VI. Nonlinear First-Order PDE

General form

$$\boxed{F(Du, u, x) = 0.}$$ $x \in \Omega \subset \mathbb{R}^n$ open

$F: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ Given

- Nonlinearity $\Rightarrow$ No simple formula for solutions
- Remarkably, we can often employ calculus to glean fairly detailed information about solutions
- Examples

① Clairaut's Eq. in differential geometry

$$x \cdot Du + f(Du) = u$$

② The eikonal Eq. in Geometric optics

$$|Du| = 1$$

③ Hamilton-Jacobi Eq. $\quad u_t + H(Du) = 0$

④ Conservation Laws. $\quad u_t + \operatorname{div} f(u) = 0$

$f: \mathbb{R} \to \mathbb{R}^n$ is given

§6.1. Quasilinean Equations

§6.1.1. The Case of 2 Independent Variables

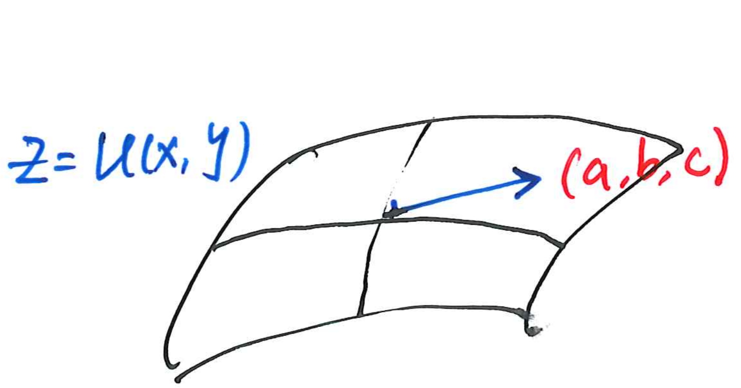
$$(*) \quad \boxed{a(u,x,y)\,u_x + b(u,x,y)\,u_y = c(u,x,y)}$$

Integral Surfaces: $u=u(x,y)$ can be represented by a surface $z=u(x,y)$ in the xyz-space:

$\Updownarrow$

The surfaces that are tangent to the direction of the vector (a,b,c) at each point on the surfaces.

$$(u_x, u_y, -1)\cdot \underbrace{(a,b,c)}_{\text{Characteristic direction}} = 0$$



Characteristic Curves:

At each point, the curves are tangent to the characteristic direction (a, b, c)

↳ Along a characteristic curve

$$\boxed{\frac{dx}{a(z,x,y)} = \frac{dy}{b(z,x,y)} = \frac{dz}{c(z,x,y)} = dt}$$

$\Updownarrow$

Characteristic ODE
$$\begin{cases} \dfrac{dx}{dt} = a(z,x,y) \\ \dfrac{dy}{dt} = b(z,x,y) \\ \dfrac{dz}{dt} = c(z,x,y) \end{cases}$$

If $a, b, c \in C^1$.

↳ $\boxed{\text{ODE Theory}}$ $\underline{\exists}$ $\underline{1}$ $(x(t), y(t), z(t))$ through each point (x_0, y_0, z_0) for any $t \in (-\delta, \delta)$, for some $\delta > 0$.

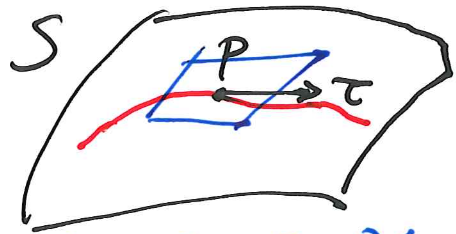
Theorem 6.1

$$\boxed{\text{Surface } S: z = u(x,y) \text{ formed as a union of characteristic curves}}$$

$$\Updownarrow$$

$$\boxed{S: z = u(x,y) \text{ is an integral surface}}$$

Proof



" $\Rightarrow$ " $\forall P \in S$,

$\exists$ a characteristic curve $\gamma \subset S$, $P \in \gamma$.

$\hookrightarrow$ The tangent $\tau = (a, b, c)$ to γ at P lies in the tangent plane of S at P.

$\hookrightarrow$ $\tau = (a, b, c)$ is the Char. direction

$\hookrightarrow$ S is the integral surface

Proof (Conti.)

" $\Leftarrow$ " **claim**: $\forall P = (x_0, y_0, \underset{\overset{\parallel}{u(x_0,y_0)}}{z_0}) \in S$

$\hookrightarrow$ $\underline{\exists\ 1}$ characteristic curve

$\gamma = (x(t), y(t), z(t))$

with $(x, y, z)|_{t=t_0} = (x_0, y_0, z_0)$

$\hookrightarrow$ $\underline{\gamma}$ lies completely on $\underline{S}$

$\Downarrow$

> Through every point $P \in S$, there passes a characteristic curve contained in S.

Proof of Claim. $\gamma = (x(t), y(t), z(t))$

with

$$\begin{cases} \dfrac{dx}{dt} = a(z, x, y), \\ \dfrac{dy}{dt} = b(z, x, y), \\ \dfrac{dz}{dt} = c(z, x, y), \\ (x, y, z)|_{t=t_0} = (x_0, y_0, z_0) \end{cases}$$

Let

$$\begin{cases} U(t) = z(t) - u(x(t), y(t)) \\ U(t_0) = 0 \quad \leftarrow \boxed{P \in S} \end{cases}$$

$\hookrightarrow$

$$\begin{cases} \dfrac{dU}{dt} = c(U + u(x,y), x, y) \\ \qquad - u_x(x,y)\, a(U + u(x,y), x, y) \\ \qquad - u_y(x,y)\, b(U + u(x,y), x, y) \\ U(t_0) = 0 \end{cases}$$

$\xrightarrow{\text{Uniqueness}}$ $U(t) \equiv 0$.

$\hookrightarrow$ $z(t) = u(x(t), y(t))$

$\hookrightarrow$ $\gamma \subset S$.

$\Rightarrow$ A general solution $u = u(x,y)$ of $(*)$ is the integral surface $z = u(x,y)$ formed as the union of characteristic curves.

Cauchy Problem

$$(\maltese)\ \begin{cases} a(u,x,y)\,u_x + b(u,x,y)\,u_y = c(u,x,y) \\ u(f(s), g(s)) = h(s) \end{cases}$$

where Γ: $x = f(s)$, $y = g(s)$, $z = h(s)$

is a curve in xyz-space

$\Longleftrightarrow$ $\boxed{\text{Seek an integral surface containing the given curve } \Gamma.}$

n.b.h.d

Let $\begin{cases} (f(s), g(s), h(s)) \in C^1(O(s_0)) \\ P_0 = (x_0, y_0, z_0) = (f(s_0), g(s_0), h(s_0)) \\ a(z,x,y),\ b(z,x,y),\ c(z,x,y) \in C^1(O(P_0)) \end{cases}$

Theorem 6.2 $\exists\, 1$ solution of $(*)$

$$u(x,y) \in C^1(O(x_0,y_0))$$

provided that

$$J := \begin{vmatrix} f'(s_0) & g'(s_0) \\ a(z_0,x_0,y_0) & b(z_0,x_0,y_0) \end{vmatrix} \neq 0$$

Proof 1. Local Existence

Consider

$$\begin{cases} \dfrac{dx}{dt} = a(z,x,y), \\ \dfrac{dy}{dt} = b(z,x,y), \\ \dfrac{dz}{dt} = c(z,x,y), \\ (x,y,z)\big|_{t=0} = (f(s), g(s), h(s)) \end{cases}$$

$\xRightarrow{\text{ODE Theory}}$ $\exists\, 1$ solution

$$\begin{cases} x = X(s,t), \\ y = Y(s,t), \\ z = Z(s,t). \end{cases}$$

s.t. $(X(s,t), Y(s,t), Z(s,t)) \in C^1(O(s_0,0))$

$\forall\ (x, y) \in O(x_0, y_0)$ n.b.h.d of (x_0, y_0)

Consider

$$\begin{cases} X(s, t) = x, \\ Y(s, t) = y, \\ X(s_0, 0) = x_0, \\ Y(s_0, 0) = y_0, \end{cases}$$

Since $J = \begin{vmatrix} X_s(s_0, 0), & Y_s(s_0, 0) \\ X_t(s_0, 0), & Y_t(s_0, 0) \end{vmatrix} \neq 0.$

$\xrightarrow[\text{function Thm}]{\text{Implicit}}$ $\exists\, 1.\ (S(x, y), T(x, y)) \in C^1(O(x_0, y_0))$ s.t.

$$\begin{cases} s = S(x, y), \\ t = T(x, y), \\ (S(x_0, y_0), T(x_0, y_0)) = (s_0, 0) \end{cases}$$

Define

$$u = u(x, y) = z(S(x, y), T(x, y))$$

$\hookrightarrow$ u is a solution of (*).

2. <u>Uniqueness</u>: Follows from <u>Claim</u>

Any integral surface through Γ has to contain the characteristic curve through the points of Γ, hence has to contains

$$\Sigma: \quad x = X(s,t), \quad y = Y(s,t), \quad z = Z(s,t).$$

$\hookrightarrow$ locally has to be identical with the surface Σ

$\square$

Remarks

① $J \neq 0$ is essential for the local existence

② Special Case: Linear PDE

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y).$$

↳ The system of 3 characteristic ODEs reduces to

$$\begin{cases} \dfrac{dx}{dt} = a(x,y) \\ \dfrac{dy}{dt} = b(x,y) \end{cases} \quad \text{or} \quad \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

Called Characteristic projections

or more commonly, "Characteristics"

i.e. projections onto the xy-plane of the characteristic curves in the xyz-space.

Examples

1. $$\begin{cases} u_y + c\,u_x = 0, \\ u|_{y=0} = h(x) \end{cases}$$

Initial curve Γ: $x = s,\ y = 0,\ z = h(s)$

Char. ODE

$$\begin{cases} \dfrac{dx}{dt} = c, \\ \dfrac{dy}{dt} = 1, \\ \dfrac{dz}{dt} = 0, \\ (x, y, z)\big|_{t=0} = (s, 0, h(s)) \end{cases}$$

$$\Rightarrow \begin{cases} x = X(s,t) = ct + s \\ y = Y(s,t) = t \\ z = Z(s,t) = h(s) \end{cases}$$

$$\left.\begin{matrix} x = ct + s \\ y = t \end{matrix}\right] \rightarrow s = x - cy$$

$$\Rightarrow \quad z = h(x - cy)$$

$$\Rightarrow \quad \boxed{u(x,y) = h(x - cy)}$$ Solution

Examples (Conti).

2. $\begin{cases} u_y + u u_x = 0, \\ u|_{y=0} = h(x) \in C^1 \end{cases}$

Initial curve Γ: $x = s,\ y = 0,\ z = h(s)$.

Char. ODE
$$\begin{cases} \dfrac{dx}{dt} = z, \\ \dfrac{dy}{dt} = 1, \\ \dfrac{dz}{dt} = 0, \\ (x, y, z)|_{t=0} = (s, 0, h(s)) \end{cases}$$

$$\Rightarrow \begin{cases} x = X(s,t) = s + h(s)t \\ y = Y(s,t) = t \\ z = Z(s,t) = h(s) \end{cases}$$

$$\begin{cases} t = y \\ s + h(s)y = x \end{cases} \xrightarrow{|t| \ll 1} s = S(x,y)$$

$$\Rightarrow \boxed{u(x,y) = h(S(x,y))}$$

$$\boxed{u = h(x - uy)}$$

Nonlinearity

§6.1.2. The Case of n Indept. Variables

$$\boxed{\sum_{j=1}^{n} a_j(u, x_1, \cdots, x_n)\, u_{x_j} = c(u, x_1, \cdots, x_n) \qquad n \geq 3}$$

Characteristic Curves

$$\begin{cases} \dfrac{dx_j}{dt} = a_j(z, x_1, \cdots, x_n), & j = 1, 2, \cdots, n \\ \dfrac{dz}{dt} = c(z, x_1, \cdots, x_n). \\ x_j\big|_{t=0} = f_j(s_1, \cdots, s_{n-1}) \\ z\big|_{t=0} = h(s_1, \cdots, s_{n-1}) \end{cases}$$

where $u(f_1, \cdots, f_n) = h$ (Initial condition)

$\xrightarrow[\text{Theory}]{\text{LODE}}$

$$\begin{cases} x_j = X_j(s_1, \cdots, s_{n-1}, t), & j = 1, 2, \cdots, n \\ z = Z(s_1, \cdots, s_{n-1}, t) \end{cases}$$

If $J = \begin{vmatrix} \frac{\partial f_1}{\partial s_1}, & \cdots, & \frac{\partial f_n}{\partial s_1} \\ - & - & - \\ \frac{\partial f}{\partial s_{n-1}} & \cdots, & \frac{\partial f_n}{\partial s_{n-1}} \\ a_1, & \cdots, & a_n \end{vmatrix}_{(s_1^0, \cdots, s_{n-1}^0, 0)} \neq 0$

$\overset{\text{IFT}}{\Longrightarrow}$

$$\begin{cases} s_j = S_j(x_1, \cdots, x_n), & j = 1, \cdots, n-1 \\ t = T(x_1, \cdots, x_n) \end{cases}$$

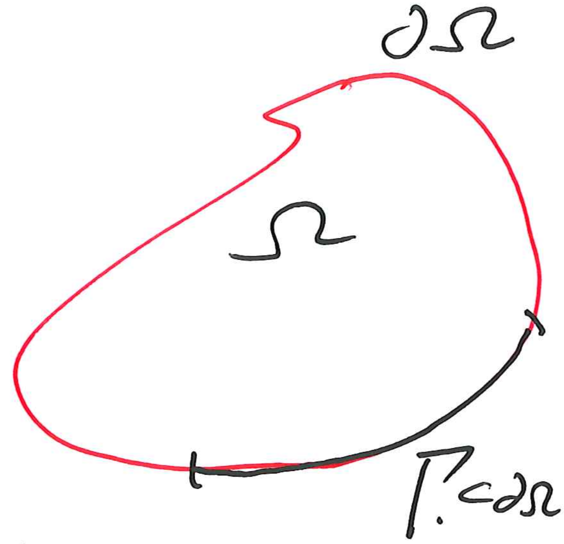
$$\hookrightarrow \boxed{u(x) = Z\big(S_1(x), \cdots S_{n-1}(x), T(x)\big)}$$

is a solution of the problem

$$\begin{cases} \sum_{j=1}^{n} a_j(u, x)\, u_{x_j} = C(u, x) \\ u(f_1, f_2, \cdots, f_n) = h \end{cases}$$

§6.2 General Nonlinear First-Order PDE.

$$(*) \quad \boxed{F(Du, u, x) = 0}$$



<u>Problem</u>. Find solutions of $(*)$ in Ω, subject to the boundary condition

$$u|_{\Gamma} = g(x)$$

Read ① §3.1 – §3.3 in Evans's book

② Chapter 2 in Courant-Hilbert's book, Vol. II

VII. Introduction to Conservation Laws

$$\begin{cases} u_t + F(u)_x = 0 & \mathbb{R} \times (0,\infty) \\ u|_{t=0} = g \end{cases}$$

$\begin{cases} F: \mathbb{R} \longrightarrow \mathbb{R} \\ g: \mathbb{R} \longrightarrow \mathbb{R} \end{cases}$ are given

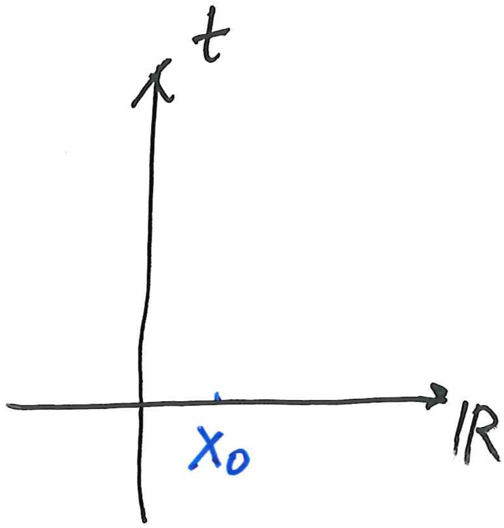
$u: \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R}$ is the unknown

§7.1 Method of Characteristics

$$u_t + F'(u)\, u_x = 0, \qquad u|_{t=0} = g(x)$$

Characteristic ODEs

$$\begin{cases} \dfrac{dx}{d\tau} = F'(z(\tau)), \\[2mm] \dfrac{dt}{d\tau} = 1, \qquad \longleftrightarrow \quad \tau = t \\[2mm] \dfrac{dz(\tau)}{d\tau} = 0, \end{cases}$$



$$\hookrightarrow \begin{cases} \dfrac{dx(t)}{dt} = F'(z(t)), \\[2mm] \dfrac{dz(t)}{dt} = 0, \\[2mm] (x, z)|_{t=0} = (x_0, g(x_0)) \end{cases}$$

$$\hookrightarrow \begin{cases} z(t) = g(x_0), \\ x(t) = F'(g(x_0))\, t + x_0 \end{cases}$$

$$\hookrightarrow u(x(t), t) = z(t) = g(x(t) - tF'(u(x(t), t)))$$

$$\hookrightarrow \boxed{u = g(x - tF'(u))}$$

$$\boxed{u = g(x - tF'(u))}$$

Implicit Function Thm

If $\boxed{1 + t\,g'(x - tF'(u)) \cdot F''(u) \neq 0}$

$\hookrightarrow$ $\exists\,1$ classical solution $u = u(x, t)$.

Ex1. $\begin{cases} F''(u) \geq 0 & \text{(Convex)} \\ g'(x) \geq 0 \end{cases}$

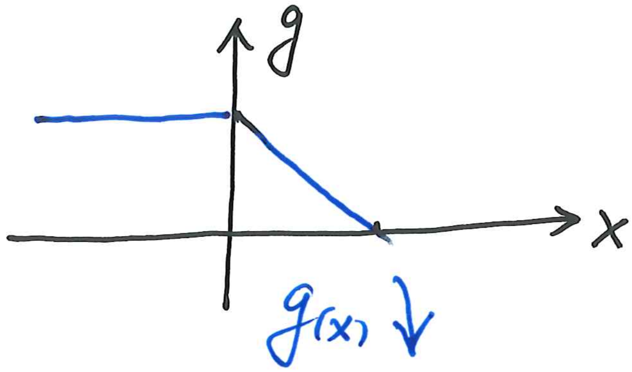
$\hookrightarrow$ The problem $\begin{cases} u_t + F(u)_x = 0, \\ u|_{t=0} = g(x) \end{cases}$

has a unique global classical solution

* But, in general, $\exists$ only local-in-time classical solution.

Ex 2 $\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, & \mathbb{R}\times(0,\infty) \\ u|_{t=0} = g(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & x \ge 1 \end{cases} \end{cases}$

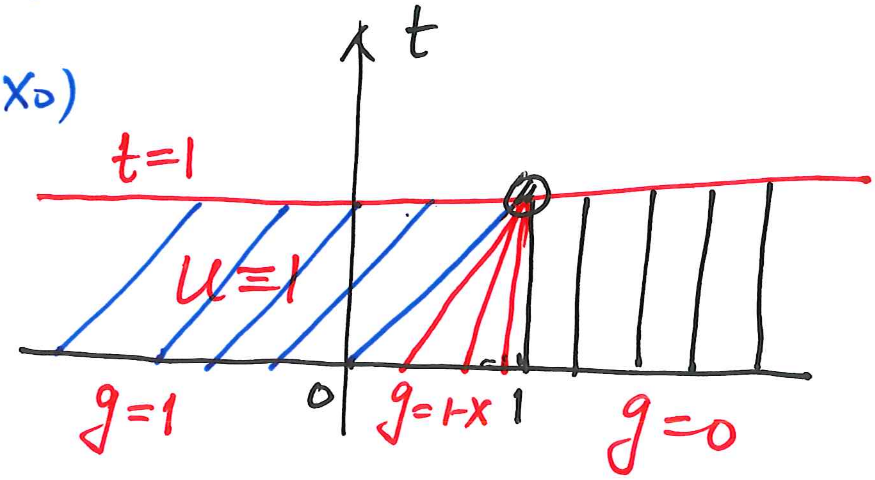
$F(u) = \frac{u^2}{2}$



$F'(u) = u$

↳ $\forall\, x_0 \in \mathbb{R}$

$$\begin{cases} x(t) = x_0 + g(x_0)\,t \\ u = z(t) = g(x_0) \end{cases}$$



$0 \le x_0 \le 1$: $\begin{cases} x(t) = x_0 + (1-x_0)t = x_0(1-t) + t \\ u = z(t) = 1 - x_0 \end{cases}$

↳ $x_0 = \frac{x-t}{1-t}$

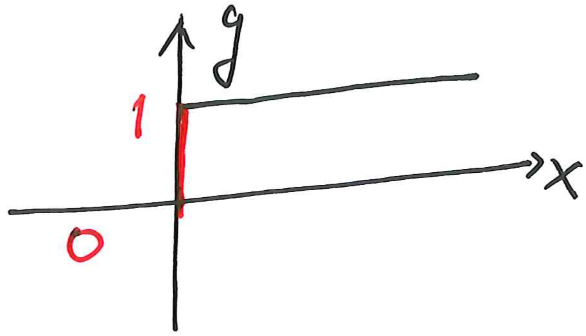
↳ $\boxed{u = 1 - \frac{x-t}{1-t} = \frac{1-t-(x-t)}{1-t} = \frac{1-x}{1-t}}$

↳

$$u(x,t) = \begin{cases} 1 & x \le t,\ 0 \le t \le 1 \\ \frac{1-x}{1-t} & t \le x \le 1,\ 0 \le t \le 1 \\ 0 & x \ge 1,\ 0 \le t \le 1. \end{cases}$$

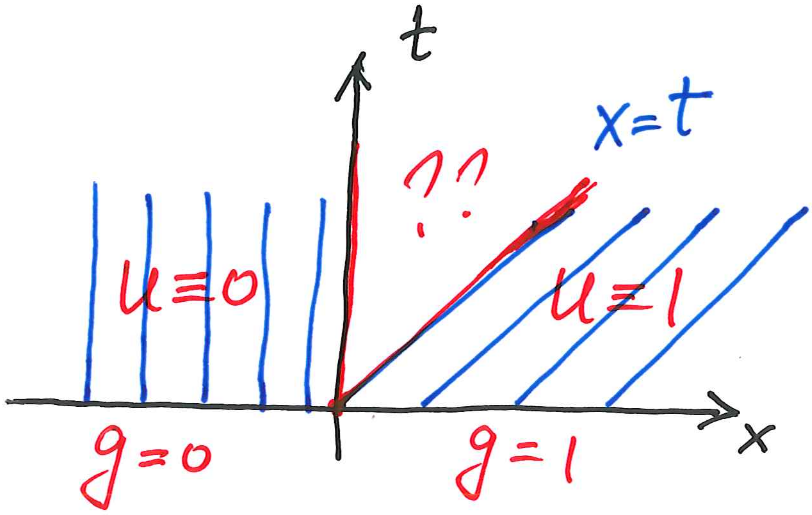
$\overset{?}{\Rightarrow}$ $\boxed{t > 1?}$??

Ex3
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \mathbb{R}\times(0,\infty), \\ u|_{t=0} = g(x) = \begin{cases} 0 & x<0, \\ 1 & x>0 \end{cases} \end{cases}$$



$\forall\, x_0 \in \mathbb{R}$

$$\begin{cases} x(t) = x_0 + g(x_0)t, \\ u = z(t) = g(x_0) \end{cases}$$



* No points of the sector $\{0<x<t\}$ can be reached by the characteristics originating from the x-axis and carrying the initial data.

— fails to provide any information within the sector $\{0<x<t\}$.

§7.2. Shocks, Entropy Condition

$$\begin{cases} u_t + F(u)_x = 0, \\ u|_{t=0} = g(x) \end{cases}$$

* PDE does not make sense unless u is differentiable

? $\exists$ a way to interpret a less regular function as "solving" the Cauchy problem ?

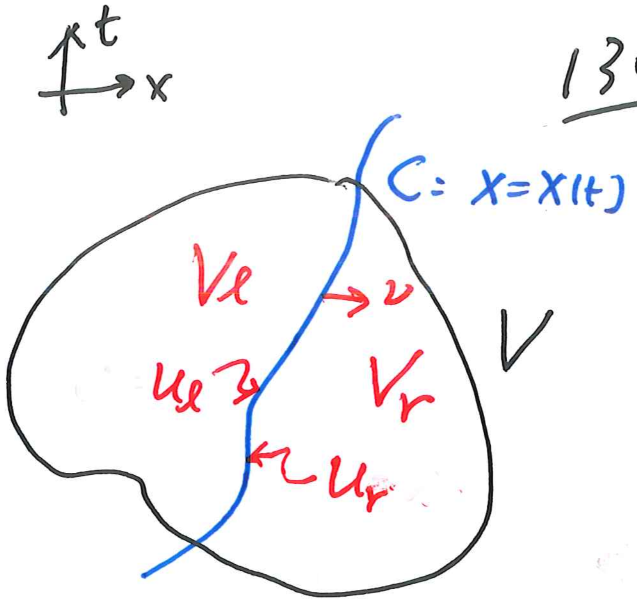
Integral Solutions (Weak Solutions)

$u \in L^\infty(\mathbb{R} \times (0,\infty))$:

$$(*) \quad \boxed{\int_0^\infty \int_{-\infty}^\infty (u \varphi_t + F(u)\varphi_x)\, dx\, dt + \int_{-\infty}^\infty g(x)\varphi(x,0)\, dx = 0}$$

for any test function $\varphi \in C_c^1(\mathbb{R} \times [0,\infty))$.

Q: What is the implication of this definition?



- $V \subset \mathbb{R}\times(0,\infty)$ open
- $V = V_\ell \cap V_r$
- $u = u(x,t)$ is smooth on either side of C

$\nu = (\nu^1, \nu^2) = \dfrac{(1, -X'(t))}{\sqrt{1+(X'(t))^2}}$

↳ $$\boxed{u_t + F(u)_x = 0 \qquad \forall (x,t) \in V_\ell \text{ or } V_r}$$

But the traces of u on the sides of C: $\boxed{u_\ell \neq u_r}$

$\forall \varphi \in C_c^1(V)$, i.e. $\operatorname{supp}\varphi \subset\subset V$.

$$0 = \int_0^\infty \int_{-\infty}^{\infty} (u\varphi_t + F(u)\varphi_x)\,dx\,dt$$

$$= \underbrace{\iint_{V_\ell} (u\varphi_t + F(u)\varphi_x)\,dx\,dt}_{\parallel} + \underbrace{\iint_{V_r} (u\varphi_t + F(u)\varphi_x)\,dx\,dt}_{\parallel}$$

$$-\iint_{V_\ell} (u_t + F(u))\varphi\,dx\,dt + \int_{\partial V_\ell} (u_\ell \nu^2 + F(u_\ell)\nu^1)\varphi\,ds \qquad -\int_{\partial V_r} (u_r \nu^2 + F(u_r)\nu^1)\varphi\,ds$$

$\hookrightarrow \displaystyle\int_C \left[(F(u_\ell)-F(u_r))\nu^1 + (u_\ell - u_r)\nu^2 \right] \varphi \, dS = 0$

$\hookrightarrow (F(u_\ell)-F(u_r))\nu^1 + (u_\ell - u_r)\nu^2 = 0$ along C

$$\hookrightarrow \boxed{x'(t)(u_\ell - u_r) = F(u_\ell) - F(u_r) \quad \text{along } C}$$

Rankine-Hugoniot Jump Condition along the discontinuity

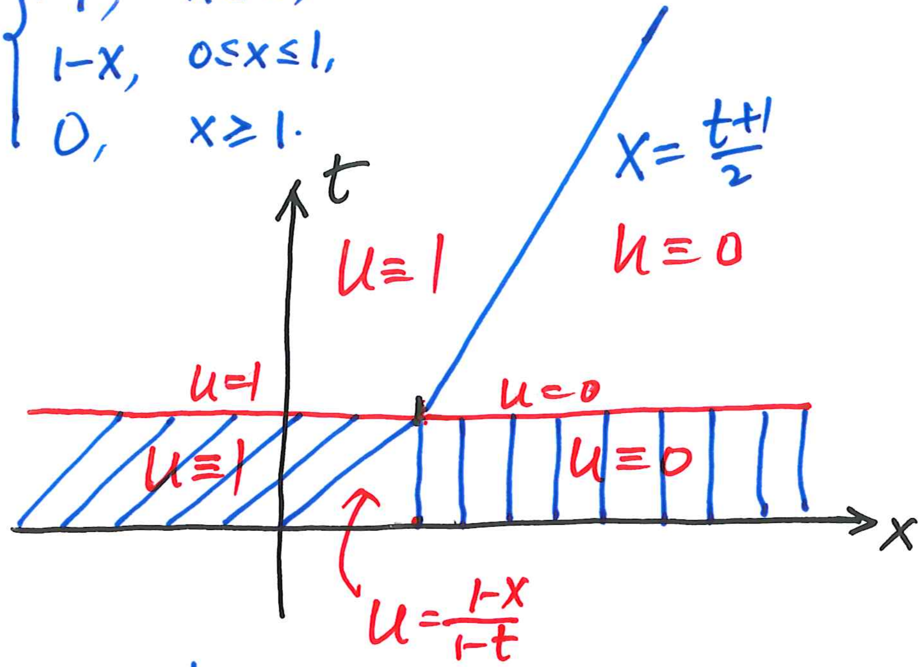
$$\boxed{\sigma[u] = [F(u)]}$$

$\sigma = x'(t)$: The speed of the discontinuity curve

$[u] = u_r - u_\ell$: Jump in u across C.

$[F(u)] = F(u_r) - F(u_\ell)$: Jump in $F(u)$ across C.

Ex 2. $\begin{cases} F(u) = \dfrac{u^2}{2} \\ g(x) = \begin{cases} 1, & x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & x \ge 1. \end{cases} \end{cases}$



$$\sigma = \frac{[F(u)]}{[u]} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2} = \frac{dx(t)}{dt}, \quad x(1) = 1,$$

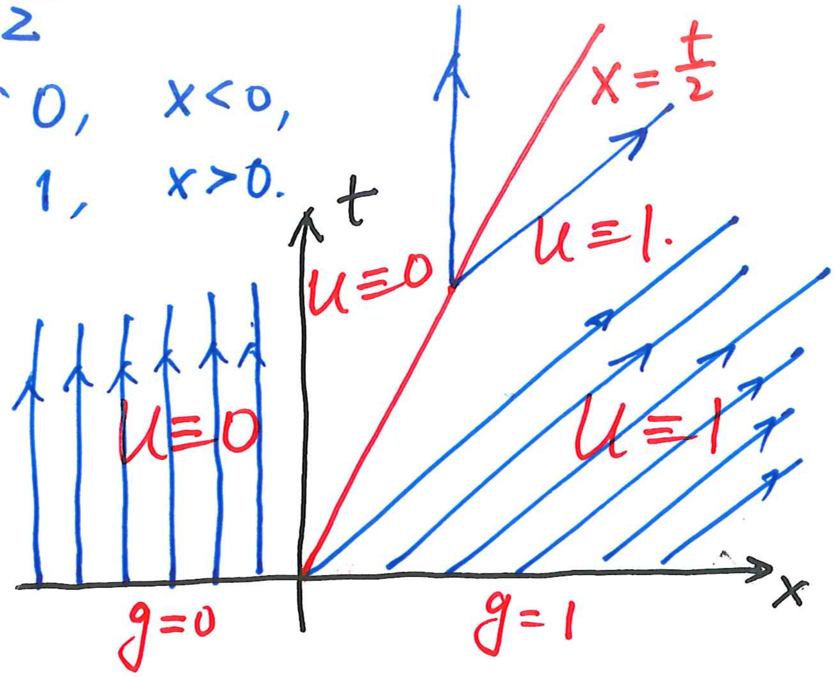
$\hookrightarrow$ $\boxed{x = x(t) = \dfrac{t+1}{2}}$

$$u(x,t) = \begin{cases} 1, & x < \frac{t+1}{2}, \\ 0, & x > \frac{t+1}{2}. \end{cases}$$

is an integral solution for $t > 1$

$\hookrightarrow$ global solution

Ex 3.
$$\begin{cases} F(u) = \dfrac{u^2}{2} \\ g(x) = \begin{cases} 0, & x<0, \\ 1, & x>0. \end{cases} \end{cases}$$

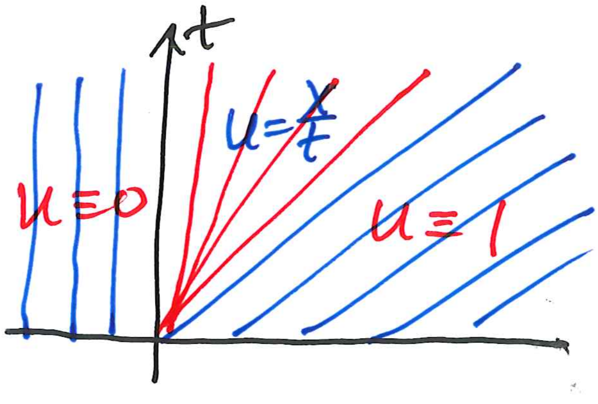


$$\begin{cases} \dfrac{dx}{dt} = \dfrac{[F(u)]}{[u]} = \dfrac{\frac{1}{2}-0}{1-0} = \dfrac{1}{2} \\ x(0)=0 \end{cases} \Longrightarrow \quad x = x(t) = \frac{1}{2}t$$

↳ $u_1(x,t) = \begin{cases} 0, & x<\frac{t}{2}, \\ 1, & x>\frac{t}{2} \end{cases}$ is an integral solution.

? physical ?

Another Solution



$$u_2(x,t) = \begin{cases} 1, & x>t, \\ \frac{x}{t}, & 0<x<t, \\ 0, & x<0 \end{cases}$$

is also an integral solution!!!

* Nonuniqueness of integral solutions!!

↳ Require additional criteria to single out the physically relevant solution.

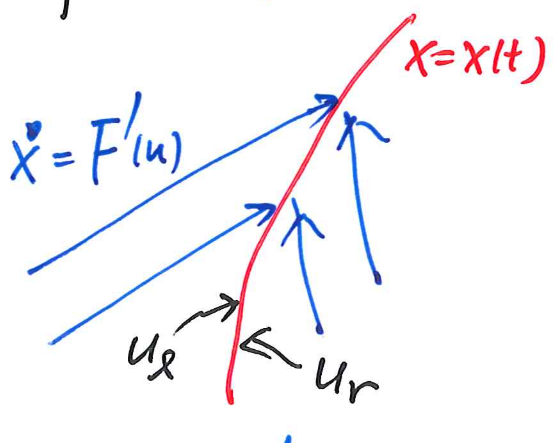
Entropy Condition, Entropy Solutions

Requirement (Stability): Entropy Solutions

Piecewise smooth integral solutions with the property that, if we move backwards in time t along any characteristic, we will never encounter any discontinuity curve for u.

$$\boxed{F'(u_\ell) > \sigma > F'(u_r)}$$

Lax Entropy Condition



Shocks: A curve of discontinuity for u satisfying $\begin{cases} \text{R-H Condition} \\ \text{Lax Entropy Condition} \end{cases}$

If $F''(u) > 0 \rightarrow F'(u) \nearrow u \Longrightarrow \boxed{u_\ell > u_r}$

Ex 2

$$u(x,t) = \begin{cases} 1 & x \le t,\ 0 \le t \le 1 \\ \frac{1-x}{1-t} & t \le x \le 1,\ 0 \le t \le 1 \\ 0 & x \ge 1 \quad 0 \le t \le 1 \\ \hline 1 & x < \frac{t+1}{2},\ t > 1 \\ 0 & x > \frac{t+1}{2},\ t > 1 \end{cases} \qquad 0 \le t \le 1$$

is an entropy solution.

Ex 3

$$u_1(x,t) = \begin{cases} 0, & x < \frac{t}{2}, \\ 1, & x > \frac{t}{2}. \end{cases}$$

is NOT an entropy solution.

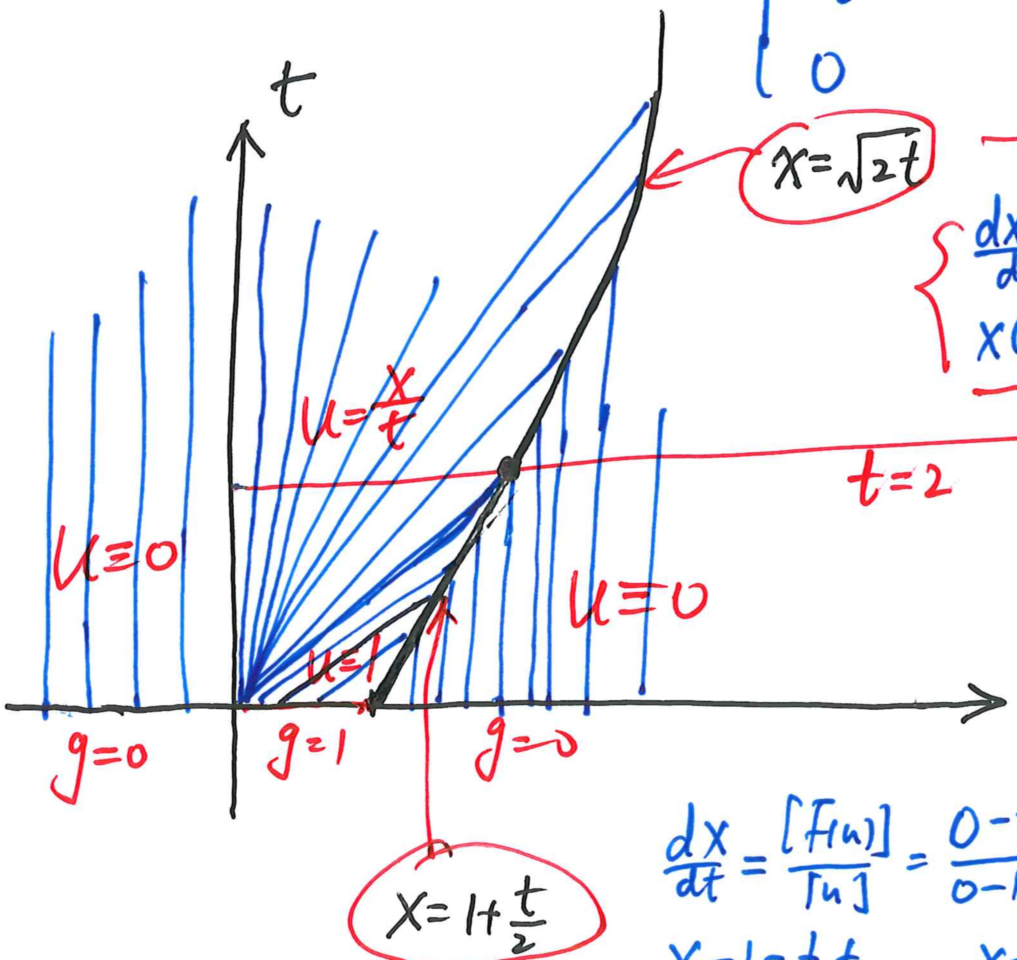
Ex 4
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ u\big|_{t=0} = g(x) = \begin{cases} 0 & x<0 \\ 1 & 0\le x\le 1 \\ 0 & x>1 \end{cases} \end{cases}$$

$\forall\, x_0 \in \mathbb{R}$

Characteristics
$$\begin{cases} X(t) = x_0 + g(x_0)\,t \\ u = g(x_0) \end{cases}$$

$0 \le t \le 2$.
$$u(x,t) = \begin{cases} 0 & x<0 \\ \frac{x}{t} & 0<x<t \\ 1 & t<x<1+\frac{t}{2} \\ 0 & x>1+\frac{t}{2} \end{cases}$$

$t>2$
$$u(x,t) = \begin{cases} 0 & x<0 \\ \frac{x}{t} & 0<x<\sqrt{2t} \\ 0 & x>\sqrt{2t} \end{cases}$$



$$\begin{cases} \frac{dX(t)}{dt} = \frac{\frac{1}{2}\left(\frac{X(t)}{t}\right)^2}{\frac{X(t)}{t}} = \frac{X(t)}{2t} \\ X(2) = 2 \end{cases}$$
$\rightarrow \quad X(t) = \sqrt{2t}$

$$\frac{dx}{dt} = \frac{[F(u)]}{[u]} = \frac{0-\frac{1}{2}}{0-1} = \frac{1}{2}$$
$x - 1 = \frac{1}{2}t, \quad x = 1 + \frac{1}{2}t$

§7.3 Lax-Oleinik formula

$$(*)\quad \begin{cases} u_t + F(u)_x = 0, \\ u|_{t=0} = g(x) \end{cases} \qquad F(0)=0 \ (\text{W.O.L.G})$$

Theorem 7.1 Assume $\begin{cases} F''(u) > 0. \\ g \in L^\infty(\mathbb{R}) \end{cases}$

Denote $\begin{cases} G = (F')^{-1} \\ L(q) = qG(q) - F(G(q)) = F^* \end{cases}$

$$\overset{\displaystyle ||}{\max_{p\in\mathbb{R}} \{pq - F(p)\}}$$

$\hookrightarrow$

(i) $\forall t>0$, $\exists$ a unique point $y(x,t)$ for all but at most countable many values of $x \in \mathbb{R}$ such that

$$\min_{y\in\mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + \int_0^y g(z)\,dz \right\} = tL\left(\frac{x-y(x,t)}{t}\right) + \int_0^{y(x,t)} g(z)\,dz$$

(ii) The mapping $x \longmapsto y(x,t)$ is nondecreasing

(iii). $\forall t > 0$

$$u(x,t) = \frac{\partial}{\partial x}\left[\min_{y\in\mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + \int_0^y g(z)\,dz \right\} \right] = G\left(\frac{x-y(x,t)}{t}\right)$$

for a.e. x.

(iv) $u(x,t)$ is an integral solution of $(*)$.

(v) $\exists$ $C>0$ s.t.

$$u(x+z,t)-u(x,t) \le \frac{C}{t} z, \quad \forall t>0,\ x,z\in\mathbb{R},\ z>0$$

§7.4. Entropy Solutions, Uniqueness

Entropy solutions. $u \in L^\infty(\mathbb{R}\times(0,\infty))$. of (*):

(i) $\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + F(u)\varphi_x)\,dx\,dt + \int_{-\infty}^\infty g(x)\varphi(x,0)\,dx = 0$

$\forall$ $\varphi \in C_c^1(\mathbb{R}\times[0,\infty))$

(ii) $u(x+z,t)-u(x,t) \le C(1+\frac{1}{t})z$

for some constant $C \ge 0$, a.e. $x, z \in \mathbb{R}$, $t>0$, $z>0$.

Theorem 7.2 (Uniqueness of entropy solutions).
If F is convex and smooth.

$\hookrightarrow$ $\exists$ 1 entropy solution of (*)

— up to a set of measure zero.

Riemann's Problem

$$u|_{t=0} = g(x) = \begin{cases} u_\ell, & \text{if } x<0, \\ u_r, & \text{if } x>0 \end{cases}$$

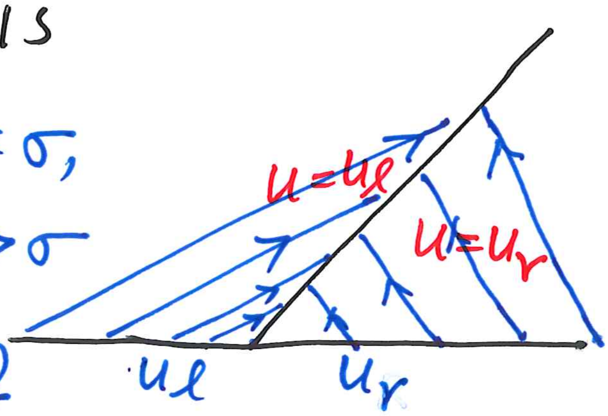
Assume: $F \in C^2$, $F''(u) > 0$, $G = (F')^{-1}$

Theorem 7.3 (Solution of the Riemann Problem)

(i) $u_\ell > u_r$: The unique entropy solution of the Riemann problem is

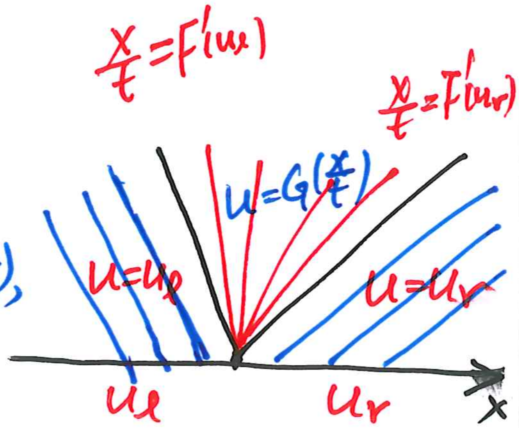
$$u(x,t) = \begin{cases} u_\ell, & \frac{x}{t} < \sigma, \\ u_r, & \frac{x}{t} > \sigma \end{cases}$$

$$\sigma = \frac{[F(u)]}{[u]} = \frac{F(u_r) - F(u_\ell)}{u_r - u_\ell}$$



(ii) $u_\ell < u_r$: The unique entropy solution of the Riemann Problem is

$$u(x,t) = \begin{cases} u_\ell, & \frac{x}{t} < F'(u_\ell), \\ G(\frac{x}{t}), & F'(u_\ell) < \frac{x}{t} < F'(u_r), \\ u_r, & \frac{x}{t} > F'(u_r) \end{cases}$$



§7.5 Large Time Behavior

$F \in C^2$, $F''(u) \geq \theta > 0$, $F(0) = 0$ (W.L.O.G)

Asymptotics in L^∞-norm: $g \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$

$\hookrightarrow$ $\exists$ $C > 0$, s.t.

$$\boxed{|u(x,t)| \leq \frac{C}{\sqrt{t}}} \quad \forall x \in \mathbb{R}, t > 0$$

Decay to N-Wave

$\operatorname{supp} g \subset\subset \mathbb{R}$,

$p = -2 \min_{y \in \mathbb{R}} \int_{-\infty}^{y} g \, dx > 0$,

$q = 2 \max_{y \in \mathbb{R}} \int_{y}^{\infty} g \, dx > 0$

$\hookrightarrow$ $\exists$ $C > 0$, s.t.

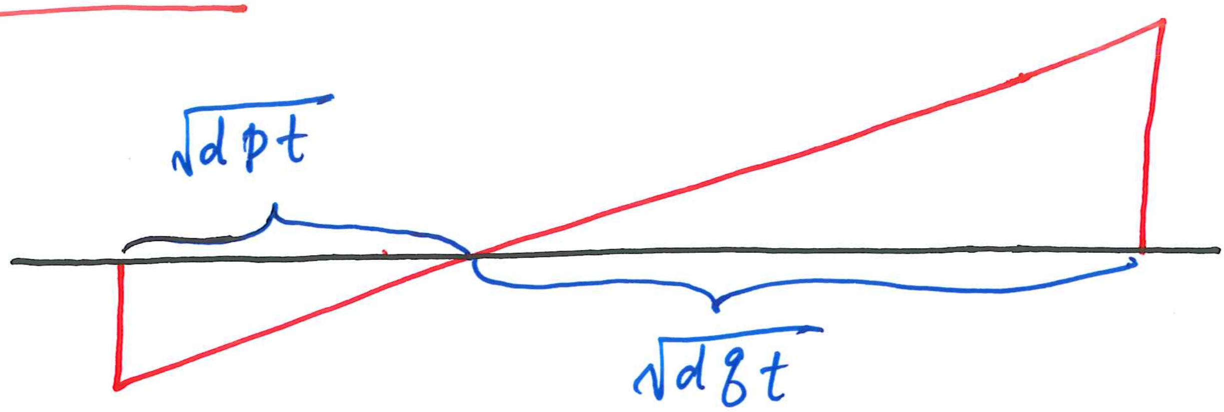
$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)| \, dx \leq \frac{C}{\sqrt{t}}, \quad \forall t > 0$$

where the N-wave:

$$N(x,t) = \begin{cases} \frac{1}{d}\left(\frac{x}{t} - \sigma\right), & \text{if } -\sqrt{dpt} < x - \sigma t \leq \sqrt{dqt} \\ 0, & \text{otherwise} \end{cases}$$

$\sigma = F'(0)$, $d = F''(0)$ $\longrightarrow$ $G'(\sigma) = \frac{1}{d}$

N-Wave



Read: { §3.3–3.4 in Evans's book
Chapter 11

More References

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2. C. M. Dafermos. Hyperbolic Conservation Laws in Continuum physics
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