) \leq 0$. Since $F_s(I) \leq \tilde{\varepsilon}$ implies $L(I) \leq \tilde{\varepsilon}$, the higher order term Q(I) can always be bounded by the linear term L(I). Then we can easily conclude that $F(J) \leq F(I)$.

Case 3 (near the wedge vertex): Λ covers a part of $\partial \Omega_{\Delta x}$, and $S_*(\sigma_{(k-1)})$ emanates from $\{C_{k-1}\}$ and enters Λ . Then, from our construction, we find $\Omega_J = \Omega_I \setminus \{C_k\}, S_*(\sigma_{(k)})$ emanates from $\{C_k\}$ and crosses $J, \sigma^I = \sigma_{(k-1)}$,

and $\sigma^J=\sigma_{(k)}.$ Moreover, there is no weak wave crossing I' or J'. Then we have

$$F(J) - F(I) \le -K_0^* |\omega_k|.$$

Since

$$|\sigma^J - \sigma_0| - |\sigma^I - \sigma_0| \le |\sigma^J - \sigma^I| \le |K_{bs}||\omega_k| + M|\omega_k|^2,$$

we can further choose C^* suitably small and $\tau > 0$ such that

$$F_s(J) - F_s(I) \le C^* |\sigma^J - \sigma^I| + F(J) - F(I) \le -\tau |\omega_k|.$$

Case 4 (near the strong 1-shock): Λ lies in the interior of $\Omega_{\Delta x}$, and the strong 1-shock $S_*(\sigma_{(k-1)})$ enters Λ . Then $S_*(\sigma_{(k)})$ is generated from the inside of Λ , $\sigma^I = \sigma_{(k-1)}$, and $\sigma^J = \sigma_{(k)}$.



Fig. 11. Case 4: near the strong 1-shock

Let δ_3 and δ_2 be the weak waves going out of Λ through J', and let $\alpha_3, \alpha_2, \beta_1$, and β_2 be the weak waves entering Λ through I', as shown in Fig. 11. Then

$$\begin{split} L_1(J) - L_1(I) &= \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1| - (|\beta_1| + \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1|) = -|\beta_1|, \\ L_2(J) - L_2(I) &= (|\delta_2| + \sum_{\gamma_2 \text{ crosses } I_0} |\gamma_2|) - (|\alpha_2| + |\beta_2| + \sum_{\gamma_2 \text{ crosses } I_0} |\gamma_2|) \\ &\leq |K_{s2}||\beta_1| + M|\alpha_3||\beta_2|, \end{split}$$

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$$L_{3}(J) - L_{3}(I) = (|\delta_{3}| + \sum_{\gamma_{3} \text{ crosses } I_{0}} |\gamma_{3}|) - (|\alpha_{3}| + \sum_{\gamma_{3} \text{ crosses } I_{0}} |\gamma_{3}|)$$

$$\leq |K_{s3}||\beta_{1}| + M|\alpha_{3}||\beta_{2}|,$$

where we have used the estimates in Proposition 4.

This case is more complicated, which requires a careful calculation of Q(J) - Q(I). For simplicity, for any weak wave γ , we denote

$$Q(\gamma, I_0) = |\gamma| \sum \{ |\gamma_j| : \gamma_j \text{ and } \gamma \text{ approach} \}.$$

Then

$$\begin{aligned} Q(J) - Q(I) &= Q(I_0) + Q(\delta_3, I_0) + Q(\delta_2, I_0) - (Q(I_0) + |\beta_1| |\alpha_2| + |\alpha_3| |\beta_1| \\ &+ |\alpha_3| |\beta_2| + Q(\alpha_2, I_0) + Q(\alpha_3, I_0) + Q(\beta_1, I_0) + Q(\beta_2, I_0)) \\ &\leq - (|\beta_1| |\alpha_2| + |\alpha_3| |\beta_1| + |\alpha_3| |\beta_2|) + \left(|\widetilde{K_{s3}}| + |\widetilde{K_{s2}}| - 1 \right) Q(\beta_1, I_0) \\ &+ Q(M |\alpha_3| |\beta_2|, I_0) \\ &\leq (-1 + ML(I_0)) |\alpha_3| |\beta_2| + (-|\alpha_2| - |\alpha_3| + ML(I_0)) |\beta_1|. \end{aligned}$$

Again, since $L(I_0) \leq \tilde{\varepsilon}$ sufficiently small, then

$$Q(J) - Q(I) \le -\frac{1}{2}|\alpha_3||\beta_2| + ML(I_0)|\beta_1|.$$

Therefore, we have

$$\begin{split} F(J) - F(I) &\leq (-1 + K_2^* |K_{s2}| + K_3^* |K_{s3}|) \, |\beta_1| + M |\alpha_3| |\beta_2| \\ &+ K \left(-\frac{1}{2} |\alpha_3| |\beta_2| + M L(I_0) |\beta_1| \right) \\ &\leq -\frac{1}{4} |\beta_1| + M L(I_0) |\beta_1| - \frac{1}{8} |\alpha_3| |\beta_2| \\ &\leq -\frac{1}{8} (|\beta_1| + |\alpha_3| |\beta_2|), \end{split}$$

where we have chosen suitably large K and used the fact that $L(I_0) \leq \tilde{\varepsilon}$. Furthermore, since $|\sigma^J - \sigma^I| \leq |K_{s1}| |\beta_1| + M |\alpha_3| |\beta_2|$, we can further choose C^* suitably small such that

$$F_s(J) - F_s(I) \le C^* |\sigma^J - \sigma^I| + F(J) - F(I) \le -\frac{1}{16} |\beta_1| - \frac{1}{16} |\alpha_3| |\beta_2|.$$

Again, we have $F(J) \leq F(I)$.

Then, from Lemma 4.1, there exists $\tilde{\epsilon} > 0$ such that, when $F(I) < \tilde{\epsilon}$, we have $|U - U_+| < \epsilon$.

Let I_k be the k-mesh curve lying in $\{(j-1)\Delta x \leq x \leq j\Delta x\}$. From Proposition 5, we obtain the following theorem for any $k \geq 1$.

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Theorem 1. Suppose that the function g(x) satisfies (63). Let $\varepsilon, \tilde{\varepsilon}, \hat{\varepsilon}(\varepsilon), K$, and C^* be the constants specified in Proposition 5. If the induction hypotheses $C_1(k-1), C_2(k-1)$, and $C_3(k-1)$ hold and if $F_s(I_{k-1}) \leq \tilde{\varepsilon}$, then

$$|U_{\Delta x,\theta}|_{\Omega^+_{\Delta x,k}} - U_+| < \varepsilon, \quad U_{\Delta x,\theta}|_{\Omega^-_{\Delta x,k}} = U_-, \quad |\sigma_k - \sigma_0| < \hat{\varepsilon}(\varepsilon),$$

and

$$F_s(I_k) \le F_s(I_{k-1}). \tag{64}$$

Moreover, we obtain

Theorem 2. There exists $\varepsilon > 0$ such that, if $TV(g'(\cdot)) < \varepsilon$, then, for any $\theta \in \prod_{k=0}^{\infty} (-1,1)$ and every $\Delta x > 0$, the modified Glimm scheme defines a family of global approximate solutions $U_{\Delta x,\theta}$ and the corresponding family of approximate strong 1-shocks $\chi_{\Delta x,\theta}$ in $\Omega_{\Delta x,\theta}$ which satisfy $C_1(k-1), C_2(k-1), C_3(k-1)$, and (64) for any $k \ge 1$. In addition,

$$TV\{U_{\Delta x,\theta}(k\Delta x-,\cdot):(-\infty,y_k]\} \le CTV(g'(\cdot))$$

for any $k \ge 0$ and

$$|\chi_{\Delta x,\theta}(x+h) - \chi_{\Delta x,\theta}(x)| \le (|\sigma_0| + M)|h| + 2\Delta x$$

for any $x \ge 0$ and h > 0, where the constant C depends only on the bound M, K, C^* , and K_i^* , i = 0, 2, 3.

4.3. Estimates on the Approximate Shock-Fronts

We use the notations and estimates in the previous section and define

$$\sigma_{\Delta x,\theta}(x) = \sigma_{(k)}$$
 if $x \in (k\Delta x, (k+1)\Delta x]$.

From Proposition 4, we have

Lemma 9. There exists a constant M, independent of $\Delta x, \theta$, and $U_{\Delta x, \theta}$, such that

$$TV\{\sigma_{\Delta x,\theta} : [0,\infty)\} = \sum_{k=0}^{\infty} |\sigma_{(k+1)} - \sigma_{(k)}| \le M.$$

Proof. For any $k \ge 1$ and any interaction diamond $\Lambda \subset \{(k-1)\Delta x \le x \le (k+1)\Delta x\}$, define

$$E_{\Delta x,\theta}(\Lambda) = \begin{cases} 0 & \text{for Case 1} \\ |\omega_k| + |\alpha_2| + |\alpha_3| & \text{for Case 2} \\ |\omega_k| & \text{for Case 3} \\ |\beta_1| & \text{for Case 4} \end{cases}$$

and

$$Q_{\Delta x,\theta}(\Lambda) = \begin{cases} Q(\Lambda) & \text{for Case 1,} \\ 0 & \text{for Case 2,} \\ |\omega|^2 & \text{for Case 3,} \\ |\alpha_3||\beta_2| & \text{for Case 4.} \end{cases}$$

Then

$$\sum_{\Lambda} E_{\Delta x,\theta}(\Lambda) \le \sum_{\Lambda} \frac{1}{\varepsilon'} (F(I) - F(J)) \le \frac{1}{\varepsilon'} F(0) := \tilde{M}$$

and

$$\sum_{\Lambda} Q_{\Delta x,\theta}(\Lambda) \le \tilde{M},$$

where $\varepsilon' = \sup_{\Lambda} \max\{K_0^* - |K_{b0}|, K_2^* - |K_{b2}|, K_3^* - |K_{b3}|, K_0^*, K_{s2}, K_{s3}\}.$ From Proposition 4, we know that, for some M > 0,

$$\sum_{k=0}^{\infty} |\sigma_{(k+1)} - \sigma_{(k)}| \le \sum_{\Lambda} (\widetilde{K_{s1}} E_{\Delta x, \theta}(\Lambda) + MQ_{\Delta x, \theta}) \le (\widetilde{K_{s1}} + 1)M \le M,$$

where $\widetilde{K_{s1}}$ is the constant in (57).

5. Global Entropy Solutions

In this section we establish the convergence of approximate solutions to a global entropy solution and show the nonlinear stability and asymptotic behavior of the strong shock emanating from the wedge vertex under the BV wedge perturbation.

5.1. Convergence of Approximate Solutions

Following the above discussions, we can extend $U_{\Delta x,\theta}$ by the constant $U_{k,0}$ continuously across the approximate shock-front to the whole strip $\{k\Delta x < x < (k+1)\Delta x\}$ for each $k \leq 0$.

Let the line x = a > 0 intersects $\partial \Omega_{\Delta x} = \bigcup \{C_{k-1}C_k, k \ge 1\}$ at the point $(a, p_a^{\Delta x})$. Similar to [29], by Theorem 2, we can prove

Lemma 10. For any h > 0 and $x \ge 0$, there exists a constant M > 0 independent of $\Delta x, \theta$, and h such that

$$\int_{-\infty}^{0} |U_{\Delta x,\theta}(x+h,y+p_{x+h}^{\Delta x}) - U_{\Delta x,\theta}(x,y+p_{x}^{\Delta x})| dy \le M|h|.$$

Denote

$$J(\theta, \Delta x, \phi) = \sum_{k=1}^{\infty} \int_{-\infty}^{0} \phi(k\Delta x, y + y_k) \cdot (U_{k\Delta x + , \theta} - U_{k\Delta x - , \theta}) dy$$

for $\phi \in C_0^{\infty}(\mathbb{R}^2; \mathbb{R}^3)$. Following the steps in [10], we have

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Lemma 11. There exists a null set $N \subset \prod_{k=0}^{\infty}(-1,1)$ and a subsequence $\{\Delta x_j\}_{j=1}^{\infty} \subset \{\Delta x\}$, which tends to 0, such that

$$J(\theta, \Delta x_j, \phi) \longrightarrow 0$$
 when $\Delta x_j \to 0$

for any $\theta \in \Pi_{k=0}^{\infty}(-1,1) \backslash N$ and $\phi \in C_0^{\infty}(\mathbb{R}^2;\mathbb{R}^3)$.

To establish the main theorem, we need to estimate the jumps of the approximate strong shock-fronts. Let

$$d_k = \frac{\sigma_{(k-1)}\Delta x - (y_k - y_{k-1}) + \Delta y}{\Delta y}$$

Then, by the choice of Δx and $\{y_k\}$ and by Lemma 4, we find that $d_k \in (0, 1)$ which depends only on $\{\theta_l\}_{l=1}^{k-1}$. Thus, we define

$$I(x, \Delta x, \theta) = \sum_{k=1}^{[x/\Delta x]} I_k(\Delta x, \theta),$$

where $I_k(\Delta x, \theta) = \mathbf{1}_{(-1,d_k)}(\theta_k)(d_k-1)\Delta y + \mathbf{1}_{(d_k,1)}(\theta_k)(d_k+1)\Delta y, \mathbf{1}_A$ denotes the characteristic function of the set A, and $[x/\Delta x]$ denotes the largest integer less than or equal to $x/\Delta x$. Notice that $I_k(\Delta x, \theta)$ is the jump of the function $y = \chi_{\Delta x,\theta}(x)$ at $x = k\Delta x$ and is a measurable function of $(\Delta x, \theta)$, which depends only on $U_{\Delta x,\theta}|_{\{0 \le x \le k\Delta x\}}$ and $\{\theta_l\}_{l=0}^k$.

Lemma 12. (i) For any $x \ge 0, \Delta x > 0$, and $\theta \in \prod_{k=0}^{\infty}(-1, 1)$, we have

$$\chi_{\Delta x,\theta}(x) = I(x, \Delta x, \theta) + \int_0^x \sigma_{\Delta x,\theta}(s) ds;$$

(ii) There exist a null set N_1 and a subsequence $\{\Delta_l\}_{l=1}^{\infty} \subset \{\Delta x_j\}_{j=1}^{\infty}$ such that

$$\int_0^\infty e^{-x} |I(x, \Delta_l, \theta)|^2 dx \longrightarrow 0 \qquad \text{when } \Delta_l \to 0$$

for any $\theta \in \Pi_{k=0}^{\infty}(-1,1) \setminus N_1$.

Proof. Part (i) can be obtained by a direct calculation. We will focus only on part (ii). As in [10], let $d\theta = \prod_{k=0}^{\infty} (d\theta_k/2)$. Then, for any l > j, we have

$$\int I_l I_j d\theta = \int \Pi_{i=1}^{l-1} d\theta_i (I_j \int I_l d\theta_l) = 0$$

Therefore, we can deduce

$$\int |I(x,\Delta x,\theta)|^2 d\theta = \sum_{k=1}^{[x/\Delta x]} \int |I_k(\Delta x,\theta)|^2 d\theta \le 4 |\frac{\Delta y}{\Delta x}|^2 x \Delta x.$$

Then, by choosing a subsequence $\{\Delta_l\}_{l=1}^{\infty} \subset \{\Delta x_j\}_{j=1}^{\infty}$ with $\sum_{l=0}^{\infty} \Delta_l < \infty$ as in Lemma 11, we arrive at (ii).

By Theorem 2 and Lemmas 9 and 10–11, we have

Theorem 3 (Existence and Stability). There exist $\varepsilon > 0$ and C > 0such that, if (10) holds, then, for each $\theta \in (\prod_{k=0}^{\infty}(-1,1)) \setminus (N \cup N_1)$, there exist a subsequence $\{\Delta_l\}_{l=1}^{\infty}$ of mesh sizes with $\Delta_l \to 0$ as $l \to \infty$ and a pair of functions $U_{\theta} \in L^{\infty}(\Omega; O_{\varepsilon}(U_+))$ and $\chi_{\theta} \in Lip([0,\infty))$ with $\chi_{\theta}(0) = 0$ such that

(i) $U_{\Delta_l,\theta}(x,\cdot)$ converges to $U_{\theta}(x,\cdot)$ in $L^1(-\infty, g(x))$ for every x > 0, and U_{θ} is a global entropy solution of problem (6) and (8)–(9) in Ω and satisfies (11)–(12);

(ii) $\chi_{\Delta_l,\theta}$ converges to χ_{θ} uniformly in any bounded x-interval;

(iii) $\sigma_{\Delta_l,\theta}$ converges to $\sigma_{\theta} \in BV([0,\infty))$ a.e. with $|\sigma_{\theta} - \sigma_0| \leq \hat{\varepsilon} < \epsilon$ and

$$\chi_{\theta}(x) = \int_0^x \sigma_{\theta}(t) dt.$$

In addition, if θ is equidistributed, then $\chi_{\theta}(x) < g(x)$ for any x > 0 with (13) and the Rankine-Hugoniot conditions a.e. along the curve $\{y = \chi_{\theta}(x)\}$.

The proof of (i) and (ii) and the convergence proof of $\sigma_{\Delta_k,\theta}$ in (iii) can be carried out in the same way as in the standard cases (see [2,10,11,28]) by using the structure of the approximate solutions. In particular, for any $\varphi \in C_0^{\infty}(\mathbb{R}^2; \mathbb{R}),$

$$\begin{split} &\int_{\Omega_{\Delta_x,\theta}} (\rho^{\Delta x,\theta} u^{\Delta x,\theta} \varphi_x + \rho^{\Delta x,\theta} v^{\Delta x,\theta} \varphi_y) dx dy \\ &= \int_{\Omega} \chi_{\Omega_{\Delta x,\theta}} (\rho^{\Delta x,\theta} u^{\Delta x,\theta} \varphi_x + \rho^{\Delta x,\theta} v^{\Delta x,\theta} \varphi_y) dx dy \end{split}$$

weak-star converges, hence the initial condition is satisfied by the trace theorem for BV functions (cf. [26]). Similarly, the boundary condition can be shown to be satisfied. The equality in (iii) can be deduced from Lemma 12 and the result on the convergence of $\{\chi_{\Delta_l,\theta}\}$ and $\{\sigma_{\Delta_l,\theta}\}$.

5.2. Asymptotic Behavior of the Strong Shock

As in Theorem 3, let $\theta \in (\Pi_{k=0}^{\infty}(-1,1)) \setminus (N \cup N_1)$ be equidistributed, and let U_{θ} be the solution and χ_{θ} its shock-front, respectively. By Theorem 3, we conclude that the solution U_{θ} contains at most countable shock-fronts and countable points of wave interactions; Moreover, we can modify the solution U_{θ} such that U_{θ} is continuous except the shock curves and the points of wave interactions (see [9,11,19]). Then we have

Lemma 13. The solution U_{θ} and its strong shock-front χ_{θ} satisfy

$$\lim_{x \to \infty} TV\{\arctan\left(v_{\theta}(x, \cdot)/u_{\theta}(x, \cdot)\right) : (\chi_{\theta}(x), g(x))\} = 0$$

and

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$$\lim_{x \to \infty} TV\{\rho_{\theta}(x, \cdot) : (\chi_{\theta}(x), g(x))\} = 0.$$

Proof. Let $\{\Delta_l\}$ be the sequence given as in Theorem 3, and let $E_{\Delta_l,\theta}(\Lambda)$ and $Q_{\Delta_l,\theta}(\Lambda)$ be the quantities defined in Lemma 9. As in [11], we denote by $dE_{\Delta_l,\theta}$ and $dQ_{\Delta_l,\theta}$ the measures assigning to $E_{\Delta_l,\theta}(\Lambda)$ and $Q_{\Delta_l,\theta}(\Lambda)$ to the center of Λ , respectively.

The boundedness of $E_{\Delta_l,\theta}(\Lambda)$ and $Q_{\Delta_l,\theta}(\Lambda)$ in Lemma 9 implies the compactness of $\{dE_{\Delta_l,\theta}\}$ and $\{dQ_{\Delta_l,\theta}\}$. Then we can select their subsequences (still denoted by themselves) so that $\Delta_l \to 0$ and the limits:

$$dE_{\Delta_l,\theta} \to dE_{\theta}$$
 and $dQ_{\Delta_l,\theta} \to dQ_{\theta}$

exist in the weak-star topology in the measure space, and the limits are finite on Ω . Therefore, for any $\delta > 0$, we can choose $x_{\delta} > 0$ independent of $\{U_{\Delta_l,\theta}\}$ and $\{\Delta_l\}$ such that, for any l > 0,

$$\sum_{k \ge [x_{\delta}/\Delta_{l}]} E_{\lambda,\theta}(\Lambda_{k,n}) < \delta, \quad \sum_{k \ge [x_{\delta}/\Delta_{l}]} Q_{\lambda,\theta}(\Lambda_{k,n}) < \delta.$$

Moreover, let $X_{\delta}^1 = (x_{\delta}, y_{\delta}^1)(or X_{\delta}^3 = (x_{\delta}, y_{\delta}^3))$ be the point lying in the $\chi_{\Delta_l,\theta}$ (or $\partial \Omega_{\Delta_k}$ resp.). Let $\chi^3_{\Delta_l,\theta}$ be the minimum approximate 3characteristics in $U_{\Delta_l,\theta}$ emanating from the point X_{δ}^1 , and $\chi^1_{\Delta_l,\theta}$ the maximum approximate 1-characteristics in $U_{\Delta_l,\theta}$ emanating from the point X_{δ}^3 . From the construction of the approximate solutions, we have

$$|\chi^{j}_{\Delta_{l},\theta}(x+h) - \chi^{j}_{\Delta_{l},\theta}(x)| \le M(|h| + \Delta_{l}), \qquad j = 1, 3,$$

for some constant M > 0 independent of Δx and θ . Then, for $\theta \in (\prod_{k=0}^{\infty} (-1, 1)) \setminus (N \cup N_1)$, we can select a subsequence (still denoted by) $\{\Delta_l\}_{l=1}^{\infty}$ such that

 $\chi^j_{\Delta_l,\theta} \to \chi^j_{\theta}$ uniformly on every bounded interval as $\Delta_l \to 0$

for some $\chi^j_{\theta} \in Lip$ with $(\chi^j_{\theta})'$ bounded.

Let the characteristics $y = \chi_{\theta}^3(x)$ and $y = \chi_{\theta}^1(x)$ intersect $\partial \Omega$ and $y = \chi_{\theta}(x)$, respectively, at $(t_{\delta}^3, \chi_{\theta}(t_{\delta}^3))$ and $(t_{\delta}^1, \chi_{\theta}(t_{\delta}^1))$ for some t_{δ}^3 and t_{δ}^1 . Then, since the flow angle $\arctan(v/u)$ and the density ρ are invariant across the contact discontinuities, by the approximate conservation laws for the weak j-waves, j = 1, 3, we can deduce in the same way as in [11] that

$$TV\{\arctan\left(v_{\Delta_l,\theta}(x-,\cdot)/u_{\Delta_l,\theta}(x-,\cdot)\right) : (\chi_{\Delta_l,\theta}(x), g_l(x))\} \le C\,\delta$$

and

$$TV\{\rho_{\Delta_l,\theta}(x-,\cdot) : (\chi_{\Delta_l,\theta}(x), g_l(x))\} \le C\delta$$

for $x > 2(t_{\delta}^1 + t_{\delta}^3)$, where C > 0 is independent of $\delta, x, U_{\Delta_l,\theta}$, and Δ_l .

Thus, taking the limit as $\Delta_l \to 0$ and using Theorem 3 and the regularity of U_{θ} yields that, for $x > 2(t_{\delta}^1 + t_{\delta}^3)$,

$$TV\{\arctan(v_{\theta}(x-,\cdot)/u_{\theta}(x-,\cdot) : (\chi_{\theta}(x), g_{l}(x))\} \le C\delta$$

and

$$TV\{\rho_{\theta}(x-,\cdot) : (\chi_{\theta}(x), g_l(x))\} \leq C \delta.$$

Theorem 4. (i) Let $\omega_{\infty} = \lim_{x \to \infty} \arctan(g'(x+))$. Then

$$\lim_{x \to \infty} \sup\{|\arctan\left(v_{\theta}(x, y)/u_{\theta}(x, y)\right) - \omega_{\infty}| : \chi_{\theta}(x) < y < g(x)\} = 0$$

(ii) There exist constants ρ_{∞} and σ_{∞} such that

 $\lim_{x \to \infty} \sup\{ |\rho_{\theta}(x, y) - \rho_{\infty}| : \chi_{\theta}(x) < y < g(x) \} = 0, \lim_{x \to \infty} \sup|\sigma_{\theta}(x) - \sigma_{\infty}| = 0.$

Proof. Let $U_{l,\theta} = U_{\Delta_l,\theta}$, $\sigma_{l,\theta} = \sigma_{\Delta_l,\theta}$, and $\chi_{l,\theta} = \chi_{\Delta_l,\theta}$, where Δ_l is chosen as in the proof of Lemma 13. Following the construction of the approximate solutions, we conclude that, for every x > 0,

$$\arctan(v_{l,\theta}/u_{l,\theta})|_{\Gamma_k} = \arctan\left(\frac{y_{k+1}-y_k}{\Delta x}\right) = \arctan(g'(\eta_k))$$

for some $\eta_k \in [k\Delta x, (k+1)\Delta x)$. Then, choosing x_{δ} so that $|g'(x+)-g'(\infty)| < \delta$ for $x > x_{\delta}$, we have

 $\begin{aligned} \sup\{|\arctan(v_{l,\theta}(x,y)/u_{l,\theta}(x,y)) - \omega_{\infty}| &: \chi_{\theta}(x) < y < g(x)\} \\ &\leq TV\{\arctan(v_{l,\theta}(x,\cdot)/u_{l,\theta}(x,\cdot)) &: (\chi_{\theta}(x),g(x))\} + M\delta \quad \text{for } x > 2x_{\delta}. \end{aligned}$

Therefore, taking the limit as $\Delta_l \to 0$, by Theorem 3 and Lemma 13, and by the regularity of U_{θ} , we can deduce part (i).

Moreover, from Theorem 3, we also have

$$\sigma_{\theta} \in BV(\mathbb{R}_+), \ |\sigma_{\theta} - \sigma_0| \le \hat{\varepsilon} < \varepsilon, \qquad G(\sigma_{\theta}) \in BV(\mathbb{R}_+; O_{\varepsilon}(U_+)).$$

Let $\sigma_{\infty} = \lim_{x \to \infty} \sigma_{\theta}(x+)$ and $U_{\infty} = \lim_{x \to \infty} G(\sigma_{\theta}(x))$. Then part (ii) follows from Lemma 13.

6. Extension to the Adiabatic Euler Flows past Lipschitz Wedges

In this section, we turn to the adiabatic Euler equations (1) for steady supersonic flows, which can be written in the following conservation form:

$$W(U)_x + H(U)_y = 0, \qquad U = (u, v, p, \rho)$$
(65)

with

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$$W(U) = (\rho u, \rho u^2 + p, \rho uv, \rho u(h + \frac{u^2 + v^2}{2})), H(U) = (\rho v, \rho uv, \rho v^2 + p, \rho v(h + \frac{u^2 + v^2}{2})),$$

and $h = \frac{\gamma p}{(\gamma - 1)\rho}$. As in Section 1, the problem of supersonic Euler flows governed by (65) past Lipschitz wedges can be formulated as problem (8)–(9) for system (65) in the region below the lower edge Γ of the wedge.

Definition 4 (Entropy Solutions). A *BV* function U = U(x, y) is called an entropy solution of problem (65) and (8)–(9) provided that

(i) U is a weak solution of (65) and satisfies

 $(u, v) \cdot \mathbf{n}|_{y=g(x)} = 0$ in the trace sense;

(ii) U satisfies the entropy inequality:

$$(\rho uS)_x + (\rho vS)_y \le 0 \tag{66}$$

in the sense of distributions in Ω including the boundary.

6.1. Riemann Problems and Riemann Solutions

The eigenvalues of system (65) are the solutions of the fourth order polynomial equation:

$$(v - \lambda u)^2 ((v - \lambda u)^2 - c^2 (1 + \lambda^2)) = 0,$$

where $c^2 = \gamma p / \rho$. Then we have

$$\lambda_j = \frac{uv + (-1)^j c \sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 4; \qquad \lambda_i = v/u, \ i = 2, 3.$$
(67)

When the flow is supersonic (i.e., $u^2 + v^2 > c^2$), system (65) is hyperbolic and the corresponding eigenvectors for $u \neq 0$ are

$$\mathbf{r}_{j} = \kappa_{j}(-\lambda_{j}, 1, \rho(\lambda_{j}u - v), \rho(\lambda_{j}u - v)/c^{2})^{\top}, \ j = 1, 4, \mathbf{r}_{2} = (1, v/u, 0, 0)^{\top}, \qquad \mathbf{r}_{3} = (0, 0, 0, 1)^{\top},$$

where κ_j are chosen so that $\mathbf{r}_j \cdot \nabla \lambda_j = 1$ since the *j*-characteristic fields are genuinely nonlinear, j = 1, 4. Note that the second and third characteristic fields are always linearly degenerate: $\mathbf{r}_j \cdot \nabla \lambda_j = 0, j = 2, 3$.

6.1.1. Wave Curves in the Phase Space Similarly as in Section 2, the contact discontinuity curves $C_i(U_0)$ through U_0 are

$$C_i(U_0): \quad p = p_0, \quad w = v/u = v_0/u_0, \quad i = 2, 3,$$
 (68)

which describe compressible vortex sheets. Moreover, the rarefaction wave curves $R_j(U_0)$ in the phase space through U_0 are

$$R_j(U_0): \quad dp = c^2 d\rho, \quad du = -\lambda_j dv, \quad \rho(\lambda_j u - v) dv = dp, \qquad j = 1, 4.$$
(69)

It is easy to check that $\frac{d\lambda_j}{d\rho}$ along $R_j(U_0), j = 1, 4$, satisfy

$$\frac{d\lambda_1}{d\rho}|_{R_1(U_0)} < 0, \quad \frac{d\lambda_4}{d\rho}|_{R_4(U_0)} > 0.$$

Similarly, the Rankine-Hugoniot conditions for (65) are

$$\sigma[\rho u] = [\rho v],\tag{70}$$

$$\sigma[\rho u^2 + p] = [\rho u v],\tag{71}$$

$$\sigma[\rho uv] = [\rho v^2 + p],\tag{72}$$

$$\sigma[\rho u(h + \frac{u^2 + v^2}{2})] = [\rho v(h + \frac{u^2 + v^2}{2})].$$
(73)

Then we have

$$(v_0 - \sigma u_0)^2 \left((v_0 - \sigma u_0)^2 - \overline{c}_0^2 (1 + \sigma^2) \right) = 0,$$

where
$$\overline{c}_0^2 = \frac{c_0^2}{b_0} \frac{\rho}{\rho_0}$$
 and $b_0 = \frac{\gamma+1}{2} - \frac{\gamma-1}{2} \frac{\rho}{\rho_0}$. This implies
 $\sigma = \sigma_j := \frac{u_0 v_0 + (-1)^j \overline{c}_0 \sqrt{u_0^2 + v_0^2 - \overline{c}_0^2}}{u_0^2 - \overline{c}_0^2}, j = 1, 4, \quad \sigma = \sigma_i = v_0/u_0, i = 2, 3.$

Plugging σ_i , i = 2, 3, into (70)–(73), we get the same $C_i(U_0)$, i = 2, 3, as defined in (68); while plugging σ_j , j = 1, 4, into (70)–(73), we get the j^{th} -shock wave curve $S_j(U_0)$ through U_0 :

$$S_{j}(U_{0}): \quad [p] = \frac{c_{0}^{2}}{b_{0}}[\rho], \quad [u] = -\sigma_{j}[v], \quad \rho_{0}(\sigma_{j}u_{0} - v_{0})[v] = [p], \qquad j = 1, 4.$$
(74)

Notice that $S_j(U_0)$ contacts with $R_j(U_0)$ at U_0 up to second-order and

$$\frac{d\sigma_1}{d\rho}|_{S_1(U_0)} < 0, \qquad \frac{d\sigma_4}{d\rho}|_{S_4(U_0)} > 0.$$
(75)

Similarly, Lemma 2.1 still holds for system (65), which implies that the entropy inequality (66) is equivalent to (32) or

$$\begin{aligned} \lambda_j(back) < \sigma_j < \lambda_j(front), \qquad j = 1, 4, \\ \sigma_1 < \lambda_{2,3}(back), \qquad \lambda_{2,3}(front) < \sigma_4. \end{aligned}$$

We only show here the equivalence between (66) and (32) when the back state $U_+ = (u_+, 0, p_+, \rho_+)$ with $u_+ > 0$ so that $\sigma < 0$. First, the entropy condition (66) is equivalent to

$$\sigma[\rho u(\ln p - \gamma \ln \rho)] < [\rho v(\ln p - \gamma \ln \rho)].$$
(76)

From (70), we know

$$\sigma\rho u = \sigma\rho_+ u_+ + \rho v. \tag{77}$$

Plugging (77) into (76), we get $\ln(p/p_+) - \gamma \ln(\rho/\rho_+) > 0$. That is,

$$p/p_{+} > (\rho/\rho_{+})^{\gamma}.$$
 (78)

On the other hand, from (70) and (73), we have the Bernoulli law:

$$h + (u^2 + v^2)/2 = h_+ + u_+^2/2,$$
(79)

which implies

$$\frac{p}{p_{+}} = H(t) := \frac{(\gamma - 1) - (\gamma + 1)t}{(\gamma - 1)t - (\gamma + 1)} \quad \text{with} \ t = \rho/\rho_{+}.$$

Then the function $G(t) := H(t)/t^{\gamma}$ is strictly decreasing in t, since

$$G'(t) = \frac{\gamma(1-\gamma^2)(t+1)^2}{t^{\gamma+1}((\gamma-1)t - (\gamma+1))^2} < 0$$

Since G(1) = 1 and G(t) > G(1) from (78), we conclude t < 1, which implies $\rho < \rho_+$.

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6.1.2. Lateral Riemann Problem Again, the simplest case of problem (65) and (8)–(9) is $g \equiv 0$. It can be shown that, if $g \equiv 0$, then problem (65) admits an entropy solution that consists of the constant state U_{-} and a constant state U_{+} , with $U_{+} = (u_{+}, 0, p_{+}, \rho_{+})$ and $u_{+} > c_{+} > 0$ in the subdomain of Ω separated by a straight shock emanating from the vertex. That is to say that the state ahead of the shock-front is U_{-} , while the state behind the shock-front is U_{+} (see Fig. 4). When the angle between the flow direction of the front state and the wedge boundary at a boundary vertex is larger than π , then an entropy solution contains a rarefaction wave that separates the front state from the back state (see Fig. 5).

6.1.3. Riemann Problem Involving Only Weak Waves Consider the following initial value problem:

$$\begin{cases} W(U)_x + H(U)_y = 0, \\ U|_{x=x_0} = \underline{U} = \begin{cases} U_a, \ y > y_0, \\ U_b, \ y < y_0, \end{cases}$$
(80)

where U_b and U_a are constant states. As before, we can parameterize the physically admissible elementary solution curve in a neighborhood of U_+ , $O_{\varepsilon}(U_+)$, by $\alpha_j \mapsto \Phi_j(\alpha_j; U_b)$, with $\Phi \in C^2$, $\Phi_j|_{\alpha_j=0} = U_b$, and $\frac{\partial \Phi_j}{\partial \alpha_j}|_{\alpha_j=0} = \mathbf{r}_j(U_b)$.

Denote $\Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U_b) = \Phi_4(\alpha_4, \Phi_3(\alpha_3, \Phi_2(\alpha_2, \Phi_1(\alpha_1; U_b))))$. Then we have

Lemma 14. There exists $\varepsilon > 0$ such that, for any states $U_a, U_b \in O_{\varepsilon}(U_+)$, problem (80) admits a unique admissible solution consisting of four elementary waves. In addition, state U_a can be represented by $U_a = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U_b)$ with $\Phi|_{\alpha_1=\alpha_2=\alpha_3=\alpha_4=0} = U_b$ and $\frac{\partial \Phi}{\partial \alpha_i}|_{\alpha_1=\alpha_2=\alpha_3=\alpha_4=0} = \mathbf{r}_i(U_b), i = 1, 2, 3, 4.$

Similar to the argument for Lemma 2.3, we have

Lemma 15. At state $U_+ = (u_+, 0, p_+, \rho_+)$ with $u_+ > 0$, $\kappa_1(U_+) = \kappa_4(U_+) > 0$, which implies $\kappa_j(U) > 0$, j = 1, 4, for any state $U \in O_{\varepsilon}(U_+)$ since κ_j are continuous, j = 1, 4.

6.1.4. Riemann Problem Involving a Strong 1-Shock For simplicity, we use notation $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to denote that $U_a = \varPhi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U_b)$ throughout this section. For any $U \in S_1(U_-)$, we also use $\{U_-, U\} = (\sigma, 0, 0, 0)$ to denote the 1-shock that connects U_- and U with speed σ . Then we have

Lemma 16. Let $\{U_{-}, U_{+}\} = (\sigma_0, 0, 0, 0), \rho_{+} > \rho_{-}$, and $\gamma > 1$. Then

$$\sigma_0 < 0, \qquad u_+ < u_- < (1+1/\gamma)u_+.$$

The slight difference for proving this lemma from that of Lemma 4 lies in that \overline{c}_0 and \overline{c}_- are different for system (65). However, we still have $c_0 > \overline{c}_0$ and $c_- < \overline{c}_-$, and hence there is no problem to carry out the same steps.

Lemma 17. There exists a neighborhood $O_{\varepsilon}(U_+)$ of U_+ such that the shock polar $S_1(U_-) \cap O_{\varepsilon}(U_+)$ can be parameterized by the shock speed σ as

$$\sigma \to G(\sigma)$$

with $G \in C^2$ near σ_0 and $G(\sigma_0) = U_+$.

Lemma 18. For states U_{-} and U_{+} in the unperturbed solution,

$$P := u_{+}(h_{-} + \frac{u_{-}^{2} + v_{-}^{2}}{2} - \frac{u_{+}^{2}}{2}) + (\frac{c_{+}^{2}}{\gamma - 1} + u_{+}^{2})(u_{+} - u_{-}) > 0.$$

Proof. From the Bernoulli law (79), we have

$$P = \frac{c_+^2}{\gamma - 1}(2u_+ - u_-) + u_+^2(u_+ - u_-).$$
(81)

Following the same steps as in Lemma 4, we have the following facts from Lemma 16: $u_+(u_- - u_+) < c_+^2/\gamma$, $u_- < (1 + 1/\gamma)u_+$, and $u_- < 2u_+$, which implies

$$P > \frac{\gamma u_+(u_- - u_+)}{\gamma - 1} \left((1 + 1/\gamma)u_+ - u_- \right) > 0.$$

Lemma 19. Let $A = \nabla_U H(U_+) - \sigma_0 \nabla_U W(U_+)$. Then

$$det A > 0, \ det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+} > 0, det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, AG_{\sigma}(\sigma_0))|_{U=U_+} > 0.$$

Proof. A direct calculation shows that

$$A = \begin{pmatrix} -\sigma_0 \rho_+ & \rho_+ & 0 & -\sigma_0 u_+ \\ -2\sigma_0 \rho_+ u_+ & \rho_+ u_+ & -\sigma_0 & -\sigma_0 u_+^2 \\ 0 & -\sigma_0 \rho_+ u_+ & 1 & 0 \\ -\sigma_0 \rho_+ (\frac{c_+^2}{\gamma - 1} + \frac{3}{2}u_+^2) & \rho_+ (\frac{c_+^2}{\gamma - 1} + \frac{u_+^2}{2}) - \frac{\gamma}{\gamma - 1}\sigma_0 u_+ - \frac{1}{2}\sigma_0 u_+^3 \end{pmatrix},$$

and

$$\mathbf{r}_{j}(U_{+}) = \frac{(-1)^{j-1}\kappa_{j}(U_{+})}{c_{+}\sqrt{u_{+}^{2} - c_{+}^{2}}} (c_{+}^{2}, (-1)^{j-1}c_{+}\sqrt{u_{+}^{2} - c_{+}^{2}}, \rho_{+}u_{+}c_{+}, \rho_{+}u_{+})^{\top}, \ j = 1, 4,$$

$$\mathbf{r}_{2}(U_{0}) = (1, 0, 0, 0)^{\top}, \qquad \mathbf{r}_{3}(U_{0}) = (0, 0, 0, 1)^{\top}.$$

Hence, we have

$$A\mathbf{r}_{j}(U_{+}) = \frac{\kappa_{j}(U_{+})\rho_{+}(\lambda_{j+} - \sigma_{0})}{\lambda_{j+}} (1, u_{+}, u_{+}\lambda_{j+}, (\frac{c_{+}^{2}}{\gamma - 1} + \frac{u_{+}^{2}}{2}))^{\top} \text{ for } j = 1, 4,$$

$$A\mathbf{r}_{2}(U_{+}) = -\sigma_{0}\rho_{+}(1, 2u_{+}, 0, \frac{c_{+}^{2}}{\gamma - 1} + \frac{3}{2}u_{+}^{2})^{\top}, \quad A\mathbf{r}_{3}(U_{+}) = -\sigma_{0}u_{+}(1, u_{+}, 0, \frac{u_{+}^{2}}{2})^{\top},$$

and

$$AG_{\sigma}(\sigma_0) = W(U_+) - W(U_-) = -\frac{\rho_- v_-}{\sigma_0} (1, u_-, \sigma_0 u_-, h_- + \frac{u_-^2 + v_-^2}{2})^{\top}.$$

By Lemmas 16 and 18, we find

$$\det A = \frac{\sigma_0^2 \rho_+^2 u_+^2}{\gamma - 1} (\lambda_{1+}^2 - \sigma_0^2) (u_+^2 - c_+^2) > 0,$$

$$\det(A\mathbf{r}_{4}, A\mathbf{r}_{3}, A\mathbf{r}_{2}, A\mathbf{r}_{1})|_{U=U_{+}} = \frac{(\kappa_{4}(U_{+}))^{2}\sigma_{0}^{2}\rho_{+}^{3}u_{+}^{3}c_{+}^{2}}{(\gamma - 1)\lambda_{1+}\lambda_{4+}}(\sigma_{0} - \lambda_{4+})(\sigma_{0} - \lambda_{1+})(\lambda_{4+} - \lambda_{1+}) > 0,$$

and

$$\det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, AG_{\sigma}(\sigma_0))|_{U=U_+}$$

= $\frac{\kappa_4(U_+)\sigma_0\rho_-v_-\rho_+^2u_+}{\lambda_{4+}}(\sigma_0 - \lambda_{4+})(u_+\lambda_{4+}P + \sigma_0u_-Q) > 0,$

since P > 0 and $Q = -c_{+}^{2}/(\gamma - 1) < 0$.

6.2. Estimates on Wave Interactions and Reflections

Now we make essential estimates as in Section 3. The interaction estimates are similar and the corresponding Figs. 5–7 are the same except that 2-contact discontinuities and 3-waves in Section 3 are now replaced by 2, 3-contact discontinuities and 4-waves, respectively.

6.2.1. Estimates on Weak Wave Interactions First we have

Proposition 6. Suppose that $U_b, U_m, U_a \in O_{\varepsilon}(U_+)$ are three states with $\{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, \beta_4), \text{ and } \{U_b, U_a\} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ (cf. Fig. 5). Then

$$\gamma_i = \alpha_i + \beta_i + O(1) \triangle(\alpha, \beta),$$

where $\triangle(\alpha,\beta) = (|\alpha_4| + |\alpha_3| + |\alpha_2|)|\beta_1| + |\alpha_4|(|\beta_2| + |\beta_3|) + \sum_{j=1,4} \triangle_j(\alpha,\beta)$ with $\triangle_j(\alpha,\beta) = \begin{cases} 0, & \alpha_j \ge 0, \ \beta_j \ge 0, \\ |\alpha_j||\beta_j|, & otherwise. \end{cases}$

 $\det(\frac{\partial \Phi(\gamma_4, \gamma_3, \gamma_2, \gamma_1; U_l)}{\partial(\gamma_4, \gamma_3, \gamma_2, \gamma_1)})|_{\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0} = \det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U = U_+} > 0,$

then, by the implicit function theorem, there exists $(\gamma_4, \gamma_3, \gamma_2, \gamma_1)$ as a C^2 -function of $(\beta_4, \beta_3, \beta_2, \beta_1, \alpha_4, \alpha_3, \alpha_2, \alpha_1; U_b)$ so that

$$\Phi(\beta_4,\beta_3,\beta_2,\beta_1;\Phi(\alpha_4,\alpha_3,\alpha_2,\alpha_1;U_b)) = \Phi(\gamma_4,\gamma_3,\gamma_2,\gamma_1;U_b).$$

Then we follow the proof of Proposition 3.1 to arrive at the result.

6.2.2. Estimates on the Weak Wave Reflections on the Boundary We use the same notations as in Section 3.2 for $C_k(a_k, b_k)$ with $a_{k+1} > a_k > 0$, $\omega_{k,k+1}$, ω_k , Ω_k , Γ_k , and the outer normal vector \mathbf{n}_k to Γ_k (cf. Fig. 7). Then we consider the initial-boundary value problem with \underline{U} a constant state:

$$\begin{cases} (65) & \text{in } \Omega_{k+1}, \\ U|_{x=a_k} = \underline{U}, \\ (u, v) \cdot \mathbf{n}_{k+1} = 0 \text{ on } \Gamma_{k+1}. \end{cases}$$

Proposition 7. Let $\{U_b, U_m\} = (\alpha_4, \alpha_3, \alpha_2, 0)$ and $\{U_m, U_k\} = (0, 0, 0, \beta_1)$ with

$$(u_k, v_k) \cdot \mathbf{n}_k = 0.$$

Then there exists U_{k+1} such that

$$\{U_b, U_{k+1}\} = (0, 0, 0, \delta_1)$$
 and $(u_{k+1}, v_{k+1}) \cdot \mathbf{n}_{k+1} = 0.$

Furthermore,

$$\delta_1 = \beta_1 + K_{b4}\alpha_4 + K_{b3}\alpha_3 + K_{b2}\alpha_2 + K_{b0}\omega_k,$$

where K_{b4}, K_{b3}, K_{b2} , and K_{b0} are C^2 -functions of $(\alpha_4, \alpha_3, \alpha_2, \beta_1, \omega_k; U_b)$ satisfying

$$\begin{split} K_{b4}|_{\{\omega_k = \alpha_4 = \alpha_3 = \alpha_2 = \beta_1 = 0, U_b = U_+\}} &= 1, \\ K_{b2}|_{\{\omega_k = \alpha_4 = \alpha_3 = \alpha_2 = \beta_1 = 0, U_b = U_+\}} &= K_{b3}|_{\{\omega_k = \alpha_4 = \alpha_3 = \alpha_2 = \beta_1 = 0, U_b = U_+\}} = 0, \end{split}$$

and K_{b0} is bounded.

Since

$$\begin{aligned} &\frac{\partial}{\partial \delta_1} (\varPhi(0,0,0,\delta_1;U_b) \cdot (\mathbf{n}_{k+1},0,0))|_{\{\delta_1=0,U_b=U_+,\omega_{k,k+1}=0\}} \\ &= \kappa_1(U_+)(-\lambda_{1+},1,\rho_+u_+\lambda_{1+},\frac{\rho_+u_+\lambda_{1+}}{c_+^2}) \cdot (0,1,0,0) > 0, \end{aligned}$$

we know from the implicit function theorem that δ_1 can be solved as a C^2 -function of $(\alpha_4, \alpha_3, \alpha_2, \beta_1, \omega_{k-1,k}, \omega_k; U_b)$ such that

 $\Phi(0,0,0,\beta_1;\Phi(\alpha_4,\alpha_3,\alpha_2,0;U_b)) \cdot (\mathbf{n}_k,0,0) = \Phi(0,0,0,\delta_1;U_b) \cdot (\mathbf{n}_{k+1},0,0).$ (82)

Then, following the argument in the proof of Proposition 3.2 yields the results.

6.2.3. Estimate on the Boundary Perturbation of the Strong Shock We have

Proposition 8. For $\varepsilon > 0$ sufficiently small, there exists $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon) < \varepsilon$ so that $G(O_{\hat{\varepsilon}}(\sigma_0)) \subset O_{\varepsilon}(U_+)$ and, when $|\omega_k| < \varepsilon$, the following equation

$$G(\sigma) \cdot (\mathbf{n}_k, 0, 0) = 0 \tag{83}$$

admits a unique solution $\sigma_k \in O_{\hat{\varepsilon}}(\sigma_0)$. Moreover, we have

$$\sigma_{k+1} = \sigma_k + K_{bs}\omega_k + O(1)|\omega_k|^2, \tag{84}$$

where $|K_{bs}|$ is bounded.

Proof. It suffices to find a solution $\sigma = \sigma(h)$ to the following equation:

$$G(\sigma) \cdot (-\sin h, \cos h, 0, 0) = 0.$$
(85)

Differentiating both sides of (85) in σ , following the part of the calculation we had in Lemma 19, and denoting by (A_{ij}^*) the co-factor matrix of (a_{ij}) , we obtain

$$\frac{\partial}{\partial \sigma} (G(\sigma) \cdot (-\sin h, \cos h, 0, 0))|_{\{\sigma = \sigma_0, h = 0\}} = \frac{1}{\det A} (A_{12}^*, A_{22}^*, A_{32}^*, A_{42}^*) \cdot AG_{\sigma}(\sigma_0) > 0.$$

By the implicit function theorem, we can find a unique C^2 -function $\sigma = \sigma(h)$ with $\sigma(0) = \sigma_0$, which solves (85) in some neighborhood of $(\sigma, h) = (\sigma_0, 0)$. Then $\sigma(\Omega_j) = \sigma_j, j = k, k + 1$, and, by Taylor's expansion formula, we have the desired estimates (84).

6.2.4. Estimates on the Interaction Between the Strong Shock and Weak Waves

Proposition 9. Let $U_m, U_a \in O_{\varepsilon}(U_+)$ with

$$\{G(\sigma), U_m\} = (0, \alpha_2, \alpha_3, \alpha_4), \qquad \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, 0),$$

Then there exists a unique $(\sigma', \delta_2, \delta_3, \delta_4)$ such that the Riemann problem (80) with $U_b = U_-$ admits an admissible solution consisting of a strong 1shock, two contact discontinuities of strengths δ_2 and δ_3 , and a weak 4-wave of strength δ_4 :

$$\{U_-, U_a\} = (\sigma', \delta_2, \delta_3, \delta_4).$$

Moreover,

$$\sigma' = \sigma_0 + K_{s1}\beta_1 + O(1)\Delta, \quad \delta_2 = \alpha_2 + \beta_2 + K_{s2}\beta_1 + O(1)\Delta,$$

$$\delta_4 = \alpha_4 + K_{s4}\beta_1 + O(1)\Delta, \quad \delta_3 = \alpha_3 + \beta_3 + K_{s3}\beta_1 + O(1)\Delta,$$

where

$$|K_{s4}| < 1$$
 and $|K_{s3}|, |K_{s2}|, and |K_{s1}|$ are bounded,

and $\Delta = |\alpha_3||\beta_1| + |\alpha_2||\beta_1| + |\alpha_4||\beta_1| + |\alpha_4||\beta_2| + |\alpha_4||\beta_3|$. Furthermore, we can write the estimates in a more precise fashion:

$$\begin{split} \delta_4 &= \alpha_4 + \widetilde{K_{s4}}\beta_1 + O(1)\tilde{\Delta}, \qquad \delta_3 &= \alpha_3 + \widetilde{K_{s3}}\beta_1 + O(1)\tilde{\Delta}, \\ \delta_2 &= \alpha_2 + \widetilde{K_{s2}}\beta_1 + O(1)\tilde{\Delta}, \qquad \sigma' = \sigma + \widetilde{K_{s1}}\beta_1 + O(1)\tilde{\Delta}, \end{split}$$

where

$$|\widetilde{K_{s4}}| < 1, \qquad |\widetilde{K_{s3}}| + |\widetilde{K_{s2}}| + |\widetilde{K_{s1}}| \le M,$$

for some M > 0 and $\tilde{\Delta} = |\alpha_4| |\beta_3| + |\alpha_4| |\beta_2|$.

Proof. We first show that there exists a unique solution $(\sigma', \delta_2, \delta_3, \delta_4)$ as a function of $(\sigma, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3)$ to

$$\Phi(0,\beta_3,\beta_2,\beta_1;\Phi(\alpha_4,\alpha_3,\alpha_2,0;G(\sigma))) = \Phi(\delta_4,\delta_3,\delta_2,0;G(\sigma')).$$
(86)

By Proposition 6, there exists $(\gamma_4, \gamma_3, \gamma_2, \gamma_1)$ such that

$$\Phi(0,\beta_3,\beta_2,\beta_1;\Phi(\alpha_4,\alpha_3,\alpha_2,0;G(\sigma))) = \Phi(\gamma_4,\gamma_3,\gamma_2,\gamma_1;G(\sigma))$$
(87)

with

$$\gamma_1 = \beta_1 + O(1)\Delta, \ \gamma_2 = \beta_2 + \alpha_2 + O(1)\Delta, \ \gamma_3 = \alpha_3 + \beta_3 + O(1)\Delta, \ \gamma_4 = \alpha_4 + O(1)\Delta.$$

Thus, (86) can be reduced to

$$\Phi(\gamma_4, \gamma_3, \gamma_2, \gamma_1; G(\sigma)) = \Phi(\delta_4, \delta_3, \delta_2, 0; G(\sigma')).$$
(88)

Furthermore, Lemma 19 implies

$$\det\left(\frac{\partial\Phi(\delta_4,\delta_3,\delta_2,0;G(\sigma'))}{\partial(\delta_4,\delta_3,\delta_2,\sigma')}\right)|_{\{\delta_4=\delta_3=\delta_2=0,\sigma'=\sigma_0\}}$$
$$=\frac{1}{\det A}\det(A\mathbf{r}_4(U_+),A\mathbf{r}_3(U_+),A\mathbf{r}_2(U_+),AG_{\sigma}(\sigma_0))>0.$$

Therefore, the implicit function theorem implies that $(\delta_4, \delta_3, \delta_2, \sigma')$ can be solved as a C^2 -function of $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \sigma)$ uniquely:

$$\sigma' = \sigma'(\gamma_4, \gamma_3, \gamma_2, \gamma_1, \sigma), \qquad \delta_i = \delta_i(\gamma_4, \gamma_3, \gamma_2, \gamma_1, \sigma), \quad i = 2, 3, 4.$$

Using identity (45) in Lemma 7, we find

$$\sigma' = K_{s1}\gamma_1 + \sigma, \qquad \delta_i = K_{si}\gamma_1 + \gamma_i, \qquad i = 2, 3, 4,$$

where $K_{s1} = \int_0^1 \partial_{\gamma_1} \sigma'(\gamma_3, \gamma_2, \lambda \gamma_1, \sigma) d\lambda$ and $K_{si} = \int_0^1 \partial_{\gamma_1} \delta_i(\gamma_4, \gamma_3, \gamma_2, \lambda \gamma_1, \sigma) d\lambda$. When $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 0$, it is clear that $|\frac{\partial \sigma'}{\partial \gamma_1}|$ and $|\frac{\partial \delta_i}{\partial \gamma_1}|$, i = 2, 3, 4, are bounded. We can further claim the important fact that

$$\left|\frac{\partial \delta_4}{\partial \gamma_1}\right| < 1$$
 when $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 0.$

This can be shown as follows: Differentiate (88) with respect to γ_1 and let $\gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 0$. Then we have

$$\mathbf{r}_1(U_+) = \mathbf{r}_4(U_+)\frac{\partial\delta_4}{\partial\gamma_1} + \mathbf{r}_3(U_+)\frac{\partial\delta_3}{\partial\gamma_1} + \mathbf{r}_2(U_+)\frac{\partial\delta_2}{\partial\gamma_1} + G_\sigma(\sigma_0)\frac{\partial\sigma'}{\partial\gamma_1}$$

Multiplying both sides by A defined in Lemma 19, we have

$$\begin{aligned} |\frac{\partial \delta_4}{\partial \gamma_1}| &= |\frac{\det(A\mathbf{r}_1(U_+), A\mathbf{r}_2(U_+), A\mathbf{r}_3(U_+), AG_{\sigma}(\sigma_0))}{\det(A\mathbf{r}_4(U_+), A\mathbf{r}_2(U_+), A\mathbf{r}_3(U_+), AG_{\sigma}(\sigma_0))}| \\ &= |\frac{\lambda_{4+} + \sigma_0}{\lambda_{4+} - \sigma_0}| \cdot |\frac{\sigma_0 u_- Q - u_+ \lambda_{4+} P}{\sigma_0 u_- Q + u_+ \lambda_{4+} P}| < 1, \end{aligned}$$

since $P > 0, Q < 0, \sigma_0 < 0$, and $\lambda_{4+} > 0$. Combining with the estimates we had on $\gamma_1, \gamma_2, \gamma_3$, and γ_4 , we complete the proof.

6.3. Approximate Solutions

Similarly to Section 4, we can construct the globally defined, modified Glimm approximate solutions $U_{\Delta x,\theta}$ in the approximate domains (see Fig. 7):

$$\Omega_{\Delta x} = \bigcup_{k \ge 0} \Omega_{\Delta x, k}$$

with

$$\Omega_{\Delta x,k} = \{ (x,y) : (k-1)\Delta x < x \le k\Delta x, \ y = y_{k-1} + (x - (k-1)\Delta x) \tan(\omega_{k-1,k}) \}$$

under the Courant-Friedrichs-Lewy type condition:

$$\frac{\Delta y - m\Delta x}{\Delta x} < |\sigma_0| + \max_{j=1,4} (\sup_{O_{\varepsilon}(U_+)} |\lambda_j(U)|) \quad \text{with } m \text{ as defined in (63).}$$

Lemma 20. (i) If $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $U_b, U_a \in O_{\varepsilon}(U_+)$, then

$$|U_b - U_a| \le s_1(|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|),$$

where $s_1 = \max_{1 \le i \le 4} (\sup_{U \in O_{\varepsilon}(U_+)} |\partial_{\alpha_i} \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U)|);$ (ii) For any $\sigma \in O_{\hat{\varepsilon}}(\sigma_0)$ with $G(O_{\hat{\varepsilon}}(\sigma_0)) \subset O_{\varepsilon}(U_+)$ for $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon)$,

$$|G(\sigma) - G(\sigma_0)| \le s_2 |\sigma - \sigma_0|,$$

where $s_2 = \sup_{\sigma \in O_{\hat{\varepsilon}(\sigma_0)}} |G'_{\sigma}(\sigma)|.$

We need to establish the estimates on $U_{\Delta x,\theta}$ on a class of space-like curves, *j*-mesh curves J as introduced in Definition 2. To achieve this, we now define the Glimm-type functional.

Definition 5. We define

$$F_s(J) = C^* |\sigma^J - \sigma_0| + F(J),$$

with

$$F(J) = L(J) + KQ(J),$$

$$L(J) = K_0^* L_0(J) + L_1(J) + K_2^* L_2(J) + K_3^* L_3(J) + K_4^* L_4(J),$$

$$Q(J) = \sum \{ |\alpha_i| |\beta_j| : \text{ both } \alpha_i \text{ and } \beta_j \text{ cross } J \text{ and approach} \},$$

and

$$L_0(J) = \sum \{ |\omega(C_k)| : C_k \in \Omega_J \}, \quad L_j(J) = \sum \{ |\alpha_j| : \alpha_j \text{ crosses } J \}, \ 1 \le j \le 4,$$

where K and C^* are to be defined later, while $\Omega_J = \{C_k \in J^+ \cap \partial \Omega_{\Delta x} : k \geq 0\}$ is the set of the corner points C_k lying in J^+ , σ^J stands for the speed of the strong shock crossing J, and $K_0^*, K_2^*, K_3^*, K_4^*$ are constants that satisfy the following conditions:

$$|K_0^* > |K_{b0}|, \quad |K_{b4}| < K_4^* < \frac{1}{|K_{s4}|}, \quad |K_{b3}| < K_3^* < \frac{1 - |K_{s4}|K_4^*}{|K_{s3}|},$$

and

$$|K_{b2}| < K_2^* < \frac{1 - |K_{s3}|K_3^* - |K_{s4}|K_4^*}{|K_{s2}|}$$

which can be achieved from our discussions of the properties of K_{bi} and K_{si} , $0 \le i \le 4$, as in the propositions in Section 6.2.

Now we prove the decreasing property of our functional F_s . We have

Proposition 10. Suppose that the wedge boundary function g(x) satisfies (63), and I and J are two k-mesh curves such that J is an immediate successor of I. Suppose that

$$|U_{\Delta x,\theta}|_{I\cap (\Omega^+_{\Delta x,k-1}\cup \Omega^+_{\Delta x,k})} - U_+| < \varepsilon, \quad |\sigma^I - \sigma_0| < \hat{\varepsilon},$$

where $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon)$ is defined in Proposition 8 and Lemma 20. Then there exist constants $\tilde{\varepsilon} > 0, K > 0$, and $C^* > 1$, depending only on the system in (1) and states U_- and U_+ , such that, if $F_s(I) < \tilde{\varepsilon}$, then

$$F_s(J) \le F_s(I)$$

and hence

$$|U_{\Delta x,\theta}|_{J\cap (\Omega^+_{\Delta x,k-1}\cup \Omega^+_{\Delta x,k})} - U_+| < \varepsilon, \quad |\sigma^J - \sigma_0| < \hat{\varepsilon}.$$

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Proof. Let Λ be the diamond that is formed by I and J. We can always assume that $I = I_0 \cup I'$ and $J = J_0 \cup J'$ such that $\partial \Lambda = I' \cup J'$. As in the proof of Proposition 4.1, we divide our proof in four cases depending on the location of the diamond.

Case 1 (interior weak-weak interaction). Denote $Q(\Lambda) = \Delta(\alpha, \beta)$ as defined in Proposition 6. Then, for some constant M > 0,

$$L(J) - L(I) = (1 + K_2^* + K_3^* + K_4^*)MQ(\Lambda),$$

and, since $L(I_0) < \tilde{\varepsilon}$ from $F_s(I) < \tilde{\varepsilon}$,

$$Q(J) - Q(I) = (ML(I_0) - 1)Q(\Lambda) \le -\frac{1}{2}Q(\Lambda).$$

Hence, we have

$$F(J) - F(I) = \left(\left(1 + K_2^* + K_3^* + K_4^* \right) M - K/2 \right) Q(\Lambda) \le -\frac{1}{4} Q(\Lambda),$$

by choosing suitably large K.

Case 2 (near the boundary). Then $\Omega_J = \Omega_I \setminus \{C_k\}$ for certain k and $\sigma^I = \sigma^J$. Let δ_1 be the weak 1-wave going out of Λ through J', and β_1 , α_2 , and α_3 be the weak waves entering Λ through I', similarly as shown in Fig. 10. Then

$$L_{0}(J) - L_{0}(I) = -|\omega_{k}|,$$

$$L_{i}(J) - L_{i}(I) = \sum_{\gamma_{i} \text{ crosses } I_{0}} |\gamma_{i}| - (|\alpha_{i}| + \sum_{\gamma_{i} \text{ crosses } I_{0}} |\gamma_{i}|) = -|\alpha_{i}|, \quad i = 2, 3, 4,$$

$$L_{1}(J) - L_{1}(I) = |\delta_{1}| - |\beta_{1}| \leq |K_{b4}||\alpha_{4}| + |K_{b3}||\alpha_{3}| + |K_{b2}||\alpha_{2}| + |K_{b0}||\omega_{k}|,$$

where the last step is from Proposition 7. Thus,

$$L(J) - L(I) \le (|K_{b0}| - K_0^*) |\omega_k| + (|K_{b2}| - K_2^*) |\alpha_2| + (|K_{b3}| - K_3^*) |\alpha_3| + (|K_{b4}| - K_4^*) |\alpha_4|.$$

From our requirement in Definition 5, we get $L(J) - L(I) \leq 0$. Since $F_s(I) \leq \tilde{\varepsilon}$ implies $L(I) \leq \tilde{\varepsilon}$, the higher order term Q(I) can always be bounded by the linear term L(I). Then we can easily conclude that $F(J) \leq F(I)$.

Case 3 (near the wedge vertex). Then, from our construction, we find that $\Omega_J = \Omega_I \setminus \{C_k\}$, and $S_*(\sigma_{(k)})$ emanates from C_k and crosses $J, \sigma^I = \sigma_{(k-1)}$, and $\sigma^J = \sigma_{(k)}$. Moreover, there is no weak wave crossing I' or J'. Then we have

$$F(J) - F(I) \le -K_0^* |\omega_k|.$$

Since $|\sigma^J - \sigma_0| - |\sigma^I - \sigma_0| \le |\sigma^J - \sigma^I| \le |K_{bs}| |\omega_k| + M |\omega_k|^2$ and $|K_{bs}|$ is bounded, we can further choose suitably small C^* and $\tau > 0$ such that

$$F_s(J) - F_s(I) \le C^* |\sigma^J - \sigma^I| + F(J) - F(I) \le -\tau |\omega_k|.$$

Case 4 (near the strong 1-shock). Then $S_*(\sigma_{(k)})$ is generated from the inside of Λ , $\sigma^I = \sigma_{(k-1)}$, and $\sigma^J = \sigma_{(k)}$. Let δ_4, δ_3 , and δ_2 be the weak waves going out of Λ through J', and let $\alpha_4, \alpha_3, \alpha_2, \beta_1, \beta_2$, and β_3 be the weak waves entering Λ through I', similarly as shown in Fig. 11.

Then

$$L_{1}(J) - L_{1}(I) = \sum_{\gamma_{1} \text{ crosses } I_{0}} |\gamma_{1}| - (|\beta_{1}| + \sum_{\gamma_{1} \text{ crosses } I_{0}} |\gamma_{1}|) = -|\beta_{1}|,$$

$$L_{i}(J) - L_{i}(I) \leq |K_{si}||\beta_{1}| + M|\alpha_{4}||\beta_{2}| + |\alpha_{4}||\beta_{3}|, \quad i = 2, 3,$$

$$L_{4}(J) - L_{4}(I) \leq |K_{s4}||\beta_{1}| + M(|\alpha_{4}||\beta_{2}| + |\alpha_{4}||\beta_{3}|),$$

where we have used the estimates in Proposition 9.

Again, this case is much more complicated and requires a careful calculation of Q(J) - Q(I). For simplicity, for any weak wave γ , we denote

$$Q(\gamma, I_0) = |\gamma| \sum \{ |\gamma_j| : \gamma_j \text{ and } \gamma \text{ approach}, \gamma_j \text{ crosses } I_0 \}.$$

Then

$$\begin{aligned} Q(J) - Q(I) &\leq -(|\alpha_4||\beta_1| + |\alpha_4||\beta_2| + |\alpha_4||\beta_3| + |\beta_1||\alpha_2| + |\beta_1||\alpha_3|) \\ &+ (\sum_{i=2}^4 |\widetilde{K_{si}}| - 1)Q(\beta_1, I_0) + Q(M(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|), I_0) \\ &= (-1 + ML(I_0))(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|) + (ML(I_0) - \sum_{i=2}^4 |\alpha_i|)|\beta_1|. \end{aligned}$$

Since $L(I_0) < \tilde{\varepsilon}$ from $F_s(I) < \tilde{\varepsilon}$, then

$$Q(J) - Q(I) \le -\frac{1}{2} \left(|\alpha_4| |\beta_2| + |\alpha_4| |\beta_3| \right) + ML(I_0) |\beta_1|.$$

Therefore, we have

$$F(J) - F(I) \leq \left(-1 + \sum_{i=2}^{4} K_{i}^{*} |K_{si}|\right) |\beta_{1}| + M(|\alpha_{4}||\beta_{2}| + |\alpha_{4}||\beta_{3}|) + K(-\frac{1}{2}(|\alpha_{4}||\beta_{2}| + |\alpha_{4}||\beta_{3}|) + ML(I_{0})|\beta_{1}|) \leq -\frac{1}{8}(|\beta_{1}| + |\alpha_{4}||\beta_{2}| + |\alpha_{4}||\beta_{3}|),$$

where we have chosen suitably large K and used the fact that $L(I_0) < \tilde{\varepsilon}$.

Furthermore, since

$$\sigma^{J} - \sigma^{I} \leq |K_{s1}||\beta_{1}| + M(|\alpha_{4}||\beta_{2}| + |\alpha_{4}||\beta_{3}|),$$

we can further choose suitably small C^* such that

$$F_s(J) - F_s(I) \le C^* |\sigma^J - \sigma^I| + F(J) - F(I) \le -\frac{1}{16} |\beta_1| - \frac{1}{16} (|\alpha_4| |\beta_2| + |\alpha_4| |\beta_3|).$$

Again we have $F(J) \leq F(I)$. Then, from Lemma 6.7, there exists $\tilde{\epsilon} > 0$ such that, when $F(I) < \tilde{\epsilon}$, we have $|U - U_+| < \epsilon$.

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Then the same argument as in Section 4 yields the following theorem.

Theorem 5 (Existence and Stability). There exist $\varepsilon > 0$ and C > 0such that, if (10) holds, then, for each $\theta \in (\Pi_{k=0}^{\infty}(-1,1)) \setminus (N \cup N_1)$, there exist a sequence $\{\Delta_l\}_{l=1}^{\infty}$ of mesh sizes with $\Delta_l \to 0$ as $l \to \infty$ and a pair of functions $U_{\theta} \in O_{\varepsilon}(U_+)$ and $\chi_{\theta} \in Lip(\mathbb{R}_+)$ with $\chi_{\theta}(0) = 0$ such that

(i) $U_{\Delta_l,\theta}(x,\cdot)$ converges to $U_{\theta}(x,\cdot)$ in $L^1(-\infty, g(x))$ for every x > 0, and U_{θ} is a global entropy solution of problem (1) and (8)–(9) in Ω and satisfies (11)–(12);

(ii) $\chi_{\Delta_l,\theta}$ converges to χ_{θ} uniformly in any bounded x-interval;

(iii) $\sigma_{\Delta_l,\theta}$ converges a.e. to $\sigma_{\theta} \in BV(\mathbb{R}_+)$ with $|\sigma_{\theta} - \sigma_0| < \hat{\varepsilon}$ and $\chi_{\theta}(x) = \int_0^x \sigma_{\theta}(t) dt$.

In addition, if θ is equidistributed, then $\chi_{\theta}(x) < g(x)$ for any x > 0 with (13) and the Rankine-Hugoniot conditions a.e. along the curve $\{y = \chi_{\theta}(x)\}$.

Furthermore, let $\theta \in (\Pi_{k=0}^{\infty}(-1,1)) \setminus (N \cup N_1)$ be equidistributed, and let U_{θ} be the solution and χ_{θ} its shock-front, respectively. By Theorem 5, we find that the solution U_{θ} contains at most countable shock-fronts and countable points of wave interactions; Moreover, we can modify the solution U_{θ} such that U_{θ} is continuous except the shock curves and the points of wave interactions (cf. [9]). Then we have

Theorem 6. (i) Let $\omega_{\infty} = \lim_{x \to \infty} \arctan(g'(x+))$. Then

$$\lim_{x \to \infty} \sup\{|\arctan\left(v_{\theta}(x, y)/u_{\theta}(x, y)\right) - \omega_{\infty}| : \chi_{\theta}(x) < y < g(x)\} = 0.$$

(ii) There exist constants p_{∞} and σ_{∞} such that

$$\lim_{x \to \infty} \sup\{ |p_{\theta}(x, y) - p_{\infty}| : \chi_{\theta}(x) < y < g(x) \} = 0$$

and

$$\lim_{x \to \infty} |\sigma_{\theta}(x) - \sigma_{\infty}| = 0.$$

Theorem 6 can be proved in the same way as Theorem 4.

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