

Part I. Hyperbolic Systems of
First-Order Equations

2. Linear Theory

Spaces of Functions

Evans. 2nd Edition
Pages 253-309

Space H^{-1} and H_0^1

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$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{H^1}$$

$$\left\{ u \mid \exists u_k \in C_0^\infty(\Omega) \text{ s.t. } \|(u_k - u, Du_k - Du)\|_{L^2} \xrightarrow{k \rightarrow \infty} 0 \right\}$$

$$H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$$

$$H^{-1}(\Omega) = (H_0^1(\Omega))^* \quad \text{Dual space}$$

$$\left\{ \begin{array}{l} f \in H^{-1}(\Omega) \Leftrightarrow |\langle f, u \rangle| < \infty \\ \forall u \in H_0^1(\Omega) \\ \|f\|_{H^{-1}(\Omega)} = \sup \left\{ |\langle f, u \rangle| \mid u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1 \right\} \end{array} \right.$$

Thm (Characterization of H^1)

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$$f \in H^1(\Omega)$$

$$\Rightarrow (i) \exists f^0, f^1, \dots, f^n \in L^2(\Omega) \text{ s.t.}$$

$$(*) \quad \langle f, v \rangle = \int_{\Omega} (f^0 v + \sum_{i=1}^n f^i v_{x_i}) dx$$

$$\forall v \in H_0^1(\Omega).$$

$$(ii) \quad \|f\|_{H^1(\Omega)} = \inf \left\{ \left(\int_{\Omega} \sum_{i=0}^n |f^i|^2 dx \right)^{1/2} \mid \begin{array}{l} f \text{ satisfies} \\ (*) \text{ for} \\ f^0, \dots, f^n \in L^2(\Omega) \end{array} \right\}$$

Notation We write

$$f = f^0 - \sum_{i=1}^n \frac{\partial}{\partial x_i} f^i$$

whenever (*) holds.

Weak derivatives

Spaces Involving Time

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X — real Banach space
with norm $\|\cdot\|_X$

Space $L^p(0, T; X)$, $1 \leq p \leq \infty$

$u: [0, T] \rightarrow X$ measurable functions

$$\left\{ \begin{array}{l} \|u\|_{L^p(0, T; X)} := \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}} < \infty \\ \|u\|_{L^\infty(0, T; X)} := \operatorname{ess\,sup}_{0 \leq t \leq T} \|u(t)\|_X < \infty \end{array} \right. \quad 1 \leq p < \infty$$

Space $C([0, T]; X)$:

$u: [0, T] \rightarrow X$ continuous function

$$\|u\|_{C([0, T]; X)} := \max_{0 \leq t \leq T} \|u(t)\|_X < \infty$$

Sobolev Space $W^{1,p}(0, T; X)$

//

$$\left\{ u \in L^p(0, T; X) \mid \|u\|_{W^{1,p}(0, T; X)} < \infty \right\}$$

$$\|u\|_{W^{1,p}(0, T; X)} = \begin{cases} \left(\int_0^T (\|u(t)\|_X^p + \|u'(t)\|_X^p) dt \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{0 \leq t \leq T} (\|u(t)\|_X + \|u'(t)\|_X) & p = \infty \end{cases}$$

$v = u' \in L^1(0, T; X)$ is the weak derivative of $u \in L^1(0, T; X)$ provided

$$\int_0^T \phi'(t) u(t) dt = - \int_0^T \phi(t) v(t) dt$$

$$\forall \phi \in C_c^\infty(0, T)$$

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Thm $u \in W^{1,p}(0, T; X)$ for some
 $1 \leq p \leq \infty$

\Rightarrow

(i) $u \in C([0, T]; X)$

(after possibly being redefined on a set
of measure zero)

$$(ii) \quad u(t) = u(s) + \int_s^t u'(\tau) d\tau$$

for all $0 \leq s \leq t \leq T$.

$$(iii) \quad \max_{0 \leq t \leq T} \|u(t)\|_X \leq \underbrace{C}_{\substack{\text{only} \\ T}} \|u\|_{W^{1,p}(0, T; X)}$$

$$\underline{\text{Thm}} \quad \left\{ \begin{array}{l} u \in L^2(0, T; H_0^1(\Omega)) \\ u' \in L^2(0, T; H^1(\Omega)) \end{array} \right.$$

\Rightarrow

(i) $u \in C([0, T]; L^2(\Omega))$

(after possibly being modified on a set of measure zero)

(ii) The mapping $t \mapsto \|u(t)\|_{L^2(\Omega)}^2$ absolutely continuous

$$\left\{ \begin{array}{l} \frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 = 2 \langle u'(t), u(t) \rangle \\ \text{a.e. } 0 \leq t \leq T \end{array} \right.$$

(iii) $\max_{0 \leq t \leq T} \|u(t)\|_{L^2(\Omega)}$

$$\leq \underbrace{C}_{\text{only}} \left(\|u\|_{L^2(0, T; H_0^1(\Omega))} + \|u'\|_{L^2(0, T; H^1(\Omega))} \right)$$

T

Thm (Mappings into Better Spaces)

- Ω open, bdd; $\partial\Omega$ smooth
- m — nonnegative integer
- $u \in L^2(0, T; H^{m+2}(\Omega))$
- $u' \in L^2(0, T; H^m(\Omega))$



(i) $u \in C([0, T]; H^{m+1}(\Omega))$

(after possibly being redefined on a set of measure zero)

(ii) $\max_{0 \leq t \leq T} \|u(t)\|_{H^{m+1}(\Omega)}$

$\leq C \left(\|u\|_{L^2(0, T; H^{m+2}(\Omega))} + \|u'\|_{L^2(0, T; H^m(\Omega))} \right)$

Exercise

Hyperbolic Systems of First-Order Equations

$$(*) \begin{cases} U_t + \sum_{j=1}^n B_j U_{x_j} = f, & \mathbb{R}^n \times (0, \infty) \\ U|_{t=0} = g \end{cases}$$

- $U = (u^1, \dots, u^m)^T \in \mathbb{R}^m$
- $B_j = B_j(x, t)$ — $m \times m$ Matrices.
- $f : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^m$ Given
- $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ Given.

$$U_t + \sum_{j=1}^n B_j(x,t) U_{x_j} = f$$

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Hyperbolicity: For each $y \in \mathbb{R}^n$

the $m \times m$ Matrix

$$B(x,t; y) = \sum_{j=1}^n y_j B_j(x,t)$$

is Diagonalizable for each $x \in \mathbb{R}^n, t \geq 0$

\Rightarrow For each x, y, t ,

$B(x,t; y)$ has m real eigenvalues

$$\lambda_1(x,t; y) \leq \lambda_2(x,t; y) \leq \dots \leq \lambda_m(y)$$

and corresponding m linearly

independent eigenvectors $\{r_j(x,t; y)\}_{j=1}^m$

* No hypothesis concerning $\{r_j(x,t; y)\}_{j=1}^m$

$$u_t + \sum_{j=1}^n B_j(x,t) u_{x_j} = f$$

Symmetric Hyperbolic Systems:

For each $x \in \mathbb{R}^n$, $t \geq 0$,

$B_j(x,t)$, $j=1, \dots, m$, are all symmetric $m \times m$ matrices

Strict Hyperbolicity:

For each $x, y \in \mathbb{R}^n$, $y \neq 0$, $t \geq 0$.

$$B(x,t; y) = \sum_{j=1}^n y_j B_j(x,t)$$

has m distinct real eigenvalues:

$$\lambda_1(x,t; y) < \lambda_2(x,t; y) < \dots < \lambda_m(x,t; y)$$

Motivation (B_j constant, $f=0$)

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Look for a Plane Wave Solution with the form

$$U(x, t) = w(y \cdot x - \sigma t) \underline{r}(y), \quad x \in \mathbb{R}^n, t \geq 0$$

for some direction $y \in \mathbb{R}^n$, velocity $\frac{\sigma}{|y|}$ ($\sigma \in \mathbb{R}$)
profile $w: \mathbb{R} \rightarrow \mathbb{R}$, where $\underline{r}(y) \in \mathbb{R}^m$

$$\Rightarrow \left[\frac{w'(y \cdot x - \sigma t)}{\sigma} \left(-\sigma I + \underbrace{\sum_{j=1}^n y_j B_j}_{B(y)} \right) \underline{r}(y) = 0 \right]$$

\Rightarrow $\underline{r}(y)$ is an eigenvector of $B(y)$
corresponding to the eigenvalue σ .

\hookrightarrow The Hyperbolicity Condition requires that

\exists m distinct plane wave solutions
for each direction y

with form

$$\left\{ (y \cdot x - \lambda_k(y)t) \underline{r}_k(y) \right\}_{k=1}^m$$

* The eigenvalues for $|y|=1$ are
called the wave speeds

Systems of First-Order
Symmetric Hyperbolic PDE
with Variable Coefficients

$$(*) \begin{cases} U_t + \sum_{j=1}^n B_j U_{x_j} = f & \mathbb{R}^n \times (0, T) \\ U|_{t=0} = g \end{cases}$$

- The matrices $B_j(x, t)$ are symmetric,
 $j=1, 2, \dots, n, x \in \mathbb{R}^n, 0 \leq t \leq T$

- $B_j \in C^2$ with

$$|B_j| + |D_{x,t} B_j| + |D_{x,t}^2 B_j| \leq C, \quad j=1, 2, \dots, n$$

- $\begin{cases} g \in H^1(\mathbb{R}^n; \mathbb{R}^m) \\ f \in H^1(\mathbb{R}^n \times (0, T); \mathbb{R}^m) \end{cases}$

Weak Solutions

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Bilinear Form: $U, V \in H^1(\mathbb{R}^n; \mathbb{R}^m)$

$$B[u, v; t] = \int_{\mathbb{R}^n} \sum_{j=1}^n (B_j u_{x_j}) \cdot v \, dx$$

$0 \leq t \leq T.$

Definition. We say $u \in L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m))$
with $u' \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$
is a weak solution of (*), provided

(i) $(u', v) + B[u, v; t] = (f, v)$
 $\forall v \in H^1(\mathbb{R}^n; \mathbb{R}^m)$
a.e. $0 \leq t \leq T.$

Inner product in $L^2(\mathbb{R}^n; \mathbb{R}^m)$

(ii) $u(0) = g.$

Then in Spaces involving time

$\hookrightarrow u \in C([0, T]; L^2(\mathbb{R}^n; \mathbb{R}^m))$

\Rightarrow The initial condition (ii)
makes sense

Vanishing Viscosity Method

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Construct approximate solutions

$$u^\varepsilon = u^\varepsilon(x, t)$$

by the parabolic system

$$(*) \begin{cases} u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + \sum_{j=1}^n B_j u_{x_j}^\varepsilon = f & \mathbb{R}^n \times (0, T) \\ u^\varepsilon|_{t=0} = g^\varepsilon = \eta_\varepsilon * g \end{cases}$$

Ideas.

1. $\forall \varepsilon > 0$, (*) has a unique smooth solution $u^\varepsilon = u^\varepsilon(x, t)$ s.t.

$$|u^\varepsilon| \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty.$$

2. ? $u^\varepsilon \longrightarrow u$ $\varepsilon \rightarrow 0$?

↑ Weak solution

? Which sense

Heat Equation

$$\begin{cases} W_t - \Delta W = f \\ W|_{t=0} = 0 \end{cases}$$

$$\Rightarrow W(x, t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) dy$$

$x \in \mathbb{R}^n, t > 0$

$$\max_{0 \leq t \leq T} \|W(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}$$

$$\leq C \|f\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))}.$$

$$W \in L^2(0, T; H^2(\mathbb{R}^n; \mathbb{R}^m))$$

$$W' \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$$

Formal. $\int U$ is smooth
 $\int U \rightarrow 0$ as $|x| \rightarrow \infty$ sufficiently rapidly.

$$\begin{aligned} \int_{\mathbb{R}^n} f^2 dx &= \int_{\mathbb{R}^n} (U_t - \Delta U)^2 dx \\ &= \int_{\mathbb{R}^n} (U_t^2 - 2\Delta U U_t + (\Delta U)^2) dx \\ &= \int_{\mathbb{R}^n} (U_t^2 + \underbrace{2DU \cdot DU_t}_{\frac{d}{dt}(|DU|^2)} + (\Delta U)^2) dx. \end{aligned}$$

$$\int_0^t \int_{\mathbb{R}^n} 2DU \cdot DU_t dx ds = \int_{\mathbb{R}^n} |DU|^2 dx \Big|_{s=0}^{s=t}$$

$$\begin{aligned} \int_{\mathbb{R}^n} (\Delta U)^2 dx &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} U_{x_i x_i} U_{x_j x_j} dx \\ &= - \sum_{i,j=1}^n \int_{\mathbb{R}^n} U_{x_i x_i x_j} U_{x_j} dx \\ &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} U_{x_i x_j} U_{x_i x_j} = \int_{\mathbb{R}^n} |D^2 U|^2 dx \end{aligned}$$

\Rightarrow

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} |DU|^2 dx + \int_0^T \int_{\mathbb{R}^n} (U_t^2 + |D^2 U|^2) dx dt \\ \leq C \int_0^T \int_{\mathbb{R}^n} f^2 dx dt. \end{aligned}$$

Thm (Existence of Approximate Solutions) ³⁵

For each $\varepsilon > 0$, \exists 1 u^ε of (A) with

$$(V) \begin{cases} u^\varepsilon \in L^2(0, T; H^3(\mathbb{R}^n; \mathbb{R}^m)) \\ u^{\varepsilon'} \in L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m)) \end{cases}$$

Proof.

1. Set $X = L^\infty(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m))$.

For each $v \in X$, consider the Cauchy problem

$$\begin{cases} \underline{u_t - \varepsilon \Delta u = f - \sum_{j=1}^n B_j v_{x_j}} \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m)) \\ u|_{t=0} = g^\varepsilon \end{cases}$$

$$\Rightarrow \exists 1 \text{ solution } \begin{cases} u \in L^2(0, T; H^2(\mathbb{R}^n; \mathbb{R}^m)) \\ u' \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m)) \end{cases}$$

Similarly, let $\tilde{v} \in X$ and let \tilde{u} solve

$$\begin{cases} \tilde{u}_t - \varepsilon \Delta \tilde{u} = f - \sum_{j=1}^n B_j \tilde{v}_{x_j} & \mathbb{R}^n \times (0, T) \\ \tilde{u}|_{t=0} = g^\varepsilon \end{cases}$$

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2. Set $\hat{u} = u - \tilde{u}$, $\hat{v} = v - \tilde{v}$

$$\Rightarrow \begin{cases} \hat{u}_t - \varepsilon \Delta \hat{u} = - \sum_{j=1}^n B_j \hat{v}_{x_j} & \mathbb{R}^n \times (0, T) \\ \hat{u}|_{t=0} = 0 \end{cases}$$

Linear Theory

$$\begin{aligned} & \hookrightarrow \max_{0 \leq t \leq T} \|\hat{u}(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} \\ & \leq C(\varepsilon) \left\| \sum_{j=1}^n B_j \hat{v}_{x_j} \right\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))} \\ & \leq C(\varepsilon) \|\hat{v}\|_{L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m))} \\ & \leq C(\varepsilon) T^{\frac{1}{2}} \max_{0 \leq t \leq T} \|\hat{v}(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)}. \end{aligned}$$

$$\Rightarrow \|\hat{u}\|_X \leq C(\varepsilon) T^{\frac{1}{2}} \|\hat{v}\|_X$$

3. If T is so small that

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$$C(\varepsilon)T^{1/2} \leq \frac{1}{2}.$$

(2.)

\Rightarrow

$$\|u - \tilde{u}\|_X \leq \frac{1}{2} \|v - \tilde{v}\|_X.$$

Banach Fixed Point Thm

\hookrightarrow The mapping $v \mapsto u$

has a unique fixed pt.

\Rightarrow

$$u = u^\varepsilon \text{ solves (A) if } C(\varepsilon)T^{1/2} = \frac{1}{2}.$$

If $T > \frac{1}{2}C$, then we choose $T_1 C(\varepsilon)^2 = \frac{1}{4}$.

and repeat the above argument on the intervals

$[0, T_1], [T_1, 2T_1], \dots$

the assertion (v) follows from the

Regularity theory of the nonhomogeneous heat equation.

Energy Estimates Uniform in $\varepsilon > 0$ 38

$\varepsilon \rightarrow 0$? We need some uniform estimates.

Thm (Energy Estimates). $\exists C \sim n, B_j$ s.t.

$$\begin{aligned} \max_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} + \|u^{\varepsilon'}\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))} \\ \leq C \left(\|g\|_{H^1(\mathbb{R}^n; \mathbb{R}^m)} + \|f\|_{L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m))} \right. \\ \left. + \|f'\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))} \right) \end{aligned}$$

for each $\varepsilon > 0$.

Proof

$$\begin{aligned} 1. \frac{d}{dt} \left(\frac{1}{2} \|u^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right) &= (u^\varepsilon, u_t^\varepsilon) \\ &= (u^\varepsilon, f - \sum_{j=1}^n B_j u_{x_j}^\varepsilon + \varepsilon \Delta u^\varepsilon) \\ &= (u^\varepsilon, f) - \underbrace{(u^\varepsilon, \sum_{j=1}^n B_j u_{x_j}^\varepsilon)}_{\| \|} + \frac{\varepsilon (u^\varepsilon, \Delta u^\varepsilon)}{\| \|} \\ &\leq \|u^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \|f\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 - \varepsilon \|Du^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq 0 \end{aligned}$$

2. If $U \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$

39.

$$(U, \sum_{j=1}^n B_j U_{x_j}) = \int_{\mathbb{R}^n} (B_j U_{x_j}) \cdot U \, dx$$

$$\int_{\mathbb{R}^n} ((B_j U)_{x_j} \cdot U - (B_j, x_j U) \cdot U) \, dx$$

$$\int_{\mathbb{R}^n} ((B_j U) \cdot U)_{x_j} \, dx - \int_{\mathbb{R}^n} (B_j, x_j U) \cdot U \, dx$$

$$- \int_{\mathbb{R}^n} (B_j U) \cdot U_{x_j} \, dx$$

// B_j symmetric

$$- \int_{\mathbb{R}^n} (B_j U_{x_j}) \cdot U \, dx = 0$$

$$\Rightarrow (U, \sum_{j=1}^n B_j U_{x_j}) = \left[\frac{1}{2} \int_{\mathbb{R}^n} ((B_j U) \cdot U)_{x_j} \, dx \right] - \frac{1}{2} \int_{\mathbb{R}^n} (B_j, x_j U) \cdot U \, dx$$

$$\Rightarrow |(U, \sum_{j=1}^n B_j U_{x_j})| \leq \frac{1}{2} \int_{\mathbb{R}^n} |(B_j, x_j U) \cdot U| \, dx$$

$$\leq C \|U\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

By Approximation

$$|(U^\varepsilon, \sum_{j=1}^n B_j U_{x_j}^\varepsilon)| \leq C \|U^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

3. (1) + (2)

↳

$$\frac{d}{dt} (\|u^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2)$$

$$\leq C \left(\|u^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \|f\|_{L(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))}^2 \right)$$

Gronwall
Ineq.

$$\max_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

$$\leq C \left(\underbrace{\|g^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2}_{\lambda} + \|f\|_{L(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))}^2 \right)$$

$$\|g\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

$$\leq C \left(\|g\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \|f\|_{L(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))}^2 \right)$$

4. Fix $k \in \{1, 2, \dots, n\}$.

Write $U^k = U_{x_k}^\varepsilon$

$$\Rightarrow \begin{cases} (U^k)_t - \varepsilon \Delta U^k + \sum_{j=1}^n B_j U_{x_j}^k \\ = f_{x_k} - \sum_{j=1}^n B_{j, x_k} U_{x_j}^\varepsilon, \mathbb{R}^n \times (0, T). \\ U^k|_{t=0} = g_{x_k}^\varepsilon \end{cases}$$

Reasoning as above, we find

$$\begin{aligned} (*)_k \frac{d}{dt} (\|U^k\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2) \\ \leq C (\|DU^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \|Df\|_{L^2(\mathbb{R}^n; M^{m \times n})}^2) \end{aligned}$$

$$\frac{\sum_{k=1}^n (*)_k}{\hookrightarrow} \frac{d}{dt} (\|DU^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2)$$

$$\stackrel{\text{Gronwall}}{\leq} C (\|DU^\varepsilon\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \|Df\|_{L^2(0, T; L^2(\mathbb{R}^n; M^{m \times n}))}^2)$$

$$\stackrel{\text{Ineq.}}{\implies} \begin{aligned} \max_{0 \leq t \leq T} \|DU^\varepsilon(t)\|_{L^2(\mathbb{R}^n; M^{m \times n})}^2 \\ \leq C (\|Dg\|_{L^2(\mathbb{R}^n; M^{m \times n})}^2 + \|Df\|_{L^2(0, T; L^2(\mathbb{R}^n; M^{m \times n}))}^2) \end{aligned}$$

5. Next, Set $U = U^{\varepsilon'}$

$$\begin{cases} U_t - \varepsilon \Delta U + \sum_{j=1}^n B_j U_{x_j} = f_t - \sum_{j=1}^n B_{j,t} U_{x_j}, \\ U|_{t=0} = f - \sum_{j=1}^n B_j g_{x_j}^{\varepsilon} + \varepsilon \Delta g^{\varepsilon}, \end{cases} \mathbb{R}^n \times (0, T),$$

\Rightarrow

$$\max_{0 \leq t \leq T} \|U^{\varepsilon'}(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

$$\leq C \left(\|Dg\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \varepsilon^2 \|D^2 g^{\varepsilon}\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \right. \\ \left. + \|f(0)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 + \|f'\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))}^2 \right)$$

\wedge

$$C \left(\|f\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))}^2 \right. \\ \left. + \|f'\|_{L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))}^2 \right).$$

$$\|D^2 g^{\varepsilon}\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 \leq \frac{C}{\varepsilon^2} \|Dg\|_{L^2(\mathbb{R}^n; M^{m \times n})}^2$$

$$g^{\varepsilon} = \chi_{\varepsilon} * g$$

\square

Existence of Weak Solutions

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Thm. \exists a weak solution of (*).

Proof.

1. Energy Estimates

$$\begin{aligned} \hookrightarrow \exists & \begin{cases} \varepsilon_k \rightarrow 0 \\ u \in L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m)) \\ \text{with } u' \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m)) \end{cases} \\ \text{s.t.} & \end{aligned}$$

$$\begin{cases} u^{\varepsilon_k} \rightharpoonup u \text{ weakly in } L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m)) \\ u^{\varepsilon_k'} \rightharpoonup u' \text{ in } L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m)) \end{cases}$$

2. Choose $v \in C^1([0, T]; H^1(\mathbb{R}^n; \mathbb{R}^m))$.

\hookrightarrow

$$\int_0^T (u^{\varepsilon'}', v) dt + \int_0^T B[u^{\varepsilon}, v; t] dt$$

$$= \int_0^T (f, v) dt + \varepsilon \int_0^T (\Delta u^{\varepsilon}, v) dt$$

(V-1)

$$\underline{\varepsilon = \varepsilon_k \rightarrow 0}$$

$$\begin{aligned} \hookrightarrow \int_0^T (u', v) dt + \int_0^T B[u, v; t] dt \\ = \int_0^T (f, v) dt \end{aligned}$$

(V-2).

Valid for all $v \in C([0, T], H^1(\mathbb{R}^n; \mathbb{R}^m))$.

$$\Rightarrow (u', v) + B[u, v; t] = (f, v)$$

a.e. t for each $v \in H^1(\mathbb{R}^n; \mathbb{R}^m)$.3. Assume now $v(T) = 0$

$$\begin{aligned} (V-1) \Rightarrow - \int_0^T (u^\varepsilon, v') dt + \int_0^T B[u^\varepsilon, v; t] dt \\ = \int_0^T (f, v) dt + \varepsilon \int_0^T (\Delta u^\varepsilon, v) dt \\ + (g^\varepsilon, v(0)) \end{aligned}$$

$$\underline{\varepsilon = \varepsilon_k \rightarrow 0}$$

$$\begin{aligned} \hookrightarrow - \int_0^T (u, v') dt + \int_0^T B[u, v; t] dt \\ = \int_0^T (f, v) dt + (g, v(0)) \end{aligned}$$

(V-2)

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$$\begin{aligned} \hookrightarrow & -\int_0^T (u, v') dt + \int_0^T B[u, v; t] dt \\ & = \int_0^T (f, v) dt + (u(0), v(0)) \end{aligned}$$

$$\Rightarrow (g, v(0)) = (u(0), v(0))$$

$$\forall v(0) \in H^1(\mathbb{R}^n; \mathbb{R}^m)$$

$$\Rightarrow u(0) = g.$$

Uniqueness of Weak Solutions

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Thm A weak solution is unique.

Proof. It suffices to show that the only weak solution with $f = g = 0$ is $u \equiv 0$.

Definition of Weak solutions

$$(u', v) + B[u, v; t] = 0. \quad \forall v \in H^1(\mathbb{R}^n; \mathbb{R}^m) \\ \text{for a.e. } 0 \leq t \leq T.$$

Choose $v = u$

\hookrightarrow

$$(u', u) + B[u, u; t] = 0$$

$$\frac{d}{dt} (\|u(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2) \leq C \|u(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

$$\Rightarrow \frac{d}{dt} (\|u(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2) \leq C \|u(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2$$

Gronwall
Ineq. \rightarrow $\|u(t)\|_{L^2(\mathbb{R}^n; \mathbb{R}^m)}^2 = 0, \quad 0 \leq t \leq T.$

Remarks

1. The methods developed above also apply more general systems with the form:

$$(**) \begin{cases} B_0 U_t + \sum_{j=1}^n B_j U_{x_j} = f \\ B_j(x, t) \text{ are symmetric, } j=0, 1, \dots, n. \end{cases}$$

2. Symmetric hyperbolic systems of form (**) generalize the second-order hyperbolic PDE.

$$U_{tt} - \sum_{i,j=1}^n a_{ij} U_{x_i x_j} = 0, \quad a_{ij} = a_{ji}$$

$$U = (U^1, \dots, U^{n+1}) = (U_{x_1}, U_{x_2}, \dots, U_{x_n}, U_t)$$

$$B_0 = \begin{pmatrix} a_{11} & \dots & a_{1n} & 0 \\ & \ddots & & \vdots \\ & & a_{nn} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}_{(n+1) \times (n+1)}$$

$$B_j = \begin{pmatrix} 0 & \dots & 0 & -a_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -a_{nj} \\ -a_{1j} & \dots & -a_{nj} & 0 \end{pmatrix}_{(n+1) \times (n+1)} \quad j=1, \dots, n.$$

Hyperbolic Systems

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With Constant Coefficients

$$U_t + \sum_{j=1}^n B_j U_{x_j} = 0 \quad \mathbb{R}^n \times (0, \infty)$$

- $U = (U^1, \dots, U^m)^T \in \mathbb{R}^m$
- B_j — $m \times m$ constant Matrices

Hyperbolicity: For each $y \in \mathbb{R}^n$

the $m \times m$ Matrix $B(y) := \sum_{j=1}^n y_j B_j$ is Diagonalizable

$\Rightarrow B(y)$ has m real eigenvalues

$$\lambda_1(y) \leq \lambda_2(y) \leq \dots \leq \lambda_m(y)$$

and corresponding m linearly independent eigenvectors $\{r_j(y)\}_{j=1}^m$.

* No hypothesis concerning $\{r_j(y)\}_{j=1}^m$

* No assumption of symmetry for $\{B_j\}_{j=1}^m$

Fourier Transform Methods

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Fourier Transform on L^1 : $u \in L^1(\mathbb{R}^n)$

$$\hat{u}(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix \cdot y} u(x) dx, \quad y \in \mathbb{R}^n$$

The Inverse Fourier Transform

$$\check{u}(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot y} u(x) dx, \quad y \in \mathbb{R}^n$$

? on L^2 ?

Thm (Plancherel's Thm): $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

$$\Rightarrow \begin{cases} \hat{u}, \check{u} \in L^2(\mathbb{R}^n) \\ \|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \end{cases}$$

Proof.

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1. Basic Facts

$$(i) \quad u, w \in L^1(\mathbb{R}^n) \Rightarrow \widehat{u}, \widehat{w} \in L^\infty(\mathbb{R}^n)$$

$$(ii) \quad \int_{\mathbb{R}^n} u(x) \widehat{w}(x) dx = \int_{\mathbb{R}^n} \widehat{u}(y) w(y) dy$$

$$(iii) \quad \int_{\mathbb{R}^n} e^{i x \cdot y - t |x|^2} dx = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|y|^2}{4t}}, \quad t > 0$$

$$\hookrightarrow \text{for } u_\varepsilon(x) = e^{-\varepsilon |x|^2}, \quad \varepsilon > 0$$

$$\Rightarrow \widehat{u}_\varepsilon(x) = \frac{e^{-\frac{|y|^2}{4\varepsilon}}}{(2\varepsilon)^{\frac{n}{2}}}$$

2. For $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, Set $v(x) := \overline{u(-x)}$.

Define $w := u * v \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

$$\widehat{w} = (2\pi)^{\frac{n}{2}} \widehat{u} \widehat{v} \in L^\infty(\mathbb{R}^n). \quad (\text{check?})$$

Also

$$\widehat{v}(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i x \cdot y} \overline{u(-x)} dx = \overline{\widehat{u}(y)}$$

$$\Rightarrow \widehat{w} = (2\pi)^{\frac{n}{2}} |\widehat{u}|^2$$

3. w is continuous

$$\Rightarrow \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \underbrace{\Phi_\varepsilon(x)}_{\parallel} w(x) dx = (2\pi)^{\frac{n}{2}} w(0)$$

$$\underbrace{\frac{1}{(2\varepsilon)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\varepsilon}}}_{\parallel} \underbrace{\hat{v}_\varepsilon(x)}_{\wedge}$$

$$\int_{\mathbb{R}^n} \underbrace{\Phi_\varepsilon(x)}_{\parallel} dx = (2\pi)^{\frac{n}{2}}$$

$$\downarrow$$

$$(2\pi)^{\frac{n}{2}} \delta_0(x)$$

(ii) \Rightarrow

$$\int_{\mathbb{R}^n} \hat{w}(y) \underbrace{e^{-\varepsilon|y|^2}}_{\parallel} dy = \int_{\mathbb{R}^n} w(y) \hat{v}_\varepsilon(y) dy$$

$$\int_{\mathbb{R}^n} \hat{w}(y) dy \stackrel{\varepsilon \rightarrow 0}{=} (2\pi)^{\frac{n}{2}} w(0)$$

$\parallel \hat{w}$ is summable

$$(2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} |\hat{u}|^2 dy \stackrel{\parallel}{=} (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} |u|^2 dx$$

* The proof for \hat{u} is similar.

Plancherel's Thm

↳ Definition of Fourier Transform on L^2

For $u \in L^2(\mathbb{R}^n)$. Choose $\{u_k\}_{k \in \mathbb{N}} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$
s.t. $u_k \rightarrow u$ in $L^2(\mathbb{R}^n)$

⇒
$$\|\widehat{u_k} - \widehat{u_j}\|_{L^2(\mathbb{R}^n)} = \|\widehat{u_k - u_j}\|_{L^2(\mathbb{R}^n)}$$
$$= \|u_k - u_j\|_{L^2(\mathbb{R}^n)} \xrightarrow{k, j \rightarrow \infty} 0$$

⇒ $\exists \widehat{u} \in L^2$ s.t.
$$\widehat{u_k} \rightarrow \widehat{u} \text{ in } L^2(\mathbb{R}^n)$$

Define

$$\widehat{u} = \lim_{k \rightarrow \infty} \widehat{u_k} \text{ in } L^2(\mathbb{R}^n)$$

↗ choice of $\{u_k\}_{k \in \mathbb{N}}$

* Similarly for \check{u}

Well-defined

Useful Properties of Fourier Transform

$$u, v \in L^2(\mathbb{R}^n)$$

$$\Rightarrow (i) \int_{\mathbb{R}^n} u \bar{v} dx = \int_{\mathbb{R}^n} \hat{u} \overline{\hat{v}} dy$$

$$(ii) \widehat{D^\alpha u} = (iy)^\alpha \hat{u}$$

for each multiindex α such that

$$D^\alpha u \in L^2(\mathbb{R}^n)$$

$$(iii) \text{ If } u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

$$\Rightarrow (u * v)^\wedge = (2\pi)^{\frac{n}{2}} \hat{u} \hat{v}$$

$$(iv) (\hat{u})^\vee = u$$

Exercise

Space $H^k(\mathbb{R}^n)$, k - integer

$$H^k(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \frac{(1+|y|^k) \widehat{u}(y)}{} \in L^2(\mathbb{R}^n) \right\}$$



$$\boxed{\begin{array}{l} D^\alpha u \in L^2(\mathbb{R}^n) \\ \forall |\alpha| \leq k. \end{array}}$$

Extension to

Space $H^s(\mathbb{R}^n)$: $0 < s < \infty$

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid (1+|y|^s) \widehat{u}(y) \in L^2(\mathbb{R}^n) \right\}$$

Norm:

$$\|u\|_{H^s(\mathbb{R}^n)} = \|(1+|y|^s) \widehat{u}\|_{L^2(\mathbb{R}^n)}$$

$\hookrightarrow H^s(\mathbb{R}^n)$, $0 < s < \infty$, are Hilbert spaces

Cauchy Problem

$$\begin{cases} U_t + \sum_{j=1}^n B_j U_{x_j} = 0 & \mathbb{R}^n \times (0, \infty) \\ U|_{t=0} = g: \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{given} \end{cases}$$

Thm $g \in H^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k > \frac{n}{2} + m$

\Rightarrow The Cauchy problem has a unique solution

$$U \in C^1([0, \infty); \mathbb{R}^m).$$

Proof

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1. First assume u is a smooth solution

Apply the Fourier Transform in x for
fixed t

$$\widehat{u} = (\widehat{u}^1, \dots, \widehat{u}^m)^T$$

$$\Rightarrow \begin{cases} \widehat{u}_t + i B(y) \widehat{u} = 0, & \mathbb{R}^n \times (0, \infty) \\ \widehat{u}|_{t=0} = \widehat{g} \end{cases}$$

$$\Rightarrow \widehat{u}(y, t) = e^{-itB(y)} \widehat{g}(y)$$

$y \in \mathbb{R}^n, t \geq 0.$

$$\Rightarrow (*) \quad u(x, t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot y} e^{-itB(y)} \widehat{g}(y) dy$$

$x \in \mathbb{R}^n, t \geq 0$

* Necessary Condition

If Verified \rightarrow Uniqueness.

2. We have derived formula (*)
 Under the assumption that U is a
 smooth solution.

We now verify that, in fact,
 the function U defined by (*)
 is a solution

when $g \in H^k(\mathbb{R}^n; \mathbb{R}^m)$

\Downarrow

$$\boxed{\begin{aligned} \exists f \in L^2(\mathbb{R}^n; \mathbb{R}^m) \text{ s.t.} \\ |\widehat{g}(y)| \leq C (1+|y|^k)^{-1} |f(y)|, y \in \mathbb{R}^n \end{aligned}}$$

Main Point:

Estimate $\| e^{-itB(y)} \|$

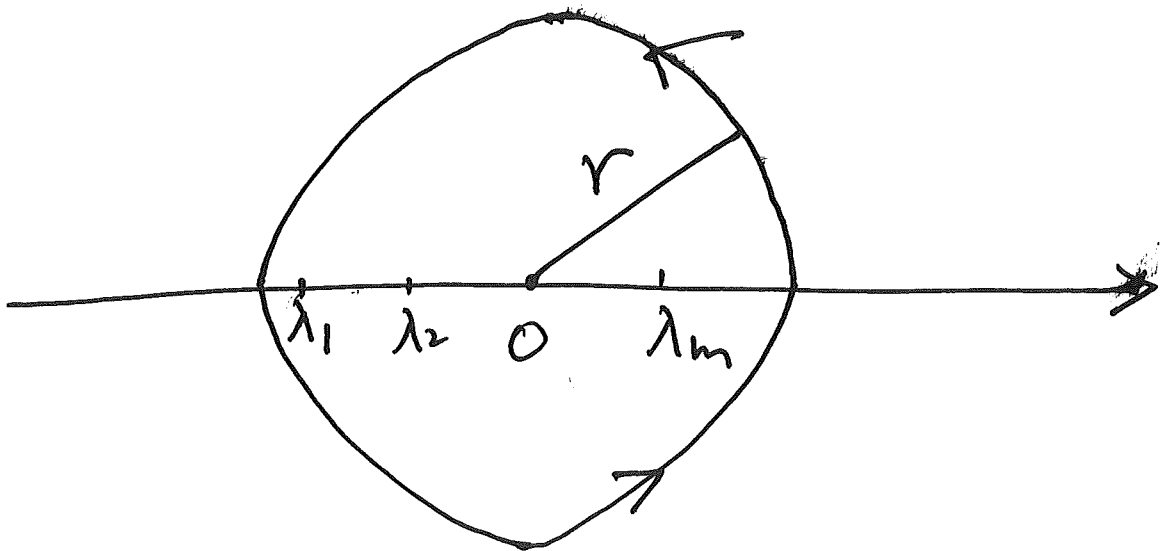
\hookrightarrow Convergence of Integral in (*)

3. Fix $y \in \mathbb{R}^n$

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Claim:
$$e^{-itB(y)} = \frac{1}{2\pi i} \int_{\Gamma} e^{-it\zeta} (\zeta I - B(y))^{-1} d\zeta$$

$\Gamma \parallel A(t, y)$



$\Gamma = \partial B(0, r), \quad r \gg 1.$

?
$$\begin{cases} \left(\frac{d}{dt} + iB(y)\right) A(t, y) = 0 \\ A(0, y) = I \end{cases}$$

$$\forall x \in \mathbb{R}^m$$

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$$B(y) A(t, y) x = \frac{1}{2\pi i} \int_{\Gamma} e^{-it\zeta} \underbrace{B(y)}_{\parallel} (\zeta I - B(y))^{-1} x d\zeta$$

$$= \frac{1}{2\pi i} \int_{\Gamma} e^{-it\zeta} (\zeta (\zeta I - B(y))^{-1} x - x) d\zeta$$

$$\parallel \int_{\Gamma} e^{-it\zeta} d\zeta = 0$$
$$\frac{1}{2\pi i} \int_{\Gamma} e^{-it\zeta} \zeta (\zeta I - B(y))^{-1} x d\zeta$$

$$\parallel -\frac{1}{i} \frac{d}{dt} A(t, y) x. \quad \forall x \in \mathbb{R}^m$$

$$\Rightarrow \left(\frac{d}{dt} + i B(y) \right) A(t, y) = 0$$

$$\forall x \in \mathbb{R}^m$$

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$$\begin{aligned} A(\infty, y)x &= \frac{1}{2\pi i} \int_{\Gamma} \underbrace{(\lambda I - B(y))^{-1}}_{\frac{1}{\lambda} I + \frac{B(y)(\lambda I - B(y))^{-1}}{\lambda}} x \, d\lambda \\ &= x + \frac{1}{2\pi i} \int_{\Gamma} \frac{B(y)(\lambda I - B(y))^{-1} x}{\lambda} \, d\lambda \end{aligned}$$

$$\text{Set } w := (\lambda I - B(y))^{-1} x$$

$$\Rightarrow \|w\|_{\mathcal{P}} \leq \frac{C}{|\lambda|}, \quad C \sim x, y$$

(Exercise).

$$\Rightarrow \frac{1}{2\pi i} \int_{\Gamma} \frac{B(y)(\lambda I - B(y))^{-1} x}{\lambda} \, d\lambda \xrightarrow{r \rightarrow \infty} 0$$

$$\Rightarrow A(\infty, y)x = x$$

$$\Rightarrow A(\infty, y) = I$$

\hookrightarrow claim

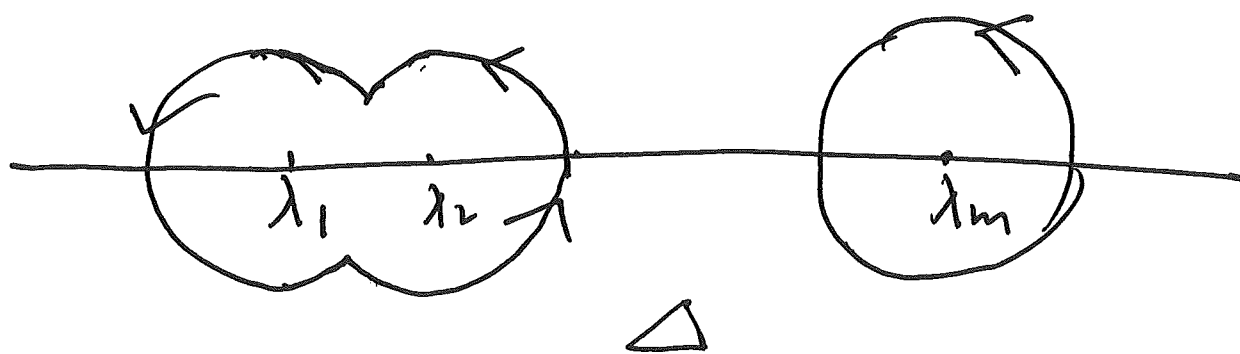
4. Define a new path Δ in the complex plane:

Fix y , draw circles

$B_i = B_i(\lambda_i(y), 1)$ of radius 1 centered at $\lambda_i(y)$, $i=1, 2, \dots, m$

Take Δ to be the bdry of $\bigcup_{k=1}^m B_k$

traversed counterclockwise



Deform the path Γ into Δ to obtain

$$e^{-itB(y)} = \frac{1}{2\pi i} \int_{\Delta} e^{-itz} (zI - B(y))^{-1} dz$$

$$(\lambda I - B(y))^{-1} = \frac{\text{Cof}(\lambda I - B(y))^T}{\det(\lambda I - B(y))}$$

The cofactor matrix

$$|\det(\lambda I - B(y))| = \left| \prod_{k=1}^m (\lambda - \lambda_k(y)) \right| \geq 1$$

if $\lambda \in \Delta$

$$\|\text{Cof}(\lambda I - B(y))^T\| \leq C \left(1 + |\lambda|^{m-1} + \|B(y)\|^{m-1} \right)$$

$$\leq C (1 + |y|^{m-1})$$

$$\uparrow |\lambda_k(y)| \leq C|y|, \quad k=1, 2, \dots, m.$$

$$\left| e^{-i\lambda t} \right| \leq e^t, \quad \forall \lambda \in \Delta$$

$$\begin{aligned} \Rightarrow \|e^{-i t B(y)}\| &\leq \frac{1}{2\pi} \left\| \int_{\Delta} e^{-i\lambda t} (\lambda I - B(y))^{-1} d\lambda \right\| \\ &\leq C e^t (1 + |y|^{m-1}), \quad \forall y \in \mathbb{R}^n \end{aligned}$$

$$\begin{aligned}
& 5. \left| \int_{\mathbb{R}^n} e^{i x \cdot y} e^{-i t B(y)} \widehat{g}(y) dy \right| \\
& \leq C \int_{\mathbb{R}^n} \| e^{-i t B(y)} \| (1+|y|^k)^{-1} |f(y)| dy \\
& \leq C e^t \int_{\mathbb{R}^n} |f(y)| (1+|y|^{m+1}) (1+|y|^k)^{-1} dy \\
& \leq C \|f\|_{L^2} \underbrace{\int_{\mathbb{R}^n} \frac{dy}{1+|y|^{2(k-m+1)}}}_{\substack{\wedge \\ \infty}}
\end{aligned}$$

$$\begin{array}{l}
\wedge \\
\infty
\end{array} \text{ if } 2(k-m+1) > n$$



$$\boxed{k > \frac{n}{2} + m - 1}$$

Similarly

$$|\nabla_x u(x,t)| < \infty \iff$$

$$\boxed{k > \frac{n}{2} + m}$$

$$\implies u \in C^1([0, \infty); \mathbb{R}^m) \lll$$

Remarks

① Uniqueness: $g \equiv 0 \Rightarrow u \equiv 0$

② Continuous Dependence.

$$g_n \rightarrow g \quad \text{in } H^k(\mathbb{R}^n; \mathbb{R}^m)$$

$$k > \frac{n}{2} + m$$

$$\Rightarrow u_n(x, t) \rightarrow u(x, t) \quad \text{in } C^1([0, \infty); \mathbb{R}^m)$$

③ We don't need

* Symmetry of the matrices $\{B_j\}_{j=1}^m$

* strict Hyperbolicity

$$\lambda_1(y) < \lambda_2(y) < \dots < \lambda_m(y)$$

$$\forall y \in \mathbb{R}^n$$

Exercise Read Lax: Chapters 2-3

pages 5-36

Higher-Order Hyperbolic Equations 65

With Constant Coefficients

$$(*) \begin{cases} P(D, \tau) u = 0 \\ \tau^k u = g_k(x) \end{cases}$$

$$D = (D_1, \dots, D_n) = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \quad \tau = \frac{\partial}{\partial t}$$

$$P(D, \tau) = \tau^m + P_1(D) \tau^{m-1} + \dots + P_m(D)$$

$P_k(D)$ is a polynomial of degree $\leq k$
in D_1, \dots, D_n .

Gårding's Hyperbolicity Condition.

$\exists c \in \mathbb{R}$ s.t

$P(i\xi, i\lambda) \neq 0$ for all $\xi \in \mathbb{R}^n$

and all λ with $\text{Im } \lambda \leq -c$.



All of the m roots λ of $P(i\xi, i\lambda) = 0$

lies in one and the same half plane $\text{Im } \lambda > -c$
of the complex number plane for all real $\xi \in \mathbb{R}^n$

See

Fritz John: PDE, P143-157.

Springer-Verlag: 1982

Solvability of (*)

Via Fourier Transform Methods.

* Higher-Order Hyperbolic Equations

\Rightarrow First-Order Hyperbolic Systems