# Differential Equations

Ex.

Physical Variable $\longleftrightarrow$ $\begin{cases} \text{Location} \\ \text{Time} \end{cases}$

(Density, velocity, temperature ....)

$\boxed{u} \longleftrightarrow \boxed{(t, x, y, z)}$

$\boxed{\text{Unknown}} \rightarrow \boxed{u}$

$u = u(t, x, y, z)$

$\boxed{\text{Physical Laws}}$

$\Rightarrow$ A Relation between $\boxed{u}$, $\boxed{\text{its Derivatives}}$ and Independent Variables $\boxed{(t, x, y, z)}$

# Differential Equations

$$(*) \quad \boxed{F(D^\alpha u,\ u,\ x) = 0, \qquad |\alpha| \le m}$$

- $x = (x_1, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$ : Indept. Variables
- $u = u(x) = u(x_1, \cdots, x_n)$ : Unknown function(s)
- $\alpha = (\alpha_1, \cdots, \alpha_n)$ : Multi-index,

  $\alpha_i \in \mathbb{Z}, \quad \alpha_i \ge 0, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n$
- $Du = \left( \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n} \right)$

  $D^\alpha u = \left( \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \right), \qquad 2 \le |\alpha| \le m$

  e.g. $|\alpha| = 2$, $\qquad D^\alpha u = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{1 \le i, j \le n}$
- $F$ : Known function of $u$, $D^\alpha u$, $x$.
- If $n = 1$ : $\qquad (*)$ is ODE
- If $n \ge 2$ : $\qquad (*)$ is PDE
- If $u \in \mathbb{R}^p \ (p \ge 2)$, $F \in \mathbb{R}^q \ (q \ge 2)$

  $\hookrightarrow$ $\qquad (*)$ is a system of ODEs or PDEs
- Order $= m$ $\qquad$ Dimension $\sim$ # of spatial variables

$$(*) \quad \boxed{F(D^\alpha u, u, x) = 0, \qquad |\alpha| \le m}$$

- We can rewrite $(*)$ as

$$\boxed{\begin{array}{l} G(D^\alpha u, u, x) = f(x) \\ \hline G(0, 0, x) = 0 \end{array}}$$

If $f(x) = 0$: Homogeneous

If $f(x) \neq 0$: Nonhomogeneous

- Linear: $F$ is linear w.r.t. $D^\alpha u$ and $u$

$$\sum_{|\alpha| \le m} A_\alpha(x) D^\alpha u = f(x)$$

- Quasilinear: $F$ is linear w.r.t. $\boxed{D^{\tilde{\alpha}} u, \ |\tilde{\alpha}| = m}$

$$\sum_{|\tilde{\alpha}| = m} A_{\tilde{\alpha}}(D^\alpha u, u, x) D^{\tilde{\alpha}} u + A_0(D^\alpha u, u, x) = 0$$

$$|\alpha| \le m-1$$

- Semilinear

$$\sum_{|\tilde{\alpha}| = m} A_{\tilde{\alpha}}(x) D^{\tilde{\alpha}} u + A_0(D^\alpha u, u, x) = 0, \quad |\alpha| \le m-1$$

- Fully Nonlinear: $F$ depends nonlinearly upon $D^{\tilde{\alpha}} u$, $|\tilde{\alpha}| = m$

$$(*) \quad \boxed{F(D^\alpha u, u, x) = 0 \qquad |\alpha| \leq m}$$

$\boxed{\text{SOLUTION}}$. $u = u(x)$ is called a $\boxed{\text{Solution}}$ of $(*)$ in $\Omega$, provided that

$$(**) \quad \boxed{F(D^\alpha u(x), u(x), x) = 0 \qquad x \in \Omega}$$

in an appropriate sense of topology

Classical Solution: $u \in C^m(\Omega)$

Weak Solution, Generalized Solution:

$(**)$ holds in a WEAK sense

Entropy Solution

Besides $(**)$, $\boxed{u = u(x)}$ satisfies an additional condition:

$\boxed{\text{the entropy condition}}$

# Linear PDEs: Models

- Transport equation: $\boxed{u_t + a u_x = 0}$

  $a$: const.

- Laplace's equation: $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

  $$\boxed{\Delta u = 0}$$

  $u$ — potential/harmonic function

- Heat equation

  $$\boxed{u_t = k \Delta u,}$$ $k > 0$ const.

  $u \sim$ temperature

- Wave equation

  $$\boxed{u_{tt} - c^2 \Delta u = 0.}$$ $c = $ const.

  $n=1$: Vibrating of strings, Propagation of sound waves in tubes

  $n=2$. Waves on the surface of shallow water, and vibrating drumhead

  $n=3$ Acoustic or light waves

- Shrödinger's Eq., Black-Scholes Eq., ....

# Nonlinear PDEs: Models

- Hopf-Burgers Eq. $\boxed{u_t + u\,u_x = 0}$

- p-system
$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0, \end{cases} \quad p'(v)<0$$

$$u \in C^2 \Longrightarrow u_{tt} + (p'(v)\,u_x)_x = 0$$

$p'(v) = -c^2 \Longrightarrow$ Wave Eq.

- Minimal Surface Eq. $z = u(x, y)$.

$$(1+u_y^2)\,u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)\,u_{yy} = 0$$

- Incompressible Euler (Navier-Stokes) Eqs.

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = 0 & (\varepsilon \Delta u) \\ \operatorname{div} u = 0 & u \in \mathbb{R}^n,\ x \in \mathbb{R}^n \end{cases}$$

$\varepsilon = 0,\ p = \text{const.}\ n = 1 \Longrightarrow$ Hopf-Burgers Eq.

- KdV Eq. (Korteweg-de Vries Eq).

$$u_t + u\,u_x + \varepsilon\,u_{xxx} = 0$$

$\varepsilon = 0 \Longrightarrow$ Hopf-Burgers Eq.

- ......

# Solving PDEs − Solvability

Experience from solving the $n^{th}$ Order algebraic Eq.

$$(*) \quad \boxed{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0}$$

$n=1, 2$: Natural

$n=3$: Niccolò Fontana Tartaglio: 1535, 1539

Gerolamo Cardano: Ars Magna, 1545

$n=3$: Lodovico Ferrari (Cardano's Student): 1540

↓ 300 Years Descartes, Newton, Euler, ...

Abel (1826): It is impossible to solve high-order eqs. ($n \geq 5$) by simple explicit algebraic operations.

Galois (1831): Galois theory for $(*)$ indicating that the higher the order is, the more difficult one solves it.

- The Same Feature for PDEs.

  Higher Order PDEs are more difficult to be solved than lower Order PDEs

## Other Difficulties

- Nonlinear PDEs are more difficult than Linear PDEs.

  In many applications, they are often approximated by linear PDEs.
- Systems of PDEs are more difficult than Single PDE.
- Higher dimensional PDEs are more difficult than lower dimensional PDEs
- For most PDEs, it is impossible to write out Explicit Formulas for Solutions, Except some classes of PDEs.

**Point** Our interest is NOT so much in being able to "write down" solns. Rather than it is in understanding Properties of Solns & Methods for finding/analyzing Solns & Various Classes of PDEs — where they're from, how solutions behave, & how different they can be from one another.

## Well-Posedness – Meaning of Solvability

Difficulty: Different PDEs have completely different behavior of solutions.

Side Conditions:

- Boundary Conditions
- Initial Conditions
- Other Side Conditions...

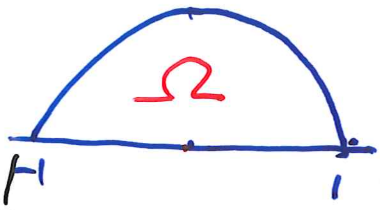
Issues.

- Sensitive, intimately connected with the form of the PDEs.
- Must be extremely careful in selecting the "Side Conditions" for the problems involving PDEs.

**Example 1.**

Cauchy Problem / IBV Problem
$$\begin{cases} \Delta u := u_{xx} + u_{yy} = 0, & (x,y) \in \Omega \\ u|_{y=0} = 0 \\ u_y|_{y=0} = a(x), & -1 < x < 1 \end{cases}$$

**Claim**: $\nexists$ a solution of this problem in general!




If $\exists$ a soln $u(x,y) \in C^2(\Omega)$,

we extend
$$u(x,y) = \begin{cases} u(x,y), & y > 0 \\ -u(x,-y), & y < 0 \end{cases} \in C^2$$

(Odd extension)

Define $v(x,y) = \int_{(0,0)}^{(x,y)} (-u_y\,dx + u_x\,dy)$

$\hookrightarrow$ $\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$ Cauchy-Riemann Eq.

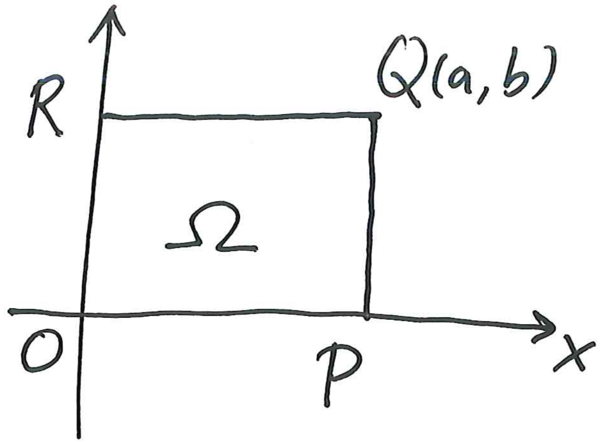
$\hookrightarrow$ $u + iv$ analytic

$\hookrightarrow$ $u(x,y)$ is real analytic

$\hookrightarrow$ $a(x) = u_y(x,0)$ Must be real analytic

## Example 2 Dirichlet Problem

$$\begin{cases} u_{xy} = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = u_0 \end{cases}$$




General Solution

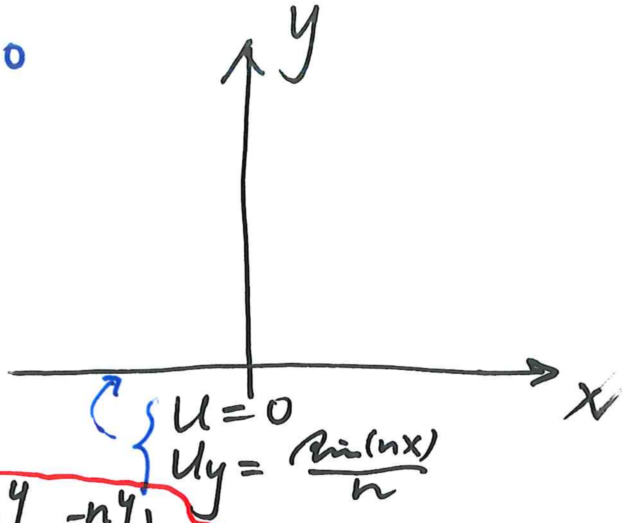
$$u(x,y) = \varphi(x) + \psi(y)$$

$$\boxed{u_0(O) + u_0(Q) = u_0(R) + u_0(P)} \qquad (*)$$

This problem has No solutions if $(*)$ fails to hold

## Example 3 Cauchy Problem

$$
(\mathrm{I}) \quad \begin{cases} \Delta u = 0 & x \in \mathbb{R},\ y > 0 \\ u|_{y=0} = 0 \\ u_y|_{y=0} = \dfrac{\sin(nx)}{n} \end{cases}
$$




$$
\boxed{u^n(x,y) = \frac{\sin(nx)\left(e^{ny} - e^{-ny}\right)}{2n^2}}
$$

$$
(\mathrm{II}) \quad \begin{cases} \Delta u = 0 \\ u|_{y=0} = 0 \\ u_y|_{y=0} = 0 \end{cases} \longrightarrow \quad u^\infty(x,y) = 0.
$$

**Phenomenon**

$$
\begin{cases} u^n|_{y=0} = u^\infty|_{y=0} \\ |u_y^n(x,0) - u_y^\infty(x,0)| = \dfrac{|\sin(nx)|}{n} \xrightarrow[n\to\infty]{} 0 \end{cases}
$$

**BUT**

$$
\lim_{n\to\infty} |u^n(x,y) - u^\infty(x,y)| = \lim_{n\to\infty} \frac{|\sin(nx)|\,|e^{ny} - e^{-ny}|}{2n^2}
$$

$$
= +\infty, \qquad \forall\, y > 0.
$$

* Instability: Arbitrarily small changes in the data

↳ Large changes in the Solution.

# More Examples

4. $\begin{cases} u_t - \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$ $(*)$

Uniqueness: A solution of $(*)$ is unique in the class of solutions with

$(\#)$ $|u(x,t)| \le M e^{C|x|^2}$, $0 \le t \le T$

for some constants $M, C$.

Tychonoff Example: Without assumption $(\#)$, $\exists$ $\infty$-many solns.

cf. Fritz John: P211-213

5. H. Lewy's Example  Fritz John P235-239

Consider $Lu = -u_x - i u_y + 2i(x+iy)u_z$

$\exists$ $F(x,y,z) \in C^\infty(\mathbb{R}^3)$ such that Eq.

$Lu = F(x,y,z)$

has $\boxed{\text{no solution}}$ whose domain is an open set $\Omega \subset \mathbb{R}^3$ with $u \in C^1(\Omega)$ and $u_x$, $u_y$, $u_z$ Hölder Continuous in $\Omega$

Well-posedness $\left\{ \begin{array}{l} \text{Existence} \\ \text{Uniqueness} \\ \text{Stability} \end{array} \right.$

? Spaces for Solutions

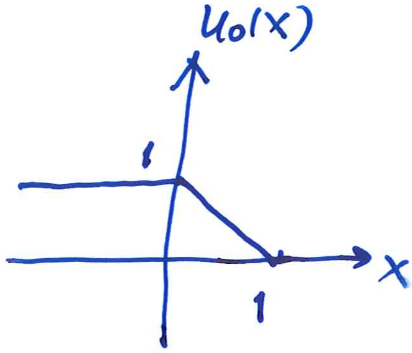
? Topology for the Stability of Solutions

Spaces:

$C^k$, $W^{k,p}$, $BV$, $\mathcal{M}$, ....

# Example 4 Hopf-Burgers Eq.

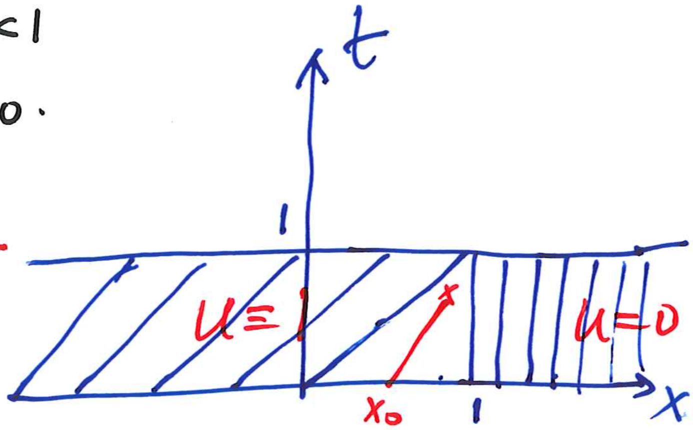
$$\begin{cases} u_t + u u_x = 0. & x \in \mathbb{R},\ t>0 \\ u|_{t=0} = u_0(x) = \begin{cases} 1 & x<0 \\ 1-x & 0<x<1 \\ 0 & x>1 \end{cases} \end{cases}$$




For $t<1$

$$u(x,t) = \begin{cases} 1 & x<t \\ \frac{1-x}{1-t} & t<x<1 \\ 0 & x>0. \end{cases}$$

$u_x(x,t)$ blowsup at $t=1$




The phenomenon is natural in Physics.

? How do we get out of the dilemma ?

Answer: Weaken the Notion of solution.

Allow the solution discontinous

starting at $t=1$

$$\boxed{\text{Shock Wave phenomenon !}}$$

# I. Ordinary Differential Equations

## Cauchy Problem

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = f(x,t) & x \in \mathbb{R}^n \\ x|_{t=0} = x_0 \end{cases}$$

## §1. Local Well-Posedness

Choose $a>0$, $b>0$, and $M>0$ s.t.

- $f$ is defined in
  $$R = \{(x,t) \in \mathbb{R}^{n+1} \mid |x-x_0| \le b,\ |t-t_0| \le a\}$$
- $f(x,t)$ is continuous in $R$
- $|f(x,t)| \le M$, for $(x,t) \in R$.

Observation: If $x \in C^1$ is a soln. with
$|x(t)-x_0| \le b$ for $|t-t_0| \le T$

$$\hookrightarrow |x(t)-x_0| = \left|\int_{t_0}^t f(x(s),s)\,ds\right| \le \int_{t_0}^t |f(x(s),s)|\,ds$$
$$\le M|t-t_0| \le MT < b$$

if $\boxed{T \le \min\{a, \frac{b}{M}\}}$

# Theorem 1.1 (Picard's Theorem)

Assume

- $|f(x,t)| \le M, \qquad (x,t) \in R$
- $|f(x,t) - f(y,t)| \le C\,|x-y|, \quad \forall \begin{matrix}(x,t)\\(y,t)\end{matrix} \in R$

(i) $\exists\, 1 \quad x = x(t) \in C^1(t_0 - T,\, t_0 + T)$ of $(*)$

provided that $T \le \min\{a, \frac{b}{M}\}$

(ii) The Cauchy problem

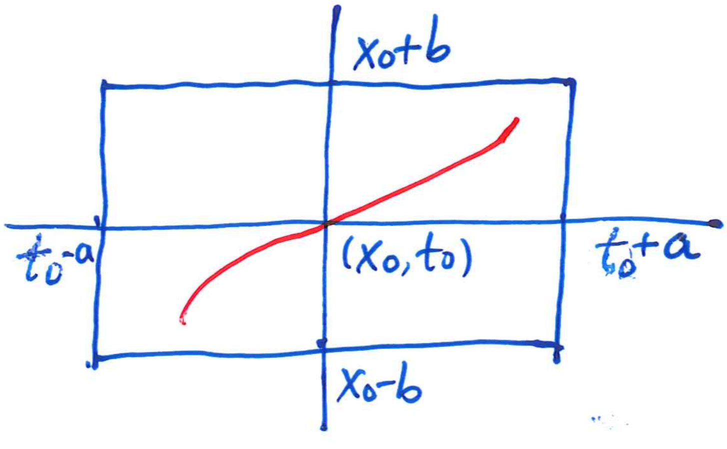
$$\begin{cases} \dfrac{dx}{dt} = f(x,t) \\ x|_{t=t_0} = y \end{cases}$$

has a unique soln $x = x(t, y)$

for $|t - t_0| M + |y - x_0| \le b, \quad |t - t_0| \le a$

and

$$|x(t,y) - x(t,z)| \le e^{C|t-t_0|}\,|y - z|$$




## Homework #1

**Remark.** The uniqueness part of Thm 1.1 remains valid for $t \geq t_0$, if the Lipschitz condition on $f$ is weakened to

$$(**) \quad \langle y-z,\ f(t,y)-f(t,z) \rangle \leq C|y-z|^2$$

When $n=1$, $(**)$ becomes

$$(***) \quad f(t,y)-f(t,z) \leq C(y-z) \quad \text{if } y \geq z$$

(One-side Lipschitz condition)

**Comparison Principle:** $n=1$. $(***)$ holds

$$\begin{cases} \dfrac{dx(t)}{dt} \geq f(x(t),t) \\ \dfrac{dy(t)}{dt} \leq f(x(t),t) \\ x(t_0) \geq y(t_0) \end{cases} \Longrightarrow \quad x(t) \geq y(t), \quad t \geq t_0$$

**Proof.** Set $R(t) = y(t) - x(t)$. If $\exists\ t \in (t_1, t_2)$ s.t.

$$\begin{cases} R(t) \leq 0 & t_0 < t < t_1 \\ R(t) > 0 & t_1 < t < t_2 \end{cases} \longrightarrow \quad R(t_1) = 0$$

$$\begin{cases} \dfrac{dR(t)}{dt} \leq f(y(t),t) - f(x(t),t) \leq C\,R(t), & t_1 < t < t_2 \\ R(t_1) = 0 \end{cases}$$

$$\Longrightarrow \quad 0 < R(t) \leq R(t_1)\,e^{C(t-t_1)} \quad \text{for } t_1 < t < t_2$$

Contradiction

## Theorem 1.2 (Global existence and uniqueness for Linear ODEs)

If $\begin{cases} f(x,t) = A(t)\,x \\ A(t) \text{ is continuous for } |t-t_0| \le a \end{cases}$

then the Cauchy problem (*) has a unique soln for $|t-t_0| \le a$:

$$x(t) = F(t)\,x_0$$

where $F(t)$ is the unique $n \times n$ matrix function determined by

$$\begin{cases} \dfrac{dF(t)}{dt} = A(t)F(t) \\ F(t_0) = I \end{cases}$$

If $\|A(t)\| \le M$, then, for $|t-t_0| \le a$

$$\begin{cases} \|F(t)\| \le e^{|t-t_0|M} \\ \|F(t)^{-1}\| \le e^{|t-t_0|M} \\ \det(F(t)) \le \exp\left(\int_{t_0}^{t} Tr(A(s))\,ds\right) \end{cases}$$

## Homework #2

# Theorem 1.3 (Peano's Theorem)

$$\begin{cases} f(x,t) & \text{is continuous in } R \\ |f(x,t)| \le M, & (x,t) \in R \end{cases}$$

$\Longrightarrow$ $\exists$ a $C^1$-solution for $|t-t_0| \le T$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\min\{a, \frac{b}{M}\}}$

Proof. 1. Extend $f$ to $\mathbb{R}^n \times [t_0-a, t_0+a]$ by defining

$$f(x,t) = f\left(x_0 + \frac{b(x-x_0)}{|x-x_0|}, t\right) \quad \text{for } |x-x_0| > b$$

2. For $\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi\left(\frac{x}{\varepsilon^n}\right)$ with

Mollifier

$$\chi(x) = \begin{cases} A e^{\frac{1}{|x|^2-1}} & |x| \le 1 \\ 0 & |x| > 1. \end{cases} \qquad \int \chi(x)\,dx = 1$$

define

$$f_\varepsilon(x,t) = f * \chi_\varepsilon(x) = \int \chi_\varepsilon(x-y) f(y,t)\,dy$$

$$\hookrightarrow \begin{cases} f_\varepsilon(x,t) & C^\infty \text{ in } x, \text{ continuous in } t \\ f_\varepsilon \longrightarrow f & \text{uniformly in } R, \text{ as } \varepsilon \to 0 \\ |f_\varepsilon| \le M & \forall \varepsilon \end{cases}$$

Proof (Conti.)

3. Consider the Cauchy problem:

$$\begin{cases} \dfrac{dx_\varepsilon(t)}{dt} = f_\varepsilon(x_\varepsilon(t), t) \\ x_\varepsilon\big|_{t=t_0} = x_0 \end{cases}$$

Thm 1.1 $\Rightarrow$ $\exists$ 1 soln $x = x_\varepsilon(t)$ for $|t-t_0| \le T$ satisfying

$$\begin{cases} |x_\varepsilon(t) - x_0| \le b & \forall \varepsilon \\ \left|\dfrac{dx_\varepsilon(t)}{dt}\right| \le |f_\varepsilon(x_\varepsilon(t), t)| \le M & \forall \varepsilon \end{cases}$$

$\hookrightarrow$ Equicontinuity

4. Ascoli-Arzela Thm

$\hookrightarrow$ $\exists$ $\{x_{\varepsilon_k}\}_{k=1}^{\infty} \subset \{x_\varepsilon\}_{\varepsilon>0}$ s.t.

$$\begin{cases} x_{\varepsilon_k}(t) \longrightarrow x(t) & \text{uniformly} \\ \dfrac{dx_{\varepsilon_k}(t)}{dt} = f_{\varepsilon_k}(x_{\varepsilon_k}(t), t) \longrightarrow f(x(t), t) & \text{uniformly} \end{cases}$$

$\hookrightarrow$
$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t), t) \\ x\big|_{t=t_0} = x_0 \end{cases}$$

* Approximation Method

# Observation

If
- $\begin{cases} f \in C(R) \\ |f(x,t)| \le M \quad \forall (x,t) \in R \end{cases}$
- $x(t)$ is any soln of $\begin{cases} \dfrac{dx(t)}{dt} = f(x(t),t), \\ x|_{t=0} = x_0 \end{cases}$ $|t-t_0| \le T$

$\hookrightarrow$ $\exists$ $f_\varepsilon(x,t)$ s.t.

(i) $\begin{cases} \dfrac{dx(t)}{dt} = f_\varepsilon(x(t),t) \\ x(0) = x_0 \end{cases}$

(ii) $f_\varepsilon$ $C^\infty$ in $x$, continuous in $t$

(iii) $f_\varepsilon \longrightarrow f$ uniformly in $R$ as $\varepsilon \to 0$

In fact, for $\widehat{f_\varepsilon}$ defined as in the proof of Theorem 1.3, define

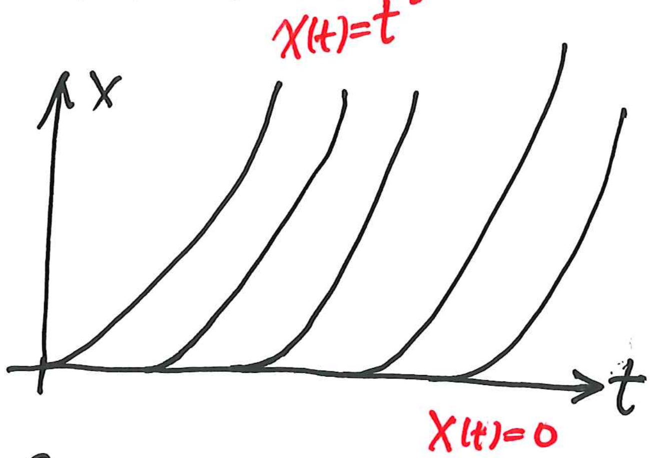
$$\boxed{f_\varepsilon(x,t) = \widehat{f_\varepsilon}(x,t) + f(x(t),t) - \widehat{f_\varepsilon}(x(t),t)}$$

**Example**:
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{3}{2} x^{\frac{1}{3}} \\ x|_{t=0} = 0 \end{cases}$$

- $f(x,t) = \frac{3}{2} x^{\frac{1}{3}}$ is continuous in $x$
  but is not Lipschitz at $x=0$

Easy to check: Two solutions

$$\begin{cases} x(t) = 0, \\ x(t) = t^{3/2} \end{cases}$$




**General solutions**

$$x_c(t) = \begin{cases} 0 & t \le c \\ (t-c)^{\frac{3}{2}} & t \ge c \end{cases}$$

Starting at $(0,0)$ lie between these two solutions, and fill up the funnel between them.

# §2 Existence of Solutions in the Large

Ex. $\begin{cases} \frac{dx}{dt} = x^2 \\ x|_{t=0} = x_0 > 0 \end{cases}$ $(*)$

Soln: $\frac{dx}{x^2} = dt$, $-\frac{1}{x} + \frac{1}{x_0} = t$

$$x(t) = \frac{x_0}{1-x_0 t} \nearrow \infty \text{ as } t \nearrow \frac{1}{x_0}$$

In general, $\cancel{\exists}$ a g

If $x_0 < 0$ $\longrightarrow$ $\exists\, 1$ global solution for $t>0$

Thm 1.4 (Lower Semicontinuity of the domain of existence for perturbations of the data)

If (i) $f(x,t)$, $\frac{\partial f(x,t)}{\partial x}$ are continuous in an open set $\Omega \subset \mathbb{R}^{n+1}$

(ii) The Cauchy problem $(*)$ has a soln $x$ with graph in $\Omega$ for $t_1 \le t \le t_2$ with $t_0 \in (t_1, t_2)$

$\Rightarrow$ $\exists$ a nbhd $U$ of $x_0$ in $\mathbb{R}^n$, s.t. $\forall\, y \in U$ $\exists\, 1$ soln $x(t,y) \in C^1([t_1, t_2] \times U)$ of $\begin{cases} \frac{dx}{dt} = f(x,t) \\ x|_{t=t_0} = y \end{cases}$

for $t_1 \le t \le t_2$.

<u>Remark</u>: We could also allow a small perturbation of $\boxed{f}$ in Thm 1.4, for which we could replace $\boxed{f}$ by $\boxed{f+\varepsilon g}$, for small $\varepsilon$, adding the Eq. $\frac{d\varepsilon}{dt}=0$

Now we allow perturbation of the Cauchy problem (*).

<u>Theorem 1.5</u> Consider the Cauchy problem

$$(**)\quad \begin{cases} \frac{dx}{dt} = a_0(t)x^2 + a_1(t)x + a_2(t) \\ x(0) = x_0 \end{cases}$$

& $a_j(t)$ are continuous function in $[0,T]$

Set $\begin{cases} a_0^+ = \max\{a_0, 0\} \\ K = \int_0^T |a_2(t)|dt \exp\left(\int_0^T |a_1(t)|dt\right) \end{cases}$

If $\begin{cases} x_0 \geq 0 \\ \int_0^T a_0^+(t)dt \exp\left(\int_0^T |a_1(t)|dt\right) < \frac{1}{x_0+K} \\ \int_0^T |a_0(t)|dt \exp\left(\int_0^T |a_1(t)|dt\right) < \frac{1}{K} \end{cases}$

$\Rightarrow$ The Cauchy Problem (**) has a Solution in $[0, T]$. satisfying

$$\frac{1}{x(T)} \geq \frac{1}{x_0+K} - \int_0^T a_0^+(t)\,dt \,\exp\left(\int_0^T |a_1(t)|\,dt\right)$$

if $x(T) \geq 0$.

or

$$\frac{1}{x(T)} \geq \frac{1}{K} - \int_0^T |a_0(t)|\,dt \,\exp\left(\int_0^T |a_1(t)|\,dt\right)$$

if $x(T) < 0$.

* $T \sim$ Life span, i.e.

$$T = \sup_{t^*}\left\{t^* \;\middle|\; x(t) \text{ exists in } [0, t^*]\right\}$$

Homework # 3

# §3 Generalized Solutions

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = f(x,t) \\ x|_{t=t_0} = x_0 \end{cases}$$

$f(x,t)$ ~ real value measurable function in
$R = \{(x,t) \in \mathbb{R}^{n+1} \mid |t-t_0| \le a,\ |x-x_0| \le b\}$

$|f(x,t)| \le M(t)$ with $M \in L^1$

**Theorem 1.6** The Cauchy problem $(*)$ has a solution in the sense that $x$ is absolutely continuous and for almost all $t \in [t_0-a, t_0+a]$,

$$\boxed{x'(t) \in F(x(t), t)}$$

$F(x,t)$ ~ The smallest closed convex set s.t. every nbhd contains the value of $f(y,t)$ for almost all $y$ in some nbhd of $x$

$$\langle x'(t), \xi \rangle \le H(x(t), t, \xi) = \lim_{\delta \to 0} \operatorname*{ess\,sup}_{|x-y|<\delta} \langle f(y,t), \xi \rangle$$

$\forall \xi \in \mathbb{R}^n$.

# References for Part I


1. R. Hörmander: Chapter 1
   Lectures on Nonlinear Hyperbolic Differential Equations
   Springer-Verlag, 1997.

2. Other standard ODE Textbooks

# II. Transport Equation

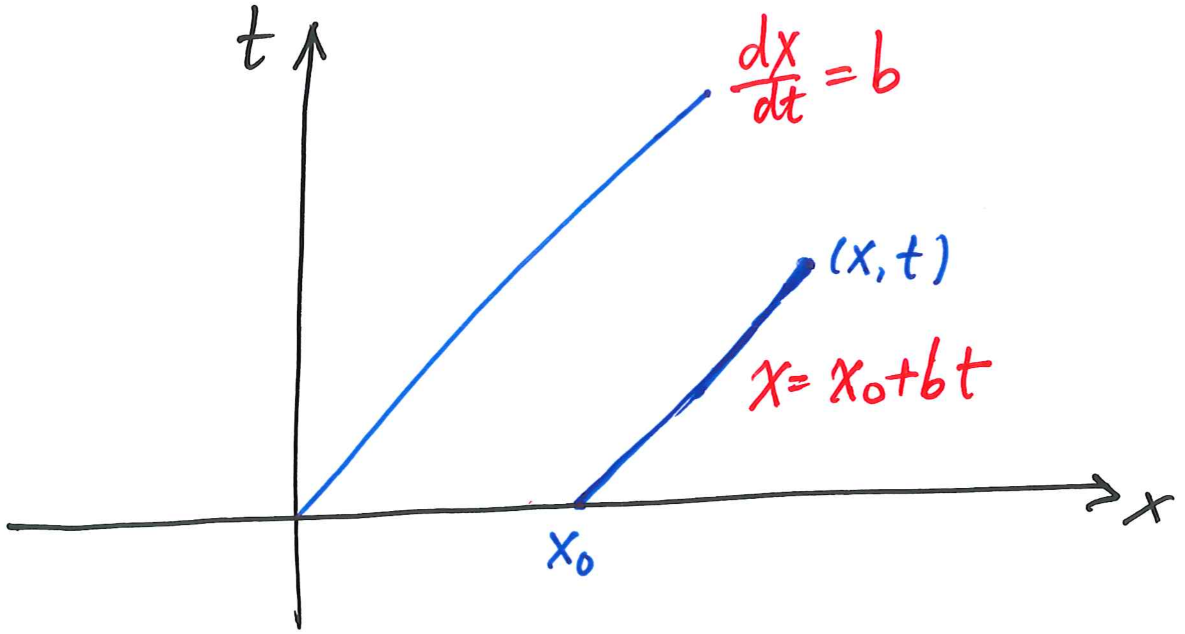
Probably the simplest PDE is

(V) $\quad u_t + b\,u_x = 0.$ $\quad b$ is a constant.

Along the direction $\frac{dx}{dt} = b$, for any solution $u = u(x,t)$ of (V), we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + b\,u_x = 0$$

$\hookrightarrow$ $u \equiv$ const. along the line with direction $(b, 1) \in \mathbb{R}^{1+1}$.



## General

$$(*) \quad u_t + b \cdot Du = 0 \qquad x \in \mathbb{R}^n, \ t \in \mathbb{R}_+$$

$b = (b_1, \cdots, b_n) \in \mathbb{R}^n$

$Du = D_x u = (u_{x_1}, \cdots, u_{x_n})$

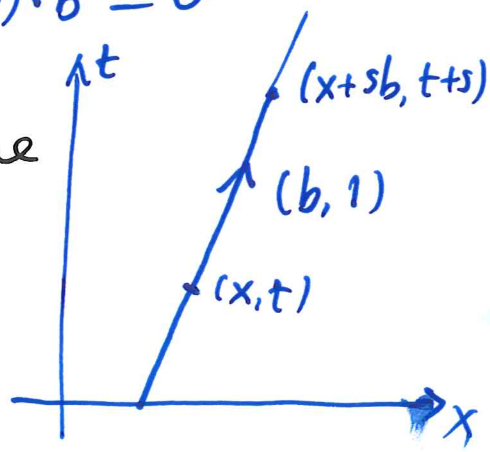
**Analysis** Assume $u \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ is a soln of $(*)$

$\forall (x,t) \in \mathbb{R}^n \times (0,\infty)$, define

$$z(s) = u(x+sb,\ t+s) = u((x,t) + s(b,1))$$

$$\frac{dz}{ds} = \frac{\partial u}{\partial t}(x+sb,\ t+s) + Du(x+sb,\ t+s) \cdot b \equiv 0.$$

↳ $u$ is const. along the line through $(x,t)$ with direction $(b,1) \in \mathbb{R}^{n+1}$




↳ If we know the value of $u$ at any point on each such line, we know its value everywhere in $\mathbb{R}^n \times (0, \infty)$.

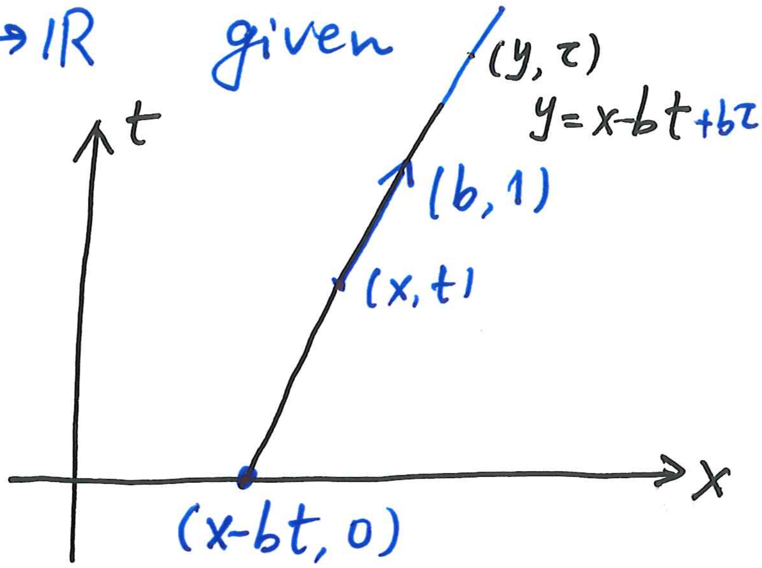
# Cauchy Problem

$$(*)\begin{cases} u_t + b\cdot Du = 0 & \mathbb{R}^n\times(0,\infty) \\ u|_{t=0} = g(x) : \mathbb{R}^n \longrightarrow \mathbb{R} & \text{given} \end{cases}$$

Analysis

$$u(x,t) = u(x-bt, 0) = g(x-bt)$$

$$\forall x\in\mathbb{R}^n,\ t>0$$




$$\begin{cases} y-x = bs \\ \tau - t = s \end{cases}$$

$$\hookrightarrow\ y = x + b(\tau - t) = (x-bt) + b\tau$$

$\hookrightarrow$ Uniqueness

If $g\in C^1 \longrightarrow u(x,t) = g(x-bt)$ satisfies $(*)$

$\hookrightarrow$ Existence of classical Soln.

If $g\notin C^1 \longrightarrow \nexists\ C^1$ solution.

But we have to accept $u(x,t) = g(x-bt)$ as a solution, which is called a **Weak Solution** of $(*)$.

# Nonhomogeneous Problem

$$(**)\begin{cases} u_t + b\cdot Du = f(x,t) & \mathbb{R}^n\times(0,\infty) \\ u|_{t=0} = g(x), & x\in\mathbb{R}^n \end{cases}$$

Analysis: For $z(s) = u(x+bs,\ t+s)$.

$$\frac{dz(s)}{ds} = u_t(x+bs,\ t+s) + Du(x+bs,\ t+s)\cdot b$$
$$= f(x+bs,\ t+s)$$

$$\int_{-t}^{0}\frac{dz(s)}{ds}\,ds = \int_{-t}^{0} f(x+bs,\ t+s)\,ds$$

$$z(0)-z(-t) = \int_0^t f(x+(s-t)b,\ s)\,ds$$

$$u(x,t) - g(x-bt)$$

$$\boxed{u(x,t) = g(x-bt) + \int_0^t f(x+(s-t)b,\ s)\,ds}$$

Verification. $u(x,t)$ solves $(**)$ indeed.

## * Method of characteristics

$$u_t + b\cdot Du = 0, \qquad u|_{t=0} = g(x)$$

$$\begin{cases} \dfrac{dx}{dt} = b \longrightarrow x = x_0 + bt \longrightarrow x_0 = x - bt \\ \dfrac{du}{dt} = 0 \longrightarrow u = g(x_0) \\ x|_{t=0} = x_0 \\ u|_{t=0} = g(x_0) \end{cases}$$

$$\boxed{u = g(x-bt)}$$

PDE $\Longleftrightarrow$ System of ODEs.

* General $b = b(x,t) \in \mathbb{R}^n$

References

1. R. J. DiPerna & P.-L. Lions. ODEs, Transport Theory and Sobolev spaces. Invent. Math. 98 (1989), 511-547
2. L. Ambrosio, Transport equation and Cauchy problem for BV vector fields. Invent. Math. 158 (2004), 227-260

# III. Laplace's Equation

$$\Delta u = 0 \qquad x \in \mathbb{R}^n$$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \quad \text{Laplacian}$$

Poisson's Eq.: $\begin{cases} -\Delta u = f \\ f: \mathbb{R}^n \to \mathbb{R} \text{ is given} \\ \quad (U) \end{cases}$

$u \in C^2$ satisfying $\Delta u = 0$: $\begin{cases} \text{Harmonic} \\ \text{Potential} \end{cases}$ function

* Ubiquity of Laplace's Eq. in Physics.

R. Feynmann, R. Leighton, & M. Sands: Lectures in Physics, Vol. II. Addison-Wesley, 1966

# Examples

## 1. Maxwell Eq.

$$\begin{cases} \text{Curl } E = 0 \\ \text{div } E = 4\pi\rho \end{cases}$$

$\rho$ — charge density

$$\text{Curl } E = 0 \xrightarrow[\text{electric potential}]{\exists\phi} E = -\nabla\phi$$

$$\hookrightarrow \Delta\phi = \text{div}(\nabla\phi) = -\text{div } E = -4\pi\rho$$

## 2. Analytic Function

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

**Cauchy-Riemann Eqs.**

$$\begin{cases} u_x = v_y \\ v_x = u_y \end{cases}$$

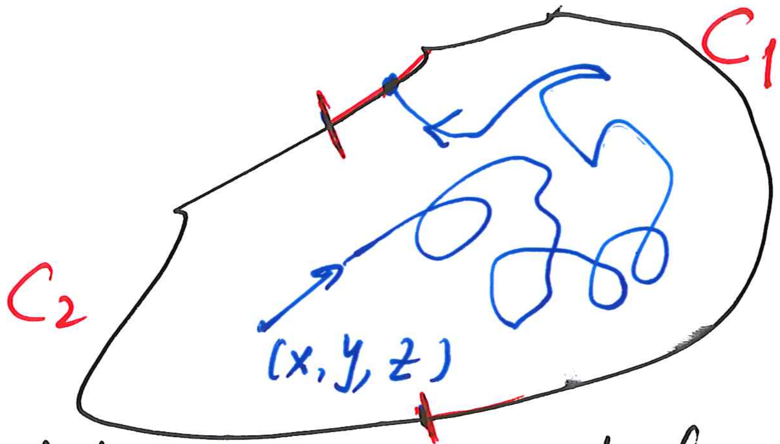
$$\hookrightarrow \begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$$

# Examples (Conti.)

3. **Brownian Motion** in a Container $D$:

Particles inside $D$ move completely randomly until they hit the bdry $\partial D$ where they stop.

$$\partial D = C_1 \cup C_2$$

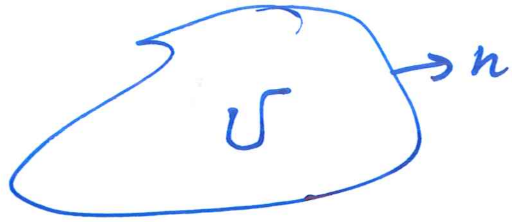



$u(x,y,z)$ – Probability that a particle which begins at the point $(x,y,z)$ and stops at some point on $C_1$.

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u|_{C_1} = 1 \\ u|_{C_2} = 0 \end{cases}$$

$\boxed{\text{Dirichlet Problem}}$

# Basic Problems




$$\Delta u = f \qquad \text{in } U$$

$$u\big|_{\partial U} = h \qquad \text{Dirichlet}$$

or

$$\frac{\partial u}{\partial n}\Big|_{\partial U} = h \qquad \text{Neumann}$$

or

$$\left(\frac{\partial u}{\partial n} + \underset{\substack{\vee\\0}}{\underline{\alpha}}\, u\right)\Big|_{\partial U} = h \qquad \text{Robin}$$

## §1 Mean-Value Property (MVP)

**Thm 3.1** If $u \in C^2$, then

$$\Delta u = 0 \iff \boxed{u(x) = ⨍_{\partial B(x,r)} u \, dS_y = ⨍_{B(x,r)} u \, dy \quad \forall x \in \mathbb{R}^n}$$

where

$$⨍_{B(x,r)} u \, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy$$

$$= \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u \, dy$$

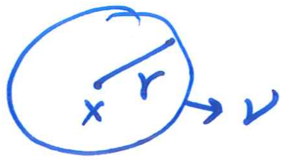
$$⨍_{\partial B(x,r)} u \, dS_y = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u \, dS$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u \, dS$$

$\alpha(n) =$ Volume of $B(0,1) \subset \mathbb{R}^n = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$

$\alpha(2) = \pi$, $\quad \alpha(3) = \dfrac{4\pi}{3}$

$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx.$




## Proof.

"⇒" 1. Set

$$\phi(r) = \fint_{\partial B(x,r)} u \, dS_y \overset{y=x+rz}{=\!=} \fint_{\partial B(0,1)} u(x+rz)\, dS_z$$

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz)\cdot z \, dS_z$$

$$\overset{y=x+rz}{=\!=} \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, dS_y$$

$$= \frac{r}{n} \fint_{\partial B(x,r)} \Delta u(y)\, dy \equiv 0$$

$\hookrightarrow$ $\phi(r) = \text{const.}$

$\hookrightarrow$ $\displaystyle \fint_{\partial B(x,r)} u(y)\, dS_y = \lim_{\varepsilon \to 0} \fint_{\partial B(x,\varepsilon)} u \, dS = u(x)$

2. Method of Shells

$\hookrightarrow$
$$\int_{B(x,r)} u \, dy = \int_0^r \left( \int_{\partial B(x,\rho)} u \, dS \right) d\rho$$

$$= u(x) \int_0^r n\alpha(n) \rho^{n-1} \, d\rho$$

$$= \alpha(n) r^n u(x)$$

$\hookrightarrow$
$$u(x) = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u \, dy = \fint_{B(x,r)} u \, dy$$

Proof (Conti.)

"$\Leftarrow$" If $\Delta u \not\equiv 0$, then $\exists$ some small ball $B(x,r) \subset \mathbb{R}^n$ s.t. $\Delta u > 0$ in $B(x,r)$
$(<)$

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\,dy \underset{(<)}{>} 0$$

* MVP has many important consequences.

# Theorem 3.2 (Maximum Principles)

$\Omega \subset \mathbb{R}^n$ open, bounded

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in $\Omega$

$\Rightarrow$

(i) Weak Maximum Principle:

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

(ii) Strong Maximum Principle:

If $\Omega$ is connected

$\exists$ a point $x_0 \in \Omega$ s.t.

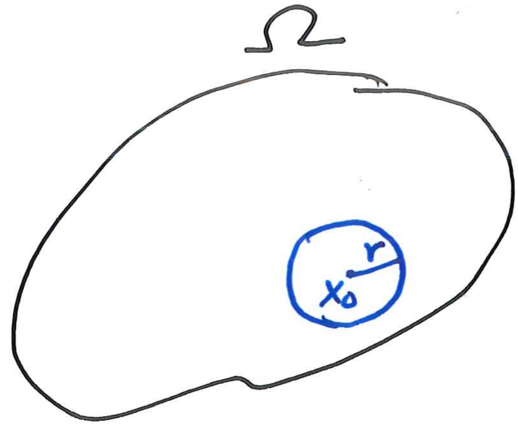
$$u(x_0) = \max_{\bar{\Omega}} u$$

$\Rightarrow$ $u = \text{const.}$ in $\Omega$

(iii) Similar assertions with "min".

(By replacing $u$ by $-u$)

Proof.




If $\exists$ a point $x_0 \in \Omega$

s.t.

$$u(x_0) = \max_{\Omega} u \equiv M$$

$\hookrightarrow$ For $0 < r < \operatorname{dist}(x_0, \partial\Omega)$.

$$M = u(x_0) = \fint_{B(x_0,r)} u \, dy \le M.$$

$\hookrightarrow$ $u \equiv M$ on $B(x_0, r)$

$\hookrightarrow$ The set

$$K \triangleq \{x \in \Omega \mid u(x) = M\}$$

is both open and relatively closed in $\Omega$

$\xrightarrow{\Omega \text{ connected}}$ $K = \Omega$

$\hookrightarrow$ (ii)

$\hookrightarrow$ (i)

# Theorem 3.3 (Uniqueness)

Let $f \in C(U)$, $g \in C(\partial U)$.

Consider the BVP $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

$\hookrightarrow$ $\exists$ at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$

Proof. If $\exists$ two solutions $u_1$, $u_2$.

$\hookrightarrow$ $u = u_1 - u_2$ satisfies

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

$\xrightarrow{\text{Thm 3.2}}$ $\max_{\bar{\Omega}} u = \min_{\bar{\Omega}} u = 0$

$\hookrightarrow$ $u \equiv 0$

$\hookrightarrow$ $u_1 \equiv u_2$

# Theorem 3.4 (Smoothness).

If $u \in C(\Omega)$ satisfies the MVP for each ball $B(x,r) \in \Omega$

$\Rightarrow$ $u \in C^\infty(\Omega)$.

Proof. Let $\eta(x)$ be a standard mollifier

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right) \qquad \begin{cases} \eta \in C^\infty \\ \eta(x) = 0 & |x| \geq 1 \\ \int \eta(x)\,dx = 1 \end{cases}$$

$$\Rightarrow u^\varepsilon = \eta_\varepsilon * u = \int \eta_\varepsilon(y)\,u(x-y)\,dy \;\boxed{\in C^\infty(\Omega_\varepsilon)}$$

$$\Omega_\varepsilon = \{x \in \Omega \mid dist(x, \partial\Omega) > \varepsilon\}$$

$$\int \eta_\varepsilon(y)\,u(x-y)\,dy = \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\,dy$$

$$= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\left(\underbrace{\int_{\partial B(x,r)} u\,dS_y}_{= n\alpha(n)\, r^{n-1} u(x)}\right) dr$$

$$= u(x)\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy = u(x)\int_{B(0,1)} \eta(y)\,dy$$

$$= u(x)$$

## Theorem 3.5 (Local Estimates on derivatives)

Assume that $u$ is harmonic in $\Omega$.

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

$\forall\, B(x_0,r) \subset \Omega$, $\forall$ multi-index $\alpha$ with $|\alpha|=k$

where $C_0 = \frac{1}{\alpha(n)}$, $C_k = \frac{(2^{n+1} n k)^k}{\alpha(n)}$, $k=1,2,\cdots$

$k=0$.

$$u(x_0) = \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} u(y)\,dy$$

$$|u(x_0)| \le \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} |u(y)|\,dy$$

$$= \frac{C_0}{r^n} \|u\|_{L^1(B(x_0,r))}. \qquad C_0 = \frac{1}{\alpha(n)}$$

By Induction $\Longrightarrow$ Estimates.

$k=1$ $\quad u_{x_i}(x_0) = ⨍_{B(x_0,\frac{r}{2})} u_{x_i}\,dx = \frac{2^n}{\alpha(n) r^n} \int_{\partial B(x_0,\frac{r}{2})} u \nu_i \, dS$

$$|u_{x_i}(x_0)| \le \frac{2^n}{\alpha(n) r^n} \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))} n\alpha(n) \left(\frac{r}{2}\right)^{n-1} = \frac{2n}{r} \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))}$$

$$\le \frac{2n}{r} C_0 \left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))} = \frac{2^{n+1} n}{\alpha(n)} \frac{1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

**Theorem 3.6** (Liouville Theorem)

$u: \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded.

$\hookrightarrow$ $u$ is constant.

**Proof.** $\forall x_0 \in \mathbb{R}^n$, $r > 0$

$$|Du(x_0)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \leq \frac{C_1}{r^{n+1}} \|u\|_{L^\infty(\mathbb{R}^n)} \alpha(n) r^n$$

$$= \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0, \quad r \to \infty$$

$\hookrightarrow$ $Du \equiv 0 \to u = \text{Const.}$

**Remark** (Stronger Result): If

$$\frac{|u(x)|}{|x|} \to 0 \quad \text{as} \quad |x| \to \infty$$

$\hookrightarrow$ $u \equiv \text{Const.}$

$\hookrightarrow$ A nonconstant harmonic function in all $\mathbb{R}^n$ must grow at least linearly at $\infty$.

**Ex.** Linear functions are harmonic

**Theorem 3.7** (Analyticity)

If $u$ is harmonic in $\Omega$

$\hookrightarrow$ $u$ is analytic in $\Omega$

(See Evans's book, Pages 31-32)

**Theorem 3.8** (Harnack's Ineq. - Simplest version).

If $u$ is harmonic and nonnegative in $B_r$

$$\hookrightarrow \begin{cases} \max\limits_{B_{\frac{r}{2}}} u \le C \min\limits_{B_{\frac{r}{2}}} u \\ c \not\sim r \end{cases}$$

$\Leftrightarrow$ The values of $u$ are all comparable in the concentric half-ball.

Proof (from Han, §4.2).

$\because u \ge 0.$

$$\hookrightarrow \frac{\partial u}{\partial x_i}(x_0) \overset{MVP}{=} \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} u \cdot \nu_i \, dS \le \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} |u|$$

$$= \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} u \, dS \overset{MVP}{=\!=} \frac{c}{\rho} u(x_0)$$

Taking $\rho = \frac{r}{2}$.

$\hookrightarrow$

$$\begin{cases} |\nabla u(x)| \leq \frac{C_1}{r} u(x) & \forall\, x \in B_{\frac{r}{2}} \\ C_1 = C_1(n) \quad \cancel{\propto} \; r \end{cases}$$

W.O.L.G. we may assume $u > 0$

(Otherwise, consider $u + \varepsilon$ instead of $u$, passing to limit $\varepsilon \to 0$ at the end of argt.)

$\hookrightarrow$ $|\nabla \log u(x)| \leq \frac{C_1}{r}$ in $B_{\frac{r}{2}}$

$\forall\, x, y \in B_{\frac{r}{2}}$,

$$\hookrightarrow \log \frac{u(x)}{u(y)} = \int_0^1 \frac{d}{ds} \log u(sx + (1-s)y)\, d$$

$$= (x-y) \cdot \int_0^1 \nabla \log u(sx + (1-s)y)\, ds$$

$$\leq |x-y| \int_0^1 |\nabla \log u(sx + (1-s)y)\, ds$$

$$\leq \frac{C_1}{r} |x-y| \leq C_1$$

$$\hookrightarrow \boxed{\frac{u(x)}{u(y)} \leq e^{C_1}} \qquad \square$$

Rm: <u>General Harnack Inequality</u>

For any connected open set $V \subset\subset \Omega$,

$\hookrightarrow$ $\exists$ $C = C(n, V) > 0$ s.t.

$$\sup_V u \leq C \inf_V u$$

for all harmonic function $u \geq 0$ in $\Omega$

$\hookrightarrow$ $$\frac{1}{C} u(y) \leq u(x) \leq C u(y) \quad \forall x, y \in V \subset\subset \Omega$$

* Since $V$ is a positive distance away from $\partial\Omega$, there is "room for the averaging effects of Laplace's Eq. to occur"!

# §2 Fundamental Solutions / Green's Function

## §2.1. Fundamental Solutions

Oberv**a**tion $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is $\boxed{\text{rotational invariant}}$

$y = Ox$ with $OO^T = O^TO = I$.

$\hookrightarrow$ $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$

$\hookrightarrow$ $\boxed{\text{Symmetry}}$

$\hookrightarrow$ Seek radial solutions

$u(x) = v(r)$, $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$

$\Delta u = 0$

$\hookrightarrow$

$$v''(r) + \frac{n-1}{r} v'(r) = 0$$

$\hookrightarrow$

$$v(r) = \begin{cases} b \log r + c & n = 2 \\ \frac{b}{r^{n-2}} + c & n \geq 3. \end{cases}$$

$b, c$ are arbitrary constants

# Fundamental Solution

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}, & n=3 \end{cases} \qquad \forall x \in \mathbb{R}^n,\ |x| \neq 0$$

$\hookrightarrow$

$$\begin{cases} -\Delta\Phi = \delta_0 & \text{in } \mathbb{R}^n \\ |D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, & x \neq 0 \\ |D^2\Phi(x)| \leq \frac{C}{|x|^n}, & x \neq 0. \end{cases}$$

$\longleftrightarrow \boxed{\int_{\partial B_r(0)} \frac{\partial\Phi}{\partial r}\, dS = -1}$

Set $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$

$\hookrightarrow$

$$\begin{cases} u \in C^2(\mathbb{R}^n) \\ -\Delta u = f & \text{in } \mathbb{R}^n \end{cases}$$

**Theorem 3.9** (Representation Formula).

Let $f \in C_c^2(\mathbb{R}^n)$.

If

$$\begin{cases} \dfrac{|u(x)|}{|x|} \to 0 \quad \text{as } |x| \to \infty \\ -\Delta u = f \end{cases}$$

Then $u$ has the form

$$\boxed{u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C \quad \forall x \in \mathbb{R}^n}$$

Proof. Set

$$w(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

$$\hookrightarrow \quad \frac{|w(x)|}{|x|} \to 0 \quad \text{as } |x| \to \infty$$

Define $v = u - w$

$$\hookrightarrow \quad \begin{cases} \Delta v = 0 \\ \dfrac{|v(x)|}{|x|} \to 0 \quad \text{as } |x| \to \infty \end{cases}$$

$\hookrightarrow$ $v = \text{const.}$ $\square$