IV. Heat Equation

$$u_t - \Delta u = 0$$

$$u_t - \Delta u = f(x,t) \qquad \text{Nonhomogeneous}$$

$f: \Omega \times [0,\infty) \to \mathbb{R}$ is given

$\Omega \subset \mathbb{R}^n$ bdd, $\Omega_T \triangleq \Omega \times [0,T]$

Cauchy Problem in $\mathbb{R}^n$

$$(CP) \begin{cases} u_t - \Delta u = f \\ u|_{t=0} = u_0(x) \end{cases}$$

Initial-Boundary Value Problem (IBVP).

$$(IBVP) \begin{cases} u_t - \Delta u = f \\ u|_{\partial\Omega} = g \\ u|_{t=0} = u_0(x) \end{cases}$$

§4.1. Uniqueness of Classical Solutions

Energy Method

Consider
$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$

Multiplying the equation by u

& Integration by parts

$\hookrightarrow$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx = 0$$

$\hookrightarrow$
$$\frac{d}{dt}\int_\Omega |u|^2 dx \leq 0$$

$\hookrightarrow$ Uniqueness

More Information

via Poincaré's Inequality

$\forall\ u \in C^1(\Omega),\quad u|_{\partial\Omega} = 0$

$\hookrightarrow$ $$\int_\Omega |u|^2 dx \leq C_\Omega \int_\Omega |\nabla u|^2 dx$$

Optimal value of C_Ω

$$\frac{1}{C_\Omega} = \min_{u|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} =$$ 1st Eigenvalue of the Laplacian with Dirichlet bdry condition

---

$\hookrightarrow$ $$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 = -\int_\Omega |\nabla u|^2 dx$$
$$\leq -\frac{1}{C_\Omega}\int_\Omega |u|^2 dx$$

$\hookrightarrow$ $$\int_\Omega |u|^2 dx \leq \int_\Omega |u_0|^2 dx\ e^{-\frac{2}{C_0}t}$$

$\xrightarrow[\text{exponentially}]{} 0$ as $t \to \infty$

Warning: The situation is different for the Neumann BC: $\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0$

<u>Theorem 4.1</u> The IBVP has at most one classical solution

$$u(x,t) \in C^{2,1}(\Omega_T) \cap C^{1,0}(\bar{\Omega})$$

$\boxed{C^2 \text{ in } x,\ C^1 \text{ in } t}$

* If $f = f(x)$, $g = g(x)$,

↳ The solution $u(x,t)$ of (IBVP)

$$u(x,t) \xrightarrow[\text{exponentially}]{} u_\infty(x), \quad t \to \infty$$

where $u_\infty(x)$ is the solution of the Stationary Problem $\begin{cases} -\Delta u_\infty = f \\ u_\infty|_{\partial\Omega} = g \end{cases}$

<u>Homework</u> What is analogue of the preceding in case of Neumann data.? $\begin{cases} -\Delta u_\infty = f \\ \frac{\partial u_\infty}{\partial \nu}\Big|_{\partial\Omega} = g \end{cases}$

Maximum Principle (Weak form)

Assume that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$

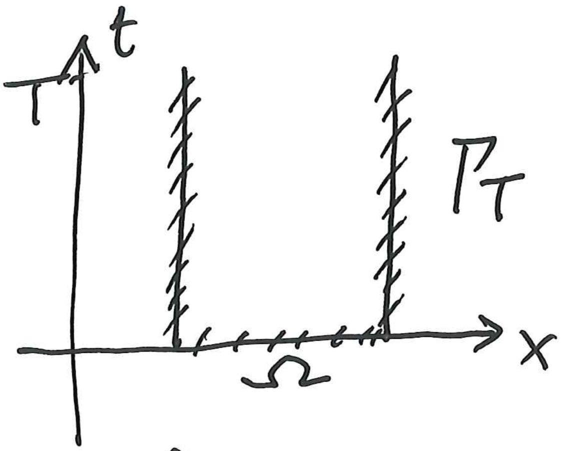
solves $u_t - \Delta u = 0$ in Ω_T.

↳ $$\boxed{\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u}$$

$\Gamma_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\})$

The parabolic boundary

(exclude the top $\Omega \times \{t=T\}$)



↳ $$\boxed{\min_{\overline{\Omega_T}} u = \min_{\Gamma_T} u}$$

Proof.

1. First assume $u_t - \Delta u < 0$ in Ω

At the interior max. pt.

$$u_t = 0, \quad Du = 0, \quad \Delta u \le 0.$$

At the max. pt. with $t = T$:

$$u_t \ge 0, \quad Du = 0, \quad \Delta u \le 0$$

↳ $u_t - \Delta u \ge 0$ Contradiction

2. If we only know $u_t - \Delta u \le 0$ in Ω

$\forall \varepsilon > 0$, Consider

$$u^{\varepsilon}(x,t) = u(x,t) - \varepsilon t.$$

$$\hookrightarrow \quad u^{\varepsilon}_t - \Delta u^{\varepsilon} < 0$$

$$\xrightarrow{\text{step 1}} \quad \max_{\Omega_T}(u - \varepsilon t) = \max_{\Gamma_T}(u - \varepsilon t)$$

$$\xrightarrow{\varepsilon \to 0} \quad \max_{\Omega_T} u = \max_{\Gamma_T} u$$

3. Similarly, apply the preceding result to $-u$

$$\hookrightarrow \quad \min_{\Omega_T} u = \min_{\Gamma_T} u$$

---

* The same arguments apply to

$$u_t - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} = f$$

$(a_{ij}(x,t))_{n\times n} \ge 0, \quad a_{ij}(x,t) = a_{ji}(x,t)$

§4.2. Fundamental Solution & Cauchy Problem

A. Fundamental Solution

Observation If $u(x,t)$ satisfies the HE

↳ So does $u(\lambda x, \lambda^2 t)$, $\forall \lambda \in \mathbb{R}$

↳ Seek a solution with the form

$$u(x,t) = v\left(\frac{|x|^2}{t}\right). \qquad t>0,\ x \in \mathbb{R}^n$$

↳ The fundamental solution.

An Easier Way: Seek

$$u(x,t) = w(t)\, v\left(\frac{r^2}{t}\right), \qquad r = |x|$$

$$\begin{aligned} \hookrightarrow 0 &= u_t - \Delta u \\ &= w'(t)\, v\left(\frac{r^2}{t}\right) \\ &\quad - \frac{w(t)}{t}\left[\underbrace{v''\left(\frac{r^2}{t}\right)\frac{4r^2}{t} + v'\left(\frac{r^2}{t}\right)\frac{r^2}{t}}_{\shortparallel} + v'\left(\frac{r^2}{2}\right)2n\right] \end{aligned}$$

Choose $v(s)$ s.t.

$$4s\,v''(s) + s\,v'(s) = 0$$

$$\Downarrow$$

$$4\,v''(s) + v'(s) = 0$$

$\hookrightarrow$ $\boxed{v(s) = e^{-\frac{s}{4}}}$

$\hookrightarrow$ $W'(t) + \frac{n}{2}\frac{W(t)}{t} = 0$

$\hookrightarrow$ $\boxed{W(t) = t^{-\frac{n}{2}}}$

$$\Longrightarrow \quad u(x,t) = \frac{a}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} + b \quad \forall a, b \in \mathbb{R}$$

Fundamental Solution

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0 \\ 0, & x \in \mathbb{R}^n, t < 0 \end{cases}$$

- $\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1$
- $\begin{cases} \Phi_t - \Delta \Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi|_{t=0} = \delta_0(x) & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$

B. Cauchy Problem

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = g(x) \end{cases}$$

Observation

$(x,t) \longrightarrow \Phi(x,t)$ solves the HE for $t>0$

$(x,t) \longrightarrow \Phi(x-y,t)$ solves the HE for $t>0$, $\forall y$

$\overset{?}{\hookrightarrow}$ $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy, \quad x \in \mathbb{R}^n, \ t>0$

Theorem 4.2 Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

$\hookrightarrow$ $$\boxed{u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy}$$

Satisfies

(i) $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$

(ii) $u_t - \Delta u = 0$

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0), \quad \forall x^0 \in \mathbb{R}^n$

Rm ① Infinite Propagation speed $\Leftarrow$ $\boxed{\text{If } \exists K \subset \mathbb{R}^n,\ |K|>0,\ g|_K > 0 \hookrightarrow u(x,t)>0}$

② If $|g| \le M \implies |u(x,t)| \le M$.

C. Nonhomogeneous Cauchy Problem

$$(*)\quad \begin{cases} u_t - \Delta u = f, & \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = 0, \end{cases}$$

Observation

- $\Phi(x,t)$ solves the HE

  $\hookrightarrow$ So does $\Phi(x-y, t-s)$, $\forall\, y \in \mathbb{R}^n$, $0<s<t$

- Fix $s \in (0,t)$.

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\,dy$$

Solves

$$(*)_s\quad \begin{cases} u_t(x,t;s) - \Delta u(x,t;s) = 0, & t>s \\ u(x,s;s) = f(x,s) \end{cases}$$

Duhamel's Principle

Build a solution of $(*)$ out of $(*)_s$

$$(\triangle)\quad u(x,t) = \int_0^t u(x,t;s)\,ds$$

$$= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\,dy\,ds$$

$$= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy\,ds$$

- Duhamel's Principle has wide applications to linear PDEs.

General Nonhomogeneous Cauchy Problem

$$\begin{cases} u_t - \Delta u = f & \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = g \end{cases}$$

The solution has the form:

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds$$

$$= \int_0^t \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy + \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{(t-s)}} f(y,s)\, dy\, ds.$$

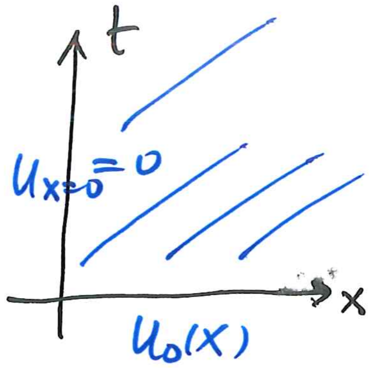
Linearity of the problem

$$* \quad u(x,t) = e^{t\Delta} g(\cdot) + \int_0^t e^{(t-s)\Delta} f(\cdot, s)\, ds.$$

? Half-space ?

Ex 1

$$\begin{cases} u_t - u_{xx} = 0 & x>0 \\ u|_{x=0} = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$



$$\hookrightarrow \quad u(x,t) = \int_0^\infty \underbrace{G(x,y,t)}_{\substack{\| \\ \Phi(x-y,t) - \Phi(x+y,t)}} u_0(y)\,dy$$

$\boxed{\text{Odd reflection}}$

Ex 2

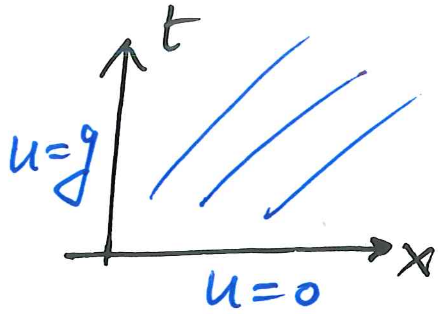
$$\begin{cases} u_t - u_{xx} = 0, & x>0 \\ u_x|_{x=0} = 0 \\ u|_{t=0} = u_0(x) \end{cases}$$

$$\hookrightarrow \quad u(x,t) = \int_0^\infty \underbrace{N(x,y,t)}_{\substack{\| \\ \Phi(x-y,t) + \Phi(x+y,t)}} u_0(y)\,dy$$

$\boxed{\text{Even reflection}}$

Ex 3

$$\begin{cases} u_t - u_{xx} = 0 \quad \text{for } x > 0 \\ u|_{x=0} = g(t) \\ u|_{t=0} = 0 \end{cases}$$



$$\begin{cases} u(x,t) = \displaystyle\int_0^t \frac{\partial G}{\partial y}(x, 0, t-s)\, g(s)\, ds. \\ G(x,y,t) = \Phi(x-y,t) - \Phi(x+y,t) \end{cases}$$

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi}\,(t-s)^{3/2}}\, e^{-\frac{x^2}{4(t-s)}}\, g(s)\, ds$$

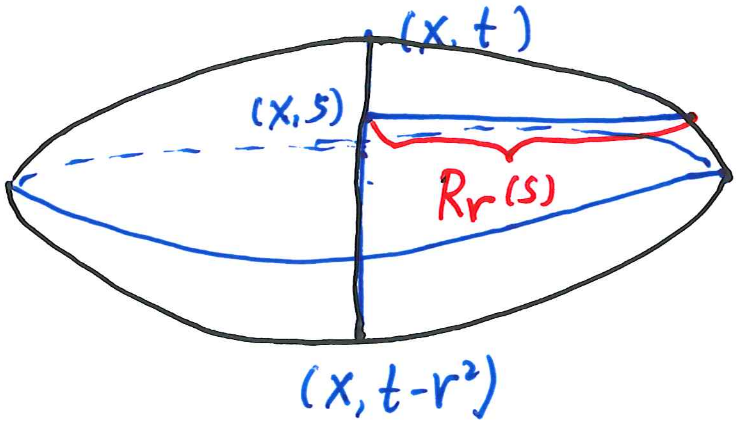
* **Homework**

§4.3 Mean-Value Formula & Properties of Solutions.

Heat ball: Fix $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \Phi(x-y,\, t-s) \geq \dfrac{1}{(4\pi)^{\frac{n}{2}} r^n} \\ s \leq t \end{array} \right\}$$

$$= \left\{ (y,s) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} t-r^2 < s < t \\ |x-y| \leq \underbrace{R_r(s)}_{\boxed{\left[-2n(t-s)\log\left(\frac{t-s}{r^2}\right)\right]^{\frac{1}{2}}}} \end{array} \right\}$$



Note

$$\iint_{E(x,t;r)} \frac{|x-y|^2}{|t-s|^2}\, dy\, ds = 2^{n+2} \pi^{\frac{n}{2}} r^n$$

$$\parallel$$

$$\iint_{E(0,0;r)} \frac{|y|^2}{s^2}\, dy\, ds$$

Theorem 4.3 (Mean-Value Formula).

Let $u \in C^{2,1}(\Omega_T)$ solve the HE.

$\hookrightarrow$

$$u(x,t) = \frac{1}{2^{n+2}\pi^{\frac{n}{2}} r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dy\, ds$$

$$\forall \; E(x,t;r) \subset \Omega_T$$

For the proof, see Evans, §2.3.2, Theorem 3

Theorem 4.4 (Strong Maximum Principle)

Assume $u \in C^{2,1}(\Omega_T) \cap C(\bar{\Omega}_T)$ solves the HE in Ω_T

$\hookrightarrow$

(i). $\max\limits_{\bar{\Omega}_T} u = \max\limits_{\Gamma_T} u$

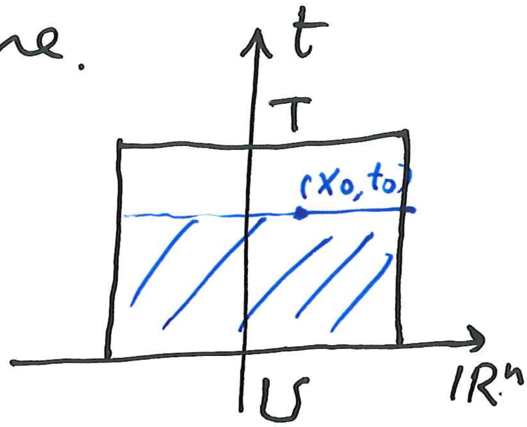
(ii). If Ω_T is connected, and $\exists\,(x_0,t_0) \in \Omega_T$

s.t. $u(x_0,t_0) = \max\limits_{\bar{\Omega}_T} u$

$\hookrightarrow$ $u \equiv \text{const.}$ in $\overline{\Omega_{t_0}}$

(iii) Similarly for $\min\limits_{\bar{\Omega}_T}$.

Remarks $\boxed{1.}$ If u attains its maximum (or minimum) at an interior point, then u must be constant at all earlier time.



(i) The initial and boundary conditions must be the same constant when $t \in [0, t_0]$.

(ii). The solution may change when $t > t_0$, provided that the boundary conditions alter after t_0.

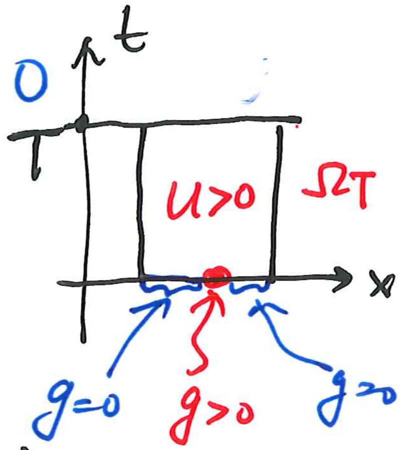
For the proof, see Evans, §2.3.2. Theorem 4.

Remarks (Conti).

2. Infinite Propagation Speed.

Consider $\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u|_{\partial\Omega \times (0,T]} = 0 \\ u|_{t=0} = g \geq 0, & \max g > 0 \end{cases}$

$\xrightarrow{\text{MP}}$ $\begin{cases} \min\limits_{\overline{\Omega}_T} u = 0. \\ \max\limits_{\overline{\Omega}_T} u = \max\limits_{t=0} g > 0 \end{cases}$



$\xrightarrow{\text{SMP}}$ $\boxed{u > 0 \text{ in } \Omega_T}$

3. Uniqueness on bdd domain

Let $g \in C(\Gamma_T)$, $f \in C(\Omega_T)$.

$\Rightarrow$ $\exists$ at most one solution $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ of the IBVP:

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega_T \\ u|_{\Gamma_T} = g \end{cases}$$

Theorem 4.5 (Maximum Principle for the Cauchy Problem)

$u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

$$(*) \quad \begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0,T) \\ u|_{t=0} = g \end{cases}$$

& satisfies the growth condition:

$$(**) \quad \boxed{u(x,t) \leq A e^{a|x|^2},} \quad x \in \mathbb{R}^n, t \in [0,T]$$

for some constants $A, a > 0$.

$$\Longrightarrow \boxed{\sup_{\mathbb{R}^n \times [0,T]} u(x,t) = \sup_{\mathbb{R}^n} g(x)}$$

$\Longrightarrow$ Uniqueness for the Cauchy Problem

Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$

Then $\exists$ at most one solution

$u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of

the Cauchy Problem. $\begin{cases} u_t - \Delta u = f, & \mathbb{R}^n \times (0,T) \\ u|_{t=0} = g \end{cases}$

Satisfying the growth condition (**)

Remark 1 For the Cauchy Problem (*) with $g \in C(\mathbb{R}^n)$.

↳ $$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

is the only bounded solution.

Remark 2. In general, there are ∞-many solutions of

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^n \times (0,T) \\ u|_{t=0} = 0. \end{cases}$$

each of which (besides $u \equiv 0$) grows very rapidly as $|x| \to \infty$

$$\left(u \sim A e^{a|x|^{2+\varepsilon}} \right)$$

↳ The growth Condition (**) provides a criterion that excludes the "wrong solutions".

Theorem 4.6 (Smoothness)

Suppose $u \in C^{2,1}(\Omega_T)$ solves the HE in Ω_T. $\Longrightarrow$ $\boxed{u \in C^\infty(\Omega_T).}$

Remark. Theorem 4.6 is valid even if u is not smooth on the boundary and the initial time

$\hookrightarrow$ Infinite propagation speed.

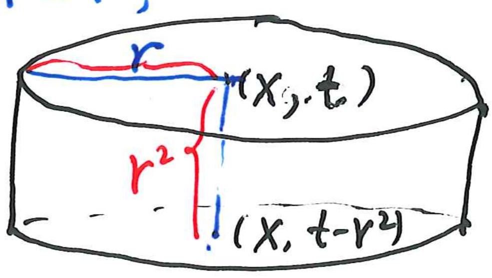
Theorem 4.7 (Estimates on derivatives)

$\forall\ k, l = 0, 1, \cdots, \quad \exists\ C_{k,l} > 0$, s.t.

$$\boxed{\max_{C(x,t;\frac{r}{2})} |D_x^k D_t^l u| \le \frac{C_{k,l}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))}}$$

for all cylinders $C(x,t;\frac{r}{2}) \subset C(x,t;r)$ and all solution u of the HE in Ω_T where

$$C(x,t;r) = \{(y,s) \mid |x-y| \le r,\ t-r^2 \le s \le t\}$$



Remark. Let u solve the HE in Ω_T

↳ for fixed t,

$x \longmapsto u(x,t)$ is analytic

However, for fixed x,

$t \longmapsto u(x,t)$ is NOT analytic in general.

Homework: Read § 2.3.2 – 2.3.3 in Evans's book

- Most results shown above are valid for general linear Parabolic Equations

$$\boxed{u_t - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} = f}$$

with

$$\begin{cases} a_{ij}(x,t) = a_{ji}(x,t) \\ \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \lambda_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \end{cases}$$

V. Wave Equation

$$u_{tt} - \Delta u = 0 \qquad x \in \mathbb{R}^n,\ t \geq 0$$

$$u_{tt} - \Delta u = f \qquad \text{Nonhomogeneous}$$

$n=1$: Vibrating string

$n=2$. Vibrating membrance

$n=3$. Elastic Solid

$\Omega \subset \mathbb{R}^n$ open.

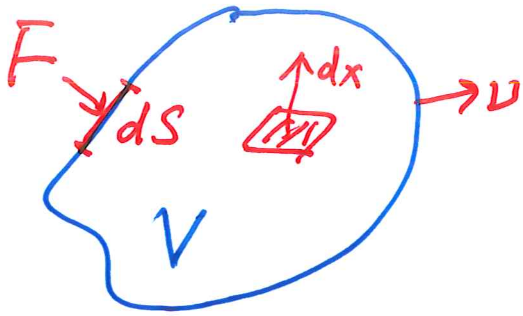
$\Omega_T = \Omega \times (0, T]$

$\Gamma_T = (\partial\Omega \times (0, T]) \times (\Omega \times \{t=0\})$

$u(x,t)$ — Displacement in some direction of the point $x \in \mathbb{R}^n$ at time $t \geq 0$.

Derivation $\forall V \subset \Omega$

$u(x,t)$ — Displacement at (x,t)



Acceleration within V

$$\int_V u_{tt}\,dx$$

ρ — density

Net Contact Force

$$-\int_{\partial V} F\cdot\nu\,dS$$

(the force acting on V through ∂V)

Newton's Second Law:

(mass) x (acceleration) = (net force)

$$\int_V \rho\, u_{tt}\,dx = -\int_{\partial V} F\cdot\nu\,dS = -\int_V \operatorname{div} F\,dx$$

Green's Thm

$\forall V \subset \Omega$

$$\boxed{\rho\, u_{tt} + \operatorname{div} F = 0}$$

For elastic bodies

$F = f(Du) \approx -c^2 Du$ when $|Du| \ll 1$.

$$u_{tt} - a^2 \Delta u = 0 \qquad a^2 = \frac{c^2}{\rho}$$

The wave Eq. when $a = 1$

§5.1 Energy Methods

A. Uniqueness for Wave Eq.

$\exists$ at most one solution $u \in C^2(\bar{\Omega}_T)$

solving $\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T \\ u|_{\Gamma_T} = g \\ u_t|_{t=0} = h \end{cases}$

Proof. If $\hat{u}$ is another such solution, then $w := u - \hat{u}$ solves

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } \Omega_T \\ w|_{\Gamma_T} = 0 \\ w_t|_{t=0} = 0 \end{cases}$$

Define the energy

$$e(t) := \frac{1}{2}\int_\Omega \left(w_t^2(x,t) + |Dw(x,t)|^2 \right) dx \qquad 0 \le t \le T$$

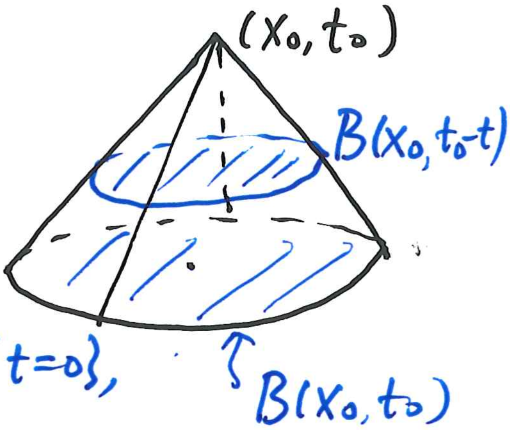
$\hookrightarrow$ $$\frac{de(t)}{dt} = \int_\Omega (w_t w_{tt} + Dw \cdot Dw_t)\, dx$$
$$= \int_\Omega w_t (w_{tt} - \Delta w)\, dx + \int_{\partial\Omega} \underbrace{w_t}_{0} Dw \cdot \nu \, dS$$
$$= 0$$

$\hookrightarrow$ $e(t) = 0 \quad 0 \le t \le T \implies Dw = w_t = 0 \xrightarrow{w|_\Gamma = 0} w \equiv 0$

B. Domain of Dependence

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$. Define the cone:

$$C = \{(x,t) \mid |x - x_0| \le t_0 - t,\ 0 \le t \le t_0\}$$



Finite Propagation Speed

If $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$,

$\hookrightarrow$ $$u|_C \equiv 0$$

That is, any "disturbance" originating outside $B(x_0, t_0)$ has no effect on the solution within C.

<u>Proof</u>. Define

$$e(t) := \frac{1}{2} \int_{B(x_0, t_0 - t)} \left(u_t^2(x,t) + |Du(x,t)|^2 \right) dx$$

$0 \le t \le t_0$

Then

$$\frac{de(t)}{dt} = \int_{B(x_0, t_0-t)} (u_t u_{tt} + Du \cdot Du_t)\, dx$$

$$- \frac{1}{2} \int_{\partial B(x_0, t_0-t)} (u_t^2 + |Du|^2)\, dS$$

$$= \int_{B(x_0,t_0-t)} u_t (\underbrace{u_{tt} - \Delta u}_{=0})\, dx$$

$$+ \int_{\partial B(x_0, t_0-t)} \frac{\partial u}{\partial \nu} u_t \, dS$$

$$- \frac{1}{2} \int_{\partial B(x_0,t_0-t)} (u_t^2 + |Du|^2)\, dS$$

$$= \int_{\partial B(x_0, t_0-t)} \Big(\underbrace{\frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2}_{\leq 0} \Big)\, dS$$

$$\left(\left|\frac{\partial u}{\partial \nu} u_t\right| \leq |u_t|\,|Du| \leq \frac{1}{2} u_t^2 + \frac{1}{2} |Du|^2 . \right)$$

$$\leq 0$$

$\Rightarrow$ $e(t) \leq e(0) = 0 \qquad \forall\ 0 \leq t \leq t_0.$

$\hookrightarrow$ $u_t = 0,\ Du \equiv 0$

$\xhookrightarrow[u|_{t=0}=0]{}$ $u \equiv 0$

§5.2 Method of Spherical Means

Scaling approach

$(x, t) \longrightarrow (\alpha x, \alpha t), \ \forall \alpha$

Invariant for the WE.

↳ Seek for solutions with the form

$u(x,t) = v\left(\frac{r}{t}\right), \quad r = |x|$

↳ Works, but it is complicated for large n.

Alternative Approach

↳ The method of spherical means.

$\Downarrow$

The multi-D problem $\Rightarrow$ 1-D problem

A. Solution for $n=1$

Cauchy Problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & \mathbb{R}\times(0,\infty) \\ u|_{t=0} = g \\ u_t|_{t=0} = h \end{cases}$$

Factorization Idea

$$(\partial_t + \partial_x)(\partial_t - \partial_x)u = u_{tt} - u_{xx} = 0$$

Set $v(x,t) = (\partial_t - \partial_x)u(x,t)$

$\hookrightarrow$ $$\begin{cases} v_t + v_x = 0 \\ v|_{t=0} = \underset{\substack{\parallel \\ u_t(x,0) - u_x(x,0)}}{\underline{S(x)}} = h(x) - g'(x) \end{cases}$$

$\hookrightarrow$ $v(x,t) = S(x-t)$

$\Rightarrow$ $$\begin{cases} u_t - u_x = S(x-t) \\ u|_{t=0} = g(x) \end{cases}$$ (Nonhomogeneous transport equation)

$\hookrightarrow$ $$u(x,t) = \int_0^t S(x+(t-\tau)-\tau)\,d\tau + g(x+t)$$

$$= \frac{1}{2}\int_{x-t}^{x+t} S(y)\,dy + g(x+t)$$

D'Alembert's Formula

$$(*) \quad \boxed{\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} \left(h(y) - g'(y)\right) dy + g(x+t) \\ &= \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy \end{aligned}}$$

Remarks

① Another approach: The above fractorization suggests:

$$\begin{cases} \xi = x+t \\ \eta = x-t \end{cases}$$

$\hookrightarrow \quad u_{\xi\eta} = 0$

$$\hookrightarrow \quad \begin{cases} u(x,t) = F(x+t) + G(x-t) \\ u|_{t=0} = g, \quad u_t|_{t=0} = h \end{cases}$$

$\Longrightarrow (*)$

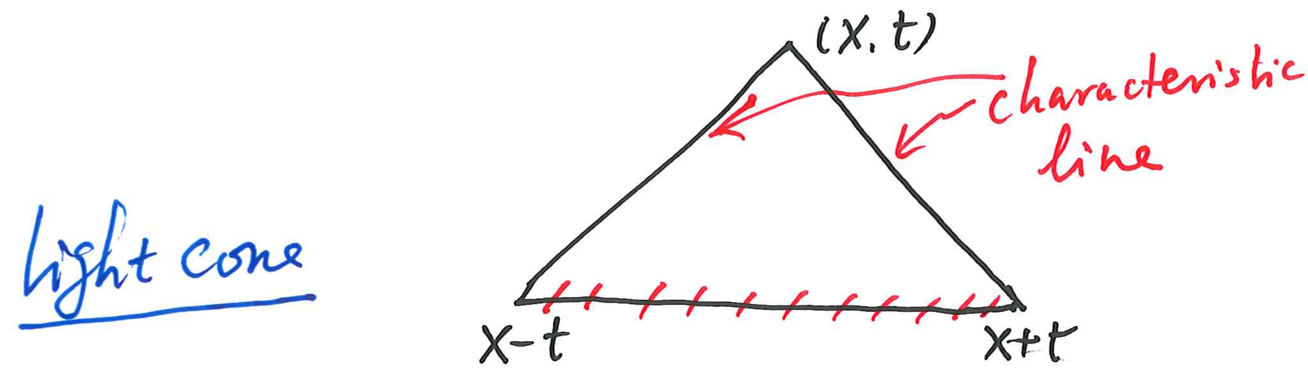
② $g \in C^k,\ h \in C^{k-1} \Longrightarrow u \in C^k$

But $u \notin C^{k+1}$ in general.

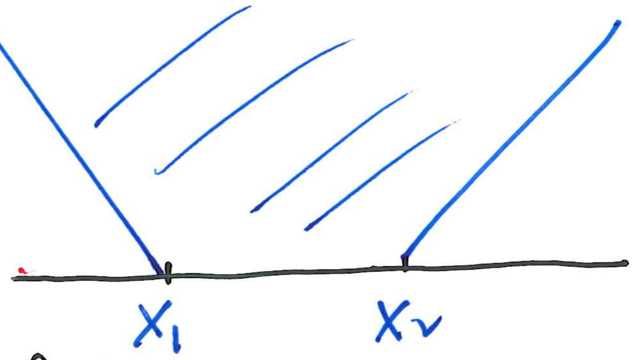
* The wave equation does NOT yield instantaneous smoothing from the initial data (different from the heat eq.)

Remarks (Conti.)

③ Domain of dependence of (x, t).



Domain of influence of (x_1, x_2)



④ Reflection Method

$$\begin{cases} u_{tt} - u_{xx} = 0, & \{x>0\}\times\{t>0\} \\ u|_{t=0} = g, \quad u_t|_{t=0} = h \\ u|_{x=0} = 0 \end{cases}$$

By the odd reflection

$$\begin{cases} u_{tt} - u_{xx} = 0, & \{x>0\}\times\{t>0\} \\ u|_{t=0} = g, \quad u_t|_{t=0} = h \\ u_x|_{x=0} = 0 \end{cases}$$

By the even reflection

B. Spherical Means $(n \geq 2)$

Let $u \in C^k(\mathbb{R}^n \times [0,\infty))$, $k \geq \frac{n+3}{2}$,

solve

$$(*) \begin{cases} u_{tt} - \Delta u = 0, & \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = g, \quad u_t|_{t=0} = h \end{cases}$$

Fix $x \in \mathbb{R}^n$, $t > 0$, $r > 0$.

(i) Define

$$\begin{cases} U(r,t;x) \triangleq \displaystyle⨍_{\partial B(x,r)} u(y,t)\,dS_y = ⨍_{\partial B(0,1)} u(x+rz,t)\,dS_z \\ G(r;x) \triangleq \displaystyle⨍_{\partial B(x,r)} g(y)\,dS_y \\ H(r;x) \triangleq \displaystyle⨍_{\partial B(x,r)} h(y)\,dS_y \end{cases}$$

(ii) Define U, G, H for $r \leq 0$ by even reflection

$$\begin{cases} U(r,t;x) = U(-r,t;x) & r \leq 0 \\ G(r;x) = G(-r,t;x) & r \leq 0 \\ H(r;x) = H(-r,t;x) & r \leq 0 \end{cases}$$

$$\begin{cases} U(0,t;x) = u(x,t) \\ G(0;x) = g(x), \quad H(0;x) = h(x) \end{cases}$$

Theorem 5.1. (Euler-Poisson-Darboux Eq.)

↳ For fixed $x \in \mathbb{R}^n$,

$U(r, t; x)$ as a function of (r, t) satisfies

$$U \in C^k(\mathbb{R} \times [0, \infty))$$

&

$$\begin{cases} \underbrace{U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0}_{\text{E-P-D}}, & \mathbb{R} \times (0, \infty) \\ U|_{t=0} = G \\ U_t|_{t=0} = H \end{cases}$$

↲ By direct calculation

C. Solution for odd $\boxed{n = 2k+1,\ k \geq 1.}$ by the method of spherical means

Define

$$
\begin{cases}
\tilde{U}(r,t) \triangleq \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(r,t;x)\right) \\
\tilde{G}(r) \triangleq \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}G(r;x)\right) \\
\tilde{H}(r) \triangleq \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}H(r;x)\right)
\end{cases}
$$

They are all odd w.r.t r ; $\tilde{U}|_{r=0} = 0$

$\hookrightarrow$ $\tilde{U}(r,t)$ satisfies

$$
\begin{cases}
\tilde{U}_{tt} - \tilde{U}_{xx} = 0 \\
\tilde{U}|_{t=0} = \tilde{G}(r), \quad \tilde{U}_t|_{t=0} = \tilde{H}(r)
\end{cases}
$$

$\hookrightarrow$

$$
\boxed{\tilde{U}(r,t) = \frac{1}{2}\left[\tilde{G}(r+t) + \tilde{G}(r-t)\right] + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(y)\,dy \qquad \forall\, r \in \mathbb{R},\ t \geq 0}
$$

On the other hand,

$$\tilde{U}(r,t) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{k-1}U(r,t;x)\right)$$

$$= \underbrace{(2k-1)!!}_{=\ \boxed{1\cdot 3\cdot 5\cdots(2k-1)}}\, r\, U(r,t;x) + \sum_{j=1}^{k-1} \beta_j^k\, r^{j+1}\frac{d^j}{dr^j}U(r,t;x)$$

$$\hookrightarrow \lim_{r\to 0}\frac{\tilde{U}(r,t)}{(2k-1)!!\, r} = \lim_{r\to 0} U(r,t;x) = u(x,t)$$

$$\|$$

$$\frac{1}{(2k-1)!!}\lim_{r\to 0}\left[\frac{\tilde{G}(t+r)-\tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_t^{t+r}\tilde{H}(y)\,dy + \frac{1}{2r}\int_{r-t}^{-t}\tilde{H}(y)\,dy\right]$$

$\tilde{G}(r)$, $\tilde{H}(r)$ Odd functions

$$\|$$

$$\frac{1}{\gamma_n}\left[\tilde{G}'(t) + \tilde{H}(t)\right]$$

where $\gamma_n = (2k-1)!! = 1\cdot 3\cdot 5\cdots(n-2)$

Representation Formula for odd Dimension n

$$(\star)\quad \begin{cases} u(x,t) = \dfrac{1}{\gamma_n}\left[\left(\dfrac{\partial}{\partial t}\right)\left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\displaystyle\fint_{\partial B(x,t)} g\, ds\right) + \left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\displaystyle\fint_{\partial B(x,t)} h\, ds\right)\right] \\ \gamma_n = 1\cdot 3\cdot 5 \cdot \cdots \cdot (n-2) \end{cases}$$

Theorem 5.2 Assume $n \geq 3$ is an integer.

$g \in C^k(\mathbb{R}^n)$, $h \in C^{k-1}(\mathbb{R}^n)$, $k \geq \frac{n+3}{2}$

$\Rightarrow$ $u = u(x,t)$ defined by $(\star)$ satisfies

(i) $u \in C^2(\mathbb{R}^n \times [0,\infty))$

(ii) $u_{tt} - \Delta u = 0$

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x^0)$ $\quad \forall x^0 \in \mathbb{R}^n$

$\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\ t>0}} u_t(x,t) = h(x^0)$.

Case: $n=3$

$\hookrightarrow$ $\gamma_3 = 1$

$$u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{|y-x|=t} g(y)\, dS_y\right) + t \fint_{|y-x|=t} h(y)\, dS_y$$

$$= t\frac{\partial}{\partial t} \fint_{\partial B(0,1)} g(x+tz)\, dS_z + \fint_{|y-x|=t} h(y)\, dS_y$$

$$= \fint_{\partial B(x,t)} Dg(y)\cdot \frac{y-x}{t}\, dS + \fint_{|y-x|=t} h(y)\, dS_y$$

$\hookrightarrow$

$$\boxed{u(x,t) = \fint_{\partial B(x,t)} \big(t h(y) + g(y) + Dg(y)\cdot(y-x)\big)\, dS_y \qquad \forall\, x\in\mathbb{R}^3,\ t>0}$$

Kirchhoff's formula.

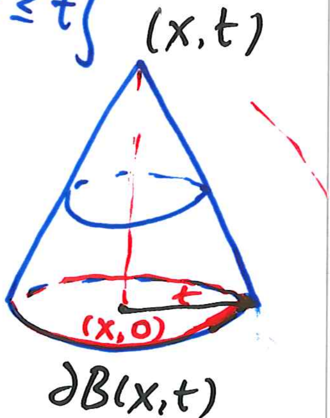
Remarks

1. From Kirchhoff's formula, the solution $u(x,t)$ is determined completely by the initial data g, h, Dg on the sphere $\partial B(x,t) = \{(y,t) \mid |y-x| = t\}$, NOT on the entire ball $B(x,t)$.

Domain of Dependence $\boxed{\partial B(x,t)}$

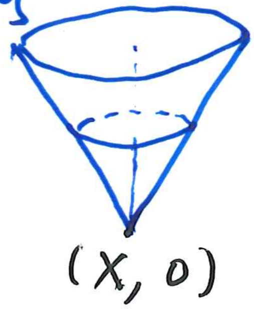
Cone of dependence $\boxed{\partial C(x,t)}$

$C(x,t) = \{(y,\tau) \mid |y-x| \le t-\tau,\ 0 \le \tau \le t\}$



Domain of Influence

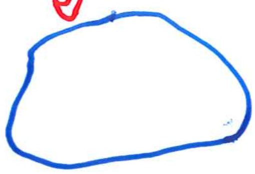
The initial disturbance at $(x_0, 0)$ affects the solution u Only on the boundary $\{(y,t) \mid |y-x_0| = t,\ t>0\}$ of the cone $\{(y,t) \mid |y-x_0| < t,\ t>0\}$



Remarks (Conti.)

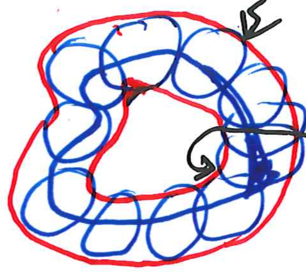
2. Phenomenon of Finite Propagation speed of the initial disturbance

perturbation



$t=0$

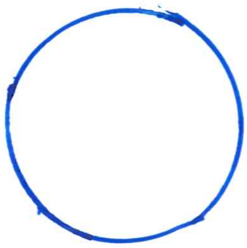
Sharp wave front



- A clean leading edge and ending edge.

$t>0$

$t=0$



$t>0$

3. $\underline{n=1}$ $\quad u \sim g$

$\underline{n=3}$ $\quad u \sim g, Dg, h$

Loss of regularity:

$u \in C^2 \Leftarrow$ Require more smoothness assumption on g and h for $n \geq 3$ odd.

D. Solution for even n by the Method of Descent

$u \in C^k$, $k \geq \frac{n+3}{2}$, solves the WE

Idea: Formulate the n-D problem as the $(n+1)$-D problem

$$\bar{x} = (\underset{\in \mathbb{R}^n}{x}, x_{n+1}) \in \mathbb{R}^{n+1}$$

$\bar{u}(x, x_{n+1}, t) \equiv u(x,t)$ is a solution of

$$\begin{cases} \bar{u}_{tt} - \Delta_{\bar{x}} \bar{u} = 0 & \text{in } \mathbb{R}^{n+1} \times (0,\infty) \\ \bar{u}|_{t=0} = g(x) \\ \bar{u}_t|_{t=0} = h(x) \end{cases}$$

$\hookrightarrow$

$$u(x,t) \equiv \bar{u}(\bar{x},t) = \frac{1}{\gamma_{n+1}} \left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} ⨍_{\partial \bar{B}(\bar{x},t)} g(y)\, dS_{\bar{y}}\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} ⨍_{\partial \bar{B}(\bar{x},t)} h(y)\, dS_{\bar{y}}\right) \right]$$

$$\gamma_{n+1} = 1 \cdot 3 \cdots (n-1).$$

Theorem 5.3 $n \geq 2$ even integer.

- $g \in C^k(\mathbb{R}^n)$, $h \in C^{k-1}(\mathbb{R}^n)$, $k \geq \frac{n+3}{2}$
- $$u(x,t) \triangleq \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right)\right]$$

$\gamma_n = 2\cdot 4 \cdots (n-2)\, n$

$\Rightarrow$

(i) $u \in C^2(\mathbb{R}^n \times (0,\infty))$

(ii) $u_{tt} - \Delta u = 0$. $\mathbb{R}^n \times (0,\infty)$

(iii) $$\left\{\begin{array}{l} \displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\, t>0}} u(x,t) = g(x^0) \\ \displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ x\in\mathbb{R}^n,\, t>0}} u_t(x,t) = h(x^0) \end{array}\right.$$ $\forall\, x^0 \in \mathbb{R}^n$

Case $n=2$

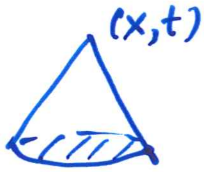
$\hookrightarrow \gamma_2 = 2$

$$u(x,t) = \frac{1}{2}\left[\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right) + t^2 \fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right]$$

$\hookrightarrow$

$$\boxed{u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{t\,g(y) + t^2 h(y) + t\,Dg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy}$$

— Poisson's Formula



Remarks

1. Domain of Dependence. $B(x,t) = \{|y-x| \le t\}$
   (NOT $\partial B(x,t)$)

2. Domain of Influence. $C(x,t) = \{|y-x| \le t,\ t>0\}$
   (Not $\partial C(x,t)$)

$\hookrightarrow$ In even dimension, the initial disturbance continue to have effects even after the leading edge of the wavefront passes.

$\nexists$ a clear ending edge of the wavefront.

§5.3 Nonhomogeneous Problem
Duhamel's Principle

$$(\ast)\quad\begin{cases} u_{tt} - \Delta u = f & \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = g(x), \quad u_t|_{t=0} = h(x) \end{cases}$$

$$\Updownarrow \qquad \underline{u = u_1 + u_2}$$

$$\underbrace{\begin{cases} u_{tt} - \Delta u = 0 \\ u|_{t=0} = g(x), \quad u_t|_{t=0} = h(x) \end{cases}}_{u_1} \oplus \underbrace{\begin{cases} u_{tt} - \Delta u = f(x,t) \\ u|_{t=0} = u_t|_{t=0} = 0 \end{cases}}_{u_2}$$

Duhamel's Principle

$u(x,t;s)$ solves $\begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0 \quad t>s \\ u|_{t=s} = 0, \quad u_t|_{t=s} = f(x,s) \end{cases}$

$$\Updownarrow \qquad \boxed{\tau = t - s}$$

$$\begin{cases} u_{\tau\tau} - \Delta u = 0, & \tau > 0 \\ u|_{\tau=0} = 0, \quad u_\tau|_{\tau=0} = f(x, t-\tau) \end{cases}$$

Set

$$(*) \quad \boxed{u_2(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n,\ t>0}$$

Theorem 5.4: $\begin{cases} f \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \times (0,\infty)) \\ u_2 \text{ is defined by } (*) \end{cases}$

$\Longrightarrow$ (i) $u_2 \in C^2(\mathbb{R}^n \times [0,\infty))$

(ii) u_2 solves $\begin{cases} u_{tt} - \Delta u = f(x,t) \\ u|_{t=0} = u_t|_{t=0} = 0 \end{cases}$

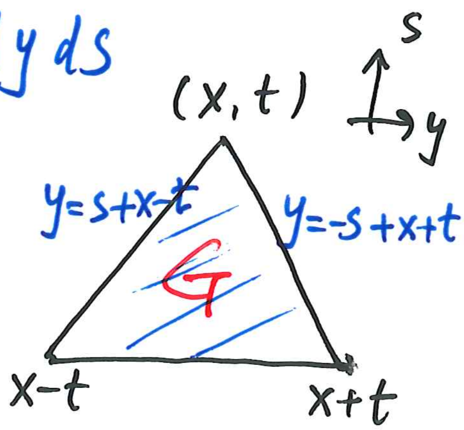
Case $n=1$:

$$u(x,t;s) = \frac{1}{2}\int_{x-\tau}^{x+\tau} f(y,s)\,dy = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)\,dy$$

$\hookrightarrow$

$$u_2(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dy\,ds$$

$$= \frac{1}{2}\iint_G f(y,s)\,dy\,ds$$



Case $n=1$ (Conti.)

Solution of $(*)$ for $n=1$:

$$\boxed{u(x,t) = \frac{1}{2}\big(g(x+t)+g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dy\,ds}$$

Case $n=3$

$$u(x,t;s) = (t-s) \fint_{\partial B(x,t-s)} f(y,s)\,dS_y$$

$$u_2(x,t) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy$$

— retarded potential

Solution of $(*)$ for $n=3$:

$$u(x,t) = \fint_{\partial B(x,t)} \big[t\,h(y) + g(y) + Dg(y)\cdot(y-x)\big]\,dS_y + \frac{t^3}{3}\fint_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy$$

Case $n=2$

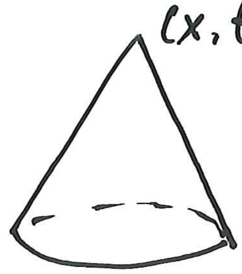
$$u(x,t;s)=\frac{1}{2}\tau^2 \fint_{B(x,\tau)} \frac{f(y,s)}{\sqrt{\tau^2-|y-x|^2}}\,dy$$

$$=\frac{1}{2}(t-s)^2 \fint_{B(x,t-s)} \frac{f(y,s)}{\sqrt{(t-s)^2-|y-x|^2}}\,dy$$

$$=\frac{1}{2\pi}\int_{B(x,t-s)} \frac{f(y,s)}{\sqrt{(t-s)^2-|y-x|^2}}\,dy$$

$$u_2(x,t)=\frac{1}{2\pi}\int_0^t\int_{B(x,t-s)} \frac{f(y,s)}{\sqrt{(t-s)^2-|y-x|^2}}\,dy\,ds$$

$$=\frac{1}{2\pi}\iiint_C \frac{f(y,s)}{\sqrt{(t-s)^2-|y-x|^2}}\,dy\,ds$$



$$C=\{(y,s)\mid |y-x|\le t-s,\ 0\le s\le t\}$$

Solution of $(*)$ for $n=2$:

$$u(x,t)=\frac{1}{2}\fint_{B(x,t)} \frac{t\,g(y)+t^2h(y)+t\,Dg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy$$

$$+\frac{1}{2\pi}\iiint_C \frac{f(y,s)}{\sqrt{(t-s)^2-|y-x|^2}}\,dy\,ds$$

I. Linear Second-Order Wave Eqs.

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u = f(x)$$

II. Nonlinear Wave Eqs

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(Du, u, x)\, u_{x_i x_j} + B(Du, u, x) = 0$$

$$u_{tt} - \operatorname{div} A(Du, u, x) + B(Du, u, x) = 0$$