Differential Equations

Ex.

Physical Variable

(Density, velocity, temperature ....)

Location
Time

$\boxed{u}$

$\boxed{(t, x, y, z)}$

Unknown

$u = u(t, x, y, z)$

Physical Laws

A Relation between

$\boxed{u}$, $\boxed{\text{its Derivatives}}$

and

Independent Variables $\boxed{(t, x, y, z)}$

Differential Equations

$$(*) \quad \boxed{F(D^\alpha u,\ u,\ x) = 0, \qquad |\alpha| \le m}$$

- $x = (x_1, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$: Indept. Variables
- $u = u(x) = u(x_1, \cdots, x_n)$: Unknown function(s)
- $\alpha = (\alpha_1, \cdots, \alpha_n)$: Multi-index,

  $\alpha_i \in \mathbb{Z}$, $\alpha_i \ge 0$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$
- $Du = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$

  $D^\alpha u = \left(\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}\right)$, $\quad 2 \le |\alpha| \le m$

  e.g. $|\alpha| = 2$, $\quad D^\alpha u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{1 \le i, j \le n}$
- F : Known function of u, $D^\alpha u$, x.
- If $n = 1$: $\quad (*)$ is ODE
- If $n \ge 2$: $\quad (*)$ is PDE
- If $u \in \mathbb{R}^p$ $(p \ge 2)$, $F \in \mathbb{R}^q$ $(q \ge 2)$

  $\hookrightarrow$ $(*)$ is a system of ODEs or PDEs
- Order $= m$ $\qquad$ Dimension $\sim$ # of spatial variables

$$(*) \qquad \boxed{F(D^\alpha u, u, x) = 0, \qquad |\alpha| \le m}$$

- We can rewrite $(*)$ as

$$\boxed{\begin{array}{l} G(D^\alpha u, u, x) = f(x) \\ \hline G(0, 0, x) = 0 \end{array}}$$

If $f(x) = 0$: Homogeneous

If $f(x) \neq 0$: Nonhomogeneous

- Linear: F is linear w.r.t. $D^\alpha u$ and u

$$\sum_{|\alpha| \le m} A_\alpha(x) D^\alpha u = f(x)$$

- Quasilinear: F is linear w.r.t. $\boxed{D^{\tilde{\alpha}} u, \ |\tilde{\alpha}| = m}$

$$\sum_{|\tilde{\alpha}| = m} A_{\tilde{\alpha}}(D^\alpha u, u, x) D^{\tilde{\alpha}} u + A_0(D^\alpha u, u, x) = 0$$

$$|\alpha| \le m-1$$

- Semilinear

$$\sum_{|\tilde{\alpha}| = m} A_{\tilde{\alpha}}(x) D^{\tilde{\alpha}} u + A_0(D^\alpha u, u, x) = 0, \quad |\alpha| \le m-1$$

- Fully Nonlinear: F depends nonlinearly upon $D^{\tilde{\alpha}} u$, $|\tilde{\alpha}| = m$

$$(*) \quad \boxed{F(D^\alpha u, u, x) = 0 \qquad |\alpha| \le m}$$

$\boxed{\text{SOLUTION}}$. $u = u(x)$ is called a $\boxed{\text{Solution}}$ of $(*)$ in Ω, provided that

$$(**) \quad \boxed{F(D^\alpha u(x), u(x), x) = 0 \qquad x \in \Omega}$$

in an appropriate sense of topology

Classical Solution: $u \in C^m(\Omega)$

Weak Solution, Generalized Solution:

(**) holds in a WEAK sense

Entropy Solution

Besides (**), $\boxed{u = u(x)}$ satisfies an additional condition:

$\boxed{\text{the entropy condition}}$

Linear PDEs: Models

- Transport equation: $\boxed{u_t + a u_x = 0}$

  a: const.

- Laplace's equation: $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

  $$\boxed{\Delta u = 0}$$

  u — potential/harmonic function

- Heat equation

  $$\boxed{u_t = k \Delta u,}$$ $k > 0$ const.

  $u \sim$ temperature

- Wave equation

  $$\boxed{u_{tt} - c^2 \Delta u = 0.}$$ $c = $ const.

  $n=1$: Vibrating of strings, Propagation of sound waves in tubes

  $n=2$. Waves on the surface of shallow water, and vibrating drumhead

  $n=3$ Acoustic or light waves

- Shrödinger's Eq., Black-Scholes Eq., ...

Nonlinear PDEs : Models

- Hopf-Burgers Eq.

$$\boxed{u_t + u\,u_x = 0}$$

- p-system

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0, \end{cases} \quad p'(v) < 0$$

$$u \in C^2 \Longrightarrow u_{tt} + (p'(v)\,u_x)_x = 0$$

$$p'(v) = -c^2 \Longrightarrow \text{Wave Eq.}$$

- Minimal Surface Eq. $\quad z = u(x, y)$.

$$(1+u_y^2)\,u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)\,u_{yy} = 0$$

- Incompressible Euler (Navier-Stokes) Eqs.

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = 0 & (\varepsilon\Delta u) \\ \operatorname{div} u = 0 & u \in \mathbb{R}^n,\ x \in \mathbb{R}^n \end{cases}$$

$$\varepsilon = 0,\ p = \text{const.}\ n = 1 \Longrightarrow \text{Hopf-Burgers Eq.}$$

- KdV Eq. (Korteweg-de Vries Eq).

$$u_t + u\,u_x + \varepsilon\,u_{xxx} = 0$$

$$\varepsilon = 0 \Longrightarrow \text{Hopf-Burgers Eq.}$$

- ......

Solving PDEs — Solvability

Experience from solving the n^{th} Order algebraic Eq.

$$(*) \quad \boxed{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0}$$

$n=1, 2$: Natural

$n=3$: Niccolò Fontana Tartaglio: 1535, 1539

Gerolamo Cardano: Ars Magna, 1545

$n=3$: Lodovico Ferrari (Cardano's Student): 1540

↓ 300 Years Descartes, Newton, Euler, ...

Abel (1826): It is impossible to solve high-order eqs. ($n \geq 5$) by simple explicit algebraic operations.

Galois (1831): Galois theory for (*) indicating that the higher the order is, the more difficult one solves it.

- The Same Feature for PDEs.

  Higher Order PDEs are more difficult to be solved than lower order PDEs

Other Difficulties

- Nonlinear PDEs are more difficult than Linear PDEs.
  In many applications, they are often approximated by linear PDEs.
- Systems of PDEs are more difficult than Single PDE.
- Higher dimensional PDEs are more difficult than lower dimensional PDEs
- For most PDEs, it is impossible to write out Explicit Formulas for Solutions, Except some classes of PDEs.

$\boxed{\text{Point}}$ Our interest is NOT so much in being able to "write down" solns. Rather than it is in understanding Properties of Solns & Methods for finding/analyzing Solns & Various classes of PDEs — where they're from, how solutions behave, & how different they can be from one another.

Well-Posedness – Meaning of Solvability

Difficulty: Different PDEs have completely different behavior of Solutions.

Side Conditions:

- Boundary Conditions
- Initial Conditions
- Other Side Conditions...

Issues.

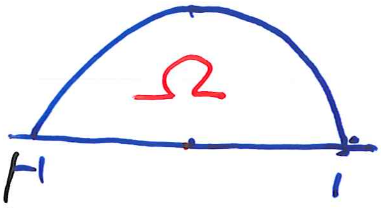
- Sensitive, intimately connected with the form of the PDEs.
- Must be extremely careful in selecting the "Side Conditions" for the problems involving PDEs.

Example 1

Cauchy Problem / IBV Problem

$$\begin{cases} \Delta u := u_{xx} + u_{yy} = 0, & (x,y) \in \Omega \\ u|_{y=0} = 0 \\ u_y|_{y=0} = a(x), & -1 < x < 1 \end{cases}$$

Claim: $\nexists$ a solution of this problem in general!



If $\exists$ a soln $u(x,y) \in C^2(\Omega)$,

we extend

$$u(x,y) = \begin{cases} u(x,y), & y > 0 \\ -u(x,-y), & y < 0 \end{cases} \in C^2$$

(Odd extension)

Define $v(x,y) = \int_{(0,0)}^{(x,y)} (-u_y\, dx + u_x\, dy)$

$\hookrightarrow$ $\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$ Cauchy-Riemann Eq.

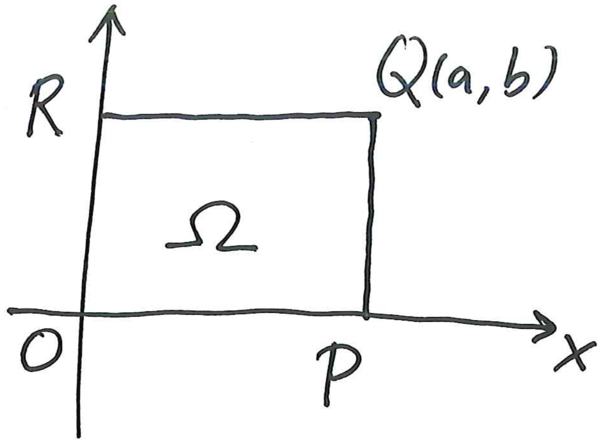
$\hookrightarrow$ $u + iv$ analytic

$\hookrightarrow$ $u(x,y)$ is real analytic

$\hookrightarrow$ $a(x) = u_y(x,0)$ Must be real analytic

Example 2 Dirichlet Problem

$$\begin{cases} u_{xy} = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = u_0 \end{cases}$$



General Solution

$$u(x,y) = \varphi(x) + \psi(y)$$

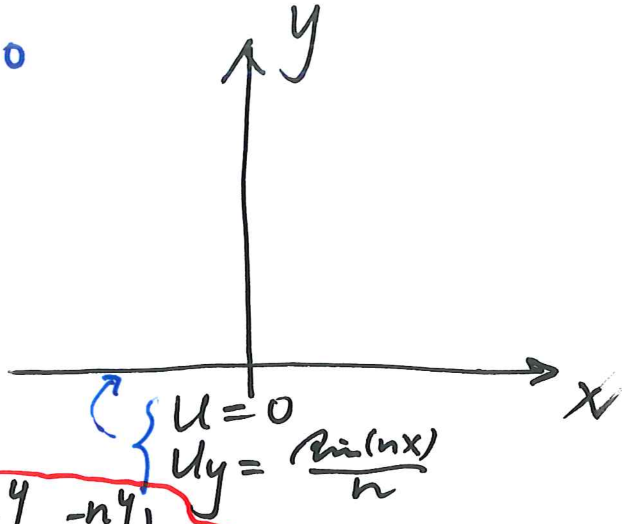
$$\boxed{u_0(O) + u_0(Q) = u_0(R) + u_0(P)} \qquad (*)$$

This problem has $\boxed{\text{No solutions}}$

if $(*)$ fails to hold

Example 3 Cauchy Problem

$$(\mathrm{I})\quad \begin{cases} \Delta u = 0 & x \in \mathbb{R},\ y > 0 \\ u|_{y=0} = 0 \\ u_y|_{y=0} = \dfrac{\sin(nx)}{n} \end{cases}$$



$$\boxed{u^n(x,y) = \frac{\sin(nx)\left(e^{ny} - e^{-ny}\right)}{2n^2}}$$

$$(\mathrm{II})\quad \begin{cases} \Delta u = 0 \\ u|_{y=0} = 0 \\ u_y|_{y=0} = 0 \end{cases} \longrightarrow \quad u^\infty(x,y) = 0.$$

Phenomenon

$$\begin{cases} u^n|_{y=0} = u^\infty|_{y=0} \\ |u_y^n(x,0) - u_y^\infty(x,0)| = \dfrac{|\sin(nx)|}{n} \xrightarrow[n\to\infty]{} 0 \end{cases}$$

BUT

$$\lim_{n\to\infty} |u^n(x,y) - u^\infty(x,y)| = \lim_{n\to\infty} \frac{|\sin(nx)|\,|e^{ny} - e^{-ny}|}{2n^2}$$

$$= +\infty, \qquad \forall\ y > 0.$$

* Instability: Arbitrarily small changes in the data

↳ Large changes in the solution.

More Examples

4. $\begin{cases} u_t - \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$ (*)

Uniqueness: A solution of (*) is unique in the class of solutions with

(#) $|u(x,t)| \le M e^{C|x|^2}$, $0 \le t \le T$

for some constants M, C.

Tychonoff Example: Without assumption (#), $\exists$ ∞-many solns.

cf. Fritz John: P211-213

5. H. Lewy's Example Fritz John P235-239

Consider $Lu = -u_x - i u_y + 2i(x+iy)u_z$

$\exists$ $F(x,y,z) \in C^\infty(\mathbb{R}^3)$ such that Eq.

$Lu = F(x,y,z)$

has $\boxed{\text{no solution}}$ whose domain is an open set $\Omega \subset \mathbb{R}^3$ with $u \in C^1(\Omega)$ and

u_x, u_y, u_z Hölder Continuous in Ω

Well-posedness $\begin{cases} \text{Existence} \\ \text{Uniqueness} \\ \text{Stability} \end{cases}$

? Spaces for Solutions

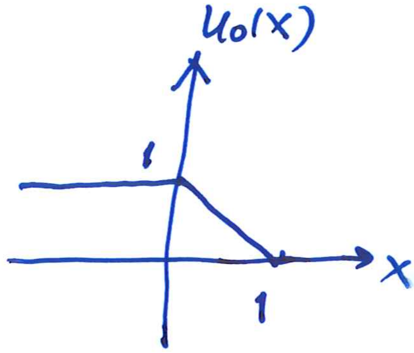
? Topology for the Stability of Solutions

Spaces:

C^k, $W^{k,p}$, BV, $\mathcal{M}$, ...

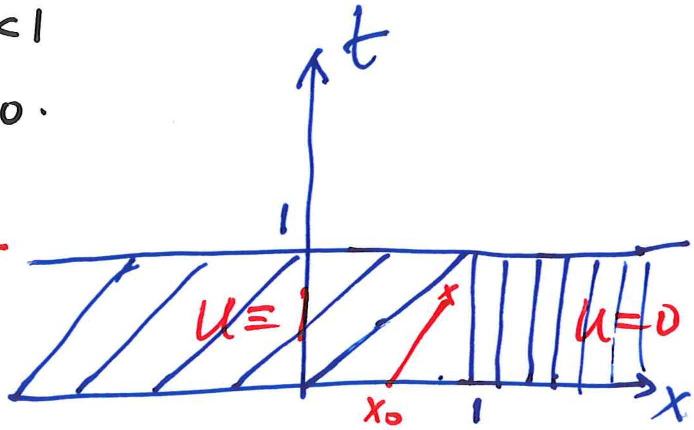
Example 4. Hopf-Burgers Eq.

$$\begin{cases} u_t + u\, u_x = 0. & x \in \mathbb{R},\ t > 0 \\ u|_{t=0} = u_0(x) = \begin{cases} 1 & x < 0 \\ 1-x & 0 < x < 1 \\ 0 & x > 1 \end{cases} \end{cases}$$



For $t < 1$

$$u(x,t) = \begin{cases} 1 & x < t \\ \frac{1-x}{1-t} & t < x < 1 \\ 0 & x > 0. \end{cases}$$



$\hookrightarrow$ $u_x(x,t)$ blows up at $t = 1$

The phenomenon is natural in Physics.

? How do we get out of the dilemma ?

Answer: Weaken the Notion of solution.

Allow the solution discontinous
starting at $t = 1$

$$\boxed{\text{Shock Wave Phenomenon !}}$$

I. Ordinary Differential Equations

Cauchy Problem

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = f(x,t) & x \in \mathbb{R}^n \\ x|_{t=0} = x_0 \end{cases}$$

§1. Local Well-Posedness

Choose $a>0$, $b>0$, and $M>0$ s.t.

- f is defined in
  $$R = \{(x,t) \in \mathbb{R}^{n+1} \mid |x-x_0| \le b,\ |t-t_0| \le a\}$$
- $f(x,t)$ is continuous in R
- $|f(x,t)| \le M$, for $(x,t) \in R$.

Observation: If $x \in C^1$ is a soln. with
$|x(t)-x_0| \le b$ for $|t-t_0| \le T$

$$\hookrightarrow\ |x(t)-x_0| = \left|\int_{t_0}^{t} f(x(s),s)\,ds\right| \le \int_{t_0}^{t} |f(x(s),s)|\,ds$$
$$\le M|t-t_0| \le \underline{MT < b}$$
if $\boxed{T \le \min\{a, \frac{b}{M}\}}$

Theorem 1.1 (Picard's Theorem)

Assume

- $|f(x,t)| \le M, \qquad (x,t) \in R$
- $|f(x,t) - f(y,t)| \le C\,|x-y|, \quad \forall \begin{matrix}(x,t)\\(y,t)\end{matrix} \in R$

(i) $\exists\, 1 \quad x = x(t) \in C^1(t_0 - T,\ t_0 + T)$ of $(*)$

provided that $T \le \min\{a, \frac{b}{M}\}$

(ii) The Cauchy problem

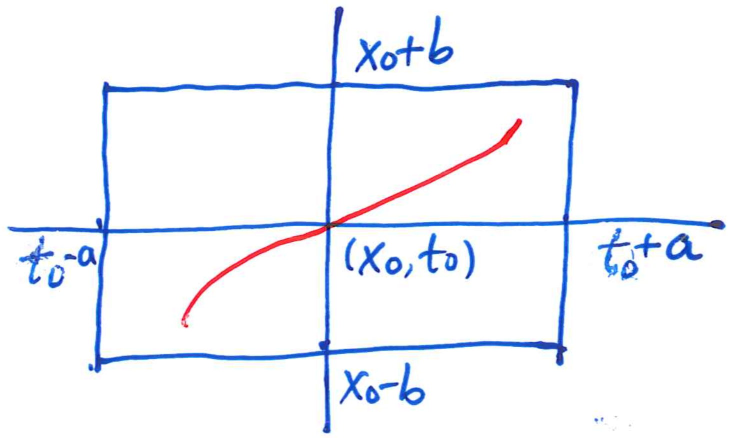
$$\begin{cases} \dfrac{dx}{dt} = f(x,t) \\ x|_{t=t_0} = y \end{cases}$$

has a unique soln $x = x(t, y)$

for $|t-t_0|M + |y-x_0| \le b, \quad |t-t_0| \le a$

and

$$|x(t,y) - x(t,z)| \le e^{C|t-t_0|}\,|y-z|$$



Homework #1

Remark The uniqueness part of Thm 1.1 remains valid for $t \geq t_0$ if the Lipschitz condition on f is weakened to

$$(**) \quad \langle y-z,\ f(t,y)-f(t,z)\rangle \leq C|y-z|^2$$

When $n=1$, $(**)$ becomes

$$(***) \quad f(t,y)-f(t,z) \leq C(y-z) \quad \text{if } y \geq z$$

(One-side Lipschitz condition)

Comparison Principle: $n=1$. $(***)$ holds

$$\left\{\begin{array}{l} \dfrac{dx(t)}{dt} \geq f(x(t),t) \\ \dfrac{dy(t)}{dt} \leq f(y(t),t) \\ x(t_0) \geq y(t_0) \end{array}\right. \Longrightarrow \quad x(t) \geq y(t), \quad t \geq t_0$$

Proof. Set $R(t) = y(t)-x(t)$. If $\exists\, t \in (t_1, t_2)$ s.t.

$$\left\{\begin{array}{ll} R(t) \leq 0 & t_0 < t < t_1 \\ R(t) > 0 & t_1 < t < t_2 \end{array}\right. \longrightarrow \quad R(t_1) = 0$$

$$\left\{\begin{array}{l} \dfrac{dR(t)}{dt} \leq f(y(t),t) - f(x(t),t) \leq C R(t), \quad t_1 < t < t_2 \\ R(t_1) = 0 \end{array}\right.$$

$$\Longrightarrow \quad 0 < R(t) \leq \underbrace{R(t_1)}_{=0} e^{C(t-t_1)} \quad \text{for } t_1 < t < t_2$$

Contradiction

Theorem 1.2 (Global existence and uniqueness for Linear ODEs)

If $\begin{cases} f(x,t) = A(t)\,x \\ A(t) \text{ is continuous for } |t-t_0| \le a \end{cases}$

then the Cauchy problem (*) has a unique soln for $|t-t_0| \le a$:

$$x(t) = F(t)\,x_0$$

where $F(t)$ is the unique $n \times n$ matrix function determined by

$$\begin{cases} \dfrac{dF(t)}{dt} = A(t)\,F(t) \\ F(t_0) = I \end{cases}$$

If $\|A(t)\| \le M$, then, for $|t-t_0| \le a$

$$\begin{cases} \|F(t)\| \le e^{|t-t_0| M} \\ \|F(t)^{-1}\| \le e^{|t-t_0| M} \\ \det(F(t)) \le \exp\left(\int_{t_0}^{t} \mathrm{Tr}(A(s))\,ds\right) \end{cases}$$

Homework #2

Theorem 1.3 (Peano's Theorem)

$$\begin{cases} f(x,t) & \text{is continuous in } R \\ |f(x,t)| \le M, & (x,t) \in R \end{cases}$$

$\Longrightarrow$ $\exists$ a C^1-solution for $|t-t_0| \le T$
$$T = \min\{a, \tfrac{b}{M}\}$$

Proof. 1. Extend f to $\mathbb{R}^n \times [t_0-a, t_0+a]$ by defining

$$f(x,t) = f\left(x_0 + \frac{b(x-x_0)}{|x-x_0|}, t\right) \quad \text{for } |x-x_0| > b$$

2. For $\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi\left(\frac{x}{\varepsilon}\right)$ with

Mollifier

$$\chi(x) = \begin{cases} A e^{\frac{1}{|x|^2-1}} & |x| \le 1 \\ 0 & |x| > 1. \end{cases} \qquad \int \chi(x)\,dx = 1$$

define

$$f_\varepsilon(x,t) = f * \chi_\varepsilon(x) = \int \chi_\varepsilon(x-y) f(y,t)\,dy$$

$\hookrightarrow$

$$\begin{cases} f_\varepsilon(x,t) & C^\infty \text{ in } x, \text{ continuous in } t \\ f_\varepsilon \to f & \text{uniformly in } R, \text{ as } \varepsilon \to 0 \\ |f_\varepsilon| \le M & \forall\, \varepsilon \end{cases}$$

Proof (Conti.)

3. Consider the Cauchy problem:

$$\begin{cases} \dfrac{dx_\varepsilon(t)}{dt} = f_\varepsilon(x_\varepsilon(t), t) \\ x_\varepsilon\big|_{t=t_0} = x_0 \end{cases}$$

Thm 1.1 $\Rightarrow$ $\exists$ 1 soln $x = x_\varepsilon(t)$ for $|t-t_0| \le T$ satisfying

$$\begin{cases} |x_\varepsilon(t) - x_0| \le b & \forall \varepsilon \\ \left|\dfrac{dx_\varepsilon(t)}{dt}\right| \le |f_\varepsilon(x_\varepsilon(t), t)| \le M & \forall \varepsilon \end{cases}$$

↳ Equicontinuity

4. Ascoli-Arzela Thm

↳ $\exists$ $\{x_{\varepsilon_k}\}_{k=1}^{\infty} \subset \{x_\varepsilon\}_{\varepsilon>0}$ s.t.

$$\begin{cases} x_{\varepsilon_k}(t) \longrightarrow x(t) & \text{uniformly} \\ \dfrac{dx_{\varepsilon_k}(t)}{dt} = f_{\varepsilon_k}(x_{\varepsilon_k}(t), t) \longrightarrow f(x(t), t) & \text{uniformly} \end{cases}$$

↳ $$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t), t) \\ x\big|_{t=t_0} = x_0 \end{cases}$$

* Approximation Method

Observation

If $\begin{cases} f \in C(R) \\ |f(x,t)| \le M \quad \forall (x,t) \in R \end{cases}$

- $x(t)$ is any soln of $\begin{cases} \frac{dx(t)}{dt} = f(x(t),t), \quad |t-t_0| \le T \\ x|_{t=0} = x_0 \end{cases}$

$\hookrightarrow$ $\exists$ $f_\varepsilon(x,t)$ s.t.

(i) $\begin{cases} \frac{dx(t)}{dt} = f_\varepsilon(x(t),t) \\ x(0) = x_0 \end{cases}$

(ii) f_ε C^∞ in x, continuous in t

(iii) $f_\varepsilon \to f$ uniformly in R as $\varepsilon \to 0$

In fact, for $\widehat{f_\varepsilon}$ defined as in the proof of Theorem 1.3, define

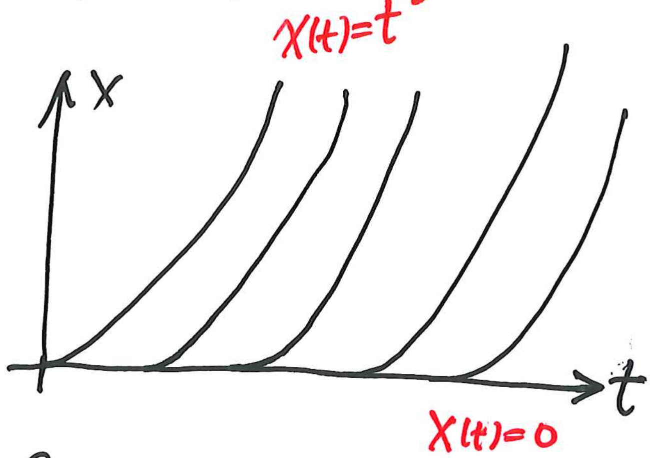
$$\boxed{f_\varepsilon(x,t) = \widehat{f_\varepsilon}(x,t) + f(x(t),t) - \widehat{f_\varepsilon}(x(t),t)}$$

Example:
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{3}{2} x^{\frac{1}{3}} \\ x|_{t=0} = 0 \end{cases}$$

- $f(x,t) = \frac{3}{2} x^{\frac{1}{3}}$ is continuous in x but is not Lipschitz at $x=0$

Easy to check: Two solutions

$$\begin{cases} x(t) = 0, \\ x(t) = t^{3/2} \end{cases}$$



General solutions

$$x_c(t) = \begin{cases} 0 & t \le c \\ (t-c)^{\frac{3}{2}} & t \ge c \end{cases}$$

Starting at (0, 0) lie between these two solutions, and fill up the funnel between them.

§2 Existence of Solutions in the Large

<u>Ex</u>. $\begin{cases} \dfrac{dx}{dt} = x^2 \\ x|_{t=0} = x_0 > 0 \end{cases}$ $\quad (*)$

<u>Soln</u>: $\dfrac{dx}{x^2} = dt, \quad -\dfrac{1}{x} + \dfrac{1}{x_0} = t$

$$x(t) = \frac{x_0}{1 - x_0 t} \nearrow \infty \quad \text{as } t \nearrow \frac{1}{x_0}$$

In general, $\nexists$ a global solution

If $x_0 < 0 \longrightarrow \exists\, 1$ global solution for $t > 0$

<u>Thm 1.4</u> (Lower semicontinuity of the domain of existence for perturbations of the data)

If (i) $f(x,t)$, $\dfrac{\partial f(x,t)}{\partial x}$ are continuous in an open set $\Omega \subset \mathbb{R}^{n+1}$

(ii) The Cauchy problem $(*)$ has a soln x with graph in Ω for $t_1 \le t \le t_2$ with $t_0 \in (t_1, t_2)$

$\Rightarrow$ $\exists$ a nbhd U of x_0 in $\mathbb{R}^n$, s.t. $\forall\, y \in U$

$\exists\, 1$ soln $x(t,y) \in C^1([t_1, t_2] \times U)$ of $\begin{cases} \dfrac{dx}{dt} = f(x,t) \\ x|_{t=t_0} = y \end{cases}$

for $t_1 \le t \le t_2$.

Remark: We could also allow a small perturbation of $\boxed{f}$ in Thm 1.4, for which we could replace $\boxed{f}$ by $\boxed{f+\varepsilon g}$ for small ε, adding the Eq. $\frac{d\varepsilon}{dt}=0$

Now we allow perturbation of the Cauchy problem (*).

Theorem 1.5 Consider the Cauchy problem

$$(**)\quad \begin{cases} \frac{dx}{dt} = a_0(t)x^2 + a_1(t)x + a_2(t) \\ x(0) = x_0 \end{cases}$$

& $a_j(t)$ are continuous function in $[0,T]$

Set $\begin{cases} a_0^+ = \max\{a_0, 0\} \\ K = \int_0^T |a_2(t)|dt \exp\left(\int_0^T |a_1(t)|dt\right) \end{cases}$

If $\begin{cases} x_0 \geq 0 \\ \int_0^T a_0^+(t)dt \exp\left(\int_0^T |a_1(t)|dt\right) < \frac{1}{x_0+K} \\ \int_0^T |a_0(t)|dt \exp\left(\int_0^T |a_1(t)|dt\right) < \frac{1}{K} \end{cases}$

$\Rightarrow$ The Cauchy Problem (**) has a Solution in $[0, T]$. satisfying

$$\frac{1}{x(T)} \geq \frac{1}{x_0+K} - \int_0^T a_0^+(t)\,dt\, \exp\left(\int_0^T |a_1(t)|\,dt\right)$$

if $x(T) \geq 0$.

or

$$\frac{1}{x(T)} \geq \frac{1}{K} - \int_0^T |a_0(t)|\,dt\, \exp\left(\int_0^T |a_1(t)|\,dt\right)$$

if $x(T) < 0$.

* $T \sim$ Life span, i.e.

$$T = \sup_{t^*} \left\{ t^* \,\middle|\, x(t) \text{ exists in } [0, t^*] \right\}$$

Homework # 3

§3 Generalized Solutions

$$(*)\ \begin{cases} \dfrac{dx}{dt} = f(x,t) \\ x|_{t=t_0} = x_0 \end{cases}$$

$f(x,t)$ ~ real value measurable function in

$$R = \{(x,t) \in \mathbb{R}^{n+1} \mid |t-t_0| \le a,\ |x-x_0| \le b\}$$

$|f(x,t)| \le M(t)$ with $M \in L^1$

Theorem 1.6 The Cauchy problem $(*)$ has a solution in the sense that x is absolutely continuous and for almost all $t \in [t_0-a, t_0+a]$,

$$\boxed{x'(t) \in F(x(t),t)}$$

$F(x,t)$ ~ The smallest closed convex set s.t. every nbhd contains the value of $f(y,t)$ for almost all y in some nbhd of x

$$\boxed{\langle x'(t), \xi\rangle \le H(x(t),t,\xi) = \lim_{\delta\to 0} \operatorname*{ess\,sup}_{|y-x(t)|<\delta} \langle f(y,t), \xi\rangle \quad \forall\, \xi \in \mathbb{R}^n.}$$

References for Part I

1. R. Hörmander: Chapter 1
   Lectures on Nonlinear Hyperbolic
   Differential Equations
   Springer-Verlag, 1997.

2. Other standard ODE Textbooks