PDE-CDT Core Course

Analysis of Partial Differential Equations-Part III

Lecture 4

EPSRC Centre for Doctoral Training in
Partial Differential Equations
Trinity Term
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Final Exam: 2 July 2015
Course format: Teaching Course (TT)

By Prof. Gui-Qiang G. Chen
Dr. Qian Wang

Hyperbolic Systems of Conservation Laws

$$\begin{cases} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \cdots, u_m) = 0, \\ \qquad\qquad \cdots\cdots \\ \frac{\partial}{\partial t} u_m + \frac{\partial}{\partial x} f_m(u_1, \cdots, u_m) = 0, \end{cases}$$

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$$

$\mathbf{u} = (u_1, \cdots, u_m)^\top \in \mathbb{R}^m$ conserved quantities

$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \cdots, f_m(\mathbf{u}))^\top$ fluxes

Hyperbolic Systems

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \qquad \mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^m$$

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \qquad \mathbf{A}(\mathbf{u}) = \nabla \mathbf{f}(\mathbf{u})$$

The system is **strictly hyperbolic** if each $m \times m$ matrix $\mathbf{A}(\mathbf{u})$ has real distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$$

Right eigenvectors $\quad \mathbf{r}_1(\mathbf{u}), \cdots, \mathbf{r}_m(\mathbf{u}) \quad$ (column vectors)
Left eigenvectors $\quad \mathbf{l}_1(\mathbf{u}), \cdots, \mathbf{l}_m(\mathbf{u}) \quad$ (row vectors)

$$\mathbf{A}\mathbf{r}_i = \lambda_i \mathbf{r}_i \qquad \mathbf{l}_i \mathbf{A} = \lambda_i \mathbf{l}_i$$

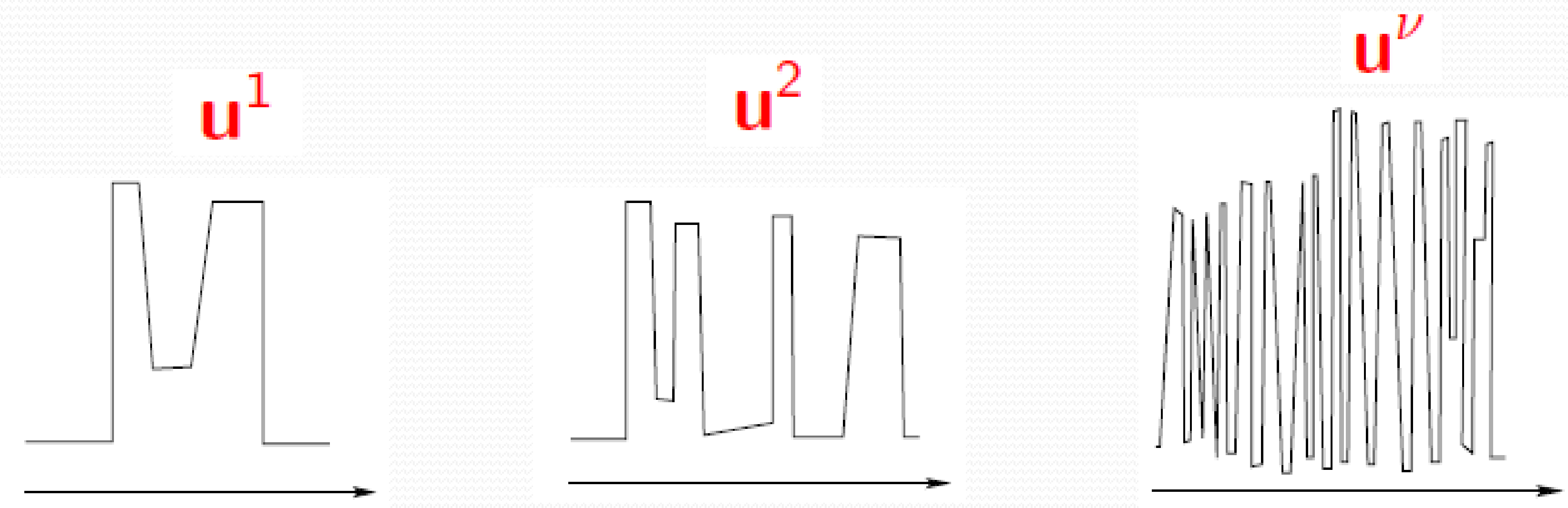
Choose the bases so that

$$\mathbf{l}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Global in Time Solutions to the Cauchy Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \mathbf{u}(x)$$

- Construct a sequence of approximate solutions $\{\mathbf{u}^\nu\}_{\nu\geq 1}$
- Show that (a subsequence) converges: $\mathbf{u}^\nu \to \mathbf{u}$ in L^1_{loc}
- Show that the limit $\mathbf{u}$ is an entropy solution.



Need: a-priori bound on the total variation (J. Glimm, 1965)

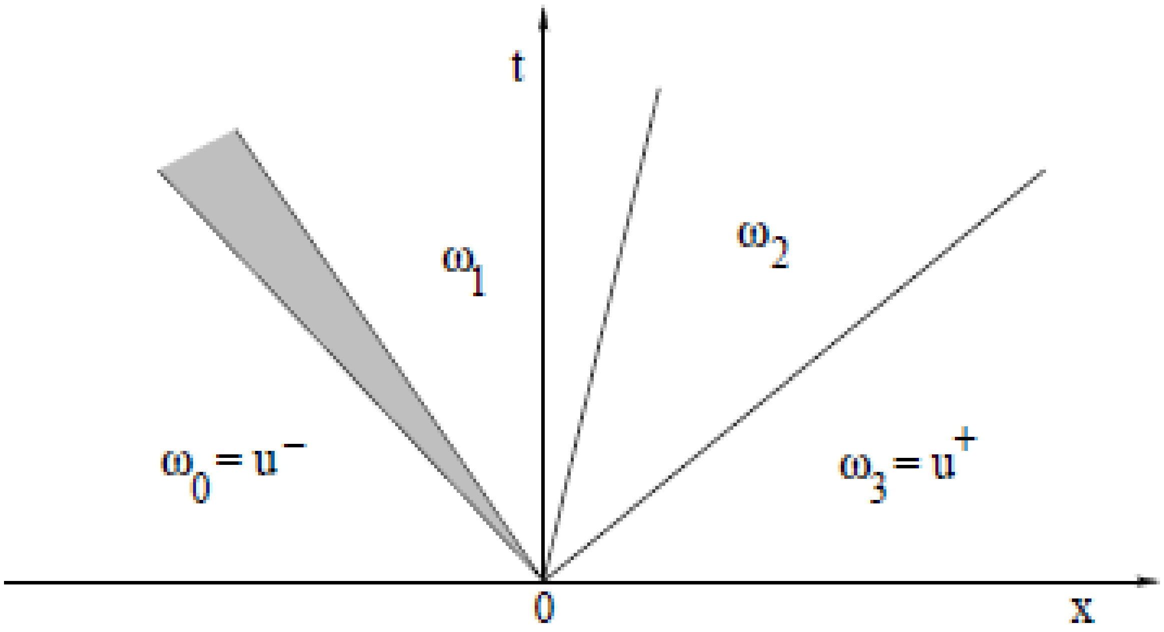
Building Block: The Riemann Problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad \mathbf{u}(0,x) = \begin{cases} \mathbf{u}^- & x < 0 \\ \mathbf{u}^+ & x > 0 \end{cases}$$

- **B. Riemann 1860:** 2×2 **Isentropic Euler equations**
- **P. Lax 1957:** $m \times m$ **systems (+ special assumptions)**
- **T.-P. Liu 1975:** $m \times m$ **systems (generic case)**

***The Riemann solutions are also the vanishing viscosity limits for general hyperbolic systems, possibly non-conservative**

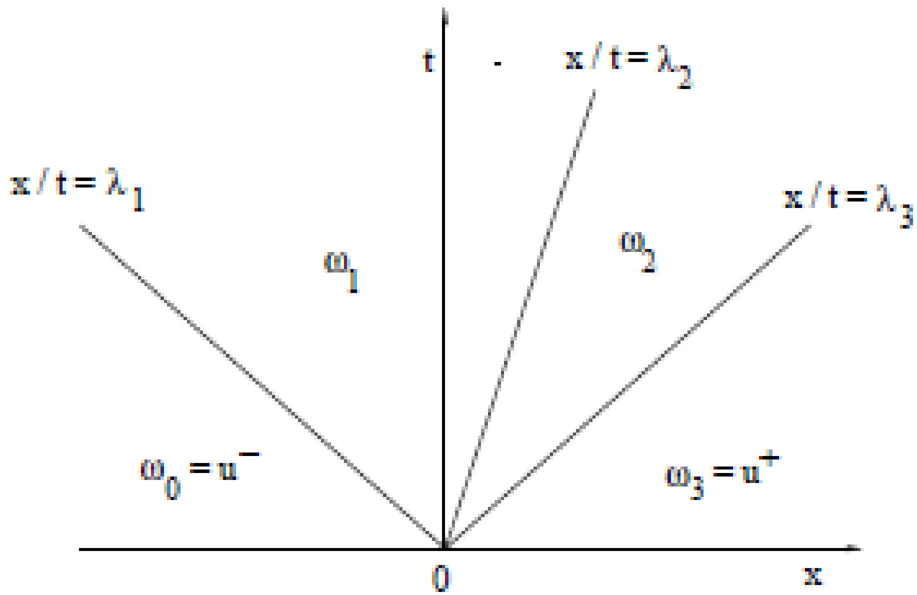
Solution to the Riemann problem



- is invariant w.r.t. rescaling symmetry: $u^\theta(t,x) \doteq u(\theta t,\, \theta x) \qquad \theta > 0$
- describes local behavior of BV solutions near each point (t_0, x_0)
- describes large-time asymptotics as $t \to +\infty$ (for small total variation)

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$



$$u^+ - u^- = \sum_{j=1}^{n} c_j r_j \qquad \text{(sum of eigenvectors of } A\text{)}$$

intermediate states : $\omega_i \doteq u^- + \sum_{j \le i} c_j r_j$

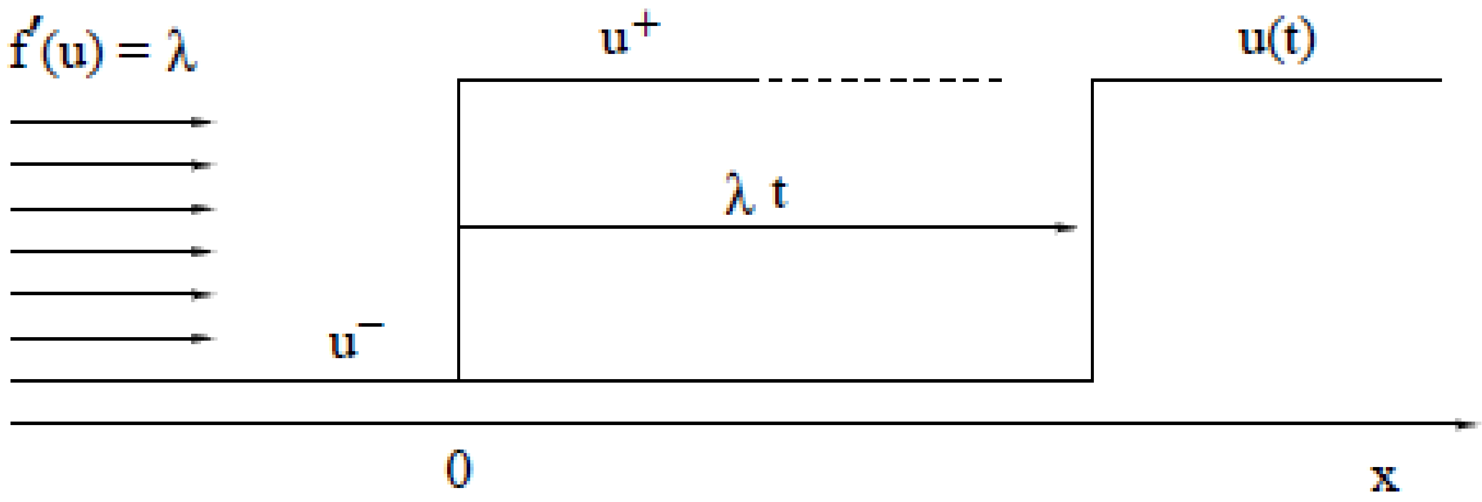
i-th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

Scalar Conservation Law

$$u_t + f(u)_x = 0 \qquad u \in \mathbb{R}$$

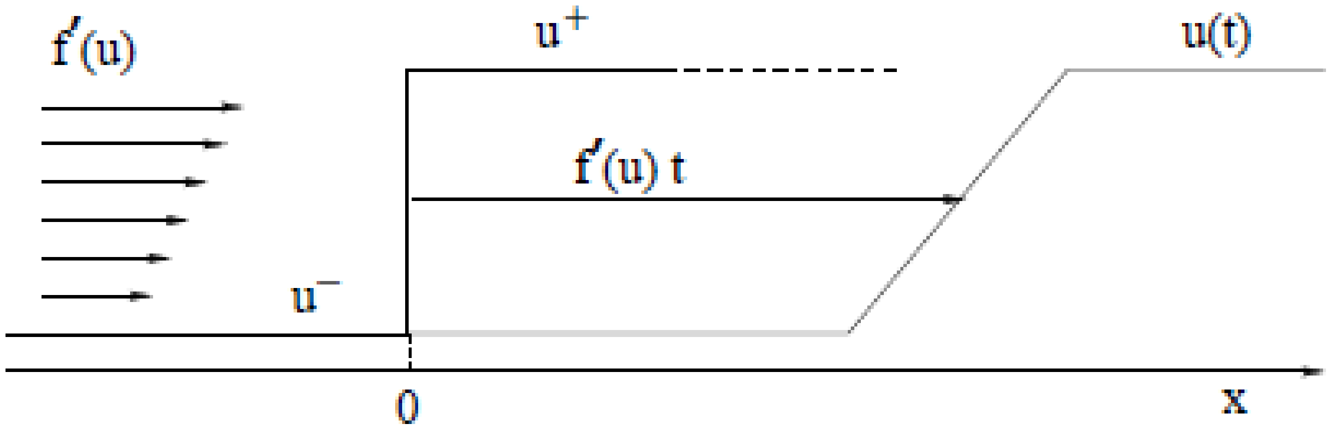
CASE 1: Linear flux: $f(u) = \lambda u$.

Jump travels with speed λ (contact discontinuity)

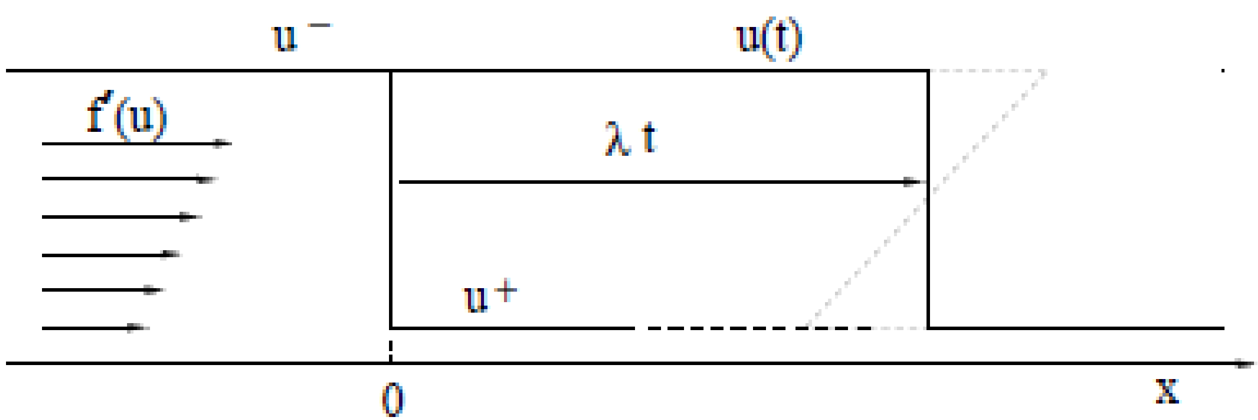


CASE 2: the flux f is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \quad \Longrightarrow \quad$ centered rarefaction wave



$u^+ < u^- \quad \Longrightarrow \quad$ stable shock



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

A class of nonlinear hyperbolic systems

$$u_t + f(u)_x = 0$$

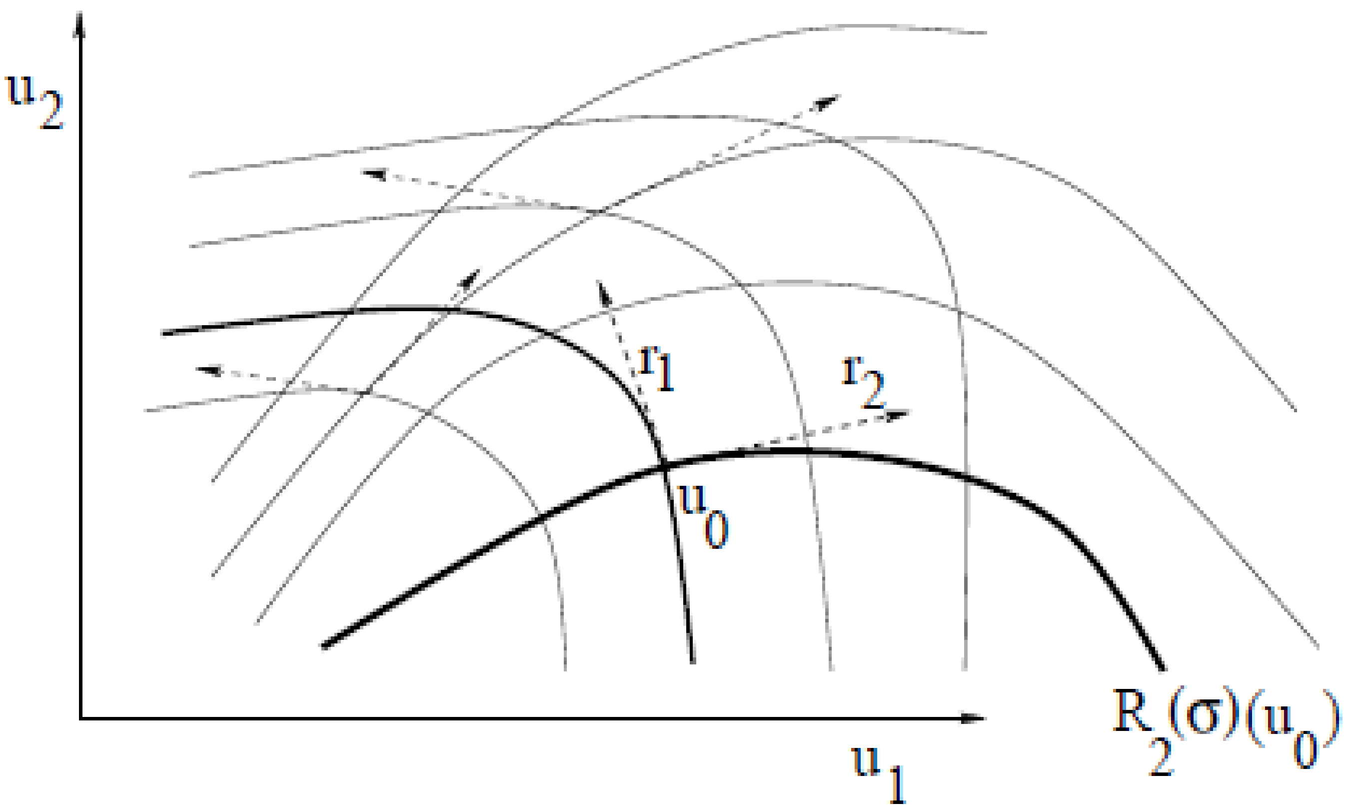
$$A(u) = Df(u) \qquad\qquad A(u)r_i(u) = \lambda_i(u)r_i(u)$$

Assumption (H) (P.Lax, 1957): Each i-th characteristic field is

- either genuinely nonlinear, so that $\nabla\lambda_i \cdot r_i > 0$ for all u
- or linearly degenerate, so that $\nabla\lambda_i \cdot r_i = 0$ for all u

genuinely nonlinear $\implies$ characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors r_i

linearly degenerate $\implies$ characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors r_i



Shock and Rarefaction curves

$$u_t + f(u)_x = 0 \qquad A(u) = Df(u)$$

i-rarefaction curve through u_0: $\sigma \mapsto R_i(\sigma)(u_0)$

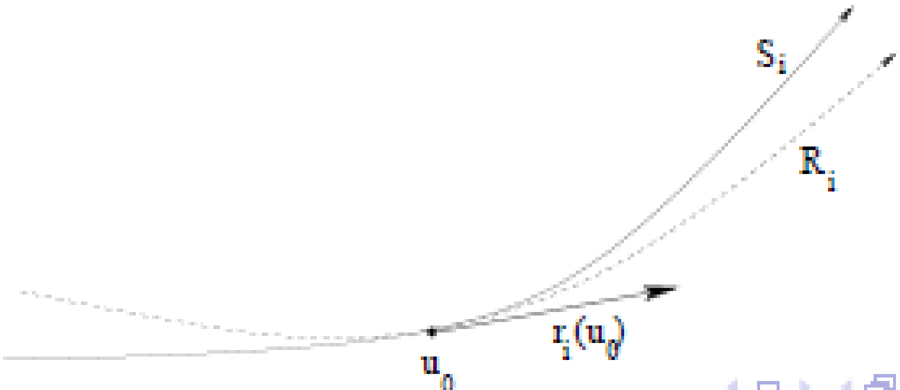
$=$ integral curve of the field of eigenvectors r_i through u_0

$$\frac{du}{d\sigma} = r_i(u), \qquad u(0) = u_0$$

i-shock curve through u_0: $\sigma \mapsto S_i(\sigma)(u_0)$

$=$ set of points u connected to u_0 by an i-shock, so that

$u - u_0$ is an i-eigenvector of the averaged matrix $A(u, u_0)$



Elementary waves

$$u_t + f(u)_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

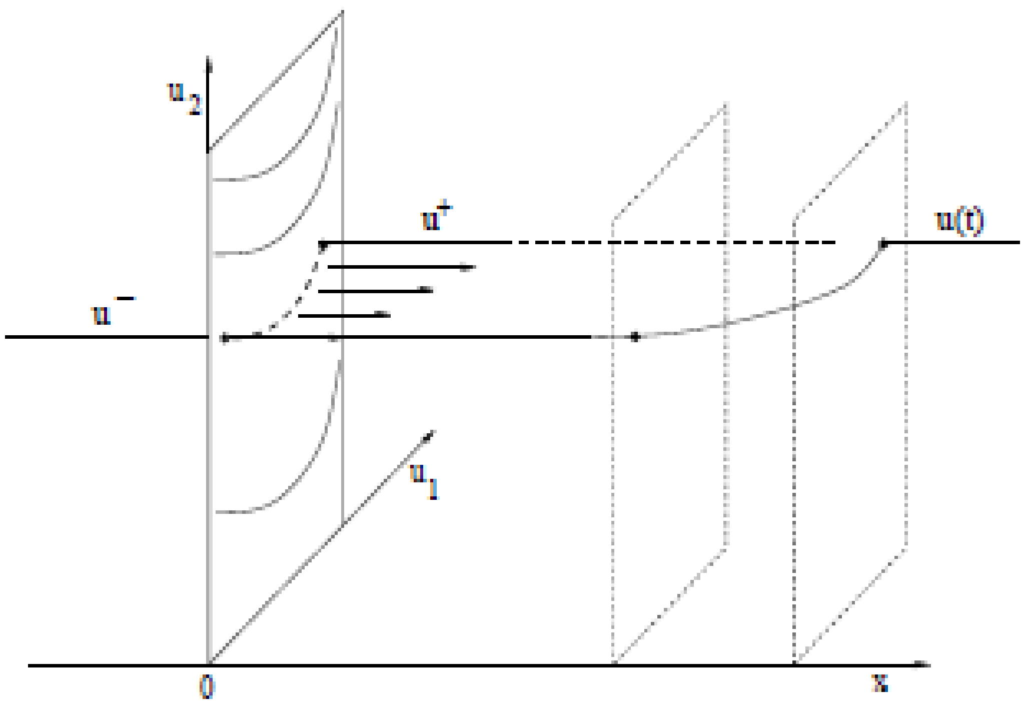
CASE 1 **(Centered rarefaction wave).** Let the i-th field be genuinely nonlinear.

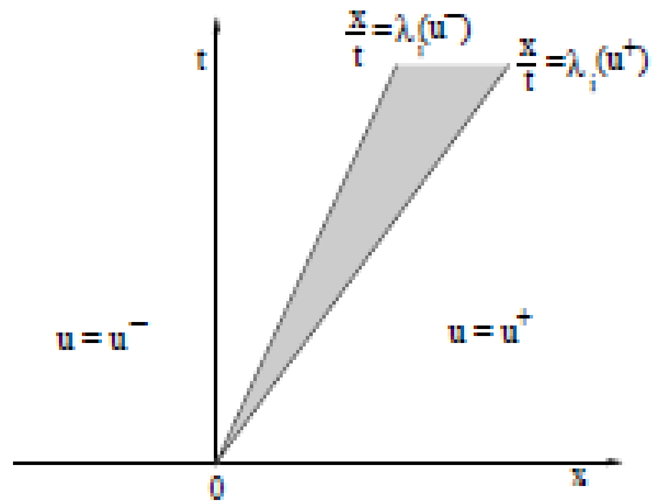
If $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$, then

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < t\lambda_i(u^-), \\ R_i(s)(u^-) & \text{if} \quad x = t\lambda_i(s) \quad s \in [0,\sigma] \\ u^+ & \text{if} \quad x > t\lambda_i(u^+) \end{cases}$$

is a weak solution of the Riemann problem

A centered rarefaction wave





CASE 2 (**Shock or contact discontinuity**). Assume that

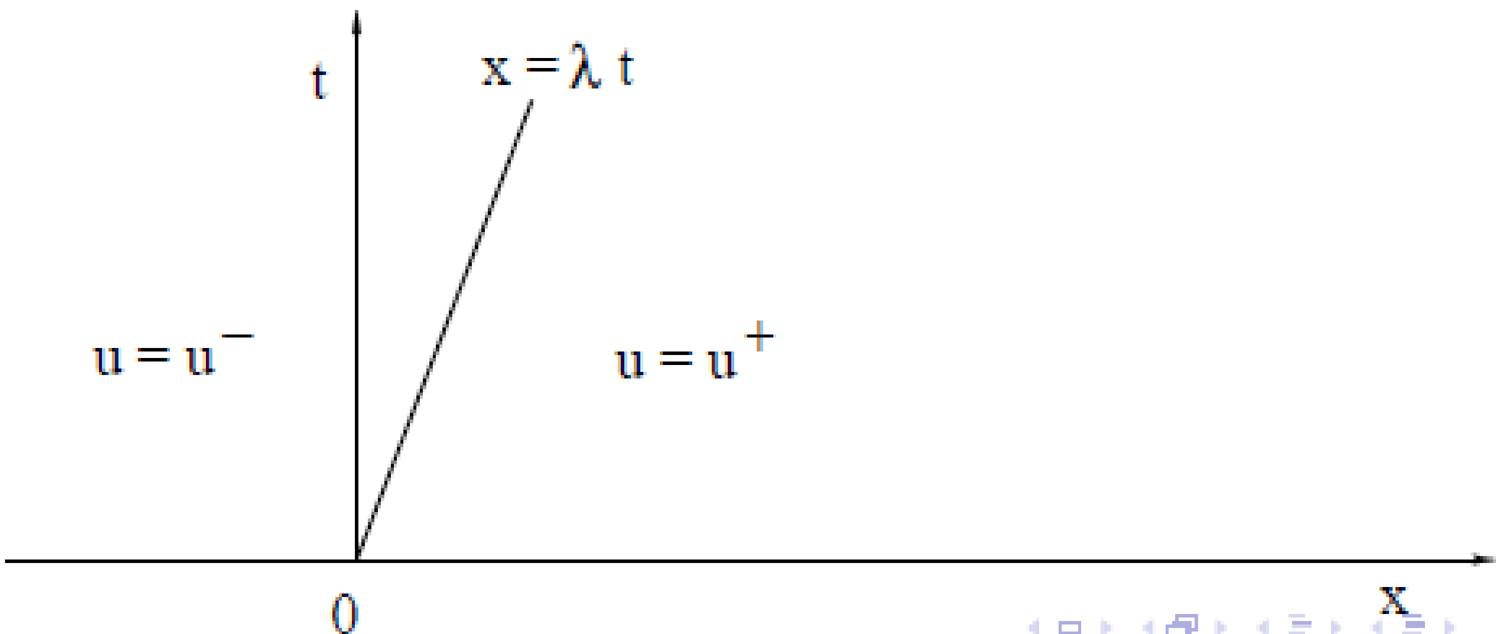
$u^+ = S_i(\sigma)(u^-)$ for some i, σ. Let $\lambda = \lambda_i(u^-, u^+)$ be the shock speed.

Then the function

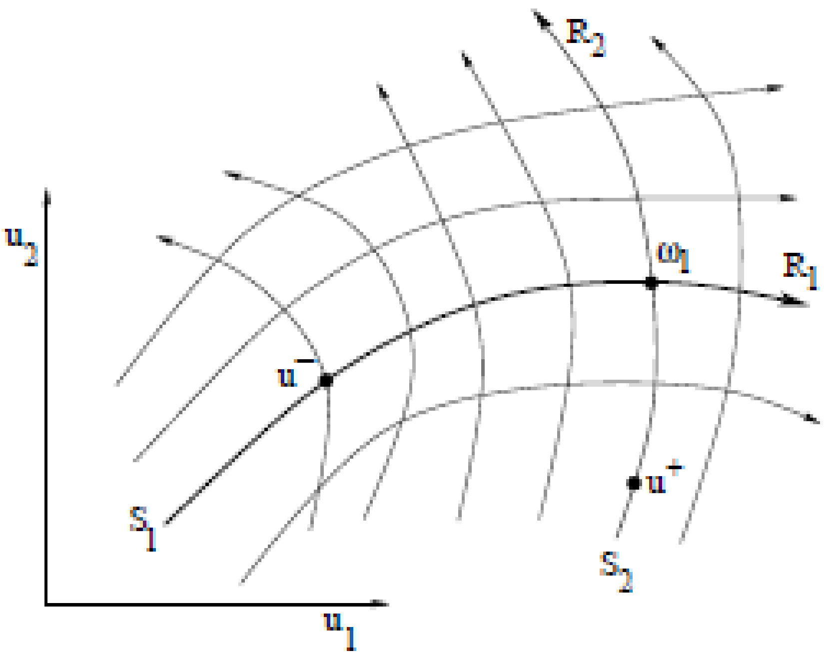
$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

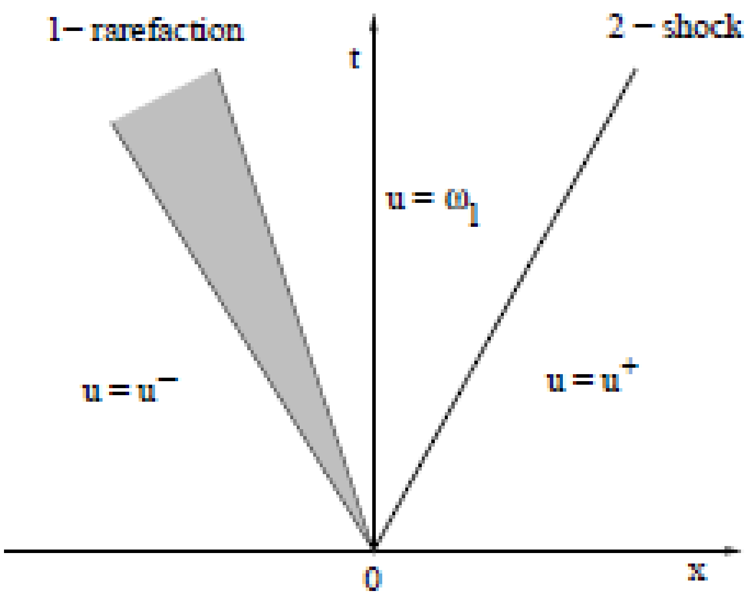
is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (i.e., it satisfies the Lax condition and the Liu condition) iff $\sigma < 0$.



Solution to a 2 x 2 Riemann problem





Solution of the general Riemann problem (P. Lax, 1957)

$$u_t + f(u)_x = 0 \qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

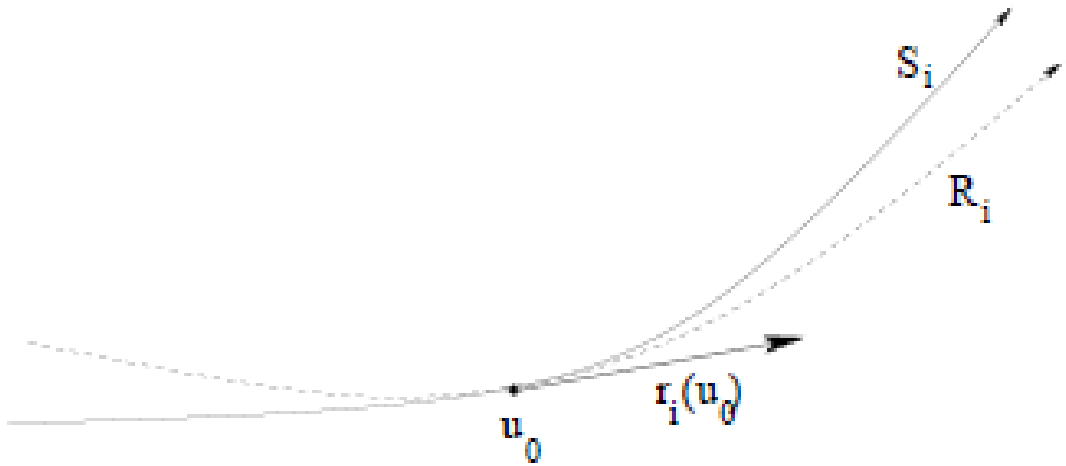
Problem: Find states $\omega_0, \omega_1, \cdots, \omega_m$ such that

$$\omega_0 = \mathbf{u}^- \qquad \omega_m = \mathbf{u}^+$$

and every couple ω_{i-1}, ω_i are connected by an elementary wave (shock or rarefaction)

$$\begin{cases} \text{either } \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \text{or } \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{cases}$$

define: $\Psi_i(\sigma)(u) = \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\ S_i(\sigma)(u) & \text{if } \sigma < 0 \end{cases}$



$$(\sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto \Psi_n(\sigma_n) \circ \cdots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)$$

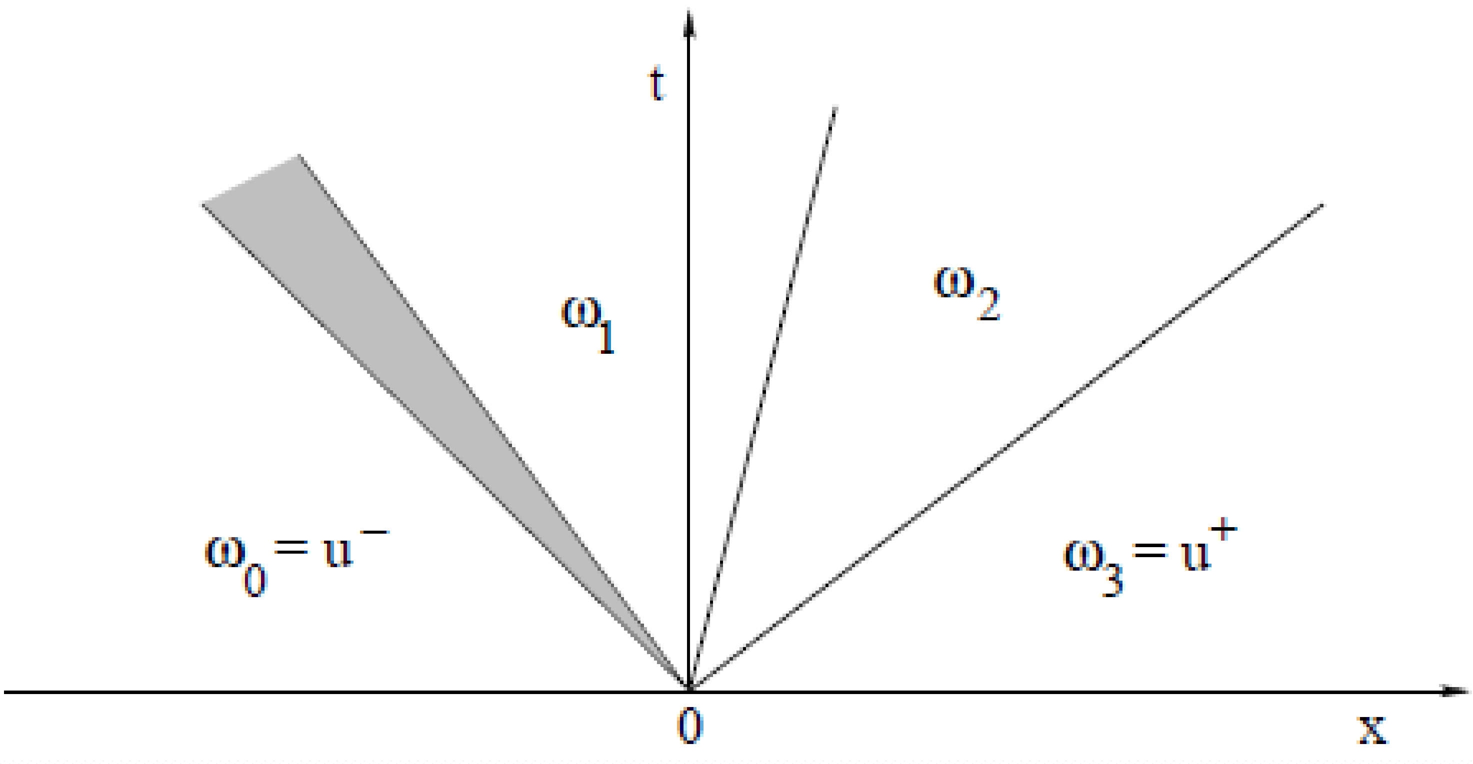
Jacobian matrix at the origin: $J \doteq \left(r_1(u^-) \middle| r_2(u^-) \middle| \cdots \middle| r_n(u^-) \right)$

always has full rank

If $|u^+ - u^-|$ is small, then the implicit function theorem yields existence and uniqueness of the intermediate states $\omega_0, \omega_1, \ldots, \omega_n$

General solution of the Riemann problem

Concatenation of elementary waves (shocks, rarefactions, or contact discontinuities)



Global solution to the Cauchy problem

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \bar{u}(x)$$

Theorem (Glimm, 1965).

Assume:

- *system is strictly hyperbolic*
- *each characteristic field is either linearly degenerate or genuinely nonlinear*

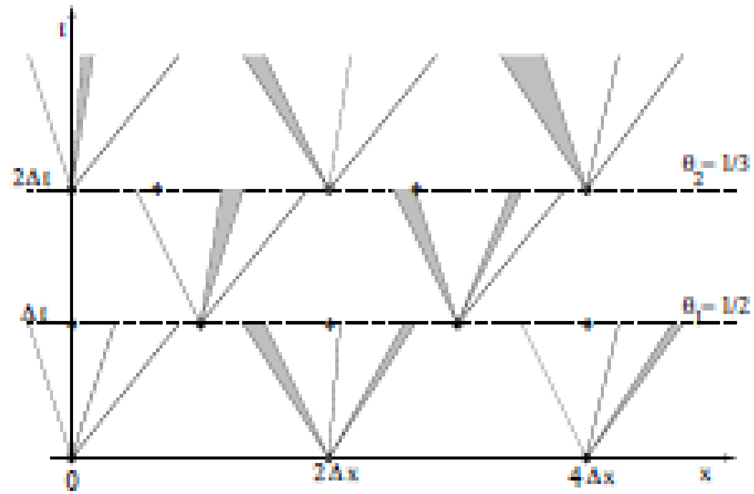
Then there exists a constant $\delta > 0$ *such that, for every initial condition* $\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ *with*

$$Tot.Var.(\bar{u}) \leq \delta,$$

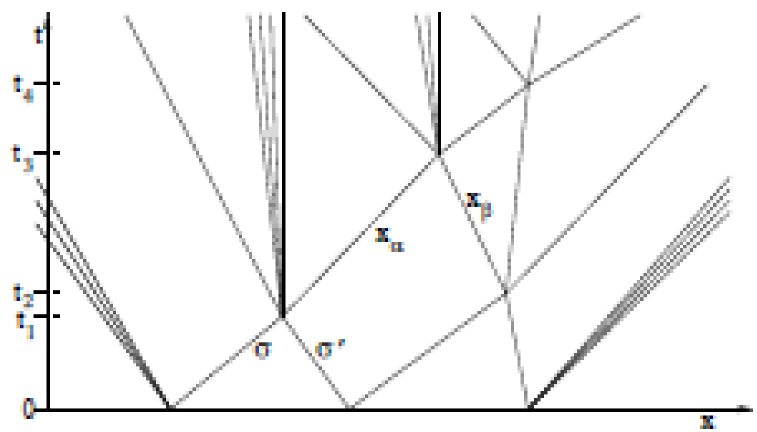
the Cauchy problem has an entropy admissible weak solution $u = u(t,x)$ *defined for all* $t \geq 0$.

Construction of a sequence of approximate solutions by piecing together solutions of Riemann problems

- on a fixed grid in t-x plane (Glimm scheme)

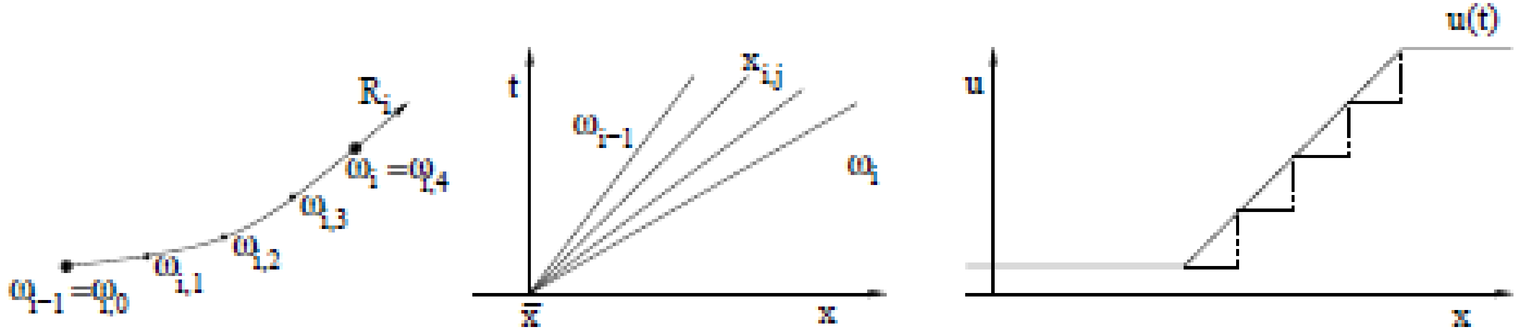


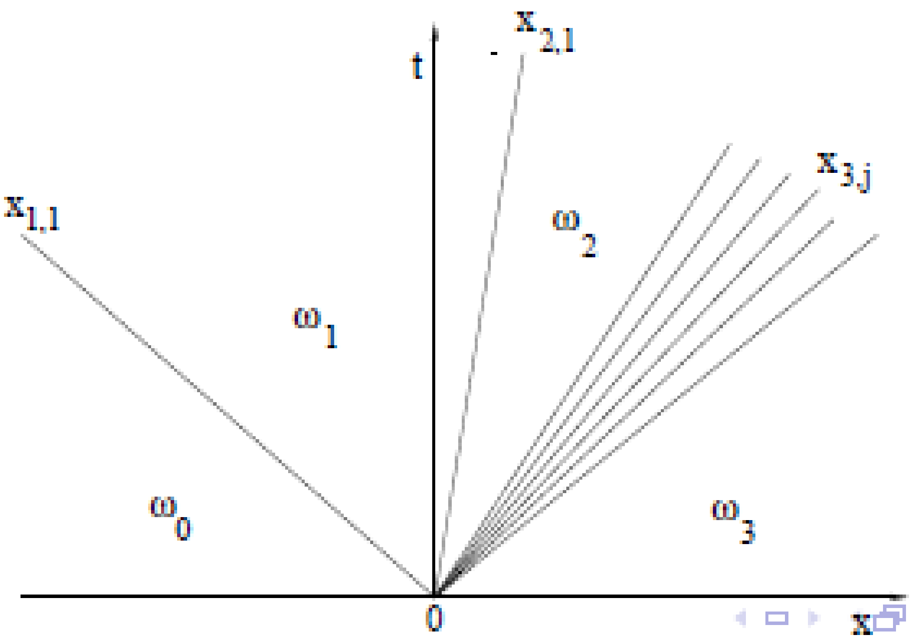
- at points where fronts interact (front tracking)



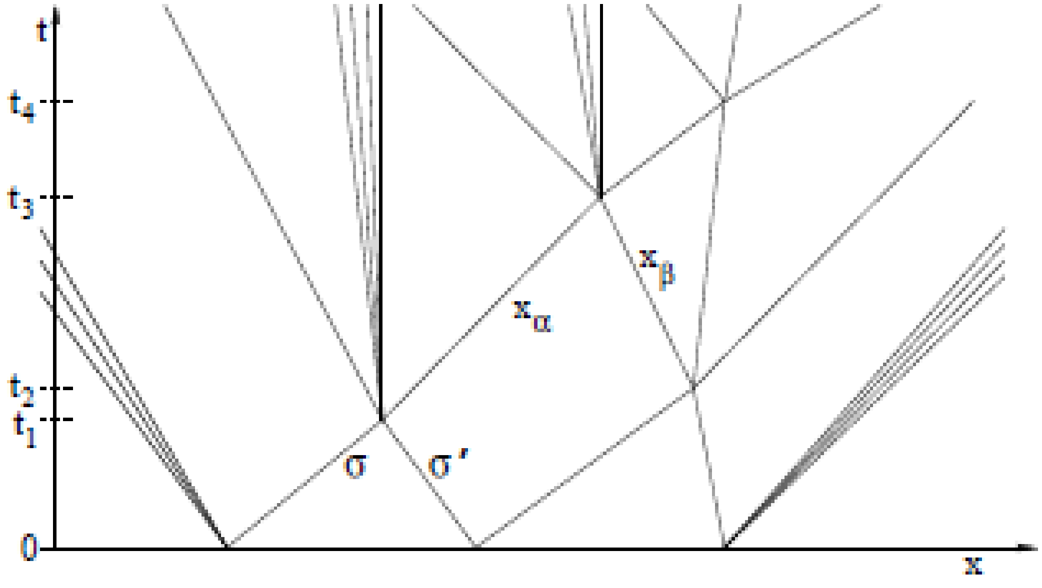
Piecewise constant approximate solution to a Riemann problem

replace centered rarefaction waves with piecewise constant rarefaction fans





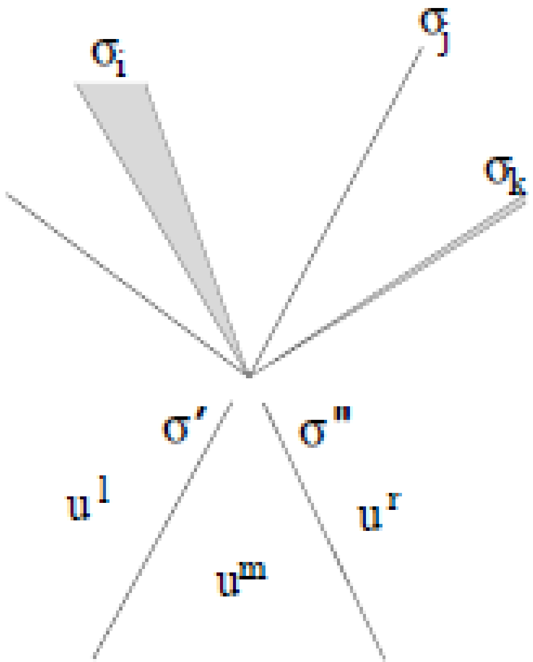
Front Tracking Approximations



- Approximate the initial data $\bar{u}$ with a piecewise constant function
- Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$
- at each time t_j where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem ...
- NEED TO CHECK: $\left\{ \begin{array}{l} \text{- total variation remains small} \\ \text{- number of wave fronts remains finite} \end{array} \right.$

Interaction estimates

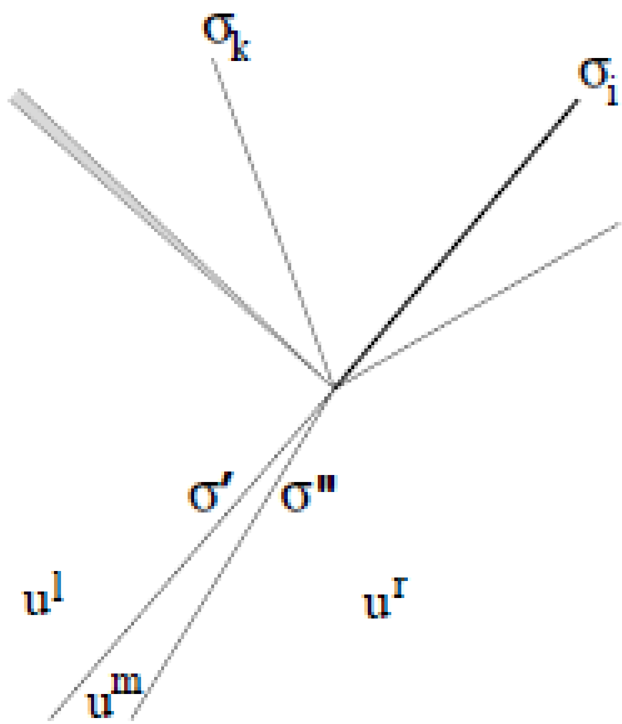
GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves σ', σ''



Incoming: a j-wave of strength σ' and an i-wave of strength σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| + \sum_{k \neq i,j} |\sigma_k| = O(1) \cdot |\sigma' \sigma''|$$



Incoming: two i-waves of strengths σ' and σ''

Outgoing: waves of strengths $\sigma_1, \cdots, \sigma_m$. Then

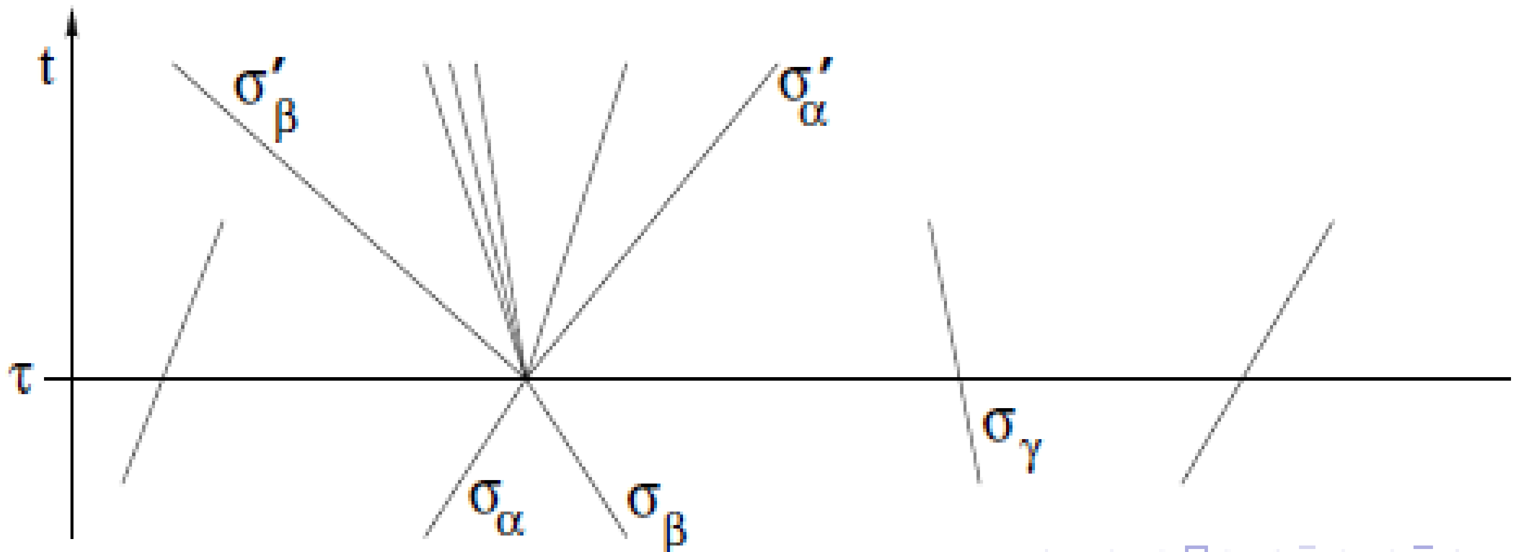
$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| \;=\; \mathcal{O}(1) \cdot |\sigma' \sigma''| \Big(|\sigma'| + |\sigma''| \Big)$$

Glimm functionals

Total strength of waves: $V(t) \doteq \sum_{\alpha} |\sigma_\alpha|$

Wave interaction potential: $Q(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|$

$\mathcal{A}$ $\doteq$ couples of *approaching* wave fronts



Changes in V, Q at time τ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) \;=\; \mathcal{O}(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) \;=\; -|\sigma_\alpha \sigma_\beta| + \mathcal{O}(1) \cdot V(\tau-)|\sigma_\alpha \sigma_\beta|$$

Choosing a constant C_0 large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

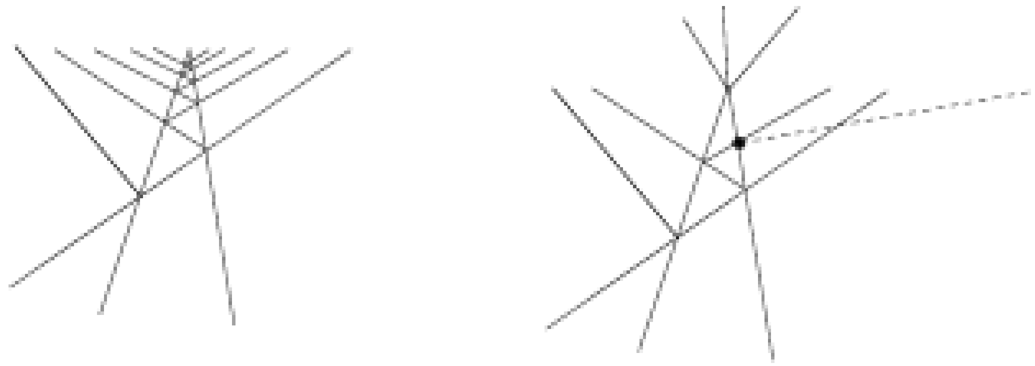
is nonincreasing, as long as V remains small

Total variation initially small $\quad\Longrightarrow\quad$ global BV bounds

$$\text{Tot.Var.}\{u(t,\cdot)\} \;\le\; V(t) \;\le\; V(0) + C_0 Q(0)$$

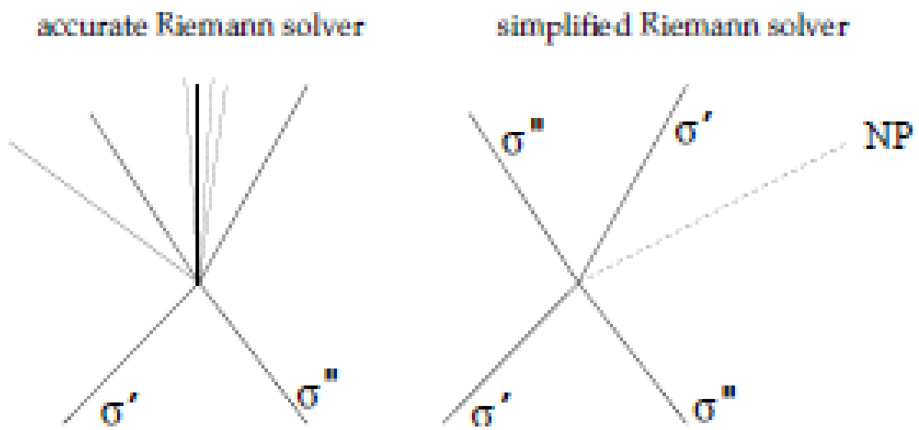
Front tracking approximations can be constructed for all $t \geq 0$

Keeping finite the number of wave fronts



At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts

The total number of fronts can become infinite in finite time



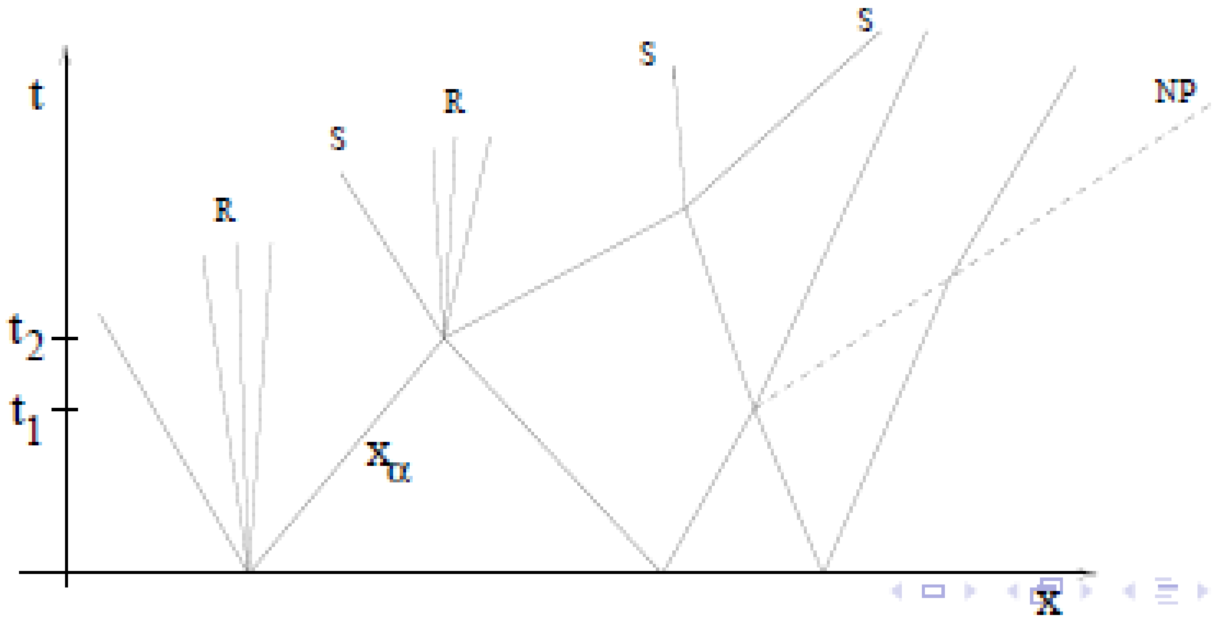
Need: a **Simplified Riemann Solver**, producing only one *"non-physical"* front

A sequence of approximate solutions

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

$(u_\nu)_{\nu\geq 1}$ sequence of approximate front tracking solutions

- initial data satisfy $\|u_\nu(0,\cdot) - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon_\nu \to 0$
- all shock fronts in u_ν are entropy-admissible
- each rarefaction front in u_ν has strength $\leq \varepsilon_\nu$
- at each time $t \geq 0$, the total strength of all non-physical fronts in $u_\nu(t,\cdot)$ is $\leq \varepsilon_\nu$



Existence of a convergent subsequence

$$\text{Tot.Var.}\{u_\nu(t,\cdot)\} \leq C$$

$$\begin{aligned}\big\|u_\nu(t) - u_\nu(s)\big\|_{\mathbf{L}^1} &\leq (t-s)\cdot[\text{total strength of all wave fronts}]\cdot[\text{maximum speed}]\\ &\leq L\cdot(t-s)\end{aligned}$$

Helly's compactness theorem $\Longrightarrow$ a subsequence converges

$$u_\nu \to u \qquad\qquad \text{in } \mathbf{L}^1_{loc}$$

Claim: $u = \lim_{\nu \to \infty} u_\nu$ is a weak solution

$$\int\int \left\{ \phi_t u + \phi_x f(u) \right\} dxdt = 0 \qquad \phi \in \mathcal{C}_c^1\Big( ]0, \infty[\times \mathbb{R} \Big)$$

Need to show:

$$\lim_{\nu \to \infty} \int\int \left\{ \phi_t u_\nu + \phi_x f(u_\nu) \right\} dxdt = 0$$

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty & \left\{ \phi_t(t,x) u_\nu(t,x) + \phi_x(t,x) f\left(u_\nu(t,x)\right) \right\} dxdt \\
&= \sum_j \int_{\partial\Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma
\end{aligned}
$$

$$
\begin{aligned}
& \limsup_{\nu\to\infty} \left| \sum_j \int_{\partial\Gamma_j} \Phi_\nu \cdot \mathbf{n} \, d\sigma \right| \\
& \leq \limsup_{\nu\to\infty} \left| \sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}} \left[\dot{x}_\alpha(t) \cdot \Delta u_\nu(t,x_\alpha) - \Delta f\left(u_\nu(t,x_\alpha)\right) \right] \phi(t,x_\alpha(t)) \right| \\
& \leq \left(\max_{t,x} \left|\phi(t,x)\right| \right) \cdot \limsup_{\nu\to\infty} \left\{ \mathcal{O}(1) \cdot \sum_{\alpha\in\mathcal{R}} \varepsilon_\nu |\sigma_\alpha| \; + \; \mathcal{O}(1) \cdot \sum_{\alpha\in\mathcal{NP}} |\sigma_\alpha| \right\} \\
& = 0
\end{aligned}
$$

The Glimm scheme

$$u_t + f(u)_x = 0 \qquad u(0,x) = \bar{u}(x)$$

Assume: all characteristic speeds satisfy $\lambda_i(u) \in [0,1]$

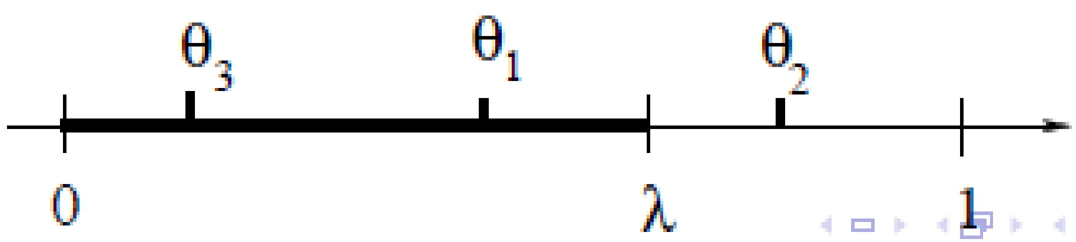
This is not restrictive. If $\lambda_i(u) \in [-M, M]$, simply change coordinates:

$$y = x + Mt, \qquad \tau = 2Mt$$

Choose:

- a grid in the t-x plane with step size $\Delta t = \Delta x$
- a sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots$ *uniformly distributed* over $[0,1]$

$$\lim_{N\to\infty} \frac{\#\{j\,;\ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad \text{for each } \lambda \in [0,1].$$

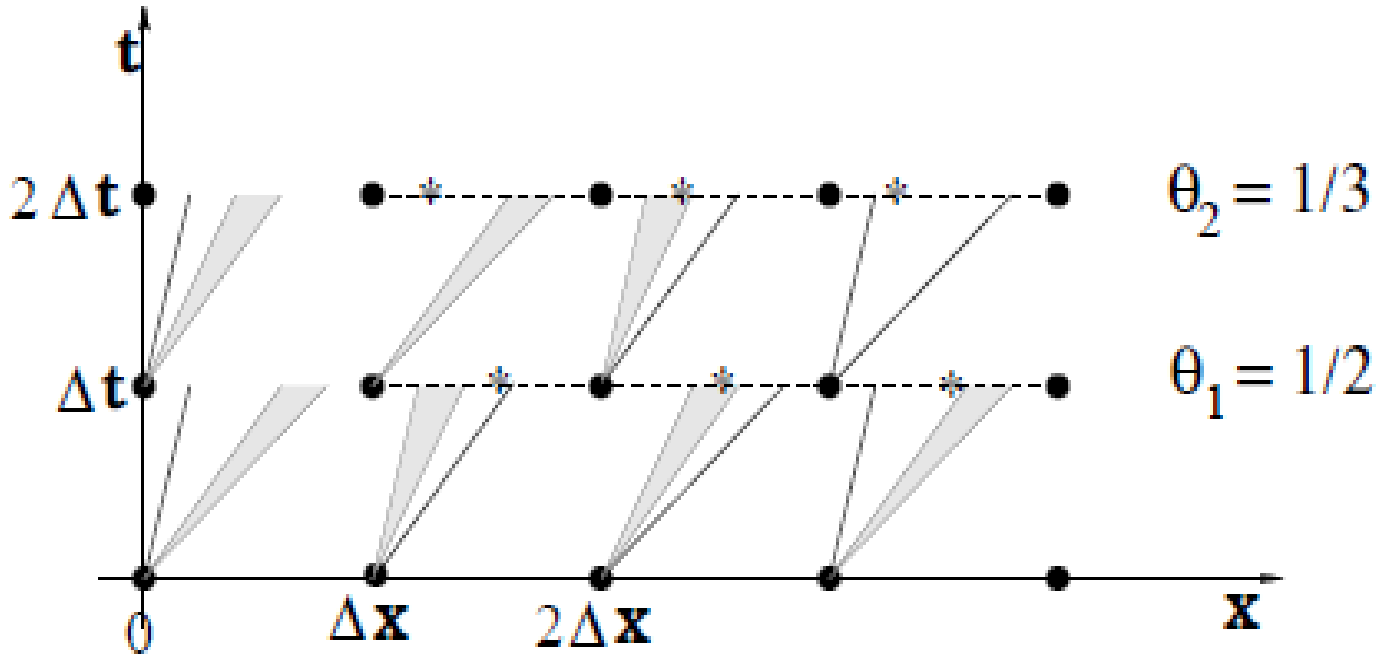


Glimm approximations

Grid points : $\quad x_j = j \cdot \Delta x, \qquad t_k = k \cdot \Delta t$

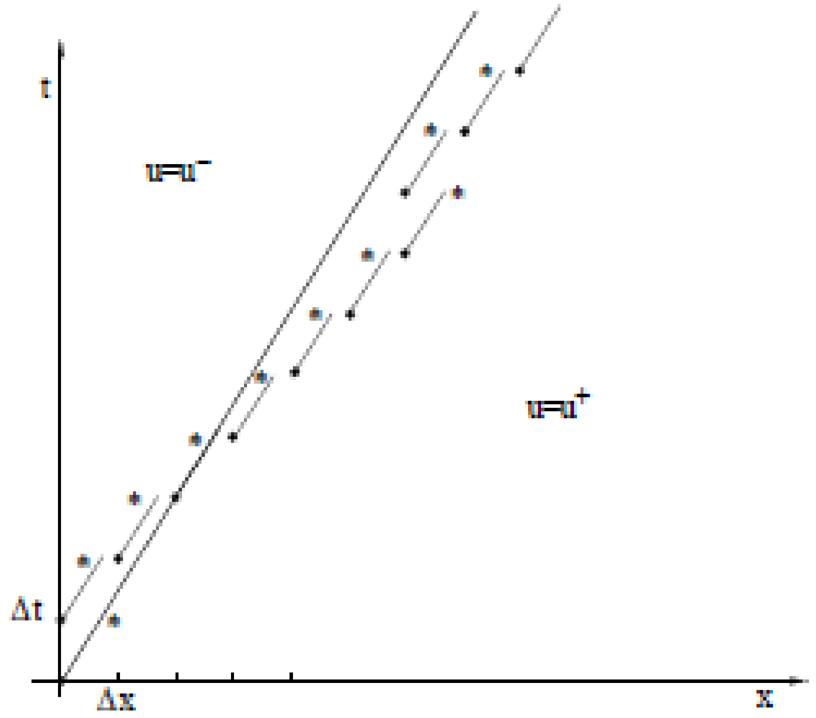
• for each $k \geq 0$, $u(t_k, \cdot)$ is piecewise constant, with jumps at the points x_j. The Riemann problems are solved exactly, for $t_k \leq t < t_{k+1}$

• at time t_{k+1} the solution is again approximated by a piecewise constant function, by a sampling technique



Example: Glimm's scheme applied to a solution containing a single shock

$$U(t,x) \;=\; \begin{cases} u^+ & \text{if} \quad x > \lambda t \\ u^- & \text{if} \quad x < \lambda t \end{cases}$$



Fix $T > 0$, take $\Delta x = \Delta t = T/N$

$$x(T) \;=\; \#\{j\,;\;\; 1 \le j \le N, \;\; \theta_j \in [0,\lambda]\;\} \cdot \Delta t$$

$$=\; \frac{\#\{j\,;\;\; 1 \le j \le N, \;\; \theta_j \in [0,\lambda]\;\}}{N} \cdot T \;\to\; \lambda T \qquad\qquad \text{as } \; N \to \infty$$

Rate of convergence for Glimm approximations

Random sampling at points determined by the equidistributed sequence $(\theta_k)_{k\geq 1}$

$$\lim_{N\to\infty} \frac{\#\{j\ ;\ \ 1\leq j\leq N,\ \ \theta_j\in[0,\lambda]\ \}}{N} = \lambda \qquad \text{for each } \lambda\in[0,1].$$

Need fast convergence to uniform distribution. Achieved by choosing:

$$\theta_1 = 0.1\,, \quad \ldots \quad, \quad \theta_{759} = 0.957\,, \quad \ldots \quad, \quad \theta_{39022} = 0.22093\,, \quad \ldots$$

Convergence rate: $$\lim_{\Delta x\to 0} \frac{\left\| u^{\text{Glimm}}(T,\cdot) - u^{\text{exact}}(T,\cdot)\right\|_{\mathbf{L}^1}}{\sqrt{\Delta x}\cdot|\ln\Delta x|} = 0$$

(A.Bressan & A.Marson, 1998)

Bressan, A.: Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem. Oxford University Press: Oxford, 2000.

Dafermos, C: Hyperbolic Conservation Laws in Continuum Physics, 3rd Edition, Springer-Verlag: Berlin, 2010.

Functional Analytic Approaches for the Existence Theory:

- Compensated Compactness
- Weak Convergence Methods
- Geometric Measure Arguments
- ……

1. **C. M. Dafermos:** **Hyperbolic Conservation Laws in Continuum Physics,** Third edition. Springer-Verlag: Berlin, 2010.
2. **B. Dacorogna**: **Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals**, Lecture Notes in Mathematics, Vol. 922, 1-120, Springer-Verlag, 1982.
3. **L. C. Evans**:. **Weak Convergence Methods for Nonlinear Partial Differential Equations**. CBMS-RCSM, 74. AMS: Providence, RI, 1990
4. **D. Serre,** La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d'espace. *J. Math. Pures Appl. (9)* 65 (1986), 423–468.
5. **The references cited therein, especially more recent references.**

Young Measures

$K \subset \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ bounded open

$u^k : \Omega \longrightarrow \mathbb{R}^m$ measurable

$u^k(y) \in K$, a.e.

$\Longrightarrow$ $\exists \{\nu_y \in \mathrm{Prob.}(\mathbb{R}^m)\}_{y \in \Omega}$ s.t.

- $\boxed{\mathrm{Supp}\, \nu_y \subset \overline{K}}$ $\forall y \in \Omega$

- $\forall f \in C(\mathbb{R}^m; \mathbb{R})$, $\exists \{u^{k_j}\}_{j=1}^{\infty} \subset \{u^k\}$.

$$\boxed{\begin{aligned} w^*\text{-}\lim f(u^{k_j}) &= \langle \nu_y(\lambda), f(\lambda) \rangle \\ &= \int f(\lambda)\, d\nu_y(\lambda) \end{aligned}}$$

- $u^{k_j} \longrightarrow u$ a.e. $\Longleftrightarrow$ $\nu_y(\lambda) = \delta_{u(y)}(\lambda)$

Dirac mass

* This theorem can be extended to more general cases.

Remarks

1. The deviation between the weak and strong convergence is measured by the spreading of the support of ν_y.

$$\boxed{\|f(w^*\text{-}\lim u^k) - w^*\text{-}\lim f(u^k)\|_{L^\infty} \leq C \sup_y (\operatorname{diam}(\operatorname{supp} \nu_y))}$$

for $f \in Lip(\mathbb{R}^m; \mathbb{R})$

$$\|f(w^*\text{-}\lim u^k) - w^*\text{-}\lim f(u^k)\|_{L^\infty}$$

$$= \|f(\langle \nu_y, \lambda \rangle) - \langle \nu_y, f(\lambda) \rangle\|_{L^\infty}$$

$$= \|\langle \nu_y, f(\lambda) - f(\langle \nu_y, \lambda \rangle) \rangle\|_{L^\infty}$$

$$\leq C \|\langle \nu_y, |\lambda - \langle \nu_y, \lambda \rangle| \rangle\|_{L^\infty}$$

$$\leq C \sup_y (\operatorname{diam}(\operatorname{supp} \nu_y)).$$

Remarks

2. The Young measure family $\{\nu_y\}_{y\in\Omega}$ can be thought of as the limiting probability distribution of the values of $\{u^k(y)\}$ near the point y as $k\to\infty$.

$\Omega \subset \mathbb{R}^n$, $y\in\Omega$.

$\exists\ \delta_0>0$ s.t. $B(y,\delta)\subset\Omega$, $0<\delta\le\delta_0$

Define

$$\langle \nu^k_{y,\delta}, \phi\rangle = \frac{1}{|B(y,\delta)|}\phi(u^k(x))dx$$

$$\forall \phi\in C_c(\mathbb{R}^m;\mathbb{R})$$

$$\Updownarrow$$

$$\nu^k_{y,\delta}(\lambda) \triangleq \frac{1}{|B(y,\delta)|}\delta_{u^k(x)}dx$$

$$\boxed{\nu_y(\lambda) = \lim_{\delta\to 0}\lim_{k\to\infty}\nu^k_{y,\delta}}$$

Weak Continuity of 2x2 Determinants

$\Omega \subset \mathbb{R}_+ \times \mathbb{R}$ bounded open

$U^k : \Omega \to \mathbb{R}^4$ measurable

$$\left\{\begin{array}{l} \text{w-}\lim_{k\to\infty} U^k = U \quad \text{in } L^2(\Omega;\mathbb{R}^4) \\ \left\{\begin{array}{l} \dfrac{\partial u_1^k}{\partial t} + \dfrac{\partial u_2^k}{\partial x} \\ \dfrac{\partial u_3^k}{\partial t} + \dfrac{\partial u_4^k}{\partial x} \end{array}\right. \quad \text{compact in } H^{-1}_{loc}(\Omega) \end{array}\right.$$

$$\Longrightarrow \quad \begin{vmatrix} u_1^k & u_2^k \\ u_3^k & u_4^k \end{vmatrix} \longrightarrow \begin{vmatrix} u_1 & u_2 \\ u_3 & u_4 \end{vmatrix} \quad \mathcal{D}'$$

Subsequentially

Another Form

$$v^k = (v_1^k, v_2^k) \in L^2(\Omega;\mathbb{R}^2)$$
$$w^k = (w_1^k, w_2^k) \in L^2(\Omega;\mathbb{R}^2)$$

$$\left\{\begin{array}{l} \text{w-}\lim_{k\to\infty} (v^k, w^k) = (v, w), \quad L^2(\Omega) \\ \left\{\begin{array}{l} \operatorname{div} v^k \\ \operatorname{curl} w^k \end{array}\right. \quad \text{compact in } H^{-1}_{loc}(\Omega) \end{array}\right.$$

$$\Longrightarrow \quad v^k \cdot w^k \longrightarrow v \cdot w \quad \mathcal{D}'.$$

Div-Curl Lemma

$\Omega \subset \mathbb{R}^n$ open, bounded

$p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$

$v^k \in L^p(\Omega; \mathbb{R}^n)$

$w^k \in L^q(\Omega; \mathbb{R}^n)$

$$\begin{cases} v^k \longrightarrow v & \text{weakly in } L^p(\Omega; \mathbb{R}^n) \\ w^k \longrightarrow w & \text{weakly in } L^q(\Omega; \mathbb{R}^n). \end{cases}$$

$$\begin{cases} \operatorname{div} v^k & \text{compact in } W^{-1,p}_{loc}(\Omega; \mathbb{R}) \\ \operatorname{curl} w^k & \text{compact in } W^{-1,q}_{loc}(\Omega; \mathbb{R}). \end{cases}$$

$\Rightarrow$

$$v^k \cdot w^k \longrightarrow v \cdot w \quad \mathcal{D}'$$

Compensated Compact Embedding Lemma

$\Omega \subset \mathbb{R}^n$ bounded open

$\hookrightarrow$

(Compact set of $W^{-1,q}_{loc}(\Omega)$)

$\cap$ (Bounded set of $W^{-1,r}_{loc}(\Omega)$)

$\subset$ (Compact set of $W^{-1,p}_{loc}(\Omega)$)

for any $1 < q \leq p < r < \infty$

2x2 Hyperbolic Systems of Conservation Laws

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \end{cases}$$

Assume

- $\exists$ a strictly convex entropy $\eta_*(u)$,

  $\nabla^2 \eta_*(u) > 0$

- $\exists$ globally defined Riemann Invariants

  $W = (W_1, W_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t.

  $\nabla W_j(u) \,/\!/\, \ell_j(u)$

If $u \in C^1$:

$$\boxed{\partial_t W_j + \lambda_j(u(W))\, \partial_x W_j = 0}$$

Entropy Equation

Entropy $\eta(u)$, **Entropy Flux** $q(u)$

$$\nabla q(u) = \nabla \eta(u) \nabla f(u)$$

$$(\lambda_j \nabla \eta - \nabla q) \cdot r_j = 0$$

$$\Rightarrow \boxed{\begin{aligned} & q_{w_j} = \lambda_j \eta_{w_j} \\ & \eta_{w_1 w_2} - \frac{\lambda_1 w_2}{\lambda_2 - \lambda_1} \eta_{w_1} + \frac{\lambda_2 w_1}{\lambda_2 - \lambda_1} \eta_{w_2} = 0 \end{aligned}}$$

Genuine Nonlinearity

$$\nabla \lambda_j(u) \cdot r_j(u) \neq 0, \qquad j = 1, 2.$$

$$\Longleftrightarrow \quad \frac{\partial \lambda_j}{\partial w_j} \neq 0, \qquad j = 1, 2,$$

Method of Vanishing Viscosity

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^2 \\ u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbb{R}) \end{cases}$$

Viscosity Approximation

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} \\ u|_{t=0} = u_0^\varepsilon(x) \longrightarrow u_0(x) \quad \text{a.e.} \end{cases}$$

$\hookrightarrow \quad u^\varepsilon = u^\varepsilon(t,x)$

- Invariant Regions or L^p Estimates

$$\|u^\varepsilon\|_{L^\infty} \le C \quad \text{or} \quad \|u^\varepsilon\|_{L^p} \le C$$

- Dissipation Estimate

$$\|\sqrt{\varepsilon}\, u_x^\varepsilon\|_{L^2} \le C \quad \not\sim \varepsilon .$$

Dissipation Estimate

$$\nabla\bar{\eta}_*(u^\varepsilon)\times\left[\ u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}\right]$$

$\hookrightarrow$

$$\varepsilon(u^\varepsilon_x)^T\nabla^2\bar{\eta}_*(u^\varepsilon)u^\varepsilon_x \geq c_0\varepsilon|u^\varepsilon_x|^2$$

$$= -\bar{\eta}_*(u^\varepsilon)_t - \bar{q}_*(u^\varepsilon)_x + \varepsilon\bar{\eta}_*(u^\varepsilon)_{xx}$$

$\hookrightarrow$

$$c_0\int_0^T\int \varepsilon|u^\varepsilon_x|^2\,dx\,dt$$

$$\leq \int\bar{\eta}_*(u^\varepsilon_0)\,dx - \int\bar{\eta}_*(u^\varepsilon(T,x))\,dx.$$

$$\leq \int\bar{\eta}_*(u^\varepsilon_0)\,dx \leq C \quad \text{indep. of } \varepsilon.$$

- $\bar{\eta}_*(u) = \eta_*(u) - \eta_*(0) - \nabla\eta(0)\,u \geq c_0 > 0$

  $\bar{q}_*(u) = q_*(u) - q_*(0) - \nabla\eta(0)(f(u) - f(0)).$

H^{-1}- Compactness

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$$
$$= \varepsilon(\nabla\eta(u^\varepsilon)\,u_x^\varepsilon)_x - \varepsilon(u_x^\varepsilon)^T\nabla^2\eta(u^\varepsilon)\,u_x^\varepsilon$$
$$= I_1^\varepsilon + I_2^\varepsilon$$

- $\|I_1^\varepsilon\|_{H^{-1}(\Omega)} \le \sqrt{\varepsilon}\,\|\sqrt{\varepsilon}\,u_x^\varepsilon\|_{L^2}\,\|\nabla\eta(u^\varepsilon)\|_{L^\infty} \le \sqrt{\varepsilon}\,C \to 0$
- $\|I_2^\varepsilon\|_{L^1(\Omega)} \le \|\nabla^2\eta(u^\varepsilon)\|_{L^\infty}\,\|\sqrt{\varepsilon}\,u_x^\varepsilon\|_{L^2}^2 \le C$

$\hookrightarrow$ I_2^ε compact in $W^{-1,q}(\Omega)$, $1<q<2$

$\hookrightarrow$ $I_1^\varepsilon + I_2^\varepsilon$ compact in $W^{-1,q}(\Omega)$, $1<q<2$

But $\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$ bounded in $W^{-1,\infty}(\Omega)$

$\hookrightarrow$ **Lemma**

$$\boxed{\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is compact in } H^{-1}_{loc}}$$

$$\forall\,(\eta, q) \in C^2$$

Commutation Identity for Young Measure. $\{\nu_{t,x}\}_{(t,x)\in\mathbb{R}_+^2}$

$$\updownarrow$$

$\{u^\varepsilon\}_{\varepsilon>0}$.

- $\operatorname{Supp}\nu_{t,x} \subset\subset \mathbb{R}^2$
- For any entropy pairs (η, q),

$$(*)\quad \left\langle \nu_{t,x}, \begin{vmatrix} \eta_1 & q_1 \\ \eta_2 & q_2 \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}, \eta_1 \rangle & \langle \nu_{t,x}, q_1 \rangle \\ \langle \nu_{t,x}, \eta_2 \rangle & \langle \nu_{t,x}, q_2 \rangle \end{vmatrix} \quad \text{a.e. } (t,x)$$

$$\overset{?}{\Longrightarrow} \quad \nu_{t,x} = \delta_{u(t,x)}(\lambda) \quad ??$$

* If $f(u) = Au$ (linear)

$\hookrightarrow$ $(*)$ is trivial.

The imbalance of $(*)$ is enforced by the nonlinearity of $f(u)$.

Proof of (*)

$\forall\ (\eta_i, q_i) \in C$, $i = 1, 2$.

$U^\varepsilon = (\eta_1(u^\varepsilon), q_1(u^\varepsilon), \eta_2(u^\varepsilon), q_2(u^\varepsilon))$ uniformly bdd

$\hookrightarrow\ \exists\ \{\varepsilon_k\}_{k=1}^\infty$, s.t $\varepsilon_k \to 0$ as $k \to \infty$.

- $U^{\varepsilon_k} \xrightarrow{*} (\langle \nu_{t,x}, \eta_1(\lambda)\rangle, \langle \nu_{t,x}, q_1(\lambda)\rangle, \langle \nu_{t,x}, \eta_2(\lambda)\rangle, \langle \nu_{t,x}, q_2(\lambda)\rangle)$
  $= U(t,x)$

- $\begin{vmatrix} U_1^{\varepsilon_k} & U_2^{\varepsilon_k} \\ U_3^{\varepsilon_k} & U_4^{\varepsilon_k} \end{vmatrix} \xrightarrow{*} \boxed{\left\langle \nu_{t,x}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda) & q_2(\lambda) \end{vmatrix} \right\rangle}$

Div-Curl

$$\begin{vmatrix} U_1 & U_2 \\ U_3 & U_4 \end{vmatrix} = \boxed{\begin{vmatrix} \langle \nu_{t,x}, \eta_1(\lambda)\rangle & \langle \nu_{t,x}, q_1(\lambda)\rangle \\ \langle \nu_{t,x}, \eta_2(\lambda)\rangle & \langle \nu_{t,x}, q_2(\lambda)\rangle \end{vmatrix}}.$$

Reduction of the Young Measure:

Scalar Conservation Laws: $u^\varepsilon \xrightarrow{*} u$, a.e.

$\hookrightarrow u(t,x) = \langle \nu_{t,x}, \lambda \rangle$

$$\boxed{\left\langle \nu_{t,x}, \begin{vmatrix} \eta_1(\lambda) & q_1(\lambda) \\ \eta_2(\lambda), & q_2(\lambda) \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu_{t,x}, \eta_1(\lambda)\rangle & \langle \nu_{t,x}, q_1(\lambda)\rangle \\ \langle \nu_{t,x}, \eta_2(\lambda)\rangle & \langle \nu_{t,x}\, q_2(\lambda)\rangle \end{vmatrix}}$$

Choose: $(\eta_1(\lambda), q_1(\lambda)) = (\lambda - u(t,x), f(\lambda) - f(u(t,x)))$

$(\eta_2(\lambda), q_2(\lambda)) = \left(f(\lambda) - f(u(t,x)), \int_{u(t,x)}^{\lambda} (f'(s))^2 ds\right)$

$$\left\langle \nu_{t,x} \begin{vmatrix} \lambda - u & f(\lambda) - f(u) \\ f(\lambda) - f(u) & \int_u^\lambda (f'(s))^2 ds \end{vmatrix} \right\rangle$$

$$= \begin{vmatrix} \underbrace{\boxed{\langle \nu_{t,x}\ \lambda - u\rangle}}_{=0} & \langle \nu_{t,x}\ f(\lambda) - f(u)\rangle \\ \langle \nu_{t,x}, f(\lambda) - f(u)\rangle & \langle \nu_{t,x}, \int_u^\lambda (f'(s))^2 ds\rangle \end{vmatrix}$$

$$\boxed{\begin{aligned} &\left\langle \nu_{t,x}, (\lambda - u)\int_u^\lambda (f'(s))^2 ds - (f(\lambda) - f(u))^2 \right\rangle \\ &+ \langle \nu_{t,x}, f(\lambda) - f(u)\rangle^2 = 0 \end{aligned}}$$

$$(\lambda-u)\int_u^\lambda (f'(s))^2 ds - (f(\lambda)-f(u))^2$$
$$= (\lambda-u)\int_u^\lambda \left(f'(s) - \frac{1}{\lambda-u}\int_u^\lambda f'(\tau)d\tau\right)^2 ds \geq 0.$$

$\Longrightarrow$

$$\begin{cases} \langle \nu_{t,x},\ f(\lambda)-f(u) \rangle = 0 \\ \langle \nu_{t,x},\ \underbrace{(\lambda-u)\int_u^\lambda \left(f'(s)-\frac{1}{\lambda-u}\int_u^\lambda f'(\tau)d\tau\right)^2 ds}_{\parallel} \rangle = 0 \end{cases}$$

$$(\lambda-u)\int_u^\lambda \left(\int_u^\lambda f''(\lambda+\theta(s-\tau))(s-\tau)d\tau\right)^2 ds$$

$\Longrightarrow$

- $\langle \nu_{t,x}, f(\lambda) \rangle = f(u(t,x))$
- If $f''(u) > 0$

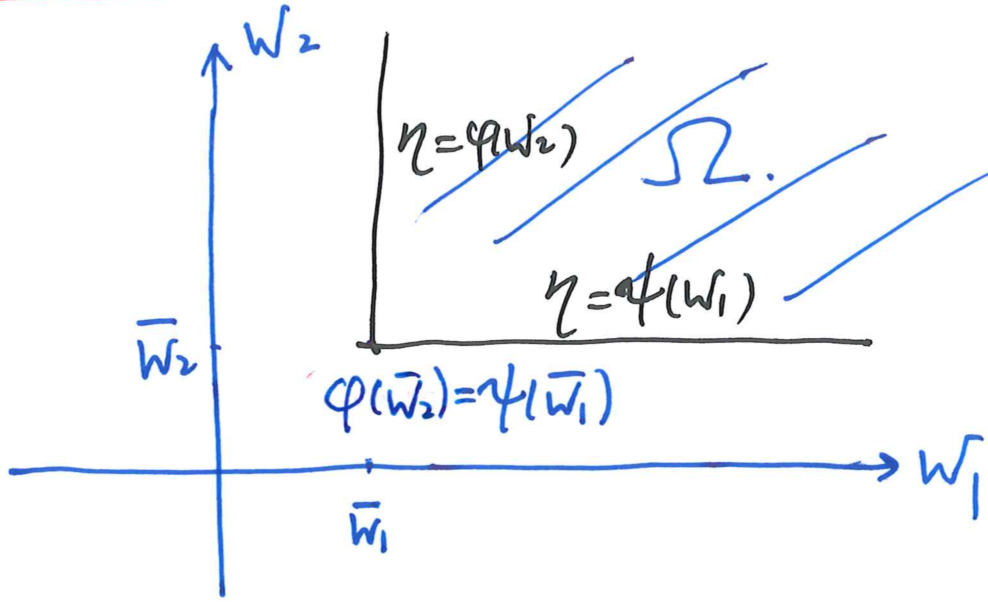
$\hookrightarrow$ $\boxed{\nu_{t,x} = \delta_{u(t,x)}}$

The Goursat Entropy Pairs for 2x2 Hyperbolic Systems of Conservation Laws

$$\eta_{w_1 w_2} - \frac{\lambda_{1 w_2}}{\lambda_2 - \lambda_1}\eta_{w_1} + \frac{\lambda_{2 w_1}}{\lambda_2 - \lambda_1}\eta_{w_2} = 0 \qquad (*)$$

$$q_{w_j} = \lambda_j \eta_{w_j} \qquad (**)$$

Goursat Problem for $(*)$



Well-posed!

Goursat Entropy Pairs

$\exists$ two families of entropy pairs:

$$\begin{cases} \eta_a(w) = I_1(w)\,a(w_1) + \int_{\bar{w}_1}^{w_1} J_1(\xi; w)\,a(\xi)\,d\xi \\ q_a(w) = K_1(w)\,a(w_1) + \int_{\bar{w}_1}^{w_1} L_1(\xi; w)\,a(\xi)\,d\xi \end{cases}$$

$$\begin{cases} \eta_b(w) = I_2(w)\,b(w_2) + \int_{\bar{w}_2}^{w_2} J_2(\xi; w)\,b(\xi)\,d\xi \\ q_b(w) = K_2(w)\,b(w_2) + \int_{\bar{w}_2}^{w_2} L_2(\xi; w)\,b(\xi)\,d\xi \end{cases}$$

where (I_i, J_i, K_i, L_i), $i = 1, 2$, are unique smooth functions and independent of $\bar{w}_1$ and $\bar{w}_2$:

$$\begin{cases} I_i(w) > 0, \qquad \begin{cases} I_1(w_1, \bar{w}_2) = 1 \\ J_1(\xi; w_1, \bar{w}_2) = 0 \end{cases} \qquad \begin{cases} I_2(\bar{w}_1, w_2) = 1 \\ J_2(\xi; \bar{w}_1, w_2) = 0 \end{cases} \\ K_i = \lambda_i I_i \\ \dfrac{\partial K_i(w)}{\partial w_1} + L_i(w_i; w) = \lambda_i(w)\left(\dfrac{\partial I_i(w)}{\partial w_i} + J(w_i; w)\right) \\ \dfrac{\partial L_i(\xi; w)}{\partial w_i} = \lambda_i(w)\,\dfrac{\partial J_i(\xi; w)}{\partial w_i} \\ \dfrac{\partial K_i(w)}{\partial w_j} = \lambda_j(w)\,\dfrac{\partial I_i(w)}{\partial w_j}, \qquad i \neq j \\ \dfrac{\partial L_i(\xi; w)}{\partial w_j} = \lambda_j(w)\,\dfrac{\partial J_i(\xi; w)}{\partial w_j} \end{cases}$$

Reduction of the Young Measure

<u>Thm</u>. If $\frac{\partial \lambda_j}{\partial w_j} \neq 0$, $j=1,2$ (Genuinely Nonlinear)

$$\Longrightarrow \quad \nu_{t,x} = \delta_{u(t,x)}$$

<u>Proof</u>. If $\nu_{t,x} \neq \delta_{u(t,x)}$, we denote $[w_1^-, w_1^+] \times [w_2^-, w_2^+]$ the smallest rectangle containing $\operatorname{supp} \nu_{t,x}$.

1. <u>Claim</u>: If $w_1^- < w_1^+$, then $\exists\ c_1(t,x)$ s.t.

$$\langle \nu, q_a \rangle = c_1 \langle \nu, \eta_a \rangle$$

$\forall\ a \in C$, $a(w_1) = 0$ when $\begin{cases} w_1 \geq \bar{w}_1 \\ \text{or} \\ w_1 \leq \bar{w}_1 \end{cases}$ for $\bar{w}_1 \in (w_1^-, w_1^+)$

- Choose $\begin{cases} a_0(w_1) = (w_1 - w_1^*)_+, & w_1^* \geq \bar{w}_1, \quad |w_1^+ - w_1^*| \ll 1 \\ a(w_1) = 0, & w_1 \geq \bar{w}_1 \end{cases}$

$\hookrightarrow \begin{cases} \eta_{a_0}(w) > 0 & \forall\ w \in \{\bar{w}_1 < w_1 < w_1^+\} \cap \operatorname{supp} \nu \\ \eta_{a_0} q_a - \eta_a q_{a_0} \equiv 0 \end{cases}$

$\hookrightarrow \quad \langle \nu, \eta_{a_0} \rangle \langle \nu, q_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, q_{a_0} \rangle$

$\hookrightarrow$ $\langle \nu, \xi_a \rangle = C_1(\overline{w}_1) \langle \nu, \eta_a \rangle$

$\forall a \in C, \quad a(w_1) = 0, \quad w_1 \geq \overline{w}_1.$

$$\left[C_1(\overline{w}_1) = \frac{\langle \nu, \xi_{a_0} \rangle}{\langle \nu, \eta_{a_0} \rangle} \right]$$

Similarly, $\forall a \in C$, $a(w_1) = 0$ when $w_1 \leq \overline{w}_1$,

$$\langle \nu, \xi_a \rangle = C_1(\overline{w}_1) \langle \nu, \eta_a \rangle$$

- <u>claim</u> $C_1(\overline{w}_1) \not\to \overline{w}_1$.

For any $\tilde{w}_1 \in (w_1^-, w_1^+)$, choose

$$\begin{cases} a_1(w_1) = 0 & w_1 \leq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \geq \overline{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 < \overline{w}_1$$

$$\begin{cases} a_1(w_1) = 0 & w_1 \geq \tilde{w}_1 \\ a_2(w_1) = 0 & w_1 \leq \overline{w}_1 \end{cases} \quad \text{for } \tilde{w}_1 > \overline{w}_1$$

$\Rightarrow$ $\eta_{a_1} \xi_{a_2} - \eta_{a_2} \xi_{a_1} \equiv 0$

$\hookrightarrow$ $C_1(\overline{w}_1) = C_1(\tilde{w}_1)$

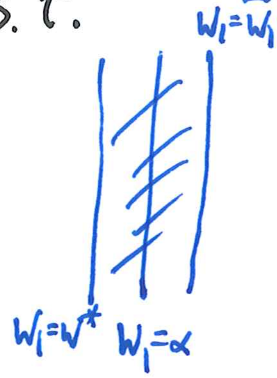
2. Claim If $W_2^- < W_2^+$, then $\exists\ C_2(t,x)$ s.t.

$$\langle \nu, \xi_b \rangle = C_2 \langle \nu, \eta_b \rangle$$

$\forall\, b \in C$, $\quad b(W_2) = 0$ when $\begin{cases} W_2 \geq \bar{W}_2 \\ \text{or } W_2 \leq \bar{W}_2 \end{cases}$ $\quad \bar{W}_2 \in (W_2^-, W_2^+)$

3. $\forall\, \alpha \in (W_1^-, W_1^+)$, choose W_1^*, $\bar{W}_1$ s.t.

$$W_1^* < \alpha < \bar{W}_1, \qquad \bar{W}_1 - W_1^* << 1$$



Choose (η_a, ξ_a): $\quad a(W_1) = (W_1 - W_1^*)_+$

Choose $(\eta_{\bar{a}}, \xi_{\bar{a}})$: $\quad \bar{a}(W_1) = (W_1 - \bar{W}_1)_-$

$$\langle \nu, \eta_a \xi_{\bar{a}} - \eta_{\bar{a}} \xi_a \rangle = \langle \nu, \eta_a \rangle \langle \nu, \xi_{\bar{a}} \rangle - \langle \nu, \eta_{\bar{a}} \rangle \langle \nu, \xi_a \rangle$$
$$= 0$$

We know that, On $\{W_1 < W_1^*\} \cup \{W_1 > \bar{W}_1\}$,

$$\eta_a \xi_{\bar{a}} - \eta_{\bar{a}} \xi_a \equiv 0$$

On $\{ w_1^* \le w_1 \le \bar{w}_1 \}$.

$$\begin{cases} \eta_a(w) = I_1(w)(w_1 - w_1^*) + \frac{1}{2} J_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \\ g_a(w) = K_1(w)(w_1 - w_1^*) + \frac{1}{2} L_1(\alpha; w)(w_1 - w_1^*)^2 + O(|w_1 - w_1^*|^3) \end{cases}$$

$$\begin{cases} \eta_{\bar{a}}(w) = I_1(w)(w_1 - \bar{w}_1) - \frac{1}{2} J_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - w_1^*|^3) \\ g_{\bar{a}}(w) = K_1(w)(w_1 - \bar{w}_1) - \frac{1}{2} L_1(\alpha; w)(w_1 - \bar{w}_1)^2 + O(|w_1 - w_1^*|^3) \end{cases}$$

$$\begin{aligned} &(\eta_a g_{\bar{a}} - \eta_{\bar{a}} g_a)(w) \\ &= \frac{1}{2}(\bar{w}_1 - w_1^*)(\bar{w}_1 - w_1)(w_1 - w_1^*)\left(I^2 \frac{\partial \lambda_1}{\partial w_1} \right)(\alpha, w_2) \\ &\quad + O((\bar{w}_1 - w_1^*)^2 (\bar{w}_1 - w_1)(w_1 - w_1^*)) \end{aligned}$$

$$\Rightarrow \quad \left\langle \nu, (\bar{w}_1 - w_1)_+ (w_1 - w_1^*)_+ \Big(\underbrace{\left(I^2 \frac{\partial \lambda_1}{\partial w_1} \right)(\alpha, w_2)}_{\neq 0} + O(|\bar{w}_1 - w_1^*|) \Big) \right\rangle = 0$$

$$\Rightarrow \quad \operatorname{Supp} \nu \cap \{ w_1^* \le w_1 \le \bar{w}_1 \} = \phi \quad \forall \begin{array}{l} w_1^* < \bar{w}_1 \\ |\bar{w}_1 - w_1^*| << 1. \end{array}$$

$$\hookrightarrow \quad \operatorname{Supp} \nu \cap \{ w_1^- < w_1 < w_1^+ \} = \phi$$

Similarly.

$$\hookrightarrow \quad \operatorname{Supp} \nu \cap \{ w_2^- < w_2 < w_2^+ \} = \phi$$

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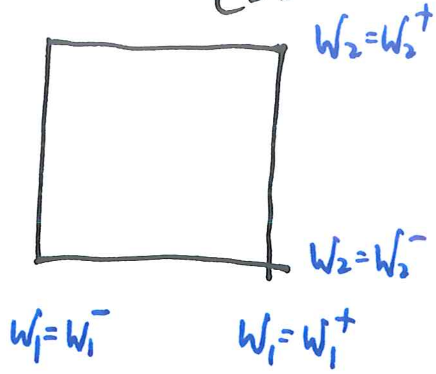

4. If

$$\text{Supp}\,\nu \cap \left(\{W_1 = W_1^{\pm}\} \cup \{W_2 = W_2^{\pm}\}\right) \neq \emptyset$$

for example

$$\text{Supp}\,\nu \cap \{W_1 = W_1^-\} \neq \emptyset$$

↳ $$\nu(\{W_1 = W_1^-\}) \neq 0$$

then we follow **Step 3** to choose

$$\alpha = W_1^-, \quad \bar{W}_1 = W_1^- + \varepsilon, \quad W_1^* = W_1^- - \varepsilon.$$

to conclude

$$\text{Supp}\,\nu \cap \{W_1^- - \varepsilon \le W_1 \le W_1^- + \varepsilon\} = \emptyset$$

for sufficiently small $\varepsilon \geq 0$

↳ Contradiction

5. Conclusion

$$\text{Supp}\,\nu \cap \left([W_1^-, W_1^+] \times [W_2^-, W_2^+]\right) = \emptyset$$

↳ Contradiction

$\Longrightarrow$ $\nu_{t,x} = \delta_{u(t,x)}$ single point support.